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ONE-SPEED NEUTRON TRANSPORT PROBLEMS IN PLANE GEOMETRY

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CHAPTER I

INTRODUCTION

The macroscopic behavior of neutrons in a system is governed
by the linearized Boltzmann or transport equation

$$\left[\frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla} + v\,\sigma(\underline{r},\underline{v}) \right] \psi(\underline{r},\underline{v},t) =$$

$$\int v'\,\sigma(\underline{v}' \to \underline{v},\underline{r})\,\psi(\underline{r},\underline{v}',t)\,d^3v' + q_c(\underline{r},\underline{v},t) \qquad (1.1)$$

where the notation is that of Reference 1. It will be assumed that
all neutrons, on the average, are of one speed and that the medium is
homogeneous. With these assumptions, the time-independent equation is

$$\left[\underline{\Omega} \cdot \underline{\nabla} + 1 \right] \psi(\underline{r},\underline{\Omega}) = c \int f(\underline{\Omega}' \to \underline{\Omega})\,\psi(\underline{r},\underline{\Omega}')\,d\underline{\Omega}' \qquad (1.2)$$

where distances are measured in units of mean free paths and c is the
mean number of secondaries per collision. The scattering function
$f(\underline{\Omega}' \to \underline{\Omega})$ is defined so that $f(\underline{\Omega}' \to \underline{\Omega})d\underline{\Omega}$ is the probability that
a neutron scattered from direction $\underline{\Omega}'$ will have velocity direction
in $d\underline{\Omega}$ about $\underline{\Omega}$ after the collision. For an isotropic medium
$f(\underline{\Omega}' \to \underline{\Omega}) = f(\underline{\Omega}' \cdot \underline{\Omega})$ and we write

$$f(\underline{\Omega}' \cdot \underline{\Omega}) = \sum_k b_k\, P_k(\underline{\Omega}' \cdot \underline{\Omega}) \qquad (1.3)$$

truncating the expansion after N terms. Assuming a one-dimensional
geometry and symmetry in the azimuthal direction about the x-axis and

applying the spherical harmonics addition theorem,[1] the transport

equation becomes

$$\left[\mu \frac{\partial}{\partial x} + 1\right] \Psi(x,\mu) = \frac{c}{2} \sum_{k=0}^{N} b_k P_k(\mu) \int_{-1}^{1} P_k(\mu') \Psi(x,\mu') d\mu'. \qquad (1.4)$$

For various problems, the angular density at any position x and angle

$\cos^{-1}\mu$, $\Psi(x,\mu)$, will be obtained from this equation.

Various exact methods have been used to solve the above inte-

grodifferential equation. Chandrasekhar,[2] with a method developed

by Ambarzumian,[3] has used Equation (1.4) to study photon transfer

ignoring polarization. The method is based upon recognizing physical

principles of invariance which the angular density must obey. These

invariance principles and the transport equation lead to solutions

for the outgoing densities from slabs of finite or infinite width

under appropriate boundary conditions. An extension of the approach

used by Ambarzumian and Chandrasekhar to inhomogeneous regions was

made by Bellman and Kalaba.[4] The method of solving the transport

equation by use of invariance concepts is known as the method of in-

variant imbedding.

Another exact approach used to solve Equation (1.4) is the

Wiener-Hopf method of Fourier transforms.[5,6] The transport equation

is first transformed into an integral equation and solutions are then

obtained for various boundary conditions.

Case[7] has obtained exact solutions of Equation (1.4) for

isotropic scattering of neutrons by still another method. The technique

consists of expanding the angular density in terms of the eigenfunctions of the homogeneous transport equation. The expansion coefficients are determined by applying boundary conditions; the method of solution is thus analogous to the eigenfunction expansions used to solve other partial differential equations of physics. Case's treatment of the transport equation has an advantage over the method of invariant imbedding since the angular density is given for any arbitrary distance inside the plane-parallel medium and may also be used to give the surface quantities obtainable from the older method. The great virtue of the Case method is that it requires the difficult mathematics, for a given class of problems, to be done only once; the solutions for various physical applications can then be found immediately.

Case used his method to solve full-space and half-space problems with isotropic scattering and briefly discussed the extension to problems in a two adjacent half-space geometry. Explicit solutions to problems in the latter geometry were obtained by Mendelson and Summerfield.[8] The method has also been used to attack problems which have no known closed form solutions. Mitsis[9] obtained a solution for the critical slab problem and results for special problems in other than slab geometry,[10] while Mendelson examined various reflected slabs.[11] References 12 to 14 also deal with problems of this type, but explicit results were not obtained.

The method of Case has also been used to examine energy-dependent problems[15-18] and time-dependent problems,[7,19,20] but we shall not be concerned with problems of this type here. Our main

concern will be with problems where the neutrons can be scattered in an anisotropic manner in the laboratory coordinate system. Mika[21] extended the completeness theorem of Case to consider problems with N'th order anisotropic scattering and Shure and Natelson[22] later showed how certain explicit results can be obtained.

In addition to the above problems in neutron and photon transport theory, a similar eigenfunction expansion technique has been used to study plasma oscillations.[23,24] The method has also been used in gas dynamics to obtain solutions to the slip-flow problem.[25] The generality of the approach to problems in linear transport theory has been discussed in Reference 26.

It is the purpose of this thesis to solve some of the simpler problems in one-speed transport theory by the method of Case. When some of these problems have been solved, it is felt that Case's method, with its inherent advantages, can then be used to solve more difficult problems not readily amenable to attack by the older methods.

Chapter II is devoted to a review of Case's method in situations were the scattering is isotropic or linearly anisotropic. Here, results are first derived for the partial-range case where $\alpha < \mu < \beta$. Although these relations have no known physical application at present, they conveniently tie together the full-range ($\alpha = 1$, $\beta = 1$) and half-range ($\alpha = 0$, $\beta = 1$) cases.

Chapter III contains the derivation of general orthogonality relations which may be used to solve problems in half-space and full-space geometries when the scattering is isotropic or linearly anisotropic.

The orthogonality relations in the full-range case are simply those obtained previously by Case[7] and Mika[21] for isotropic and linearly anisotropic scattering, respectively, and will not be considered further.

Since the isotropic scattering relations for the half-range case yield answers to half-space problems which were previously obtained,[7] no half-space applications for isotropic scattering will be explicitly exhibited. Chapter IV, however, contains the solutions to the half-space Milne,[22] albedo, constant isotropic source, and Green's function problems when the scattering is linearly anisotropic. Also included are simplified expressions for the emergent angular distributions and for the densities and net currents on the surface of the half-space. All equations in this chapter may be specialized to give the results for isotropic scattering.

In Chapter V, orthogonality relations are used to attack the slab albedo problem. Because the solution obtained is not in closed form and is consequently more involved, we will restrict our attention to the case of isotropic scattering in this chapter. In an appendix, some of the results of this chapter are used to derive new integral expressions for the X- and Y-functions of Chandrasekhar,[2] which appear in the application of invariant imbedding techniques to slab problems. In another appendix, some results are used to consider the slab albedo problem for a slab surrounded on one side by a vacuum and on the other side by a specularly and diffusely reflecting face.

In Chapter VI, we examine a very specific problem in anisotropic scattering, namely a medium which conservatively scatters

neutrons in a quadratically anisotropic manner (i.e., $N = 2$ and $c = 1$).
Although this problem is not of much interest in neutron transport
theory, it has been extensively investigated in the past with regard
to photon transport theory. It is of some interest to us, however,
since it provides additional insight into the nature of the half-space
orthogonality relations in a situation which is no more complicated
than that covered in Chapter IV where $N = 1$ and $c \leq 1$. The Milne[22]
and albedo problems are then explicitly solved.

Chapter VII contains a speculative approach for obtaining
the orthogonality relations for general-order anisotropic scattering
with $c \leq 1$.

CHAPTER II

REVIEW OF CASE'S METHOD

A. Isotropic Scattering

If the medium is assumed to scatter neutrons isotropically in the laboratory system, the homogeneous transport equation is

$$\left[\mu \frac{\partial}{\partial x} + 1\right] \psi(x,\mu) = \frac{c}{2} \int_{-1}^{1} \psi(x,\mu') \, d\mu' . \tag{2.1}$$

Separation of variables may be achieved through the _ansatz_

$$\psi(x,\mu) = e^{-x/\nu} \, \phi_{\nu}(\mu) \tag{2.2}$$

with the allowed values of the constant ν to be determined. Because the transport equation is homogeneous, the eigenfunctions may be normalized in an arbitrary manner and the normalization condition is selected such that*

$$\int_{-1}^{1} \phi_{\nu}(\mu) \, d\mu = 1 \tag{2.3}$$

For $-1 \leq \nu \leq 1$, this leads to the eigenfunctions

$$\phi_{\nu}(\mu) = \phi_{-\nu}(-\mu) = \frac{c\nu}{2} P \frac{1}{\nu - \mu} + \lambda(\nu) \, \delta(\nu - \mu) \tag{2.4}$$

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* This is permissible except for the trivial solution $\phi_{\nu}(\mu) \equiv 0$.

where $\lambda(\nu)$ is determined from (2.3) and (2.4) as

$$\lambda(\nu) = 1 - \frac{c\nu}{2} P \int_{-1}^{1} \frac{d\mu}{\nu - \mu} = 1 - c\nu \tanh^{-1}\nu , \quad -1 \leq \nu \leq 1 . \qquad (2.5)$$

$\delta(\nu-\mu)$ is the Dirac δ-function. The symbol P is a reminder that we must take the Cauchy principal value of any integral over ν or μ

For $c < 1$, the argument principle[27] shows that there are also two real eigenvalues $\pm \nu_0$ outside of the interval $-1 \leq \nu \leq 1$ with corresponding eigenfunctions $\phi_+(\mu)$ given by

$$\phi_{\pm}(\mu) = \phi_{\mp}(-\mu) = \frac{\frac{c\nu_0}{2}}{\nu_0 \mp \mu} . \qquad (2.6)$$

The eigenvalues $\pm \nu_0$ are determined from the normalization condition and are roots of the equation

$$\Lambda(\pm \nu_0) = 0 \qquad (2.7)$$

where

$$\Lambda(z) = 1 - \frac{cz}{2} \int_{-1}^{1} \frac{d\mu}{z - \mu} = 1 - cz \tanh^{-1}\frac{1}{z} . \qquad (2.8)$$

The function $\Lambda(z)$ is analytic in the complex plane excluding a cut from -1 to 1 on the real axis. For large z,

$$\lim_{z \to \infty} \Lambda(z) = \Lambda(\infty) = 1 - \frac{c}{2} \lim_{z \to \infty} \int_{-1}^{1} \frac{d\mu}{1 - \mu/z} = 1 - c . \qquad (2.9)$$

For $c = 1$, the two roots coelesce at $\nu_o = \infty$ and it is easy to verify that two linearly independent solutions of the transport equation are $\psi_1(x,\mu) = \frac{1}{2}$ and $\psi_2(x,\mu) = \frac{1}{2}(x-\mu)$. In this case, $\lim\limits_{z \to \infty} z^2 \Lambda(z) = -\frac{1}{3}$. For $c > 1$, there are again two discrete roots $\pm \nu_o$, but in this case ν_o is imaginary.

The Plemelj formulae* are of fundamental importance in the solution of the neutron transport equation by Case's method. Consider the function $F(z)$ defined by

$$F(z) = \frac{1}{2\pi i} \int_\alpha^\beta \frac{f(\mu)\,d\mu}{\mu - z} \quad . \tag{2.10}$$

$F(z)$ is analytic in the entire complex plane, cut along the real axis from α to β . For z approaching ν where $\alpha < \nu < \beta$,** the boundary values of $F(z)$ as z approaches ν from above/below are denoted by

$$F^{\pm}(\nu) = \lim_{\epsilon \to 0^+} F(\nu \pm i\epsilon) \tag{2.11}$$

and may be calculated from

$$F^{\pm}(\nu) = \frac{1}{2\pi i} \, P\!\int_\alpha^\beta \frac{f(\mu)\,d\mu}{\mu - \nu} \pm \frac{1}{2} f(\nu) \, . \tag{2.12}$$

These are the Plemelj formulas. They may also be written as

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*Reference 28, page 42.
**The behavior of $F(z)$ in the neighborhood of the endpoints α,β is discussed in Reference 28, Chapter 4.

$$\pi i \left[F^+ (\nu) + F^- (\nu) \right] = P \int_\alpha^\beta \frac{f(\mu) d\mu}{\mu - \nu} \qquad (2.13)$$

$$\left[F^+ (\nu) - F^- (\nu) \right] = f(\nu) \qquad . \qquad (2.14)$$

These equations may be used to relate $\lambda(\nu)$ to the boundary values of $\Lambda(z)$. Use of Equation (2.12) gives

$$\Lambda^{\pm} (\nu) = \lambda (\nu) \pm \frac{i \pi c \nu}{2} \qquad (2.15)$$

and addition of the two equations in (2.15) gives

$$\lambda(\nu) = \frac{1}{2} \left[\Lambda^+ (\nu) + \Lambda^- (\nu) \right] \qquad . \qquad (2.16)$$

Case[7] has shown that the functions (2.4) and (2.6) form a complete set of functions of μ for μ in the interval $[-1,1]$. The solution to (2.1) is therefore given by the expansion

$$\Psi (x,\mu) = a_+ \, \varphi_+ (\mu) e^{-x/\nu_0} + a_- \, \varphi_- (\mu) e^{x/\nu_0} + \int_{-1}^{1} A(\nu) \varphi_\nu (\mu) e^{-x/\nu} \, d\nu . \qquad (2.17)$$

Through the application of boundary conditions of a specific problem to (2.17), one ultimately arrives as an equation of the type

$$\Psi (\mu) = a_+ \, \varphi_+ (\mu) + a_- \, \varphi_- (\mu) + \int_\alpha^\beta A(\nu) \, \varphi_\nu (\mu) \, d\nu \qquad (2.18)$$

where $-1 \leq \alpha < \beta \leq 1$, $\mu \in [\alpha,\beta]$. Case has also proved this more

general "partial-range" completeness property of the eigenfunctions

$\phi_\nu(\mu)$ by directly solving the singular integral Equation (2.18).

A sufficient assumption in the proof of completeness is that $\psi(\mu)$

satisfy the Hölder condition[28] $|\psi(\mu_1) - \psi(\mu_2)| \leq M |\mu_1 - \mu_2|^p$

with $0 < p \leq 1$ for $\alpha < \mu < \beta$ and vary as $M|\mu - \alpha|^K$ or $M|\mu - \beta|^K$,

$-1 < K \leq 0$, at the endpoints.* In the proof of the completeness of

the expansion in (2.18), a general X-function, denoted here by $X_g(z)$,

is constructed such that:

1) it is analytic in the complex plane cut from α to β

2) it is non-vanishing in the finite cut plane

3) it obeys the ratio condition

$$\frac{X_g^+(\nu)}{X_g^-(\nu)} = \frac{\Lambda^+(\nu)}{\Lambda^-(\nu)} \quad , \quad \alpha < \nu < \beta \tag{2.19}$$

4) it varies as $|z - \alpha|^K$ and $|z - \beta|^K$ as $z \to \alpha$ and β ,

respectively, where $-1 < K \leq 0$.**

One function, denoted by $X_{go}(z)$, which has properties 1-3

is

$$X_{go}(z) = exp\left[\frac{1}{2\pi i}\int_\alpha^\beta \frac{\left[\ln \Lambda^+(\mu) - \ln \Lambda^-(\mu)\right] d\mu}{\mu - z}\right] . \tag{2.20}$$

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* The proof is immediately extendible to the case where $\psi(\mu)$ is any
  distribution (for example a δ-function) which is the weak limit of
  a sequence of functions satisfying the Hölder condition (Case, private
  communication).
**The function is permitted to have a weak singularity at the endpoints,
  α and β, of the cut. It may not have a zero at the endpoints, however,
  because of the manner in which the X-function is used in the construc-
  tive proof of completeness of the eigenfunction expansion.

This may also be written as

$$X_{go}(z) = exp\left[\frac{1}{\pi}\int_{\alpha}^{\beta}\frac{\Theta(\mu)\,d\mu}{\mu - z}\right]$$

(2.21)

where $\Theta(\mu) \equiv \arg \Lambda^+(\mu)$. From (2.5) and (2.15),

$$\Theta(\mu) = \tan^{-1}\frac{\frac{\pi c \mu}{2}}{1 - c\mu \tanh^{-1}\mu}$$

(2.22)

The function $\Theta(\mu)$ is an odd function of μ and has the approximate appearance shown in Figure 2.1.
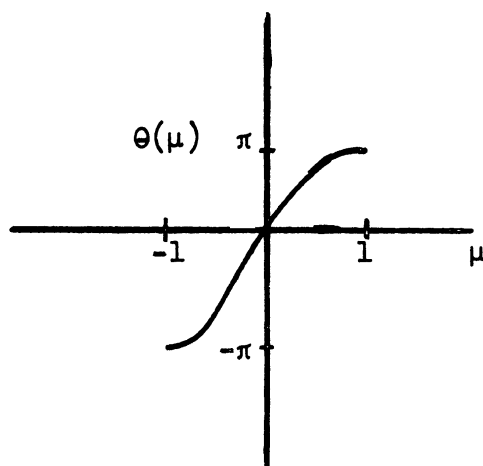
Figure 2.1. Behavior of $\Theta(\mu)$ (approximate).

Another function which obviously has properties 1-3 is

$$\frac{X_{go}(z)}{\beta - z} \quad , \quad \beta > 0 ,$$

(2.23)

since $\beta - z$ is non-vanishing in the cut plane and conditions 1 and 2 are obviously satisfied. This function also satisfies* property 4 since near the point $z = \beta$,

$$\frac{X_{go}(z)}{\beta - z} \sim \frac{exp\left[\frac{\Theta(\beta)}{\pi} \ln(\beta - z)\right]}{\beta - z} = (\beta - z)^{-1 + \frac{\Theta(\beta)}{\pi}} \qquad (2.24)$$

Likewise, the function

$$\frac{X_{go}(z)}{z - \alpha} \quad , \quad \alpha < 0 , \qquad (2.25)$$

has the properties 1-4. Any proper linear combination of (2.20), (2.23), and (2.25) also satisfies the four conditions. This linear combination, the general function $X_g(z)$, thus is given by

$$X_g(z) = X_{go}(z)\left[s_0 + \frac{s_1 h(-\alpha)}{z - \alpha} + \frac{s_2 h(\beta)}{\beta - z}\right] \quad \begin{array}{c} s_1 \neq 0 \\ s_2 \neq 0 \end{array} \qquad (2.26)$$

where s_i , $i = 0,1,2$ are arbitrary constants. We have defined $h(x)$ by

$$h(x) = \begin{cases} 1 , & x > 0 \\ 0 , & x \leq 0 . \end{cases} \qquad (2.27)$$

The requirement that $s_1 \neq 0$ and $s_2 \neq 0$ in Equation (2.26) guarantees that $X_g(z)$ will have the proper behavior at the endpoints of the cut.

*Reference 28, Chapter 4.

Define

$$\gamma_g(\nu) = \frac{c\nu}{2} \frac{X_g^-(\nu)}{\Lambda^-(\nu)} \quad , \quad \alpha < \nu < \beta \tag{2.28}$$

where

$$\gamma_{go}(\nu) = \frac{c\nu}{2} \frac{X_{go}^-(\nu)}{\Lambda^-(\nu)} \quad , \quad \alpha < \nu < \beta \tag{2.29}$$

is the particular function corresponding to $X_{go}(z)$ of (2.20). By the use of the Cauchy integral formula, it is shown in Appendix A that

$$X_g(z) - X_g(\infty) = \int_\alpha^\beta \frac{\gamma_g(\mu)\, d\mu}{\mu - z} \quad . \tag{2.30}$$

For $z = \nu$, $\alpha < \nu < \beta$, this equation gives (Appendix A)

$$\gamma_g(\nu)\lambda(\nu) - \frac{c\nu}{2} X_g(\infty) = \frac{c\nu}{2} P\!\int_\alpha^\beta \frac{\gamma_g(\mu)\, d\mu}{\mu - \nu} \quad , \quad \alpha < \nu < \beta . \tag{2.31}$$

Following Case,[7] for the special case of $\alpha = 0$ and $\beta = 1$, arbitrarily define the half-range X-function, denoted by $X(z)$, as the particular linear combination of $X_g(z)$ with $s_0 = 0$ and $s_2 = 1$ such that

$$X(z) \equiv X_g(z)\Big|_{\substack{s_0 = 0 \\ s_2 = 1}} = \frac{X_{go}(z)}{1 - z} \quad . \tag{2.32}$$

The corresponding γ-function is given by

$$\gamma(\nu) = \frac{c\nu}{2} \frac{X^-(\nu)}{\Lambda^-(\nu)} \equiv \gamma_g(\nu)\Big|_{\substack{S_0=0 \\ S_2=1}} = \frac{\gamma_{g_0}(\nu)}{1-\nu} \, , \quad 0 < \nu < 1. \quad (2.33)$$

From (2.32), $X(z) \sim -1/z$ for large z and, therefore, (2.30) is

$$X(z) = \int_0^1 \frac{\gamma(\mu)d\mu}{\mu - z} \quad (2.34)$$

and (2.31) is

$$\gamma(\nu)\lambda(\nu) = \frac{c\nu}{2} P\int_0^1 \frac{\gamma(\mu)d\mu}{\mu - \nu} \, , \quad 0 < \nu < 1 \, . \quad (2.35)$$

Also, since $\lim\limits_{z \to \infty} -zX(z) = 1$,

$$1 = \lim_{z \to \infty} \int_0^1 \frac{\gamma(\mu)d\mu}{1 - \mu/z} = \int_0^1 \gamma(\mu) \, d\mu \, . \quad (2.36)$$

Another identity of importance for the half-range case is[7]

$$X(z)X(-z) = \frac{\Lambda(z)}{(\nu_0^2 - z^2)\Lambda(\infty)} \quad (2.37)$$

so that, from (2.33),

$$\gamma(\nu) = \cfrac{\cfrac{c\nu}{2}}{(\nu_o^2 - \nu^2)\, \Lambda(\infty)\, X(-\nu)} \quad , \quad 0 < \nu < 1 . \qquad (2.38)$$

The insertion of (2.38) into (2.34) results in the equation

$$X(-z) = \frac{c}{2\, \Lambda(\infty)} \int_0^1 \frac{\nu\, d\nu}{(\nu_o^2 - \nu^2)\, X(-\nu)(\nu + z)} \qquad (2.39)$$

and this can be used for numerical evaluation of the $X(-\mu)$ function. [1]

If $\alpha = -1$ and $\beta = 1$, then the full-range X-function, also denoted by $X(z)$,* was chosen by Case[7] as the particular linear combination of $X_g(z)$ with $s_o = 0$, $s_1 = s_2 = \frac{1}{2}$ so that

$$X(z) \equiv X_g(z)\bigg|_{\substack{s_o = 0 \\ s_1 = s_2 = 1/2}} = \frac{X_{go}(z)}{1 - z^2} \quad . \qquad (2.40)$$

Corresponding to this particular function,

$$\gamma(\nu) = \frac{c\nu}{2} \frac{X^-(\nu)}{\Lambda^-(\nu)} \equiv \gamma_g(\nu)\bigg|_{\substack{s_o = 0 \\ s_1 = s_2 = 1/2}} = \frac{\gamma_{go}(\nu)}{1 - \nu^2} \quad , \quad -1 < \nu < 1 . \qquad (2.41)$$

The following relations (similar to the half-range relations) hold:

$$X(z) = \int_{-1}^1 \frac{\gamma(\mu)\, d\mu}{\mu - z} \qquad (2.42)$$

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*No confusion will arise if the half-range and full-range X-functions are both denoted by the same symbol since the full-range function is used only as an intermediate step in the derivation of Chapter III and no final answers are expressed in terms of it.

$$\gamma(\nu)\lambda(\nu) = \frac{c\nu}{2} P\int_{-1}^{1} \frac{\gamma(\mu)d\mu}{\mu-\nu} , \quad -1 < \nu < 1 \tag{2.43}$$

$$\int_{-1}^{1} \gamma(\mu) d\mu = 0 \tag{2.44}$$

$$\int_{-1}^{1} \mu \gamma(\mu) d\mu = 1 \tag{2.45}$$

$$X(z) = \frac{\Lambda(z)}{(\nu_0^2 - z^2)\Lambda(\infty)} \tag{2.46}$$

$$\gamma(\nu) = \frac{\dfrac{c\nu}{2}}{(\nu_0^2 - \nu^2)\Lambda(\infty)} , \quad -1 < \nu < 1. \tag{2.47}$$

B. Linearly Anisotropic Scattering

For this case, the homogeneous transport equation is

$$\left[\mu \frac{\partial}{\partial x} + 1\right] \psi(x,\mu) = \frac{c}{2}\int_{-1}^{1}\left[1 + b_1\mu\mu'\right] \psi(x,\mu')d\mu' . \tag{2.48}$$

Separation of variables is obtained by use of the ansatz of (2.2) with

the result

$$(\nu - \mu)\, \mathcal{G}_{\nu}(\mu) = \frac{c\nu}{2} \int_{-1}^{1} \left[1 + b_{1}\mu\mu' \right] \mathcal{G}_{\nu}(\mu')\, d\mu' . \qquad (2.49)$$

Multiplying (2.49) by $d\mu$ and integrating from -1 to 1 gives

$$\int_{-1}^{1} \mu\, \mathcal{G}_{\nu}(\mu)\, d\mu = (1-c)\nu \int_{-1}^{1} \mathcal{G}_{\nu}(\mu)\, d\mu . \qquad (2.50)$$

If the normalization condition is selected as

$$\int_{-1}^{1} \mathcal{G}_{\nu}(\mu)\, d\mu = 1 , \qquad (2.51)$$

the eigenfunctions for $-1 \leq \nu \leq 1$ are

$$\mathcal{G}_{\nu}(\mu) = \mathcal{G}_{-\nu}(-\mu) = \frac{c\nu}{2}\, P\, \frac{d(\nu\mu)}{\nu - \mu} + \lambda(\nu)\,\delta(\nu - \mu) \qquad (2.52)$$

where

$$\lambda(\nu) = 1 - \frac{c\nu}{2}\, P\int_{-1}^{1} \frac{d(\nu\mu)\, d\mu}{\nu - \mu} = 1 - \frac{c\nu}{2}\, P\int_{-1}^{1} \frac{d(\mu\mu)\, d\mu}{\nu - \mu} \qquad (2.53)$$

and where $d(\nu\mu)$ is defined by

$$d(\nu\mu) = 1 + b_{1}(1-c)\nu\mu . \qquad (2.54)$$

For $c \leq 1$, there are again two discrete roots,[21] $\pm \nu_o$, of the equation

$$\Lambda(\pm \nu_o) = 0 \tag{2.55}$$

where

$$\Lambda(z) = 1 - \frac{cz}{2} \int_{-1}^{1} \frac{d(z,\mu)d\mu}{z - \mu} = 1 - \frac{cz}{2} \int_{-1}^{1} \frac{d(\mu,\mu)d\mu}{z - \mu} . \tag{2.56}$$

The corresponding discrete modes are

$$\varphi_{\pm}(\mu) = \varphi_{\mp}(-\mu) = \frac{\frac{c\nu_o}{2} d(\pm \nu_o, \mu)}{\nu_o \mp \mu} \tag{2.57}$$

if $c \neq 1$. For $c = 1$, two linearly independent eigenfunctions which satisfy the transport equation are $\psi_1(x,\mu) = \frac{1}{2}$ and $\psi_2(x,\mu) = \frac{1}{2}(x - \frac{3\mu}{3-b_1})$ and the two discrete roots again coelesce at infinity.

The value of $\Lambda(z)$ as $z \to \infty$ will be needed later and can be found from

$$\lim_{z \to \infty} \Lambda(z) = 1 - \frac{c}{2} \lim_{z \to \infty} \int_{-1}^{1} \frac{d(\mu,\mu)d\mu}{1 - \mu/z} , \tag{2.58}$$

The result is

$$\Lambda(\infty) = (1 - c)(1 - \frac{cb_1}{3}) . \tag{2.59}$$

Also, the value of $\lambda(\nu)$ may again be related to the boundary values of $\Lambda(z)$. Use of (2.12) gives

$$\Lambda^{\pm}(\nu) = \lambda(\nu) \pm \frac{i\pi c\,\nu\,d(\nu\nu)}{2} \quad . \tag{2.60}$$

Once again the solution of the transport equation may be found by expanding the angular density in terms of the eigenfunctions of the transport equation. Since there are only two discrete eigenfunctions, the angular density may be expressed as[21]

$$\psi(x,\mu) = a_+\,\varphi_+(\mu)e^{-x/\nu_0} + a_-\,\varphi_-(\mu)e^{x/\nu_0} + \int_{-1}^{1} A(\nu)\varphi_\nu(\mu)e^{-x/\nu}\,d\nu \tag{2.61}$$

and the expansion coefficients will be found from an equation of the form of (2.18).

For the general range of $\alpha < \mu < \beta$, let $X_g(z)$ again denote a linear combination of functions analogous to (2.20), (2.23), and (2.25):

$$X_g(z) = X_{g_0}(z)\left[t_0 + \frac{t_1\,h(-\alpha)}{z-\alpha} + \frac{t_2\,h(\beta)}{\beta-z}\right] \begin{array}{l} t_1 \neq 0 \\ t_2 \neq 0 \end{array} \quad . \tag{2.62}$$

For linearly anisotropic scattering, the constants t_i , $i = 0,1,2$, may depend upon c and b_1 . This $X_g(z)$ satisfies the following identities:

$$X_g(z) - X_g(\infty) = \int_\alpha^\beta \frac{\gamma_g(\mu) \, d(\mu\mu) \, d\mu}{\mu - z} \tag{2.63}$$

$$\gamma_g(\nu)\lambda(\nu) - \frac{c\nu}{2} X_g(\infty) = \frac{c\nu}{2} P \int_\alpha^\beta \frac{\gamma_g(\mu) \, d(\mu\mu) \, d\mu}{\mu - \nu}, \quad \alpha < \nu < \beta \tag{2.64}$$

For the half-range, $0 \le \mu \le 1$, the X- and γ-functions are now defined by

$$X(z) \equiv X_g(z)\Big|_{\substack{t_0 = 0 \\ t_2 = 1}} = \frac{X_{g0}(z)}{1 - z} \tag{2.65}$$

and

$$\gamma(\nu) = \frac{c\nu}{2} \frac{X^-(\nu)}{\Lambda^-(\nu)} \equiv \gamma_g(\nu)\Big|_{\substack{t_0 = 0 \\ t_2 = 1}} = \frac{\gamma_{g0}(\nu)}{1 - \nu}, \quad 0 < \nu < 1. \tag{2.66}$$

Also, since there is only one pair of roots, $\pm \nu_0$, Equations (2.37) and (2.38) remain true. Equations (2.63) and (2.64) are now written as

$$X(z) = \int_0^1 \frac{\gamma(\mu) \, d(\mu\mu) \, d\mu}{\mu - z} \tag{2.67}$$

$$\gamma(\nu)\lambda(\nu) = \frac{c\nu}{2} P \int_0^1 \frac{\gamma(\mu) \, d(\mu\mu) \, d\mu}{\mu - \nu}, \quad 0 < \nu < 1 \tag{2.68}$$

$$1 = \int_0^1 \gamma(\mu)\, d(\mu,\mu)\, d\mu \quad . \tag{2.69}$$

As was pointed out by Shure and Natelson,[22] a technique which minimizes the number of iterations when evaluating the X-function is to use the identity

$$\Omega(-z) = 1 - \frac{cz}{2}\int_0^1 \frac{\left[\nu_0^2 - \frac{\mu^2}{\Lambda(\infty)}\right] d(\mu,\mu)\, d\mu}{(\nu_0^2 - \mu^2)(\mu + z)\,\Omega(-\mu)} \tag{2.70}$$

where

$$\Omega(-z) = \left[\frac{1}{X(0)} + z\right] X(-z) \quad . \tag{2.71}$$

This equation could also be used instead of (2.39) for the isotropic scattering case if b_1 were set equal to zero.

Equations of the form of (2.40), (2.41), (2.46), and (2.47) are valid for full-range applications. Additional full-range relations are:

$$X(z) = \int_{-1}^1 \frac{\gamma(\mu)\, d(\mu,\mu)\, d\mu}{\mu - z} \tag{2.72}$$

$$\gamma(\nu)\lambda(\nu) = \frac{c\nu}{2} P\int_{-1}^1 \frac{\gamma(\mu)\, d(\mu,\mu)\, d\mu}{\mu - \nu} \quad , \quad -1 < \nu < 1 \tag{2.73}$$

$$\int_{-1}^{1} \gamma(\mu)\, d(\mu,\mu)\, d\mu \;=\; 0 \qquad\qquad (2.74)$$

$$\int_{-1}^{1} \mu\, \gamma(\mu)\, d(\mu,\mu)\, d\mu \;=\; 1 \;. \qquad\qquad (2.75)$$

CHAPTER III

ORTHOGONALITY RELATIONS

As was pointed out in Chapter II, the angular density is given by (2.17) or (2.61); the solution of every problem of interest reduces to the determination of the expansion coefficients from the equation

$$\Psi(\mu) = a_+ \, \phi_+(\mu) + a_- \, \phi_-(\mu) + \int_\alpha^\beta A(\nu) \phi_\nu(\mu) \, d\nu; \quad \alpha \le \mu \le \beta . \quad (2.18)$$

It is the purpose of this chapter to derive orthogonality relations between the eigenfunctions $\phi_\pm(\mu)$ and $\phi_\nu(\mu)$. Use of these relations will facilitate the determination of $a_\pm$ and $A(\nu)$.

A. Isotropic Scattering

Case[7] has shown that the following orthogonality relations are true in the full-range case:

$$\int_{-1}^1 \mu \, \phi_\nu(\mu) \, \phi_{\nu'}(\mu) \, d\mu = \nu \Lambda^+(\nu) \Lambda^-(\nu) \, \delta(\nu - \nu') \quad (3.1)$$

$$\int_{-1}^1 \mu \, \phi_\pm(\mu) \, \phi_\nu(\mu) \, d\mu = 0 \quad (3.2)$$

$$\int_{-1}^1 \mu \, \phi_+(\mu) \, \phi_-(\mu) \, d\mu = 0 . \quad (3.3)$$

Equation (3.1) is an abbreviation for the statement

$$\int_{-1}^1 \mu \, \phi_\nu(\mu) \left[\int_{-1}^1 A(\nu') \, \phi_{\nu'}(\mu) \, d\nu' \right] d\mu = A(\nu) \, \nu \Lambda^+(\nu) \Lambda^-(\nu) . \quad (3.4)$$

In the evaluation of Equation (3.1), there is the term

$$\frac{c^2 \nu}{4} P \int_{-1}^{1} \frac{\mu}{\nu - \mu} \left[P \int_{-1}^{1} \frac{\nu' A(\nu') d\nu'}{\nu' - \mu} \right] d\mu \ . \tag{3.5}$$

The order of integration in this term may be interchanged by the use
of the Poincaré-Bertrand formula:[28]

$$P \int_{L} \frac{1}{\nu - \mu} \left[P \int_{L} \frac{f(\nu',\mu) d\nu'}{\nu' - \mu} \right] d\mu = P \int_{L} d\nu' \, P \int_{L} \frac{f(\nu',\mu) d\mu}{(\nu - \mu)(\nu' - \mu)}$$

$$+ \pi^2 \, f(\nu, \nu)$$

$$= \int_{L} \frac{d\nu'}{\nu - \nu'} \, P \int_{L} f(\nu',\mu) \left[\frac{1}{\nu' - \mu} - \frac{1}{\nu - \mu} \right] d\mu$$

$$+ \pi^2 \, f(\nu, \nu) \tag{3.6}$$

As was pointed out in Reference 29, a mnemonic form of (3.6) may be
written as

$$\frac{P}{\nu - \mu} \frac{P}{\nu' - \mu} = \frac{1}{\nu - \nu'} \left[\frac{P}{\nu' - \mu} - \frac{P}{\nu - \mu} \right] + \pi^2 \delta(\mu - \nu) \delta(\mu - \nu') \tag{3.7}$$

provided the integration over μ is carried out first. The order of
integration in

$$\int_{-1}^{1} \mu \, \varphi_{\nu}(\mu) \left[\int_{-1}^{1} A(\nu') \, \varphi_{\nu'}(\mu) d\nu' \right] d\mu$$

may therefore be interchanged with the use of (3.7). We will use this approach in the subsequent determination of the orthogonality relations.

The orthogonality conditions of interest are obtained by observing that (2.31) may be rewritten in the form

$$\frac{c\nu}{2} X_g(\infty) = \int_\alpha^\beta \phi_\nu(\mu) \gamma_g(\mu) d\mu \quad , \quad \alpha < \nu < \beta .$$

(3.8)

From (2.4),

$$\frac{c\nu}{2} = (\nu - \mu) \phi_\nu(\mu)$$

(3.9)

and use of (3.9) in (3.8) gives

$$(\nu - \mu) \phi_\nu(\mu) X_g(\infty) = \int_\alpha^\beta \phi_\nu(\mu') \gamma_g(\mu') d\mu' \quad , \quad \alpha < \nu < \beta .$$

(3.10)

We now seek a weight function $\gamma_g(\mu)$ which will make $\phi_\nu(\mu)$ and $\phi_{\nu'}(\mu)$, $\alpha < \nu < \beta$ and $\alpha < \nu' < \beta$, orthogonal over the range $\alpha < \mu < \beta$. The orthogonality relation and the weight function for the general-range case, $\gamma_g(\mu)$, are derived in Appendix B or can be observed by inspection of (3.10). The proper weight function for the half-range and full-range cases of interest may then be obtained as in Appendix B where additional orthogonality properties are obtained. The weight function for the half-range case may also be directly obtained without first considering the general-range case (see Reference 1).

With our knowledge of the general-range weight function, we therefore write Equation (3.10) for v and v' , multiply the first by $\phi_{v'}(\mu)\gamma_g(\mu)d\mu$ and the second by $\phi_v(\mu)\gamma_g(\mu)d\mu$, and integrate over μ . Subtraction yields

$$(v-v')\int_\alpha^\beta \phi_v(\mu)\,\phi_{v'}(\mu)\,\gamma_g(\mu)d\mu \;=\; 0\,,\quad \alpha<v<\beta\,,\;\alpha<v'<\beta \qquad (3.11)$$

provided

$$\chi_g(\infty) \neq 0\,. \qquad\qquad (3.12)$$

The normalization relation between $\phi_v(\mu)$ and $\phi_{v'}(\mu)$, $\alpha < v < \beta$ and $\alpha < v' < \beta$, may now be found. Use of (3.7) gives

$$\phi_v(\mu)\,\phi_{v'}(\mu) \;=\; \frac{1}{v-v'}\left[\frac{cv}{2}\,\phi_{v'}(\mu) - \frac{cv'}{2}\,\phi_v(\mu)\right]$$

$$+ \Lambda^+(\mu)\Lambda^-(\mu)\,\delta(\mu-v)\,\delta(\mu-v')\,, \qquad (3.13)$$

so that multiplication of (3.13) by the weight function and integration over μ yields

$$\int_\alpha^\beta \phi_v(\mu)\,\phi_{v'}(\mu)\gamma_g(\mu)d\mu \;=\; \gamma_g(v)\Lambda^+(v)\Lambda^-(v)\,\delta(v-v')\,,\quad \begin{array}{c}\alpha<v<\beta\\[2pt]\alpha<v'<\beta\end{array} \qquad (3.14)$$

where (3.8) has been used. The weight function $\gamma_g(\mu)$ for the general range is still somewhat arbitrary since it is a linear combination

of functions corresponding to those of (2.26). One can therefore select

a particular one which meets certain additional requirements (see

Appendix B).

For $\alpha = 0$ and $\beta = 1$, the eigenfunction $\phi_+(\mu)$ normally

occurs for problems where the medium of interest is the right half-

space. From (2.26) it is seen that for $s_0 = 1$ and $s_2 = \nu_0 - 1$,

a permissible $X_g(z)$ function which satisfies (3.8) and (3.12) is

$$X_g(z) = \frac{(\nu_0 - z)X_{g_0}(z)}{1 - z} \tag{3.15}$$

which is simply $(\nu_0 - z)$ times the X-function defined for the half-

space by Equation (2.32).* This would correspond to $(\nu_0 - \mu)$ times

the $\gamma(\mu)$ function of (2.33). Use of (3.8) and (3.14) and the half-

space weight function $\gamma_g(\mu) = (\nu_0 - \mu)\gamma(\mu)$, determined from Appendix

B, gives

$$\int_0^1 \phi_\nu(\mu)\,\phi_{\nu'}(\mu)(\nu_0-\mu)\gamma(\mu)\,d\mu = (\nu_0-\nu)\gamma(\nu)\Lambda^+(\nu)\Lambda^-(\nu)\delta(\nu-\nu') \tag{3.16}$$

---

*One might wonder whether it would be simpler to forget about using
the half-space X-function used by Case (with $s_0 = 0$ and $s_2 = 1$) and
simply use the X_g-function defined in (3.15) to obtain solutions to
half-space transport problems. The X_g-function varies like $+1$ as
$z \to \infty$ so Equation (2.30) would be

$$X_g(z) - 1 = \int_0^1 \frac{\gamma_g(\mu)\,d\mu}{\mu - z} \; .$$

This equation yields no auxiliary equation of the form of (2.36) and
consequently the analysis might be somewhat easier because there would
be no simplifying identity to be used. A major disadvantage of the
use of $X_g(z)$ would be that the limit of $X_g(z)$ as $c \to 1$ is infinite so
numerical values would be difficult to tabulate unless the function
$X_g(z)/\nu_0$ was considered. The particular half-range function of Case
will be retained here because of its general acceptance.

$$\int_0^1 \phi_+(\mu)\, \phi_{\nu'}(\mu)\, (\nu_0 - \mu)\, \gamma(\mu)\, d\mu = 0 \qquad (3.17)$$

where $0 < \nu < 1$, $0 < \nu' < 1$. With this choice of $\gamma_g(\mu)$, $\phi_+(\mu)$ and $\phi_\nu(\mu)$ are mutually orthogonal over the range $0 < \mu < 1$.

Other formulas of interest follow in a straight-forward manner and are shown below, for $0 < \nu < 1$ and $0 < \nu' < 1$:

$$\int_0^1 \phi_-(\mu)\, \phi_\nu(\mu)\, (\nu_0 - \mu)\, \gamma(\mu)\, d\mu = c \nu_0 \nu\, X(-\nu_0)\, \phi_-(\nu) \qquad (3.18)$$

$$\int_0^1 \phi_\pm(\mu)\, \phi_+(\mu)\, (\nu_0 - \mu)\, \gamma(\mu)\, d\mu = \mp \left(\frac{c\nu_0}{2}\right)^2 X(\pm\nu_0) \qquad (3.19)$$

$$\int_0^1 \phi_+(\mu)\, \phi_{-\nu}(\mu)\, (\nu_0 - \mu)\, \gamma(\mu)\, d\mu = \frac{c^2 \nu \nu_0}{4}\, X(-\nu) \qquad (3.20)$$

$$\int_0^1 \phi_{-\nu}(\mu)\, \phi_{\nu'}(\mu)\, (\nu_0 - \mu)\, \gamma(\mu)\, d\mu = \frac{c\nu'}{2}\, \phi_{-\nu}(\nu')\, (\nu_0 + \nu)\, X(-\nu) \qquad (3.21)$$

$$\int_0^1 \phi_\nu(\mu)\, (\nu_0 - \mu)\, \gamma(\mu)\, d\mu = \frac{c\nu}{2} \qquad (3.22)$$

$$\int_0^1 \mu\, \phi_\nu(\mu)\, (\nu_0 - \mu)\, \gamma(\mu)\, d\mu = \frac{c\nu}{2}\left[\nu - \nu_0 + \gamma^{(1)}\right] \qquad (3.23)$$

$$\int_0^1 \phi_+(\mu)(\nu_0 - \mu)\gamma(\mu)\,d\mu = \frac{c\nu_0}{2} \tag{3.24}$$

$$\int_0^1 \mu\,\phi_+(\mu)(\nu_0 - \mu)\gamma(\mu)\,d\mu = \frac{c\nu_0}{2}\gamma^{(1)}. \tag{3.25}$$

Note that $\phi_{-\nu}(\mu)$ contains no singularity. We have defined $\gamma^{(i)}$ by

$$\gamma^{(i)} = \int_0^1 \mu^i\,\gamma(\mu)\,d\mu \tag{3.26}$$

where, for isotropic scattering,

$$\gamma^{(0)} = 1 \tag{3.27}$$

from (2.36). In general, for any range α to β, the integrated moments of the γ_g-function will be defined as

$$\gamma_g^{(i)} = \int_\alpha^\beta \mu^i\,\gamma_g(\mu)\,d\mu. \tag{3.28}$$

The above half-range equations may also be used in problems where there is no absorption if they are modified by dividing by ν_0 and letting $c \to 1$ $(\nu_0 \to \infty)$. Equations (3.16) to (3.25) then reduce to the relations:

$$\int_0^1 \varphi_\nu(\mu) \, \varphi_{\nu'}(\mu) \, \gamma(\mu) \, d\mu = \gamma(\nu) \Lambda^+(\nu) \Lambda^-(\nu) \, \delta(\nu-\nu') \qquad (3.29)$$

$$\int_0^1 \varphi_\nu(\mu) \, \gamma(\mu) \, d\mu = 0 \qquad (3.30)$$

$$\int_0^1 \mu \, \varphi_\nu(\mu) \, \gamma(\mu) \, d\mu = -\frac{\nu}{2} \qquad (3.31)$$

$$\int_0^1 \varphi_{-\nu}(\mu) \, \varphi_{\nu'}(\mu) \, \gamma(\mu) \, d\mu = \frac{\nu'}{2} \, \varphi_{-\nu}(\nu') \, X(-\nu) \qquad (3.32)$$

$$\int_0^1 \varphi_{-\nu}(\mu) \, \gamma(\mu) \, d\mu = \frac{\nu}{2} \, X(-\nu) \qquad (3.33)$$

along with (3.26) and (3.27). We therefore see that the half-space weight function when $c = 1$ is simply $\gamma(\mu)$. Using the relation[5]

$$\lim_{c \to 1} \nu_0^2 (1-c) = \frac{1}{3} \, , \qquad (3.34)$$

Equation (2.38) yields the result

$$\lim_{c \to 1} \gamma(\mu) = \frac{3\mu}{2 \, X(-\mu)} \, , \quad 0 < \mu < 1. \qquad (3.35)$$

Equations (3.29) to (3.33) are sufficient to determine the expansion coefficients completely since the discrete eigenfunctions are $\frac{1}{2}$ and $\frac{1}{2}(x - \mu)$.

It is interesting to consider the full-range case as a possible check on (3.1) to (3.3). From (2.26), with $s_0 = 1$ and $s_1 = s_2 = \frac{1}{2}(v_0^2 - 1)$, a permissible $X_g(z)$ which satisfies (3.8) is

$$X_g(z) = \frac{(v_0^2 - z^2) X_{g0}(z)}{1 - z^2} . \tag{3.36}$$

This is simply $(v_0^2 - z^2)$ times the X-function defined in (2.40) for the full-range case. Therefore the full-range weight function $(v_0^2 - \mu^2)\gamma(\mu)$, derived in Appendix B, satisfies (3.8) and gives

$$\int_{-1}^{1} \varphi_\nu(\mu) \varphi_{\nu'}(\mu)(v_0^2 - \mu^2) \gamma(\mu) d\mu = (v_0^2 - \nu^2)\gamma(\nu)\Lambda^+(\nu)\Lambda^-(\nu)\delta(\nu - \nu') \tag{3.37}$$

$$\int_{-1}^{1} \varphi_\pm(\mu) \varphi_\nu(\mu) (v_0^2 - \mu^2) \gamma(\mu) d\mu = 0 \tag{3.38}$$

$$\int_{-1}^{1} \varphi_+(\mu) \varphi_-(\mu)(v_0^2 - \mu^2)\gamma(\mu) d\mu = 0 . \tag{3.39}$$

The other equation of interest is

$$\int_{-1}^{1} \varphi_\pm(\mu) \varphi_\pm(\mu)(v_0^2 - \mu^2)\gamma(\mu) d\mu = \mp \frac{c^2 v_0^3}{2} X(\pm v_0) . \tag{3.40}$$

From (2.9) and (2.47) it is seen that

$$(\nu_0^2 - \mu^2)\, \gamma(\mu) = \frac{c\mu}{2(1-c)} \tag{3.41}$$

so Equations (3.1) to (3.3) are immediately obtained and (3.40) reduces to

$$\int_{-1}^{1} \mu\, \varphi_{\pm}(\mu)\, \varphi_{\pm}(\mu)\, d\mu = \mp c \nu_0^3 (1-c)\, X(\pm\nu_0)$$

$$= \pm \frac{c\nu_0^3}{2}\left[\frac{c}{\nu_0^2 - 1} - \frac{1}{\nu_0^2}\right]. \tag{3.42}$$

This last result was also obtained by Case.[7] A few other relations for special problems are

$$\int_{-1}^{1} \mu\, \varphi_{\nu}(\mu)\, d\mu = \nu(1-c) \tag{3.43}$$

$$\int_{-1}^{1} \mu^2\, \varphi_{\nu}(\mu)\, d\mu = \nu^2(1-c) \tag{3.44}$$

$$\int_{-1}^{1} \mu\, \varphi_{\pm}(\mu)\, d\mu = \pm \nu_0(1-c) \tag{3.45}$$

$$\int_{-1}^{1} \mu^2\, \varphi_{\pm}(\mu)\, d\mu = \nu_0^2(1-c). \tag{3.46}$$

When $c \to 1$, the orthogonality relations of (3.1) to (3.3) and (3.43) to (3.46) simply reduce to the identities

$$\int_{-1}^{1} \mu \; \varphi_{\nu}(\mu) \; \varphi_{\nu'}(\mu) \; d\mu \;=\; \nu \Lambda^{+}(\nu) \Lambda^{-}(\nu) \delta(\nu - \nu') \qquad (3.47)$$

$$\int_{-1}^{1} \mu \; \varphi_{\nu}(\mu) \, d\mu \;=\; 0 \qquad (3.48)$$

$$\int_{-1}^{1} \mu^{2} \; \varphi_{\nu}(\mu) \, d\mu \;=\; 0 \qquad (3.49)$$

$$\int_{-1}^{1} \mu \, d\mu \;=\; 0 \qquad (3.50)$$

$$\int_{-1}^{1} \mu^{2} \, d\mu \;=\; \frac{1}{3} \qquad (3.51)$$

where (3.34) has been used. The weight function in the full-range case thus appears the same for $c < 1$ and $c = 1$.

Both the half-range and full-range cases are special cases of the "two-media" problem composed of two adjacent half-spaces of scattering media with different values for the mean number of secondaries per collision. These results are exhibited in Reference 29.

B. Linearly Anisotropic Scattering

In order to find orthogonality relations which parallel those of the case of isotropic scattering, one observes that (2.64) may be written as

$$\frac{c\nu}{2} X_g(\infty) = \int_\alpha^\beta \frac{\varphi_\nu(\mu)\, d(\mu,\mu)\, \gamma_g(\mu)\, d\mu}{d(\nu,\mu)} \quad, \quad \alpha < \nu < \beta. \tag{3.52}$$

From Equation (2.52),

$$\frac{c\nu}{2} = (\nu - \mu) \frac{\varphi_\nu(\mu)}{d(\nu,\mu)} \quad. \tag{3.53}$$

Therefore,

$$X_g(\infty)\, (\nu - \mu) \frac{\varphi_\nu(\mu)}{d(\nu,\mu)} = \int_\alpha^\beta \frac{\varphi_\nu(\mu')}{d(\nu,\mu')}\, d(\mu',\mu')\, \gamma_g(\mu')\, d\mu', \quad \alpha < \nu < \beta. \tag{3.54}$$

In the same manner as before, one uses (3.54) to get

$$(\nu - \nu')\int_\alpha^\beta \frac{\varphi_\nu(\mu)}{d(\nu,\mu)} \frac{\varphi_{\nu'}(\mu)}{d(\nu',\mu)}\, d(\mu,\mu)\, \gamma_g(\mu)\, d\mu = 0, \quad \begin{matrix} \alpha < \nu < \beta \\ \alpha < \nu' < \beta \end{matrix} \tag{3.55}$$

provided

$$X_g(\infty) \neq 0. \tag{3.56}$$

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Notice that this equation is of no use to us in its present form, how-
ever, since an eigenfunction expansion of the form of (2.61) does not
contain functions of the type $\phi_\nu(\mu)e^{-x/\nu}/d(\nu\mu)$ but of the type
$\phi_\nu(\mu)e^{-x/\nu}$. If (3.56) is multiplied by the symmetric function $d(\nu\ \nu')$,
however, and the identity

$$d(\nu\nu')\,d(\mu\mu') = d(\nu\mu)\,d(\nu'\mu) + b_1(1-c)(\nu-\mu)(\nu'-\mu') \qquad (3.57)$$

is used, then

$$(\nu-\nu')\left[\int_\alpha^\beta \phi_\nu(\mu)\,\phi_{\nu'}(\mu)\,\gamma_g(\mu)\,d\mu\right.$$

$$\left.+\,b_1(1-c)\int_\alpha^\beta (\nu-\mu)\frac{\phi_\nu(\mu)}{d(\nu\mu)}(\nu'-\mu)\frac{\phi_{\nu'}(\mu)}{d(\nu'\mu)}\gamma_g(\mu)\,d\mu\right] = 0. \qquad (3.58)$$

Use of (3.53) gives

$$(\nu-\nu')\left[\int_\alpha^\beta \phi_\nu(\mu)\,\phi_{\nu'}(\mu)\gamma_g(\mu)\,d\mu +b_1(1-c)\frac{c\nu}{2}\frac{c\nu'}{2}\gamma_g^{(0)}\right] = 0. \qquad (3.59)$$

The first term in (3.59) has the form we desire. The extra term
suggests that Equation (3.59) may be a bi-orthogonality relation if
properly rewritten. That is, we expect that we will be able to write

$$(\nu-\nu')\left[\int_\alpha^\beta \phi_\nu(\mu)\left[\phi_{\nu'}(\mu) + B\frac{c\nu'}{2}\right]\gamma_g(\mu)\,d\mu\right] = 0, \quad\begin{array}{l}\alpha<\nu<\beta\\ \alpha<\nu'<\beta\end{array} \qquad (3.60)$$

with B a constant to be determined. B is obtained via

$$B \int_{\alpha}^{\beta} \mathscr{P}_{\nu}(\mu) \, \Upsilon_{g}(\mu) \, d\mu \;=\; \frac{c\nu}{2} \, b_{1} (1-c) \, \Upsilon_{g}^{(0)}. \tag{3.61}$$

Use of (2.64) and

$$d(\nu/\mu) \;=\; d(\mu/\mu) + b_{1}(1-c)\mu(\nu-\mu) \tag{3.62}$$

gives

$$\int_{\alpha}^{\beta} \mathscr{P}_{\nu}(\mu) \, \Upsilon_{g}(\mu) \, d\mu \;=\; \frac{c\nu}{2} \left[X_{g}(\infty) + b_{1}(1-c) \Upsilon_{g}^{(1)} \right] \tag{3.63}$$

so Equation (3.61) becomes

$$B \;=\; \frac{b_{1}(1-c) \Upsilon_{g}^{(0)}}{X_{g}(\infty) + b_{1}(1-c) \Upsilon_{g}^{(1)}} \;. \tag{3.64}$$

Equations (3.60) and (3.64) combine to give a bi-orthogonality rela-
tion for the continuum eigenfunctions.

   Using this form of the bi-orthogonality relation, consider
the normalization condition for the continuum eigenfunctions. Use of
(3.7) gives

$$\mathscr{P}_{\nu}(\mu) \, \mathscr{P}_{\nu'}(\mu) \;=\; \frac{1}{\nu-\nu'} \left[\frac{c\nu}{2} d(\nu/\mu) \mathscr{P}_{\nu'}(\mu) - \frac{c\nu'}{2} d(\nu'/\mu) \mathscr{P}_{\nu}(\mu) \right]$$

$$+ \; \Lambda^{+}(\mu) \, \Lambda^{-}(\mu) \, \delta(\mu-\nu) \, \delta(\mu-\nu') \;. \tag{3.65}$$

Equation (3.65) can be used to give

$$\int_{\alpha}^{\beta} \varphi_{\nu}(\mu)\left[\varphi_{\nu'}(\mu) + B\frac{c\nu'}{2}\right]\Upsilon_{g}(\mu)\,d\mu = \Upsilon_{g}(\nu)\Lambda^{+}(\nu)\Lambda^{-}(\nu)\delta(\nu-\nu') \qquad (3.66)$$

provided

$$\frac{1}{\nu-\nu'}\left[\int_{\alpha}^{\beta}\frac{c\nu}{2}\,d(\nu\mu)\,\varphi_{\nu'}(\mu)\,\Upsilon_{g}(\mu)\,d\mu - \int_{\alpha}^{\beta}\frac{c\nu'}{2}\,d(\nu'\mu)\,\varphi_{\nu}(\mu)\,\Upsilon_{g}(\mu)\,d\mu\right]$$

$$+ \frac{c\nu'}{2}\,B\int_{\alpha}^{\beta}\varphi_{\nu}(\mu)\,\Upsilon_{g}(\mu)\,d\mu = 0 . \qquad (3.67)$$

The relation

$$d(\nu\mu) = d(\nu\nu') - b_{1}(1-c)\nu(\nu'-\mu) \qquad (3.68)$$

and Equations (3.61) and (3.63) reduce (3.67) to

$$\frac{1}{\nu-\nu'}\left[-\frac{c\nu}{2}\,b_{1}(1-c)\nu\int_{\alpha}^{\beta}(\nu'-\mu)\,\varphi_{\nu'}(\mu)\,\Upsilon_{g}(\mu)\,d\mu\right.$$

$$\left.+ \frac{c\nu'}{2}\,b_{1}(1-c)\nu'\int_{\alpha}^{\beta}(\nu-\mu)\,\varphi_{\nu}(\mu)\,\Upsilon_{g}(\mu)\,d\mu\right] + \frac{c\nu}{2}\,\frac{c\nu'}{2}\,b_{1}(1-c)\Upsilon_{g}^{(0)} = 0. \qquad (3.69)$$

Use of (3.53) reduces (3.69) to the identity $0 \equiv 0$.

For the half-range case, Equation (2.62) yields

$$X_{g}(z) = X_{go}(z)\left[t_{0} + \frac{t_{2}}{1-z}\right], \quad t_{2} \neq 0, \qquad (3.70)$$

where t_0 and t_2 are dependent upon c and b_1 . Since (3.56) stipulates that $X_g(\infty) \neq 0$, $t_0 \neq 0$ and (3.70) becomes

$$X_g(z) = X(z)\left[t_0 + t_2 - t_0 z\right] \tag{3.71}$$

where (2.65) has been used. Also, using (2.66),

$$\gamma_g(\nu) = \gamma(\nu)\left[t_0 + t_2 - t_0 \nu\right] , \quad 0 < \nu < 1 . \tag{3.72}$$

We wish to find t_0 and t_2 such that the continuum eigenfunction $\phi_\nu(\mu)$ is also bi-orthogonal to $\phi_+(\mu)$, i.e., that

$$\int_0^1 \phi_+(\mu)\left[\phi_\nu(\mu) + B \frac{c\nu}{2}\right] \gamma_g(\mu) d\mu = 0 . \tag{3.73}$$

Using (3.71) and (3.72), (3.64) becomes

$$B = \frac{b_1(1-c)\left[t_0 + t_2 - t_0 \bar{\nu}\right]}{t_0 + b_1(1-c)(t_0 + t_2)\bar{\nu}} , \quad \alpha = 0, \; \beta = 1 \tag{3.74}$$

where we have defined

$$\bar{\nu} = \frac{\gamma^{(1)}}{\gamma^{(0)}} = \frac{\int_0^1 \mu \, \gamma(\mu) d\mu}{\int_0^1 \gamma(\mu) d\mu} \tag{3.75}$$

and have used (2.69). Use of (3.72), (3.74) and the relation

$$d(\nu_0,\mu)d(\nu,\mu) = d(\nu_0 \nu)d(\mu,\mu) - b_1(1-c)(\nu_0-\mu)(\nu-\mu) \tag{3.76}$$

in (3.73) gives

$$\frac{2}{c\nu} d(\nu_0\nu) \left[\frac{\gamma(\nu)\lambda(\nu)[t_0+t_2-t_0\nu]}{\nu_0-\nu} \right] + \frac{c\nu}{2} P \int_0^1 \frac{d(\mu)[t_0+t_2-t_0\mu] \gamma(\mu)d\mu}{(\nu_0-\mu)(\nu-\mu)} \right]$$

$$+ \frac{b_1(1-c)[t_0+t_2-t_0\bar\nu]}{t_0+b_1(1-c)(t_0+t_2)\bar\nu} \int_0^1 \frac{d(\nu_0\mu)[t_0+t_2-t_0\mu]\gamma(\mu)d\mu}{\nu_0-\mu}$$

$$- b_1(1-c)\int_0^1 \gamma(\mu)[t_0+t_2-t_0\mu]d\mu = 0 . \tag{3.77}$$

By inspection, we see that if $t_0 = 1$ and $t_2 = \nu_0 - 1$ and if (2.68) is used, (3.77) is the identity $0 \equiv 0$. The half-space weight function $\gamma_g(\mu) = (\nu_0 - \mu)\gamma(\mu)$ is therefore identical to that for the isotropic scattering case. Equation (3.74) can therefore be rewritten in the form

$$B = \frac{b_1(1-c)(\nu_0-\bar\nu)}{d(\nu_0\bar\nu)} \quad , \quad \alpha = 0 \; , \quad \beta = 1 . \tag{3.78}$$

For linearly anisotropic scattering, the relations of interest are, for $0 < \nu < 1$ and $0 < \nu' < 1$:

$$\int_0^1 \varphi_\nu(\mu) \left[\varphi_{\nu'}(\mu) + B \frac{c\nu'}{2} \right] (\nu_0-\mu)\gamma(\mu)\,d\mu = (\nu_0-\nu)\gamma(\nu)\Lambda^+(\nu)\Lambda^-(\nu)\delta(\nu-\nu') \tag{3.79}$$

$$\int_0^1 \varphi_+(\mu) \left[\varphi_{\nu'}(\mu) + B \frac{c\nu'}{2} \right] (\nu_0-\mu)\gamma(\mu)\,d\mu = 0 \tag{3.80}$$

$$\int_0^1 \varphi_{\bar{\nu}}(\mu)\left[\varphi_+(\mu) + B\frac{c\nu_0}{2}\right](\nu_0-\mu)\gamma(\mu)\,d\mu = 0 \tag{3.81}$$

$$\int_0^1 \varphi_{\pm}(\mu)\left[\varphi_+(\mu) + B\frac{c\nu_0}{2}\right](\nu_0-\mu)\gamma(\mu)\,d\mu = \mp\left[\frac{c\nu_0}{2}\right]^2 X(\pm\nu_0)\frac{d(\nu_0\nu_0)d(\pm\nu_0\bar{\nu})}{d(\nu_0\bar{\nu})} \tag{3.82}$$

$$\int_0^1 \varphi_{-\nu}(\mu)\left[\varphi_{\bar{\nu}}(\mu) + B\frac{c\nu}{2}\right](\nu_0-\mu)\gamma(\mu)\,d\mu = c\nu_0\nu X(-\nu_0)\varphi_-(\nu)\frac{d(\nu_0\nu_0)\,d(-\nu\bar{\nu})}{d(\nu_0\bar{\nu})\,d(-\nu\nu_0)} \tag{3.83}$$

$$\int_0^1 \varphi_{-\nu}(\mu)\left[\varphi_+(\mu) + B\frac{c\nu_0}{2}\right](\nu_0-\mu)\gamma(\mu)\,d\mu = \frac{c^2\nu_0\nu}{4}X(-\nu)\frac{d(-\nu\bar{\nu})\,d(\nu_0\nu_0)}{d(\nu_0\bar{\nu})} \tag{3.84}$$

$$\int_0^1 \varphi_{-\nu}(\mu)\left[\varphi_{\nu'}(\mu) + B\frac{c\nu'}{2}\right](\nu_0-\mu)\gamma(\mu)\,d\mu = \frac{c\nu'}{2}(\nu_0+\nu)X(-\nu)\left[\varphi_{-\nu}(\nu') + B\frac{c\nu}{2}\right] \tag{3.85}$$

$$\int_0^1 \left[\varphi_{\nu}(\mu) + B\frac{c\nu}{2}\right](\nu_0-\mu)\gamma(\mu)\,d\mu = \frac{c\nu}{2}\gamma^{(0)}\frac{d(\nu_0\nu_0)\,d(\bar{\nu}\bar{\nu})}{d(\nu_0\bar{\nu})} \tag{3.86}$$

$$\int_0^1 \mu\left[\varphi_{\nu}(\mu) + B\frac{c\nu}{2}\right](\nu_0-\mu)\gamma(\mu)\,d\mu = \frac{c\nu}{2}\left[\nu + \frac{\bar{\nu}-\nu_0}{d(\nu_0\bar{\nu})}\right] \tag{3.87}$$

$$\int_0^1 \left[\varphi_+(\mu) + B\frac{c\nu_0}{2}\right](\nu_0-\mu)\gamma(\mu)\,d\mu = \frac{c\nu_0}{2}\gamma^{(0)}\frac{d(\nu_0\nu_0)\,d(\bar{\nu}\bar{\nu})}{d(\nu_0\bar{\nu})} \tag{3.88}$$

$$\int_0^1 \mu\left[\varphi_+(\mu) + B\frac{c\nu_0}{2}\right](\nu_0-\mu)\,\gamma(\mu)\,d\mu = \frac{c\nu_0}{2}\,\frac{\bar{\nu}\,d(\nu_0\nu_0)}{d(\nu_0\bar{\nu})} \quad . \tag{3.89}$$

Thus the solutions to half-range problems with linearly anisotropic scattering will be in terms of the calculated functions $X(-\mu)$ (or $\gamma(\mu)$) and $\gamma^{(i)}$, $i = 0,1,$ and the variables c and b_1.

Since

$$\lim_{c\to 1} \nu_0 B = \frac{b_1}{3}$$

$$\lim_{c\to 1}\left[\varphi_\nu(\mu) + B\frac{c\nu}{2}\right] = \varphi_\nu(\mu)$$

$$\lim_{c\to 1}\left[\varphi_+(\mu) + B\frac{c\nu_0}{2}\right] = \frac{1}{2}\left[1 + \frac{b_1}{3}\right] \quad , \tag{3.90}$$

dividing Equations (3.79) to (3.89) by ν_0 and taking the limit as $c \to 1$ gives

$$\int_0^1 \varphi_\nu(\mu)\,\varphi_{\nu'}(\mu)\,\gamma(\mu)\,d\mu = \gamma(\nu)\,\Lambda^+(\nu)\,\Lambda^-(\nu)\,\delta(\nu-\nu') \tag{3.91}$$

$$\int_0^1 \varphi_\nu(\mu)\,\gamma(\mu)\,d\mu = 0 \tag{3.92}$$

$$\int_0^1 \mu\,\varphi_\nu(\mu)\,\gamma(\mu)\,d\mu = -\frac{\nu}{2} \tag{3.93}$$

$$\int_0^1 \varphi_{-\nu}(\mu)\,\varphi_{\nu'}(\mu)\,\gamma(\mu)\,d\mu = \frac{\nu'}{2}\,\varphi_{-\nu}(\nu')\,X(-\nu) \tag{3.94}$$

$$\int_0^1 \varphi_{-\nu}(\mu)\, \gamma(\mu)\, d\mu = \frac{\nu}{2}\, \chi(-\nu) \tag{3.95}$$

$$\gamma^{(0)} = 1\,. \tag{3.96}$$

Notice that the form of these equations is identical to that of Equations (3.29) to (3.33) which were obtained for the case of isotropic scattering with $c = 1$. The above results could also have been obtained by examining (2.49) and (2.50) and observing that, for $c = 1$, the transport equations for linearly anisotropic scattering and isotropic scattering are identical.

For the full-range case, $\gamma_g(\mu) = (\nu_0^2 - \mu^2)\gamma(\mu)$ is the weight function and $X_g(\infty) = 1$ from (3.36). The moments of the gamma function are now given by $\gamma_g^{(i)} = \nu_0^2\gamma^{(i)} - \gamma^{(i+2)}$ so (3.64) becomes

$$B = \frac{b_1(1-c)\left[\nu_0^2\, \gamma^{(0)} - \gamma^{(2)}\right]}{1 + b_1(1-c)\left[\nu_0^2\, \gamma^{(1)} - \gamma^{(3)}\right]} \quad,\quad \alpha = -1, \ \beta = 1\,. \tag{3.97}$$

Use of (2.74) and (2.75) reduces (3.97) to

$$B = \frac{\gamma^{(0)}}{\gamma^{(1)}} = \frac{\int_{-1}^1 \gamma(\mu)\, d\mu}{\int_{-1}^1 \mu\, \gamma(\mu)\, d\mu} \quad,\quad \alpha = -1, \ \beta = 1\,. \tag{3.98}$$

From Equation (2.47), $\gamma(\mu)$ is an odd function of μ so

$$\gamma^{(EVEN)} = 0\,. \tag{3.99}$$

Therefore,

$$B = 0 \; , \quad \alpha = -1 \; , \quad \beta = 1 \; , \tag{3.100}$$

which means that the orthogonality form of Mika[21] is obtained. The relations are:

$$\int_{-1}^{1} \mu \, \varphi_{\nu}(\mu) \, \varphi_{\nu'}(\mu) \, d\mu = \nu \Lambda^{+}(\nu) \Lambda^{-}(\nu) \delta(\nu - \nu') \tag{3.101}$$

$$\int_{-1}^{1} \mu \, \varphi_{\pm}(\mu) \, \varphi_{\nu}(\mu) \, d\mu = 0 \tag{3.102}$$

$$\int_{-1}^{1} \mu \, \varphi_{+}(\mu) \, \varphi_{-}(\mu) \, d\mu = 0 \tag{3.103}$$

$$\int_{-1}^{1} \mu \, \varphi_{\pm}(\mu) \, \varphi_{\pm}(\mu) \, d\mu = \mp \, c \nu_0^{3} \, \Lambda(\infty) \, X(\pm \nu_0) \, d(\nu_0 \nu_0)$$

$$= N_{\pm} = \frac{c \nu_0^{2}}{2} \, d(\nu_0 \nu_0) \, \left. \frac{d \Lambda(z)}{d z} \right|_{z = \pm \nu_0} \tag{3.104}$$

$$\int_{-1}^{1} \mu \, \varphi_{\nu}(\mu) \, d\mu = \nu \, \gamma^{(1)} d(\nu_0 \nu_0)(1 - c)(1 - \frac{c b_1}{3}) \tag{3.105}$$

$$\int_{-1}^{1} \mu^{2} \, \varphi_{\nu}(\mu) \, d\mu = \nu^{2} (1 - c)(1 - \frac{c b_1}{3}) \tag{3.106}$$

$$\int_{-1}^{1} \mu \, \varphi_{\pm}(\mu) \, d\mu = \pm \nu_0 \, \gamma^{(1)} d(\nu_0 \nu_0)(1 - c)(1 - \frac{c b_1}{3}) \tag{3.107}$$

$$\int_{-1}^{1} \mu^{2} \, \varphi_{\pm}(\mu) \, d\mu = \nu_0^{2} (1 - c)(1 - \frac{c b_1}{3}) \; . \tag{3.108}$$

In the limit as $c \to 1$, the above relations take the form of Equations (3.47) to (3.51). Using (2.56), $\Lambda'(\pm \nu_0)$ in (3.104) is given by

$$\left.\frac{d\Lambda(z)}{dz}\right|_{z=\pm\nu_0} = \pm\nu_0\left[\frac{d(3\nu_0\nu_0)}{d(\nu_0\nu_0)}\left[\frac{1-\nu_0^2(1-c)}{\nu_0^2\,(\nu_0^2-1)}\right] - \frac{cb_1(1-c)d(-\nu_0\nu_0)}{(\nu_0^2-1)\,d(\nu_0\nu_0)}\right] . \tag{3.109}$$

CHAPTER IV

HALF-SPACE APPLICATIONS WITH LINEARLY ANISOTROPIC SCATTERING

In this chapter, we shall obtain explicit solutions for four
common time-independent half-space problems. These problems are the
Milne, albedo, constant isotropic source, and Green's function problems.
The solution of the Milne problem gives the neutron distribution in a
source-free half-space with no incident distribution at the surface
and a source of neutrons at infinity. The problem was originally studied
by astrophysicists who were attempting to find the emergent photon dis-
tribution from the surface of a star which was approximated by plane
geometry and a large source at the star's center. The results for this
problem have been reported earlier by Shure and Natelson,[22] who obtained
their answers without the use of the bi-orthogonality relations of
Chapter III.

For the albedo problem, we desire the angular density at
any position and direction due to an incoming distribution on the sur-
face of the half-space. This incident distribution will be assumed to
be azimuthally symmetric and expressible in terms of a Dirac delta
function. There are no other sources of neutrons for the system. Thus,
this problem is the half-space penetration problem.

The third problem concerns a constant isotropic source through-
out the half-space with the stipulation that there is no incoming dis-
tribution at the surface. This may be interpreted physically as a
half-space homogeneous reactor, surrounded by a vacuum, which is

subcritical but is operating in a steady-state condition due to its
internal source.

In the half-space Green's function problem, there is no
incident distribution on the surface and no source anywhere inside
the medium except for a single plane source at $x = x_0$ emitting
neutrons in an arbitrary direction. The solution of this problem can
be used to obtain the solution for a half-space, with no incident dis-
tribution on the surface, containing an arbitrary source.

None of these four problems is ever encountered in reality,
but the solutions can be used as approximations to some physical situ-
ations. The solutions are particularly valuable, however, when used
as a means of checking the accuracy of the approximate methods used
to solve the transport equation.

The angular density, which satisfies Equation (2.48), is

$$\Psi(x,\mu) = a_+ \varphi_+(\mu)e^{-x/\nu_0} + a_- \varphi_-(\mu)e^{x/\nu_0} + \int_{-1}^{1} A(\nu)\varphi_\nu(\mu)e^{-x/\nu}\,d\nu \ . \quad (2.61)$$

If $x \geq 0$, the boundary conditions for the half-space   a) Milne,
b) albedo,   c) constant isotropic source, and   d) Green's function
problems are given by

$$\Psi(0,\mu) = \begin{cases} 0 & , \mu \geq 0 \qquad \text{a)} \\[12pt] \delta(\mu - \mu_0) & , \mu \geq 0 \qquad \text{b)} \\[12pt] \dfrac{-q_0}{1-c} \ , \ c \neq 1, \ \mu \geq 0 & \qquad \text{c)*} \\[12pt] 0 & , \mu \geq 0 \qquad \text{d)} \end{cases} \qquad (4.1)$$

and

$$\lim_{x \to \infty} \Psi(x,\mu) = \begin{cases} \varphi_-(\mu)e^{x/\nu_0}, \ c \neq 1 & \qquad \text{a)**} \\[12pt] 0 & \qquad \text{b)} \\[12pt] 0 & \qquad \text{c)} \\[12pt] 0 & \qquad \text{d)} . \end{cases} \qquad (4.2)$$

---

*The transport equation for the constant isotropic source problem is

$$\left[\mu \frac{\partial}{\partial x} + 1\right] \overline{\Psi}(x,\mu) = \frac{c}{2}\int_{-1}^{1}\left[1 + b\mu\mu'\right]\overline{\Psi}(x,\mu')\,d\mu' + q_0$$

where $\overline{\Psi}(0,\mu) = 0$, $\mu > 0$, and $\lim\limits_{x \to \infty} \overline{\Psi}(x,\mu) = \dfrac{q_0}{1-c}$. The transformation $\overline{\Psi}(x,\mu) = \Psi(x,\mu) + q_0/(1-c)$ gives the homogeneous transport equation (2.48) for $\Psi(x,\mu)$ and the boundary conditions $\Psi(0,\mu) = \dfrac{-q_0}{1-c}$, $\mu > 0$, and $\lim\limits_{x \to \infty} \Psi(x,\mu) = 0$. Thus the constant isotropic source problem is reduced to the solution of the albedo problem with a constant incident distribution on the surface of magnitude $-q_0/(1-c)$. We will calculate the angular distribution $\Psi(x,\mu)$ and not the total angular distribution $\Psi(x,\mu) + q_0/(1-c)$.

** From Mika,[21] we know that the asymptotic distribution far away from a plane source varies as $\phi_+(\mu)e^{-x/\nu_0}$. This means that the angular density varies like $\phi_-(\mu)e^{x/\nu_0}$ when approaching the source.

An additional boundary condition which must be satisfied is

$$\Psi(x_o^+,\mu) - \Psi(x_o^-,\mu) = \begin{cases} 0 & \text{a)} \\ 0 & \text{b)} \\ 0 & \text{c)} \\ \dfrac{\delta(\mu-\mu_o)}{\mu} & \text{d)} \end{cases} \qquad (4.3)$$

which takes into account the discontinuity introduced by the source in the Green's function problem. Since boundary conditions (4.2) and (4.3) are valid for all μ , they are less restrictive than (4.1) and will be considered first. Application of (4.2) and (4.3) to (2.61) enables us to write (2.61) as

$$\Psi(x,\mu) = f(x,\mu) + a_+\, \varphi_+(\mu)e^{-x/\nu_o} + \int_0^1 A(\nu)\varphi_\nu(\mu)e^{-x/\nu}\,d\nu \qquad (4.4)$$

where

$$f(x,\mu) = \begin{cases} \varphi_-(\mu)\,e^{x/\nu_o}, \; c \neq 1 & \text{a)} \\ 0 & \text{b)} \\ 0 & \text{c)} \\ G(x_o,\mu_o \to x,\mu) & \text{d)} \end{cases} \qquad . \qquad (4.5)$$

The function $G(x_0,\mu_0 \to x,\mu)$ will be abbreviated by $G(x,\mu)$ and is the solution to the infinite-medium Green's function problem for linearly anisotropic scattering.[21] Applying (4.1) to (4.4) gives the equation (for $\mu \geq 0$)

$$
a_+ \, \varphi_+(\mu) + \int_0^1 A(\nu)\, \varphi_\nu(\mu)\, d\nu =
\begin{cases}
-\varphi_-(\mu), & c \neq 1 & \text{a)} \\[2mm]
\delta(\mu - \mu_0) & & \text{b)} \\[2mm]
\dfrac{-q_0}{1-c}, & c \neq 1 & \text{c)} \\[2mm]
-G(0,\mu) & & \text{d)} .
\end{cases}
\qquad (4.6)
$$

Use of the half-range bi-orthogonality relations from Chapter III enables us to directly obtain the expansion coefficients a_+ and $A(\nu)$ from (4.6). The results are:

$$
a_+ = R \Big/ \left[-\left(\tfrac{c\nu_0}{2}\right)^2 X(\nu_0)\, d(\nu_0 \nu_0) \right]
\qquad (4.7)
$$

$$
A(\nu) = S \Big/ \left[(\nu_0 - \nu)\, \gamma(\nu)\, \Lambda^+(\nu)\, \Lambda^-(\nu) \right]
\qquad (4.8)
$$

$$
R =
\begin{cases}
-\left(\tfrac{c\nu_0}{2}\right)^2 X(-\nu_0)\, \dfrac{d(\nu_0\nu_0)\, d(-\nu_0\bar\nu)}{d(\nu_0\bar\nu)}, & c \neq 1 & \text{a)} \\[3mm]
(\nu_0 - \mu_0)\, \gamma(\mu_0)\left[\varphi_+(\mu_0) + B\,\tfrac{c\nu_0}{2}\right] = & & \\[3mm]
\qquad \tfrac{c\nu_0}{2}\, \gamma(\mu_0)\, \dfrac{d(\bar\nu \mu_0)\, d(\nu_0\nu_0)}{d(\nu_0\bar\nu)} & & \text{b)} \\[3mm]
-\left(\tfrac{c\nu_0}{2}\right)\dfrac{q_0}{1-c}\, \dfrac{\gamma^{(0)}\, d(\bar\nu\bar\nu)\, d(\nu_0\nu_0)}{d(\nu_0\bar\nu)}, & c \neq 1 & \text{c)} \\[3mm]
-\int_0^1 G(0,\mu)(\nu_0 - \mu)\gamma(\mu)\left[\varphi_+(\mu) + B\,\tfrac{c\nu_0}{2}\right] d\mu & & \text{d)}
\end{cases}
\qquad (4.9)
$$

$$
S = \begin{cases}
- \dfrac{c\nu_0 \nu \, X(-\nu_0) \, \varphi_-(\nu) \, d(\nu_0 \nu_0) \, d(-\nu\bar{\nu})}{d(\nu_0 \bar{\nu}) \, d(-\nu\nu_0)} \;, \quad c \neq 1 & \text{a)} \\[3ex]
(\nu_0 - \mu_0) \, \gamma(\mu_0) \left[\varphi_\nu(\mu_0) + B \dfrac{c\nu}{2} \right] & \text{b)} \\[3ex]
- \left(\dfrac{c\nu}{2} \right) \dfrac{g_0}{1-c} \; \dfrac{\gamma^{(0)} \, d(\bar{\nu}\bar{\nu}) \, d(\nu_0 \nu_0)}{d(\nu_0 \bar{\nu})} \;, \quad c \neq 1 & \text{c)} \\[3ex]
- \displaystyle\int_0^1 G(0,\mu)(\nu_0 - \mu)\gamma(\mu)\left[\varphi_\nu(\mu) + B \dfrac{c\nu}{2} \right] d\mu & \text{d)} .
\end{cases}
\tag{4.10}
$$

Notice that Equations (4.9b) and (4.10b) illustrate explicitly the form of the half-range bi-orthogonality relations used to obtain them since the integration is over the function $\delta(\mu - \mu_0)$. Thus, the correct bi-orthogonal form for half-space problems may be obtained by inspection if the discrete and continuum expansion coefficients for the half-space albedo problem are already known. This is obviously true for any type of isotropic or general-order anisotropic scattering. In fact, it was this observation by Kuščer which first led to his discovery of the existence of orthogonal forms for half-space problems.

The solutions of the four problems are now complete since the angular density is known from Equations (4.4), (4.5), and (4.7) through (4.10). The neutron densities and net currents, defined by

$$
\rho(x) = \int_{-1}^{1} \psi(x,\mu) \, d\mu
$$

$$
j(x) = \int_{-1}^{1} \mu \, \psi(x,\mu) \, d\mu \;,
\tag{4.11}
$$

are easily obtained by integration of (4.4). Upon use of (2.51), one

finds that

$$\rho(x) = \int_{-1}^{1} f(x,\mu)\,d\mu \;+\; a_+ e^{-x/\nu_0} \;+\; \int_{0}^{1} A(\nu)\, e^{-x/\nu}\,d\nu$$

$$\dot{j}(x) = \int_{-1}^{1} \mu f(x,\mu)\,d\mu \;+\; (1-c)\left[a_+ \nu_0\, e^{-x/\nu_0} + \int_{0}^{1} A(\nu)\,\nu e^{-x/\nu}\,d\nu \right] . \tag{4.12}$$

The existence of the factor $e^{-x/\nu}$ in the integral term in

Equation (4.4) appears to make it impossible to evaluate the integral

analytically unless $x = 0$. The incoming distribution at the surface

is specified by the boundary conditions and hence is completely known.

It is desirable, however, to examine the distribution emerging from

the surface since the results are very important and may be greatly

simplified. Using (4.4), the emerging angular distribution may be

written as

$$\Psi(0,-\mu) = f(0,-\mu) + a_+ \varphi_-(\mu) + \int_{0}^{1} A(\nu)\varphi_{-\nu}(\mu)\,d\nu \;,\; \mu > 0, \tag{4.13}$$

where, with

$$\Psi(\mu) \equiv \Psi(0,\mu) - f(0,\mu), \tag{4.14}$$

the expansion coefficients are:

$$a_+ = -\left(\frac{2}{c\nu_0}\right)^2 \frac{\int_0^1 \Psi(\mu')\left[\varphi_+(\mu') + B\frac{c\nu_0}{2}\right](\nu_0-\mu')\,\Upsilon(\mu')\,d\mu'}{X(\nu_0)\,d(\nu_0\nu_0)}$$

$$A(\nu) = \frac{\int_0^1 \Psi(\mu')\left[\varphi_\nu(\mu') + B\frac{c\nu}{2}\right](\nu_0-\mu')\,\Upsilon(\mu')\,d\mu'}{(\nu_0-\nu)\,\Upsilon(\nu)\,\Lambda^+(\nu)\,\Lambda^-(\nu)} \qquad (4.15)$$

After simplification of the form of a_+ , we see that

$$\Psi(0,-\mu) = f(0,-\mu) - \frac{d(-\nu_0,\mu)\int_0^1 \Psi(\mu')\,d(\mu'\bar{\nu})\,\Upsilon(\mu')\,d\mu'}{(\nu_0+\mu)\,d(\nu_0\bar{\nu})\,X(\nu_0)}$$

$$+ \int_0^1 \Psi(\mu')(\nu_0-\mu')\Upsilon(\mu')\left[\int_0^1 \frac{\varphi_{-\nu}(\mu)\left[\varphi_\nu(\mu')+B\frac{c\nu}{2}\right]d\nu}{(\nu_0-\nu)\,\Upsilon(\nu)\,\Lambda^+(\nu)\,\Lambda^-(\nu)}\right]d\mu', \quad \mu>0. \quad (4.16)$$

From Appendix C, however, we have the result

$$\int_0^1 \frac{\varphi_{-\nu}(\mu)\left[\varphi_\nu(\mu')+B\frac{c\nu}{2}\right]d\nu}{(\nu_0-\nu)\,\Upsilon(\nu)\,\Lambda^+(\nu)\,\Lambda^-(\nu)}\Bigg|_{\substack{\mu>0 \\ \mu'>0}} = \frac{1}{(\nu_0+\mu)X(-\mu)}\left[\frac{d(-\mu\mu')}{\mu+\mu'} - B\right]$$

$$+ \frac{d(-\nu_0\mu)\,d(\mu'\bar{\nu})}{(\nu_0+\mu)(\nu_0-\mu')\,d(\nu_0\bar{\nu})\,X(\nu_0)} \qquad . \qquad (4.17)$$

Use of (4.17) in (4.16) yields, after some algebraic manipulation,

$$\Psi(0,-\mu) = f(0,-\mu) + \frac{1}{X(-\mu)} \int_0^1 \Psi(\mu') \, \mathcal{Y}(\mu') \left[\frac{d(\mu'\mu')}{\mu + \mu'} - \frac{d(\nu_0\nu_0) \, d(\mu'\bar{\nu})}{(\nu_0+\mu) \, d(\nu_0\bar{\nu})} \right] d\mu' \quad, \mu > 0. \quad (4.18)$$

This general result can now be specialized, by use of (4.1), (4.5), and (4.14), to give the results

$$\Psi(0,-\mu) = \begin{cases} \dfrac{c\nu_0^2 \, X(-\nu_0) \, d(\nu_0\nu_0) \, d(\bar{\nu}\mu)}{(\nu_0^2-\mu^2) \, X(-\mu) \, d(\nu_0\bar{\nu})} \quad, \ c \neq 1 & \text{a)} \\[3em] \dfrac{\frac{c\mu_0}{2} \left[d(-\mu_0\mu) - B(\mu+\mu_0) \right]}{(\mu_0+\mu)(\nu_0+\mu)X(-\mu)(1-c)(1-\frac{cb_1}{3})(\nu_0+\mu_0)X(-\mu_0)} & \text{b)} \\[3em] \dfrac{g_0}{1-c} \left[\dfrac{\mathcal{Y}^{(0)} \, d(\bar{\nu}\bar{\nu}) \, d(\nu_0\nu_0)}{X(-\mu)(\nu_0+\mu) \, d(\nu_0\bar{\nu})} - 1 \right], \ c \neq 1 & \text{c)} \\[3em] G(0,-\mu) - \dfrac{1}{X(-\mu)} \int_0^1 G(0,\mu') \, \mathcal{Y}(\mu') \left[\dfrac{d(\mu'\mu')}{\mu+\mu'} \\ \qquad - \dfrac{d(\nu_0\nu_0) \, d(\mu'\bar{\nu})}{(\nu_0+\mu) \, d(\nu_0\bar{\nu})} \right] d\mu' & \text{d)} \ . \quad (4.19) \end{cases}$$

Equation (4.19a) was obtained earlier by Shure and Natelson.[22]

Equation (4.19b) agrees with the result of Chandrasekhar[2] when the conversion equations of Appendix H are used.

The densities and net currents on the surfaces of the half-space may be obtained from either (4.18) or (4.19) after multiplication by $d\mu$ and $\mu d\mu$ and integration. The equations are

$$\rho(0) = \int_0^1 \Psi(0,\mu)\,d\mu + \int_0^1 \Psi(0,-\mu)\,d\mu$$

$$\dot{\jmath}(0) = \int_0^1 \mu\,\Psi(0,\mu)\,d\mu - \int_0^1 \mu\,\Psi(0,-\mu)\,d\mu \tag{4.20}$$

and (4.1) must be used. We will proceed by considering the general equation in (4.18) and then specializing the results to obtain specific answers to the four problems under consideration. Equations (4.18) and (4.20) combine to give

$$\rho(0) = \int_0^1 \left[\Psi(0,\mu) + f(0,-\mu) \right] d\mu$$

$$+ \int_0^1 \Psi(\mu')\,\gamma(\mu')\,d(\mu,\mu') \left[\int_0^1 \frac{d\mu}{X(-\mu)(\mu+\mu')} \right] d\mu'$$

$$- \frac{d(\nu_0 \nu_0)}{d(\nu_0 \bar{\nu})} \int_0^1 \Psi(\mu')\,\gamma(\mu')\,d(\mu',\bar{\nu}) \left[\int_0^1 \frac{d\mu}{X(-\mu)(\mu+\nu_0)} \right] d\mu' \tag{4.21}$$

$$j(0) = \int_0^1 \mu \left[\psi(0,\mu) - f(0,-\mu) \right] d\mu$$

$$- \int_0^1 \psi(\mu') \gamma(\mu') d(\mu'\mu') \left[\int_0^1 \frac{\mu \, d\mu}{X(-\mu)(\mu + \mu')} \right] d\mu'$$

$$+ \frac{d(\nu_0 \nu_0)}{d(\nu_0 \bar{\nu})} \int_0^1 \psi(\mu') \gamma(\mu') d(\mu' \bar{\nu}) \left[\int_0^1 \frac{\mu \, d\mu}{X(-\mu)(\mu + \nu_0)} \right] d\mu' . \tag{4.22}$$

Use of partial fractions shows that

$$\int_0^1 \frac{\mu \, d\mu}{X(-\mu)(\mu + \mu')} = \int_0^1 \frac{d\mu}{X(-\mu)} - \mu' \int_0^1 \frac{d\mu}{X(-\mu)(\mu + \mu')}$$

$$\int_0^1 \frac{\mu \, d\mu}{X(-\mu)(\mu \pm \nu_0)} = \int_0^1 \frac{d\mu}{X(-\mu)} \mp \nu_0 \int_0^1 \frac{d\mu}{X(-\mu)(\mu \pm \nu_0)} \tag{4.23}$$

The three integrals needed in (4.17) and (4.18) are evaluated in Appendix C and are

$$\int_0^1 \frac{d\mu}{X(-\mu)(\mu + \mu')} = \frac{2(1-c)(1 - \frac{cb_1}{3})}{c} \left[- \frac{\gamma^{(0)} d(\nu_0 \nu_0) d(\mu' \bar{\nu})}{d(\mu' \mu')} \right.$$

$$\left. + \frac{\nu_0^2 X(0)}{\mu'} \right] - \frac{1}{\gamma(\mu') d(\mu' \mu')}$$

$$\int_0^1 \frac{d\mu}{X(-\mu)(\mu \pm \nu_0)} = \frac{2(1-c)(1 - \frac{cb_1}{3})}{c} \left[-\gamma^{(0)} d(\pm \nu_0 \bar{\nu}) \pm \nu_0 X(0) \right]$$

$$\int_0^1 \frac{d\mu}{X(-\mu)} = \frac{2(1-c)(1 - \frac{cb_1}{3})}{c} \left[\nu_0^2 X(0) - \gamma^{(0)} \bar{\nu} \, d(\nu_0 \nu_0) \right] \tag{4.24}$$

where, from (2.37) and (2.59), we know that

$$\frac{1}{X(0)} = \nu_0 \left[(1-c)(1- \frac{cb_1}{3}) \right]^{1/2} .$$

(4.25)

Equations (4.23) and (4.24) reduce (4.21) and (4.22) to

$$f(0) = \int_{-1}^{1} f(0,\mu) d\mu + a_+ \frac{d(\nu_0\nu_0) X(\nu_0)}{X(0)}$$

$$+ \frac{2}{c X(0)} \int_{0}^{1} \frac{\Psi(\mu') \Upsilon(\mu') d(\mu'\mu') d\mu'}{\mu'}$$

(4.26)

$$j(0) = \int_{-1}^{1} \mu f(0,\mu) d\mu + \frac{2(1-c)}{c \Upsilon^{(0)} d(\nu_0\bar{\nu})} \int_{0}^{1} \Psi(\mu')(\nu_0 - \mu') \Upsilon(\mu') d\mu' .$$

(4.27)

Equation (4.15) was also used in the derivation of (4.26) and (C.15)

from Appendix C was used in the derivation of (4.27).

The specific results for the four half-space problems reduce to

$$f(0) = \begin{cases} \dfrac{2 \nu_0 \left[(1-c)(1- \frac{cb_1}{3}) \right]^{1/2} X(-\nu_0) d(\nu_0\nu_0)}{d(\nu_0\bar{\nu})} , \quad c \neq 1 & \text{a)} \\[20pt] \dfrac{\nu_0 d(\mu_0\mu_0) d(\nu_0\bar{\nu}) - \mu_0 d(\bar{\nu}\mu_0) d(\nu_0\nu_0)}{(\nu_0^2 - \mu_0^2) \left[(1-c)(1- \frac{cb_1}{3}) \right]^{1/2} X(-\mu_0) d(\nu_0\bar{\nu})} & \text{b)} \\[20pt] \dfrac{2 q_0}{1-c} \left[-\dfrac{1}{c} + \dfrac{1}{c \Upsilon^{(0)} d(\nu_0\bar{\nu})} \left[\dfrac{1-c}{1- \frac{cb_1}{3}} \right]^{1/2} \right] , \quad c \neq 1 & \text{c)} \end{cases}$$

$$\rho(0) = \int_{-1}^{1} G(0,\mu)\, d\mu \;+\; a_+ \frac{d(\nu_0 \nu_0)\, X(\nu_0)}{X(0)}$$

$$-\; \frac{2}{c\, X(0)} \int_{0}^{1} \frac{G(0,\mu')\, \gamma(\mu')\, d(\mu',\mu')\, d\mu'}{\mu'} \qquad\qquad \text{d)} \qquad\qquad (4.28)$$

$$\dot{j}(0) = \begin{cases} -\dfrac{2\nu_0^{2}(1-c)\, X(-\nu_0)}{\gamma^{(0)}\, d(\nu_0\bar{\nu})} \;,\; c \neq 1 & \text{a)} \\[2.5em] \dfrac{2(1-c)\, \gamma(\mu_0)(\nu_0-\mu_0)}{c\, \gamma^{(0)}\, d(\nu_0\bar{\nu})} = \dfrac{\mu_0}{(\nu_0+\mu_0)\, X(-\mu_0)\, \gamma^{(0)}\, d(\nu_0\bar{\nu})} & \text{b)} \\[2.5em] -\dfrac{2 q_0 (\nu_0 - \bar{\nu})}{c\, d(\nu_0\bar{\nu})} \;,\; c \neq 1 & \text{c)} \\[2.5em] \displaystyle\int_{-1}^{1} \mu\, G(0,\mu)\, d\mu - \dfrac{2(1-c)}{c\, \gamma^{(0)} d(\nu_0\bar{\nu})} \int_{0}^{1} G(0,\mu')(\nu_0-\mu')\, \gamma(\mu')\, d\mu' & \text{d)} \;. \; (4.29) \end{cases}$$

We have now obtained answers to all four problems which we wanted to solve in this chapter. The solution to the half-space Green's function problem, however, is not in its simplest form since it still involves the solution[7] to the infinite-medium Green's function problem. $G(x,\mu) = G(x_o,\mu_o \to x,\mu)$ and is given by

$$G(x,\mu) = \begin{cases} a_+ \varphi_+(\mu)e^{-(x-x_0)/\nu_0} + \int_0^1 A(\nu)\varphi_\nu(\mu)e^{-(x-x_0)/\nu}\,d\nu & , x > x_0 \\ \\ -a_- \varphi_-(\mu)e^{(x-x_0)/\nu_0} - \int_{-1}^0 A(\nu)\varphi_\nu(\mu)e^{-(x-x_0)/\nu}\,d\nu & , x < x_0 \end{cases} \quad (4.30)$$

where

$$G(\pm\infty,\mu) = 0$$

$$G(x_0^+,\mu) - G(x_0^-,\mu) = \frac{\delta(\mu-\mu_0)}{\mu} \quad . \quad (4.31)$$

Using the second boundary condition and the full-range orthogonality equations, (3.101) to (3.104), $G(x,\mu)$ may be written as

$$G(x,\mu) = \pm \frac{\varphi_\pm(\mu_0)\,\varphi_\pm(\mu)e^{\mp(x-x_0)/\nu_0}}{N_\pm} + \int_0^1 \frac{\varphi_{\pm\nu'}(\mu_0)\varphi_{\pm\nu'}(\mu)e^{\mp(x-x_0)/\nu'}}{\nu'\,\Lambda^+(\nu')\,\Lambda^-(\nu')}\,d\nu', \quad (4.32)$$
$$x \gtrless x_0,$$

where $N_\pm$ is given by (3.104) and (3.109). For the half-space Green's function problem with the source located in the right half-space, we therefore have

$$G(0,\mu) = - \frac{\varphi_-(\mu_0)\varphi_-(\mu)e^{-x_0/\nu_0}}{N_-} + \int_0^1 \frac{\varphi_{-\nu'}(\mu_0)\varphi_{-\nu'}(\mu)e^{-x_0/\nu'}}{\nu'\,\Lambda^+(\nu')\,\Lambda^-(\nu')}\,d\nu' \quad , \quad (4.33)$$

Equation (4.33) may be used to reduce the Green's function results of (4.9), (4.10) (4.19), (4.28), and (4.29) to the following forms:

$$R = \frac{c^2 \nu_0}{4} \frac{d(\nu_0 \nu_0)}{d(\nu_0 \bar{\nu})} \left[\frac{\nu_0 \, X(-\nu_0) \, d(-\nu_0 \bar{\nu}) \, \phi_-(\mu_0) e^{-X_0/\nu_0}}{N_-} \right. $$
$$\left. - \int_0^1 \frac{\phi_{-\nu'}(\mu_0) \, X(-\nu') \, d(-\nu'\bar{\nu}) e^{-X_0/\nu'} \, d\nu'}{\Lambda^+(\nu')\Lambda^-(\nu')} \right] \tag{4.34}$$

$$S = c\nu \left[\frac{\nu_0 \, X(-\nu_0) \, \phi_-(\nu) \, d(\nu_0 \nu_0) \, d(-\nu \bar{\nu}) \, \phi_-(\mu_0) e^{-X_0/\nu_0}}{N_- \, d(\nu_0 \bar{\nu}) \, d(-\nu\nu_0)} \right. $$
$$\left. - \frac{1}{2} \int_0^1 \frac{\phi_{-\nu'}(\mu_0)(\nu_0 + \nu') X(-\nu') \left[\phi_{-\nu'}(\nu) + B \frac{c\nu'}{2} \right] e^{-X_0/\nu'} \, d\nu'}{\nu' \Lambda^+(\nu')\Lambda^-(\nu')} \right] \tag{4.35}$$

$$\psi(0,-\mu) = - \frac{\phi_-(\mu_0) e^{-X_0/\nu_0}}{N_-} \left[\frac{c\nu_0^2 \, X(-\nu_0) \, d(\nu_0 \nu_0) \, d(\bar{\nu}\mu)}{(\nu_0^2 - \mu^2) \, X(-\mu) \, d(\nu_0 \bar{\nu})} \right]$$
$$+ \frac{1}{X(-\mu)} \int_0^1 \frac{\phi_{-\nu'}(\mu_0) \, X(-\nu') \, e^{-X_0/\nu'}}{\nu' \Lambda^+(\nu')\Lambda^-(\nu')} \left[\frac{\phi_{\nu'}(\mu) \, d(\nu'\nu')}{d(\nu'\mu)} \right.$$
$$\left. + \frac{c\nu'}{2} \frac{d(-\nu'\bar{\nu}) \, d(\nu_0 \nu_0)}{(\nu_0 + \mu) \, d(\nu_0 \bar{\nu})} \right] d\nu' \tag{4.36}$$

$$\rho(0) = - \frac{\phi_-(\mu_0) e^{-X_0/\nu_0}}{N_-} \left[\frac{2 \, X(-\nu_0) \, d(\nu_0 \nu_0)}{X(0) \, d(\nu_0 \bar{\nu})} \right]$$
$$+ \frac{1}{X(0)} \int_0^1 \frac{\phi_{-\nu'}(\mu_0) \, X(-\nu') e^{-X_0/\nu'}}{\nu' \Lambda^+(\nu')\Lambda^-(\nu')} \left[d(\nu'\nu') \right.$$
$$\left. + \frac{\nu' \, d(\nu_0 \nu_0) \, d(-\nu'\bar{\nu})}{\nu_0 \, d(\nu_0 \bar{\nu})} \right] d\nu' \tag{4.37}$$

$$j(0) = -\frac{\varphi_-(\mu_0)e^{-x_0/\nu_0}}{N_-}\left[-\frac{2\nu_0^2(1-c)X(-\nu_0)}{\gamma^{(0)}d(\nu_0\bar\nu)}\right]$$

$$-\frac{(1-c)}{\gamma^{(0)}d(\nu_0\bar\nu)}\int_0^1\frac{\varphi_{-\nu'}(\mu_0)(\nu_0+\nu')X(-\nu')e^{-x_0/\nu'}}{\Lambda^+(\nu')\Lambda^-(\nu')}d\nu' \qquad (4.38)$$

For isotropic source, the above expressions are simplified by integrating over $d\mu_0$ which amounts to replacing $\phi(\mu_0)$ by 1 wherever it appears. The preceeding equations were derived with the aid of (3.82) through (3.85) plus the following identities valid for $\xi = \nu_0$ or ν' , $0 < \nu' < 1$:

$$\int_0^1\frac{\gamma(\mu')d(\mu'\mu')\varphi_{-\xi}(\mu')d\mu'}{\mu+\mu'} = \frac{\frac{c\xi}{2}}{\xi-\mu}\left[X(-\mu)d(\xi\mu)-X(-\xi)d(\xi\xi)\right]$$

$$\int_0^1\gamma(\mu')(\nu_0-\mu')\varphi_{-\xi}(\mu')d\mu' = \frac{c\xi}{2}\left[(\nu_0+\xi)X(-\xi)-\gamma^{(0)}d(\nu_0\bar\nu)\right]$$

$$\int_0^1\gamma(\mu')d(\mu'\bar\nu)\varphi_{-\xi}(\mu')d\mu' = \frac{c\xi}{2}d(-\xi\bar\nu)X(-\xi) \qquad (4.39)$$

Another quantity of interest in neutron transport theory with linearly anisotropic scattering is the Milne problem extrapolation distance, z_0 , defined as the distance from the surface of the half-space at which the asymptotic neutron density vanishes. The asymptotic contribution to the angular density comes from the discrete modes since $e^{-d/\nu_0} \gg e^{-d/\nu}$ for $0 \le \nu \le 1$. Inserting $f(x,\mu)$ from (4.5) into (4.12) and using (2.51) gives the Milne problem asymptotic density as

$$\mathcal{f}_{as}(x) = e^{x/\nu_0} + a_+ e^{-x/\nu_0} \quad . \tag{4.40}$$

Therefore,

$$\mathcal{f}_{as}(-z_0) = 0 = e^{-z_0/\nu_0} + a_+ e^{z_0/\nu_0} \tag{4.41}$$

or

$$z_0 = - \frac{\nu_0}{2} \ln(-a_+) \quad , \tag{4.42}$$

Using (4.7) and (4.9a) in (4.42) gives

$$z_0 = - \frac{\nu_0}{2} \ln \left[- \frac{X(-\nu_0) \, d(-\nu_0 \bar{\nu})}{X(\nu_0) \, d(\nu_0 \bar{\nu})} \right] \quad . \tag{4.43}$$

It is also interesting to obtain an integral form for z_0 which does not contain the X-function. From (2.21) and (2.65), the half-space X-function is given by

$$X(z) = \frac{exp \left[\frac{1}{\pi} \int_0^1 \frac{\Theta(\mu) d\mu}{\mu - z} \right]}{1 - z} \tag{4.44}$$

where, for linearly anisotropic scattering,

$$\Theta(\mu) = arg \, \Lambda^+(\mu)$$

$$= tan^{-1} \left\{ \frac{\frac{\pi c \mu}{2} \, d(\mu\mu)}{d(\mu\mu) \left[1 - c\mu \, tanh^{-1}\mu \right] - b_1 (1-c)^2 \mu^2} \right\} \quad . \tag{4.45}$$

Using (4.45), one may show that

$$\frac{d\Theta(\mu)}{d\mu} = \frac{\pi c}{2\,\Lambda^{+}(\mu)\Lambda^{-}(\mu)}\left[d^{2}(\mu,\mu)\left[1 + \frac{c\mu^{2}}{1-\mu^{2}}\right]\right.$$
$$\left. + b_{1}(1-c)^{2}\mu^{2}\,d(-\mu,\mu)\right] \ . \tag{4.46}$$

Following the analysis of Mitsis,[10] write

$$\frac{X(-\nu_{0})}{X(\nu_{0})} = \frac{1-\nu_{0}}{1+\nu_{0}}\,exp\left[-\frac{2\nu_{0}}{\pi}\int_{0}^{1}\frac{\Theta(\mu)\,d\mu}{\mu^{2}-\nu_{0}^{2}}\right]$$
$$= \frac{1-\nu_{0}}{1+\nu_{0}}\,exp\left[\frac{2}{\pi}\int_{0}^{1}\Theta(\mu)\left[\frac{d\,\tanh^{-1}\mu/\nu_{0}}{d\mu}\right]d\mu\right] \tag{4.47}$$

and integrate by parts. Since $\Theta(0) = 0$ and $\Theta(1) = \pi$,

$$\frac{X(-\nu_{0})}{X(\nu_{0})} = \frac{1-\nu_{0}}{1+\nu_{0}}\,exp\left[2\tanh^{-1}\frac{1}{\nu_{0}} - \frac{2}{\pi}\int_{0}^{1}\tanh^{-1}\frac{\mu}{\nu_{0}}\left(\frac{d\Theta(\mu)}{d\mu}\right)d\mu\right]$$
$$= -exp\left[-\frac{2}{\nu_{0}}\left\{\frac{c\nu_{0}}{2}\int_{0}^{1}\frac{\tanh^{-1}\mu/\nu_{0}}{\Lambda^{+}(\mu)\Lambda^{-}(\mu)}\left[d^{2}(\mu,\mu)(1+\frac{c\mu^{2}}{1-\mu^{2}})\right.\right.\right.$$
$$\left.\left.\left. + b_{1}(1-c)^{2}\mu^{2}\,d(-\mu,\mu)\right]d\mu\right\}\right] \ . \tag{4.48}$$

Using (4.43) and (4.48), the Milne problem extrapolation distance may be written as

$$z_0 = -\frac{\nu_0}{2} \ln \frac{d(-\nu_0 \bar{\nu})}{d(\nu_0 \bar{\nu})}$$

$$+ \frac{c\nu_0}{2} \int_0^1 \frac{\tanh^{-1} \mu/\nu_0}{\Lambda^+(\mu) \Lambda^-(\mu)} \left[d^2(\mu\mu) \left(1 + \frac{c\mu^2}{1 - \mu^2}\right) \right.$$

$$\left. + b_1 (1-c)^2 \mu^2 d(-\mu\mu) \right] d\mu \qquad (4.49)$$

The value of $\Lambda^+(\mu)\Lambda^-(\mu)$ is given from (2.60) as

$$\Lambda^+(\mu)\Lambda^-(\mu) = \lambda^2(\mu) + \left[\frac{\pi c \mu \, d(\mu\mu)}{2}\right]^2 \qquad (4.50)$$

where $\lambda(\mu)$ is given by (2.53).

In this chapter, we have yet to consider the numerical evaluation of the quantities appearing in the preceeding equations. The value of $\gamma(\mu)$, $0 < \mu < 1$, can be related to the value of $X(-\mu)$ by Equations (2.38) and (2.59). The value of $\Lambda^+(\mu)\Lambda^-(\mu)$ is found from (4.50) and the equation

$$\lambda(\mu) = d(\mu\mu)\left[1 - c\mu \tanh^{-1} \mu\right] - b_1 (1-c)^2 \mu^2 . \qquad (4.51)$$

Thus, all results may be expressed in terms of ν_0 , $X(\pm \nu_0)$, $X(-\mu)$ for $0 \leq \mu \leq 1$, and $\gamma^{(i)}$ for $i = 0,1$. The first step is to evaluate ν_0 from (2.55) and (2.56). Then the values of $X(\pm \nu_0)$ and $X(-\mu)$ may be obtained by use of the integral equation (2.70). The constants $\gamma^{(i)}$ can then be found.

It is also possible to determine $X(\pm \nu_0)$ without the use of the integral equation (2.70). From (2.38),

$$\Lambda(\infty) X(-\nu_0) X(\nu_0) = \lim_{z \to \pm\nu_0} \frac{\Lambda(z)}{\nu_0^2 - z^2} = \mp \frac{1}{2\nu_0} \left. \frac{d\Lambda(z)}{dz} \right|_{z = \pm\nu_0} \tag{4.52}$$

where (3.109) gives the value of $\Lambda'(\pm \nu_0)$. From (4.43),

$$\frac{X(-\nu_0)}{X(\nu_0)} = -e^{-2z_0^*/\nu_0} \tag{4.53}$$

where the value of z_0^* , defined by

$$z_0^* = z_0 + \frac{\nu_0}{2} \ln \frac{d(-\nu_0\bar{\nu})}{d(\nu_0\bar{\nu})} , \tag{4.54}$$

is given by the integral in (4.49). Combining (4.52) and (4.53) yields the answer

$$X(-\nu_0) = \frac{e^{-z_0^*/\nu_0}}{\left[2(1-c)(1-\frac{cb_1}{3})\right]^{1/2}} \left[\frac{d(3\nu_0\nu_0)}{d(\nu_0\nu_0)} \left[\frac{1-\nu_0^2(1-c)}{\nu_0^2(\nu_0^2-1)} \right] - \frac{cb_1(1-c)d(-\nu_0\nu_0)}{(\nu_0^2-1)d(\nu_0\nu_0)} \right]^{1/2} . \tag{4.55}$$

For the case of isotropic scattering, this reduces to

$$X(-\nu_0) = e^{-z_0/\nu_0} \left[\frac{1-\nu_0^2(1-c)}{2\nu_0^2(\nu_0^2-1)(1-c)} \right]^{1/2} = \frac{e^{-z_0/\nu_0}}{\left[c\nu_0^2 K\right]^{1/2}} . \tag{4.56}$$

where K is tabulated in Case, et al.[5]

CHAPTER V

SLAB ALBEDO PROBLEM FOR ISOTROPIC SCATTERING

The orthogonality relations presented in Chapter III enable

one to solve several one-medium half-space and full-space geometry

problems in closed form. Four specific half-space problems were solved

in Chapter IV. Unfortunately, this program can not be accomplished in

the case of a slab and, as will be shown, the expansion coefficients

in the solution for the angular density are in terms of a Fredholm

integral equation.

The problem of interest consists of a finite plane-parallel

medium surrounded by vacuum on both sides with neutrons incident upon

one face. The incident neutrons are assumed to be azimuthally symmetric

about the x-axis. As in the half-space applications, assume that c ,

the mean number of secondaries per collision, is ≤ 1 .

The problem requires the solution of the following equations:

$$\left[\mu \frac{\partial}{\partial x} + 1 \right] \Psi(x,\mu) = \frac{c}{2} \int_{-1}^{1} \Psi(x,\mu') \, d\mu' \; , \quad 0 \leq x \leq d \tag{2.1}$$

where

$$\Psi(0,\mu) = \delta(\mu - \mu_0) \; , \; \mu > 0 \tag{5.1}$$

$$\Psi(d,\mu) = 0 \qquad , \; \mu < 0 \; . \tag{5.2}$$

The coordinate system is shown in Figure 5.1.
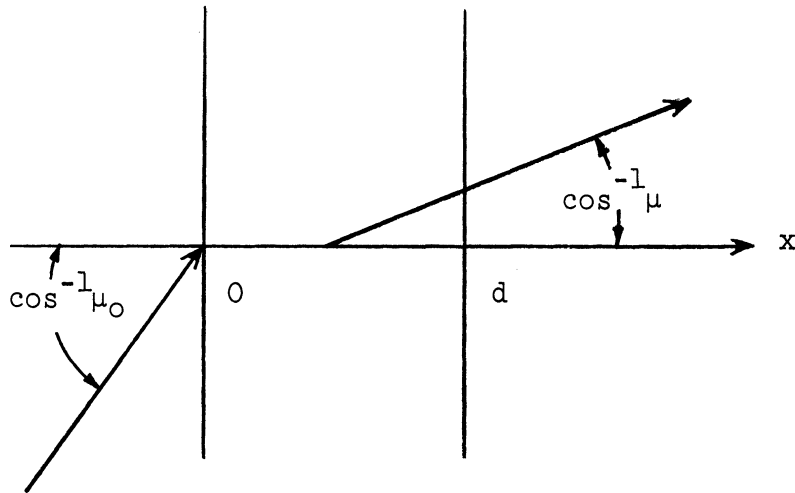
Figure 5.1. Coordinate System for Slab Geometry.

This problem has been studied by Halpern and Luneburg[30] who obtained an expression for the asymptotic density in thick slabs by means of Laplace transform techniques. Later, Kuščer[31] also obtained the asymptotic angular density, density, and net current for a thick slab by appropriately combining the solutions of the half-space Milne and albedo problems. More recently, Zelazny and Kuszell[13] used Case's method to consider a bare slab with arbitrary source distributions on each surface. Because of the generality of the problem, however, explicit solutions were not obtained.

In the present analysis, the angular density is expanded using (2.17) and the problem is reduced by use of the boundary conditions and orthogonality conditions to solving two sets of two coupled equations for appropriate linear combinations, $b_\pm$ and $B_\pm(v)$, of the two discrete and two half-range continuum coefficients. One equation in each set is a nonhomogeneous Fredholm integral equation for

$B_{\pm}(v)$ which is solved approximately by Neumann interation. This leads to explicit solutions for the angular density, scalar density, and net current in zeroth- and first-order approximations. The zeroth-order results are equivalent to those obtained by Kušcer.[31]

Equation (2.17) may be written as

$$\Psi(x,\mu) = a_+ \,\varphi_+(\mu)\, e^{-x/\nu_0} + a_- \,\varphi_-(\mu)\, e^{x/\nu_0}$$
$$+ \int_0^1 A(\nu)\, \varphi_\nu(\mu)\, e^{-x/\nu}\, d\nu \; + \int_0^1 A(-\nu)\, \varphi_{-\nu}(\mu)\, e^{x/\nu}\, d\nu \qquad (5.3)$$

where $A(\pm \nu)$ are defined for $0 \le \nu \le 1$ and are the half-range continuum mode expansion coefficients and $a_\pm$ are the discrete coefficients. It should be noted that either $\phi_\nu(\mu)$ or $\phi_{-\nu}(\mu)$ is non-singular, depending upon whether μ is negative or positive.

Using (5.1) and (5.2) gives

$$\delta(\mu - \mu_0) - a_+ \,\varphi_+(\mu) - a_- \,\varphi_-(\mu)$$
$$- \int_0^1 A(-\nu)\, \varphi_{-\nu}(\mu)\, d\nu = \int_0^1 A(\nu)\, \varphi_\nu(\mu)\, d\nu, \; \mu > 0 \qquad (5.4)$$

and

$$- a_+ \,\varphi_+(\mu)\, e^{-d/\nu_0} - a_- \,\varphi_-(\mu)\, e^{d/\nu_0}$$
$$- \int_0^1 A(\nu)\, \varphi_\nu(\mu)\, e^{-d/\nu}\, d\nu = \int_0^1 A(-\nu)\, \varphi_{-\nu}(\mu)\, e^{d/\nu}\, d\nu, \; \mu < 0. \qquad (5.5)$$

Changing variables in (5.5) from μ to $-\mu$ gives

$$- a_+ \, \varphi_-(\mu) e^{-d/\nu_0} - a_- \, \varphi_+(\mu) e^{d/\nu_0}$$

$$- \int_0^1 A(\nu) \, \varphi_{-\nu}(\mu) e^{-d/\nu} \, d\nu = \int_0^1 A(-\nu) \varphi_\nu(\mu) e^{d/\nu} \, d\nu \, , \, \mu > 0 \, . \qquad (5.6)$$

We are now faced with the problem of trying to find the expansion coefficients from two simultaneous singular integral equations for $\mu > 0$, Equations (5.4) and (5.6). The latter equation is not even in the standard form as solved by Case because of the exponential term $e^{d/\nu}$ in the integral containing the singular eigenfunction $\phi_\nu(\mu)$. We therefore resort to adding and subtracting, respectively, (5.4) and (5.6) to obtain

$$\delta(\mu - \mu_0) - b_\pm \, \varphi_+(\mu) \mp b_\pm \, e^{-d/\nu_0} \, \varphi_-(\mu)$$

$$\mp \int_0^1 B_\pm(\nu) e^{-d/\nu} \varphi_{-\nu}(\mu) \, d\nu = \int_0^1 B_\pm(\nu) \varphi_\nu(\mu) d\nu \, , \, \mu > 0 \, , \qquad (5.7)$$

where we have defined our new expansion coefficients as

$$b_\pm = a_+ \pm a_- e^{d/\nu_0}$$

$$B_\pm(\mu) = A(\mu) \pm A(-\mu) e^{d/\mu} \, . \qquad (5.8)$$

Note that the right hand side of (5.7) is, formally at any rate, a half-range expansion of the (unknown) left hand side. Indeed, the solution will be obtained in terms of the half-range X-function.

Equation (5.7) may also be obtained by using (5.3) to consider the angular densities $\psi(x,\mu)$ and $\psi(d-x,-\mu)$. Adding and subtracting the two angular densities corresponds to finding twice the symmetric density in the slab, denoted by $\psi_+(x,\mu)$, and the non-symmetric density, denoted by $\psi_-(x,\mu)$, and gives

$$\psi_\pm(x,\mu) = \psi(x,\mu) \pm \psi(d-x,-\mu)$$

$$= b_\pm\, \varphi_+(\mu)e^{-x/\nu_0} \pm b_\pm\, \varphi_-(\mu)e^{(x-d)/\nu_0}$$

$$+ \int_0^1 B_\pm(\nu)\, \varphi_\nu(\mu)e^{-x/\nu}d\nu$$

$$\pm \int_0^1 B_\pm(\nu)\, \varphi_{-\nu}(\mu)e^{(x-d)/\nu}d\nu \;. \tag{5.9}$$

The boundary conditions (5.1) and (5.2) combine to give

$$\psi_\pm(0,\mu) = \delta(\mu-\mu_0) \;,\; \mu > 0 \;. \tag{5.10}$$

Applying (5.10) to (5.9) gives (5.7). Thus, b_+ and $B_+(\nu)$ may be interpreted as the expansion coefficients for twice the symmetric angular density and b_- and $B_-(\nu)$ may be interpreted as the expansion coefficients for the non-symmetric angular density.

The orthogonality relations of Chapter III may be used to obtain b_+ and $B_+(\nu)$ from (5.7). Multiplying (5.7) by $\phi_+(\mu)(\nu_0 - \mu)\gamma(\mu)d\mu$, integrating over μ from 0 to 1 , and then using (3.17), (3.19) and (3.20) gives

$$b_{\pm}\left[-X(\nu_0) \pm e^{-d/\nu_0} X(-\nu_0)\right] = \frac{2}{c\nu_0} \gamma(\mu_0)$$

$$\mp \int_0^1 B_{\pm}(\nu) \frac{\nu}{\nu_0} e^{-d/\nu} X(-\nu) d\nu. \qquad (5.11)$$

Use of the relationship[7,9]

$$\frac{X(-\nu_0)}{X(\nu_0)} = -e^{-2z_0/\nu_0} \qquad (5.12)$$

gives

$$b_{\pm} = \frac{\frac{2}{c}\gamma(\mu_0) \mp \int_0^1 \nu B_{\pm}(\nu) e^{-d/\nu} X(-\nu) d\nu}{\nu_0 X(-\nu_0)\left[e^{2z_0/\nu_0} \pm e^{-d/\nu_0}\right]}, \qquad (5.13)$$

Here, the Milne problem extrapolation distance, z_0, is defined by

$$z_0 = \frac{c\nu_0}{2} \int_0^1 \frac{\left[1 + \frac{c\mu^2}{1-\mu^2}\right] \tanh^{-1} \mu/\nu_0 \, d\mu}{\Lambda^+(\mu)\Lambda^-(\mu)} \qquad (5.14)$$

and is tabulated.[5] Multiplication of (5.7) by $\phi_\nu(\mu)(\nu_0-\mu)\gamma(\mu)d\mu$ and integration over μ from 0 to 1 gives, upon application of (3.16), (3.17), (3.18), and (3.21):

$$B_{\pm}(\nu)(\nu_0-\nu)\gamma(\nu)\Lambda^+(\nu)\Lambda^-(\nu) = (\nu_0-\mu_0)\gamma(\mu_0)\phi_\nu(\mu_0)$$

$$\mp\, b_{\pm}\, e^{-d/\nu_0}\, c\nu_0\nu\, X(-\nu_0)\,\phi_-(\nu)$$

$$\mp\int_0^1 B_{\pm}(\nu')e^{-d/\nu'}\frac{c\nu}{2}\,\phi_{-\nu'}(\nu)(\nu_0+\nu')X(-\nu')\,d\nu' \qquad (5.15)$$

Use of partial fractions, (2.38), and (5.11) reduces (5.15) to

$$B_{\pm}(\mu) = \frac{X(-\mu)(\nu_0^2-\mu^2)(1-c)}{\Lambda^+(\mu)\Lambda^-(\mu)}\left[\frac{2\,\gamma(\mu_0)\,\phi_\mu(\mu_0)}{c\mu}\right.$$

$$-\,b_{\pm}\,\phi_+(\mu)X(\nu_0)\mp b_{\pm}\,e^{-d/\nu_0}\phi_-(\mu)X(-\nu_0)$$

$$\left.\mp\int_0^1 B_{\pm}(\nu)e^{-d/\nu}\phi_{-\nu}(\mu)X(-\nu)\,d\nu\right] \qquad (5.16)$$

Equation (5.16) may also be obtained directly, without the use of partial fractions, by use of a special set of isotropic relations which do not have $\phi_+(\mu)$ and $\phi_\nu(\mu)$ mutually orthogonal. The necessary relations are derived and applied to (5.7) in Appendix D.

The equations in (5.16) are nonhomogeneous Fredholm equations for $B_{\pm}(\mu)$ and can be solved by Neumann iteration, provided the resultant series converges. Defining

$$K_{\pm}(\nu,\mu) \equiv \mp\,\frac{X(-\mu)(\nu_0^2-\mu^2)\,\nu e^{-d/\nu}\,X(-\nu)}{\Lambda^+(\mu)\,\Lambda^-(\mu)\,(\nu+\mu)} \qquad (5.17)$$

and

$$B_{\pm}^{(0)}(\mu) = \frac{X(-\mu)(\nu_0^2-\mu^2)(1-c)}{\Lambda^+(\mu)\Lambda^-(\mu)}\left[\frac{2\,\gamma(\mu_0)\,\varphi_\mu(\mu_0)}{c\mu}\right.$$

$$\left. - b_{\pm}\,\varphi_+(\mu)X(\nu_0) \mp b_{\pm}\,e^{-d/\nu_0}\varphi_-(\mu)X(-\nu_0)\right] , \qquad (5.18)$$

Equation (5.16) becomes

$$B_{\pm}(\mu) = B_{\pm}^{(0)}(\mu) + \frac{c(1-c)}{2}\int_0^1 K_{\pm}(\nu,\mu)B_{\pm}(\nu)d\nu \quad . \qquad (5.19)$$

Following Tricomi,[32] since $K_+(\nu,\mu)$ is square integrable,
the following relation must hold for the Neumann series solution to
converge:

$$\left|\frac{c(1-c)}{2}\left[\int_0^1\int_0^1 K_{\pm}^2(\nu,\mu)\,d\nu\,d\mu\right]^{1/2}\right| < 1 \quad . \qquad (5.20)$$

Since

$$X(-\mu) \leq X(0) = \frac{1}{\nu_0\sqrt{1-c}} \quad , \qquad (5.21)$$

a conservative estimate of the above inequality is

$$\left|\frac{1}{2}\,c\,e^{-d}\sqrt{1-\ln 2}\;g(c,\mu)\big|_{MAX}\right| < 1 \quad . \qquad (5.22)$$

Here we have used the fact that

$$\frac{1}{\Lambda^+(\mu)\Lambda^-(\mu)} = g(c,\mu) = \frac{1}{\left(1 - c\mu \tanh^{-1}\mu\right)^2 + \left(\frac{\pi c \mu}{2}\right)^2} \qquad (5.23)$$

where $g(c,\mu)$ is a tabulated function.[5] For $c = 0$, the inequality is satisfied. From Reference 5,

$$g(c,\mu)\Big|_{MAX} \sim \frac{4}{\pi^2 c^2} \qquad (5.24)$$

for c very small and non-zero so that the Neumann series for $B_{\pm}(\mu)$ converges for

$$c > \frac{2\sqrt{1-\ln 2}\ e^{-d}}{\pi^2} = 0.113\ e^{-d}. \qquad (5.25)$$

Because $g(c,\mu)$ is very sharply peaked near $\mu = 1$ for small c, it is felt that the convergence is much better than the conservative estimate shown above. For other values of c, convergence is guaranteed if

$$c < \frac{3.61\ e^{d}}{g(c,\mu)\Big|_{MAX}}, \qquad (5.26)$$

Although we have no proof, it is reasonable to surmise that the Neumann series actually converges for all $c < 1$ and all d. Convergence is rapid only for reasonably large d, say $d \gtrsim 3$.

From (5.13) and (5.19) the $b_{\pm}$ and $B_{\pm}(\mu)$ may be found
to arbitrary orders of approximation. The zeroth-order solutions,
$b_{\pm}^{(0)}$ and $B_{\pm}^{(0)}(\mu)$, valid for large d , are obtained by noting
that the integral terms with $e^{-d/\nu}$ vanish more rapidly than terms
proportional to e^{-d/ν_0} . For the n'th-order approximation, use the
$B_{\pm}^{(n-1)}(\mu)$ to get the $b_{\pm}^{(n)}$ and then the $b_{\pm}^{(n)}$ and the first $n+1$
terms of the series solution of (5.19) to determine $B_{\pm}^{(n)}(\mu)$. Equa-
tion (5.8) may be used to find $a_{\pm}^{(n)}$ and $A^{(n)}(\pm\mu)$. Notice that
this scheme gives zeroth-order results which contain a continuum
component. Diffusion theory results may always be obtained by ignoring
the integral terms as was done by Mitsis[9] for his lowest order approxi-
mation to the critical slab problem.

For the zeroth-order (thick slab) approximation, we use
(5.8) and (5.13) to find that

$$a_+^{(0)} = - e^{(2z_0+2d)/\nu_0} \qquad a_-^{(0)} = \frac{\gamma(\mu_0)e^{d/\nu_0}}{c\nu_0\, X(-\nu_0)\, \sinh\left[(2z_0+d)/\nu_0\right]} \qquad (5.27)$$

Equation (5.19) reduces to $B_{\pm}(\mu) = B_{\pm}^{(0)}(\mu)$, which gives

$$A^{(0)}(\mu) = \frac{X(-\mu)(\nu_0^2-\mu^2)(1-c)}{\Lambda^+(\mu)\Lambda^-(\mu)} \left[\frac{2\,\gamma(\mu_0)\,\varphi_\mu(\mu_0)}{c\mu} \right.$$

$$\left. - a_+^{(0)}\, \varphi_+(\mu)X(\nu_0) - a_-^{(0)}\, \varphi_-(\mu)X(-\nu_0) \right]$$

$$A^{(0)}(-\mu) = - \frac{e^{-d/\mu} X(-\mu)(\nu_0^2 - \mu^2)(1-c)}{\Lambda^+(\mu)\Lambda^-(\mu)} \left[a_-^{(0)} e^{d/\nu_0} \varphi_+(\mu) X(\nu_0) \right.$$

$$\left. + a_+^{(0)} e^{-d/\nu_0} \varphi_-(\mu) X(-\nu_0) \right] . \qquad (5.28)$$

The first-order discrete coefficients are

$$\frac{a_\pm^{(1)}}{a_\pm^{(0)}} = 1 + \nu_0 I_{10} \operatorname{csch}\left[\frac{2z_0 + d}{\nu_0} \right]$$

$$+ e^{\mp(2z_0 + d)/\nu_0} \left[J_{00} + I_{20} + \nu_0 I_{10} \operatorname{coth}\left[\frac{2z_0 + d}{\nu_0} \right] \right] \qquad (5.29)$$

where I_{nm} and J_{nm} are defined as

$$I_{nm} = \frac{c(1-c)}{2} \int_0^1 \frac{\nu^n e^{-d/\nu} X^2(-\nu) \, d\nu}{(\nu + \mu)^m \Lambda^+(\nu)\Lambda^-(\nu)} , \quad \begin{array}{l} n = 0,1,2,3 \\ m = 0,1 \end{array}$$

$$J_{nm} = (1-c) \int_0^1 \frac{\nu^n e^{-d/\nu} X^2(-\nu)(\nu_0^2 - \nu^2)\varphi_\nu(\mu_0) \, d\nu}{(\nu + \mu)^m \Lambda^+(\nu)\Lambda^-(\nu)} , \quad \begin{array}{l} n = -1,0,1 \\ m = 0,1 \end{array} \qquad (5.30)$$

The first-order continuum coefficients are

$$A^{(1)}(\mu) = A^{(0*)}(\mu) + \frac{c\nu_0(1-c)X(-\mu)(\nu_0^2-\mu^2)}{2\,\Lambda^+(\mu)\Lambda^-(\mu)} \quad (X)$$

$$(X)\left[a_+^{(1)} e^{-d/\nu_0} X(-\nu_0)\left[\nu_0 I_{11} - I_{21}\right] \right.$$

$$\left. + a_-^{(1)} e^{d/\nu_0} X(\nu_0)\left[\nu_0 I_{11} + I_{21}\right] \right]$$

$$A^{(1)}(-\mu) = A^{(0*)}(-\mu) + e^{-d/\mu} \frac{c\nu_0(1-c)X(-\mu)(\nu_0^2-\mu^2)}{2\,\Lambda^+(\mu)\Lambda^-(\mu)} \quad (X)$$

$$(X)\left[a_+^{(1)} X(\nu_0)\left[\nu_0 I_{11} + I_{21}\right] \right.$$

$$\left. + a_-^{(1)} X(-\nu_0)\left[\nu_0 I_{11} - I_{21}\right] - \frac{2J_{01}\,\gamma(\mu_0)}{c\nu_0} \right] \tag{5.31}$$

where the $A^{(0*)}(\pm\mu)$ are given by (5.28) with the $a_\pm^{(0)}$ coefficients replaced by $a_\pm^{(1)}$.

The zeroth- and first-order solutions of the slab albedo problem are now complete since all of the expansion coefficients in (5.3) have been determined. The density, $\rho(x) = \int_{-1}^{1} \Psi(x,\mu')d\mu'$, and the net current, $j(x) = \int_{-1}^{1} \mu'\Psi(x,\mu')d\mu'$ are easily obtained from (5.3) upon use of the normalization condition (2.3). The equations are

$$\rho(x) = a_+ e^{-x/\nu_0} + a_- e^{x/\nu_0}$$

$$+ \int_0^1 A(\nu) e^{-x/\nu} d\nu + \int_0^1 A(-\nu) e^{x/\nu} d\nu$$

$$j(x) = (1-c)\left[a_+ \nu_0 e^{-x/\nu_0} - a_- \nu_0 e^{x/\nu_0} \right.$$

$$\left. + \int_0^1 A(\nu)\,\nu e^{-x/\nu} d\nu - \int_0^1 A(-\nu)\,\nu\, e^{x/\nu} d\nu \right] . \tag{5.32}$$

The expressions for the reflected and transmitted angular densities on the slab surfaces may be simplified by the use of Cauchy's integral formula. The results from Appendix E are (for $\mu \geq 0$):

$$\Psi(0,-\mu) = \frac{1}{X(-\mu)}\left[\frac{\gamma(\mu_0)}{\mu_0+\mu} + a_+ \, \varphi_-(\mu) X(\nu_0)\right.$$

$$\left. + a_- \, \varphi_+(\mu) X(-\nu_0) + \int_0^1 A(-\nu) \, \varphi_\nu(\mu) X(-\nu) d\nu\right]$$

$$\Psi(d,\mu) = \frac{1}{X(-\mu)}\left[a_+ e^{-d/\nu_0} \varphi_+(\mu) X(-\nu_0) + a_- e^{d/\nu_0} \varphi_-(\mu) X(\nu_0)\right.$$

$$\left. + \int_0^1 A(\nu) e^{-d/\nu} \varphi_\nu(\mu) X(-\nu) d\nu\right] . \tag{5.33}$$

In Appendix H, these results are used to correlate the $X(-\mu)$ function of Case with the $X(\mu)$ and $Y(\mu)$ functions of Chandrasekhar.[2] Equation (5.33) was derived in a different manner in Reference 33.

When d is large, asymptotic expressions for the above angular densities can easily be obtained since $e^{-d/\nu_0} \gg e^{-d/\nu}$ and the integral terms may be ignored. Use of (5.12) and (5.27) in (5.33) gives the results

$$\Psi(0,-\mu)\Big|_{\text{large } d} = \frac{\gamma(\mu_0)}{X(-\mu)}\left[\frac{1}{\mu_0+\mu} - \frac{\nu_0 \coth[(2z_0+d)/\nu_0] - \mu}{(\nu_0^2-\mu^2)}\right]$$

$$\Psi(d,\mu)\Big|_{\text{large } d} = \frac{\nu_0 \, \gamma(\mu_0)}{(\nu_0^2-\mu^2) X(-\mu) \sinh[(2z_0+d)/\nu_0]} . \tag{5.34}$$

The reflected angular density for the half-space albedo problem is

$$\lim_{d \to \infty} \psi(0,-\mu) = \frac{(\nu_0 - \mu_0)\, \gamma(\mu_0)}{(\nu_0 + \mu)\,(\mu_0 + \mu)\, X(-\mu)}$$

$$= \frac{\dfrac{c\mu_0}{2}}{(\mu_0 + \mu)\, X(-\mu_0)\,(\nu_0 + \mu_0)\,(1 - c)\, X(-\mu)\,(\nu_0 + \mu)} \tag{5.35}$$

and this agrees with the result for isotropic scattering obtained from (4.19b).

Equation (5.33) can be used to find the scalar densities and net currents on the slab surfaces. From Appendix E:

$$\rho(0) = \frac{1}{X(0)}\left[\frac{2\,\gamma(\mu_0)}{c\mu_0} + a_+ X(\nu_0) + a_- X(-\nu_0) + \int_0^1 A(-\nu)\, X(-\nu)\, d\nu\right]$$

$$\rho(d) = \frac{1}{X(0)}\left[a_+ e^{-d/\nu_0} X(-\nu_0) + a_- e^{d/\nu_0} X(\nu_0) + \int_0^1 A(\nu)\, e^{-d/\nu}\, X(-\nu)\, d\nu\right]$$

$$j(0) = -(1-c)\left[\frac{2\mu_0^2\,\gamma(\mu_0)}{c\mu_0} + a_+\,\nu_0^2\, X(\nu_0) + a_-\,\nu_0^2\, X(-\nu_0)\right.$$

$$\left. + \int_0^1 A(-\nu)\,\nu^2\, X(-\nu)\, d\nu\right]$$

$$j(d) = (1-c)\left[a_+ e^{-d/\nu_0}\,\nu_0^2\, X(-\nu_0) + a_- e^{d/\nu_0}\,\nu_0^2\, X(\nu_0)\right.$$

$$\left. + \int_0^1 A(\nu)\, e^{-d/\nu}\,\nu^2\, X(-\nu)\, d\nu\right] \quad, \tag{5.36}$$

Since $X(0) = [v_0^2(1-c)]^{-1/2}$, use of (2.38), (5.12), and (5.27) gives the asymptotic forms of the above equations (where the integral terms are ignored):

$$p(0)\Big|_{\text{large } d} = \frac{v_0 - \mu_0 \coth\big[(2z_0+d)/v_0\big]}{\sqrt{1-c}\ (v_0^2 - \mu_0^2)\, X(-\mu_0)}$$

$$p(d)\Big|_{\text{large } d} = \frac{\mu_0 \operatorname{csch}\big[(2z_0+d)/v_0\big]}{\sqrt{1-c}\ (v_0^2 - \mu_0^2)\, X(-\mu_0)}$$

$$j(0)\Big|_{\text{large } d} = \frac{\mu_0\Big[v_0 \coth\big[(2z_0+d)/v_0\big] - \mu_0\Big]}{(v_0^2 - \mu_0^2)\, X(-\mu_0)}$$

$$j(d)\Big|_{\text{large } d} = \frac{v_0\mu_0 \operatorname{csch}\big[(2z_0+d)/v_0\big]}{(v_0^2 - \mu_0^2)\, X(-\mu_0)} \ . \tag{5.37}$$

These results can be specialized to obtain the following half-space albedo problem results:

$$\lim_{d\to\infty} p(0) = \frac{1}{\sqrt{1-c}\ (v_0+\mu_0)\, X(-\mu_0)}$$

$$\lim_{d\to\infty} p(d) = 0$$

$$\lim_{d\to\infty} j(0) = \frac{\mu_0}{(v_0+\mu_0)\, X(-\mu_0)}$$

$$\lim_{d\to\infty} j(d) = 0 \ . \tag{5.38}$$

The results in (5.37) agree with those obtained by Kuscer.[31] Kuscer observed that the neutron density in a thick slab could be written as a combination of the asymptotic densities from the half-space albedo and Milne problems, denoted by $\rho_a(x)$ and $\rho_m(x)$, in the forms

$$\rho(x)\Big|_{\text{large } d} = \rho_a(x) - M\rho_m(x) , \qquad x \sim 0$$

$$\rho(x)\Big|_{\text{large } d} = N\rho_m(d-x) , \qquad x \sim d \qquad (5.39)$$

where M and N are constants which must be determined. The values of M and N are obtained by observing that the two equations are equal somewhere in the interior of the slab. Using the half-space results from Chapter IV with isotropic scattering (where $d(pq) = 1$ and $\gamma^{(o)} = 1$) in conjunction with (5.12), we have

$$\left[\frac{2\gamma(\mu_0)e^{-2z_0/\nu_0}}{c\nu_0 X(-\nu_0)} \right] e^{-x/\nu_0} - M\left[(-e^{-2z_0/\nu_0})e^{-x/\nu_0} + e^{x/\nu_0} \right]$$

$$= N\left[(-e^{-2z_0/\nu_0})e^{-(d-x)/\nu_0} + e^{(d-x)/\nu_0} \right] . \qquad (5.40)$$

Comparing the coefficients of e^{-x/ν_0} and e^{x/ν_0} yields the result

$$N = M e^{(2z_0+d)/\nu_0} = \frac{\gamma(\mu_0)}{c\nu_0 \, X(-\nu_0) \sinh\left[(2z_0+d)/\nu_0\right]}$$

$$= \frac{\mu_0}{2\nu_0 \, X(-\nu_0) \, (\nu_0^2 - \mu_0^2)(1-c) X(-\mu_0) \sinh\left[(2z_0+d)/\nu_0\right]} \qquad (5.41)$$

One can immediately obtain the results of (5.37) for $\rho(0)\big|_{\text{large d}}$ and $\rho(d)\big|_{\text{large d}}$ using the isotropic forms of (5.28). Kuščer also noted that analogous relations are valid for $\psi(x,\mu)$ and $j(x)$. In fact, the results of (5.34) and (5.37) are obtained by using (4.19), (4.29), and the equations

$$\psi(0,-\mu)\big|_{\text{large d}} = \psi_a(0,-\mu) - M\psi_m(0,-\mu)$$

$$\psi(d,\mu)\big|_{\text{large d}} = N\psi_m(0,-\mu)$$

$$j(0)\big|_{\text{large d}} = j_a(0) - M j_m(0)$$

$$j(d)\big|_{\text{large d}} = -N j_m(0) . \qquad (5.42)$$

(The minus sign in the last equation is necessary to take the reversal $\mu \rightarrow -\mu$ into account.)

The zeroth-order results are obtained from the "large d" results by the addition of the integral terms. Use of (5.28) and (5.30) to evaluate the integral terms yields the answers

$$\rho^{(0)}(0) = \rho(0)\Big|_{\text{large } d} - \frac{\nu_0^2 \mu_0 I_{00} \operatorname{csch}\left[(2z_0+d)/\nu_0\right]}{\sqrt{1-c} \; X(-\mu_0)(\nu_0^2 - \mu_0^2)}$$

$$\rho^{(0)}(d) = \rho(d)\Big|_{\text{large } d} + \frac{\mu_0\left[\begin{array}{c} \nu_0 J_{-10} + \nu_0 I_{10} \\ + \nu_0^2 I_{\infty} \coth\left[(2z_0+d)/\nu_0\right] \end{array}\right]}{\sqrt{1-c} \; (\nu_0^2 - \mu_0^2) X(-\mu_0)}$$

$$j^{(0)}(0) = j(0)\Big|_{\text{large } d} + \frac{\mu_0 \nu_0 I_{20} \operatorname{csch}\left[(2z_0+d)/\nu_0\right]}{(\nu_0^2 - \mu_0^2) X(-\mu_0)}$$

$$j^{(0)}(d) = j(d)\Big|_{\text{large } d} + \frac{\mu_0\left[\begin{array}{c} J_{10} + I_{30} \\ + \nu_0 I_{20} \coth\left[(2z_0+d)/\nu_0\right] \end{array}\right]}{(\nu_0^2 - \mu_0^2) X(-\mu_0)} \qquad . \tag{5.43}$$

In order to estimate the correction terms in (5.43) due to the integral terms, we need to asymptotically evaluate I_{n0} and J_{n0}. Use of the identity

$$\tanh^{-1} \mu = \frac{1}{2} \ln \frac{1+\mu}{1-\mu}$$

and (5.23) enables us to use (5.30) to write

$$I_{n0} = \frac{c(1-c)}{2} \int_0^1 \frac{\nu^n e^{-d/\nu} X^2(-\nu) \, d\nu}{\left(1 - \frac{c\nu}{2} \ln \frac{1+\nu}{1-\nu}\right)^2 + \left(\frac{\pi c \nu}{2}\right)^2} \quad , \; n = 0,1,2,3. \tag{5.44}$$

For $c = 0$ or 1 , $I_{n0} \equiv 0$. For any c , the major contribution to the integral occurs for $v \approx 1$. Following closely the work of Case et al.,[5] set $v = [1+x]^{-1}$ and observe that

$$I_{n0} \approx \frac{c(1-c)}{2} X^2(-1) e^{-d} \int_0^\infty \frac{e^{-dx} \, dx}{\left(1 - \frac{c}{2} \ln \frac{2}{x}\right)^2 + \left(\frac{\pi c}{2}\right)^2} \qquad . \qquad (5.45)$$

This may be rewritten as

$$I_{n0} \approx \frac{2(1-c)}{\pi^2 c} X^2(-1) \frac{e^{-d}}{d} \int_0^\infty \frac{e^{-y} \, dy}{1 + \left(\frac{1}{\pi} \ln \frac{y}{x_0 d}\right)^2} \;, \quad c \neq 0, \qquad (5.46)$$

where $x_0 = 2e^{-2/c}$. We therefore have

$$I_{n0} \approx \frac{2(1-c)}{\pi^2 c} X^2(-1) \frac{e^{-d}}{d} f(\lambda) \;, \quad c \neq 0, \qquad (5.47)$$

where

$$f(\lambda) = \int_0^\infty \frac{e^{-y} \, dy}{1 + \left(\frac{\lambda - \ln y}{\pi}\right)^2} \qquad (5.48)$$

and $\lambda = \ln x_0 d = \ln 2d - 2/c$. The function $f(\lambda)$ is tabulated[5] and its ultimate asymptotic behavior for all c is[5]

$$f(\lambda) \approx \frac{\pi^2}{(\ln d)^2} \left[1 + \frac{\frac{4}{c} - 2.54}{\ln d} + \cdots \right] . \qquad (5.49)$$

Equations (5.47) and (5.49) may be combined to yield the asymptotic result

$$I_{n0} \approx \frac{2(1-c)}{c} \chi^2(-1) \frac{e^{-d}}{d\,(\ln d)^2} \left[1 + \frac{\frac{4}{c} - 2.54}{\ln d} + \cdots \right] \begin{array}{l} n = 0,1,2,3 \\ c \neq 0 \end{array} \quad (5.50)$$

The evaluation of J_{n0} may be achieved by using (2.4) and (5.30) to obtain

$$J_{n0} = (1-c)\mu_o^n\, e^{-d/\mu_o}\, \chi^2(-\mu_0)(\nu_o^2-\mu_o^2)g(c,\mu_0)\lambda(\mu_0) + S \quad (5.51)$$

where

$$S = \frac{c(1-c)}{2} P\int_0^1 \frac{\nu^{n+1}\, e^{-d/\nu}\, \chi^2(-\nu)(\nu_o^2-\nu^2)\, g(c,\nu)d\nu}{\nu - \mu_o} \quad (5.52)$$

For $c = 0$ or 1, $S \doteq 0$. If μ_0 is small, the asymptotic evaluation of (5.52) can be carried out in the same manner as before since the integrand is not singular over the region where the major contribution to the integral occurs. After setting $v = (1+x)^{-1}$, we have

$$S \approx \frac{2(1-c)}{\pi^2 c} \chi^2(-1)(\nu_o^2-1) e^{-d}\int_0^\infty \frac{e^{-dx}\, dx}{\left[1-\mu_0(1+x)\right]\left[1 + \left(\frac{1}{\pi}\ln\frac{x}{x_0}\right)^2\right]}, \quad c \neq 0. \quad (5.53)$$

If $\dfrac{\mu_0 x}{1-\mu_0} \ll 1$, then

$$\frac{1}{1-\mu_0(1+x)} \approx \frac{1+\dfrac{\mu_0 x}{1-\mu_0}}{1-\mu_0} \approx \frac{e^{\mu_0 x/(1-\mu_0)}}{1-\mu_0}$$

so

$$S \approx \frac{2(1-c)}{\pi^2 c} \frac{X^2(-1)(\nu_0^2-1)}{1-\mu_0} e^{-d} \int_0^\infty \frac{e^{-Dx}\, dx}{1+(\frac{1}{\pi}\ln\frac{x}{x_0})^2} \quad , c \neq 0 \qquad (5.54)$$

where

$$D = d - \frac{\mu_0}{1-\mu_0} \quad . \qquad (5.55)$$

If $D \gg 1$, the integral in (5.54) equals $\dfrac{1}{D} f(\Lambda)$ where

$\Lambda = \ln 2(d - \dfrac{\mu_0}{1-\mu_0}) - \dfrac{2}{c}$. One therefore has

$$S \approx \frac{2(1-c)}{c} \frac{X^2(-1)(\nu_0^2-1)}{1-\mu_0} \frac{e^{-d}}{D(\ln D)^2}\left[1+\frac{\frac{4}{c}-2.54}{\ln D}+\cdots\right], \ c \neq 0 . \qquad (5.56)$$

If μ_0 is such that the above restrictions are not satisfied, it may

be written in a form convenient for numerical computation as

$$S = W(\mu_0)\ln\frac{1-\mu_0}{\mu_0} + \int_0^1 \frac{W(\nu)-W(\mu_0)}{\nu-\mu_0}\, d\nu \qquad (5.57)$$

where

$$W(\nu) = \frac{c(1-c)}{2} \nu^{n+1} e^{-d/\nu} X^2(-\nu)(\nu_0^2 - \nu^2) \, g(c,\nu) \, . \tag{5.58}$$

In addition to the previous results already derived in this chapter, a few specialized answers may also be obtained from the analysis. We will now proceed to the determination of:

1. The expansion coefficients for the half-space albedo problem for isotropic scattering.

2. The transmission and reflection coefficients for a thick slab.

3. The current through a slab when $c = 1$.

4. The expansion coefficients for a bare slab with a constant surface source (plane source) on one face.

1. In the limit as the slab thickness becomes infinite, all approximations reduce to the zeroth-order approximation and, from (5.27) and (5.28), the expansion coefficients are:

$$a_-^{(0)} = A^{(0)}(-\mu) = 0$$

$$a_+^{(0)} = -\frac{2 \, \gamma(\mu_0)}{c \nu_0 \, X(\nu_0)}$$

$$A^{(0)}(\mu) = \frac{\gamma(\mu_0)\left[\varphi_\mu(\mu_0) + \frac{\mu}{\nu_0} \varphi_+(\mu)\right]}{\gamma(\mu) \, \Lambda^+(\mu) \Lambda^-(\mu)}$$

$$= \frac{(\nu_0 - \mu_0) \, \gamma(\mu_0) \, \varphi_\mu(\mu_0)}{(\nu_0 - \mu) \, \gamma(\mu) \, \Lambda^+(\mu) \Lambda^-(\mu)} \, . \tag{5.59}$$

These results may also be obtained from Equations (4.7) to
(4.10).

2. The transmission coefficient for a slab is defined as

$$
T = \frac{\int_0^1 \mu \Psi(d,\mu)d\mu}{\int_0^1 \mu \Psi(0,\mu)d\mu} = \frac{j(d)}{\mu_0} \tag{5.60}
$$

and the reflection coefficient (commonly called the "albedo" in half-space problems) is given by

$$
R = - \frac{\int_{-1}^0 \mu \Psi(0,\mu)d\mu}{\int_0^1 \mu \Psi(0,\mu)d\mu} = 1 - \frac{j(0)}{\mu_0} \tag{5.61}
$$

The values of T and R may be found from (5.43) for a thick slab.

3. When c = 1 , the current is constant. From (5.27),
(5.28), and (5.32), we find that

$$
\lim_{c \to 1} j^{(0)}(x) = \frac{\mu_0}{(2z_0+d)X(-\mu_0)} , \tag{5.62}
$$

This zeroth-order result for the current agrees with the answer obtained by Kuscer[31] when Equation (H.5) of Appendix H is used. Using
(5.29) and (5.31), one obtains

$$
\lim_{c \to 1} j^{(1)}(x) = \frac{\mu_0(1+K)}{(2z_0+d)X(-\mu_0)} \tag{5.63}
$$

where the correction term to the zeroth-order approximation is given by

$$K = \frac{1}{3} \int_0^1 \frac{e^{-d/\nu} X^2(-\nu)\left[\mathcal{P}_\nu(\mu_0) + \nu/(2z_0+d)\right] d\nu}{\Lambda^+(\nu)\Lambda^-(\nu)}$$

$$= \lim_{c\to 1}\left[J_{00} + \frac{2 I_{10}}{3(2z_0+d)c(1-c)}\right] . \tag{5.64}$$

4. The expansion coefficients for the bare slab with an isotropic surface source on one face may be obtained from the previous results by integration. Using the Equation (D.7), we observe that

$$\int_0^1 \mathcal{P}_\mu(\mu_0)\, \Upsilon(\mu_0)\, d\mu_0 = \int_0^1 J_{0m}\, \Upsilon(\mu_0)\, d\mu_0 = 0 . \tag{5.65}$$

Also, from (2.36),

$$\Upsilon^{(0)} = \int_0^1 \Upsilon(\mu_0)\, d\mu_0 = 1 . \tag{5.66}$$

Integrating (5.27), (5.28), (5.29) and (5.31) over μ_0 , one therefore obtains

$$\bar{a}_+^{(0)} = -e^{(2z_0+2d)/\nu_0}\, \bar{a}_-^{(0)} = \frac{e^{d/\nu_0}}{c\,\nu_0\, X(-\nu_0)\,\sinh\left[(2z_0+d)/\nu_0\right]}$$

$$\bar{A}^{(0)}(\mu) = -\frac{X(-\mu)(\nu_0^2-\mu^2)(1-c)}{\Lambda^+(\mu)\Lambda^-(\mu)}\left[\bar{a}_+^{(0)}\,\mathcal{P}_+(\mu)\, X(\nu_0)\right.$$

$$\left. + \bar{a}_-^{(0)}\,\mathcal{P}_-(\mu)\, X(-\nu_0)\right]$$

$$\bar{A}^{(0)}(-\mu) = -e^{-d/\mu} \frac{X(-\mu)(\nu_0^2 - \mu^2)(1-c)}{\Lambda^+(\mu)\Lambda^-(\mu)} \left[\bar{a}_-^{(0)} e^{d/\nu_0} \varphi_+(\mu) X(\nu_0) \right.$$

$$\left. + \bar{a}_+^{(0)} e^{-d/\nu_0} \varphi_-(\mu) X(-\nu_0) \right]$$

$$\frac{\bar{a}_\pm^{(1)}}{\bar{a}_\pm^{(0)}} = 1 + \nu_0 I_{10} \operatorname{csch}\left[\frac{2z_0 + d}{\nu_0}\right]$$

$$+ e^{\mp(2z_0 + d)/\nu_0}\left[I_{20} + \nu_0 I_{10} \operatorname{coth}\left[\frac{2z_0 + d}{\nu_0}\right]\right]$$

$$\bar{A}^{(1)}(\mu) = \bar{A}^{(0*)} + \frac{c\nu_0(1-c)X(-\mu)(\nu_0^2 - \mu^2)}{2\Lambda^+(\mu)\Lambda^-(\mu)} \quad (x)$$

$$(x)\left[\bar{a}_+^{(1)} e^{-d/\nu_0} X(-\nu_0)\left[\nu_0 I_{11} - I_{21}\right] \right.$$

$$\left. + \bar{a}_-^{(1)} e^{d/\nu_0} X(\nu_0)\left[\nu_0 I_{11} + I_{21}\right] \right]$$

$$\bar{A}^{(1)}(-\mu) = \bar{A}^{(0*)}(-\mu) + e^{-d/\mu}\frac{c\nu_0(1-c)X(-\mu)(\nu_0^2 - \mu^2)}{2\Lambda^+(\mu)\Lambda^-(\mu)} \quad (x)$$

$$(x)\left[\bar{a}_+^{(1)} X(\nu_0)\left[\nu_0 I_{11} + I_{21}\right] \right.$$

$$\left. + \bar{a}_-^{(1)} X(-\nu_0)\left[\nu_0 I_{11} - I_{21}\right] \right] \quad . \qquad (5.67)$$

The analysis of the slab problem in this chapter can be used for different boundary conditions. In Appendix F, other slab problems are considered where there is an incident distribution on one face and a specularly or diffusely reflecting boundary at the other face.

QUADRATICALLY ANISOTROPIC SCATTERING IN
A NONABSORBING MEDIUM

As was mentioned in Chapter I, problems with this type of scattering are not of much interest in neutron transport theory, but are examined in photon transport theory because the scattering function is a generalization of the Rayleigh scattering function.[2] Because of our assumption that the transport equation and boundary conditions possess azimuthal symmetry, however, we shall derive only the azimuthally-independent part of the angular distribution, i.e., the part which is obtained after averaging the complete solution over the azimuthal angle. Some discussion of the transport equation subject to azimuthally-dependent boundary conditions is found in Appendix F.

Using (1.4) with $N = 2$ and $c = 1$,

$$\left[\mu \frac{\partial}{\partial x} + 1\right] \Psi(x,\mu) = \frac{1}{2} \int_{-1}^{1} \left[1 + b_1 \mu \mu' + b_2 P_2(\mu) P_2(\mu')\right] \Psi(x,\mu') d\mu', \quad (6.1)$$

Once again we use the <u>ansatz</u>

$$\Psi(x,\mu) = \phi_\nu(\mu) e^{-x/\nu} \quad (6.2)$$

to separate the variables in (6.1) and obtain

$$(\nu - \mu) \phi_\nu(\mu) = \frac{\nu}{2} \int_{-1}^{1} \left[1 + b_1 \mu \mu' + b_2 P_2(\mu) P_2(\mu')\right] \phi_\nu(\mu') d\mu'. \quad (6.3)$$

Multiplying (6.3) by dµ and µdµ and integrating shows that

$$\int_{-1}^{1} \mu \, \varphi_\nu(\mu) \, d\mu = 0$$

$$\int_{-1}^{1} \mu^2 \, \varphi_\nu(\mu) \, d\mu = 0 \tag{6.4}$$

and so

$$\int_{-1}^{1} P_2(\mu') \, \varphi_\nu(\mu') \, d\mu' = -\frac{1}{2} \int_{-1}^{1} \varphi_\nu(\mu') \, d\mu' . \tag{6.5}$$

Therefore, (5.3) reduces to

$$(\nu-\mu) \, \varphi_\nu(\mu) = \frac{\nu}{2} f(\mu) \int_{-1}^{1} \varphi_\nu(\mu') \, d\mu' \tag{6.6}$$

where $f(\mu)$ is defined by the equation

$$f(\mu) = 1 - \frac{b_2}{2} P_2(\mu) . \tag{6.7}$$

We again impose the normalization condition

$$\int_{-1}^{1} \varphi_\nu(\mu) \, d\mu = \int_{-1}^{1} \Psi(0,\mu) \, d\mu = 1 \tag{6.8}$$

which enables us to write (6.6) as

$$\varphi_\nu(\mu) = P \frac{\frac{\nu}{2} f(\mu)}{\nu-\mu} + \lambda(\nu) \, \delta(\nu-\mu) . \tag{6.9}$$

The value of $\lambda(\nu)$ may be obtained from the two preceeding equations and is

$$\lambda(\nu) = 1 - \frac{\nu}{2} P \int_{-1}^{1} \frac{f(\mu) \, d\mu}{\nu-\mu} . \tag{6.10}$$

Equation (6.10) may also be written as

$$\lambda(\nu) = \frac{\Lambda^+(\nu) + \Lambda^-(\nu)}{2}$$

(6.11)

where

$$\Lambda^{\pm}(\nu) = \lambda(\nu) \pm \frac{i\pi\nu f(\nu)}{2} \quad , \quad -1 < \nu < 1$$

(6.12)

and where $\Lambda(z)$ is given by

$$\Lambda(z) = 1 - \frac{z}{2} \int_{-1}^{1} \frac{f(\mu)\, d\mu}{z - \mu} \quad .$$

(6.13)

The equation $\Lambda(z) = 0$ is satisfied only in the limit of large z since

$$\Lambda(z) = 1 - \frac{1}{2} \int_{-1}^{1} f(\mu) \left[1 + \frac{\mu}{z} + \frac{\mu^2}{z^2} + \cdots \right] d\mu$$

$$\sim - \frac{1}{3z^2} \left[1 - \frac{b_2}{5} \right] \quad .$$

(6.14)

This means that there are two roots at $\nu = \infty$ in a manner similar to the case of isotropic and linearly anisotropic scattering with $c = 1$. The two linearly independent discrete eigenfunctions of the transport equation are found directly from the Boltzmann equation as

$$\psi_1(x,\mu) = \frac{1}{2}$$

$$\psi_2(x,\mu) = \frac{1}{2} \left[x - \frac{3\mu}{3 - b_1} \right] \quad .$$

(6.15)

Since all the eigenfunctions of the homogeneous transport equation are now determined, we can use the completeness theorem[21] to write the angular density of (6.1) as

$$\psi(x,\mu) = \frac{a_1}{2} + \frac{a_2}{2}\left(x - \frac{3\mu}{3-b_1}\right) + \int_{-1}^{1} A(\nu)\,\varphi_\nu(\mu)\,e^{-x/\nu}\,d\nu \ . \qquad (6.16)$$

There is an additional difficulty encountered in the special case of c = 1 which we have not previously considered. Substituting (6.15) into (6.8) shows that the normalization condition is satisfied by $\psi_1(x,\mu)$ but is not satisfied by $\psi_2(x,\mu)$. Therefore, we are still lacking a normalization condition for $\psi_2(x,\mu)$. From Shure and Natelson,[22] we realize that when c = 1 , the steady-state homogeneous transport equation satisfies the equation

$$j(x) = \int_{-1}^{1} \mu\,\psi(x,\mu)\,d\mu = CONSTANT \equiv J \ , \qquad (6.17)$$

Thus, this is the normalization condition we seek. Using (6.16) and (6.17) gives

$$J = \frac{a_1}{2}\int_{-1}^{1} \mu\,d\mu + \frac{a_2}{2}\int_{-1}^{1} \mu\left(x - \frac{3\mu}{3-b_1}\right)d\mu$$

$$+ \int_{-1}^{1} A(\nu)\,e^{-d/\nu}\left[\int_{-1}^{1} \mu\,\varphi_\nu(\mu)\,d\mu\right]d\nu \ . \qquad (6.18)$$

By use of (6.4), this reduces to

$$J = -\frac{a_2}{3-b_1} \ . \qquad (6.19)$$

Now re-examine Equation (6.1). It may be rewritten as

$$\left[\mu \frac{\partial}{\partial x} + 1\right] \psi(x,\mu) = \frac{1}{2}\int_{-1}^{1}\left[1 + b_2 P_2(\mu) P_2(\mu')\right]\psi(x,\mu')d\mu'$$

$$+ \frac{b_1 J\mu}{2} \quad . \tag{6.20}$$

It is necessary to make (6.20) a homogeneous equation. This may be done by the substitution of

$$\psi(x,\mu) = \psi'(x,\mu) + \frac{b_1 Jx}{2} \tag{6.21}$$

which reduces (6.20) to

$$\left[\mu \frac{\partial}{\partial x} + 1\right]\psi'(x,\mu) = \frac{1}{2}\int_{-1}^{1}\left[1 + b_2 P_2(\mu)P_2(\mu')\right]\psi'(x,\mu')d\mu' . \tag{6.22}$$

Therefore, the transport equation for quadratically anisotropic scattering with $c = 1$ and b_1 not necessarily zero may be reduced to the same equation with $b_1 = 0$ by the use of (6.21).

The two problems we will solve in this chapter are the half-space a) Milne and b) albedo problems for quadratically anisotropic scattering with $c = 1$ and b_1 not necessarily equal to zero. The boundary conditions for the angular density, $\psi(x,\mu)$, are

$$\psi(0,\mu) = \begin{cases} 0 & , \mu > 0 \quad \text{a)} \\ \\ \delta(\mu - \mu_0) & , \mu > 0 \quad \text{b)} \end{cases} \tag{6.23}$$

$$\Psi(x,\mu) = \begin{cases} \frac{1}{2}\left(x - \frac{3\mu}{3-b_1}\right) & \text{AS } x \to \infty & \text{a)}* \\ \\ \text{FINITE} & \text{AS } x \to \infty & \text{b)} \end{cases} \qquad (6.24)$$

where $\Psi(x,\mu)$ is given by (6.16). Applying (6.24) to (6.16) shows that $\Psi(x,\mu)$ may be written as

$$\Psi(x,\mu) = f(x,\mu) + \frac{a_1}{2} + \int_0^1 A(\nu)\,\varphi_\nu(\mu')\,e^{-x/\nu}\,d\nu \qquad (6.25)$$

where

$$f(x,\mu) = \begin{cases} \frac{1}{2}\left(x - \frac{3\mu}{3-b_1}\right) & \text{a)} \\ \\ 0 & \text{b).} \end{cases} \qquad (6.26)$$

The net currents are therefore given by

$$J = \begin{cases} -\dfrac{1}{3-b_1} & \text{a)} \\ \\ 0 & \text{b)}** . \end{cases} \qquad (6.27)$$

---

*The slowest growing mode for large x is $\Psi_2(x,\mu)$, arbitrarily normalized by setting $a_2 = 1$.

**Perhaps the logical order of (6.24b) and (6.27b) should be reversed. Indeed there is no a priori reason to demand that Ψ be finite as $x \to \infty$. On the other hand, since every neutron entering the medium eventually leaves at the interface, there is indeed no net current. The discrete mode Ψ_2 is the only one which can contribute to the current so the boundary condition on Ψ as $x \to \infty$ is selected to make the net current vanish.

If we consider the solution to the transport equation where $b_1 = 0$, Equation (6.22), the angular density is now given by

$$\Psi'(x,\mu) = f(x,\mu) + \frac{a_1}{2} + \int_0^1 A(\nu)\,\varphi_\nu(\mu)\,e^{-x/\nu}\,d\nu \,. \tag{6.28}$$

The $\Psi'(x,\mu)$ is related to $\Psi(x,\mu)$ (for the case where b_1 is not necessarily zero) by

$$\Psi(x,\mu) = \Psi'(x,\mu) - \begin{cases} \dfrac{b_1 x}{2(3-b_1)} & \text{a)} \\[2ex] 0 & \text{b)} \end{cases} \tag{6.29}$$

where (6.21) and (6.27) have been used. Now that the two problems we wish to solve have been formulated, we will proceed to the various identities satisfied by the X-function and obtain the orthogonality relations which will enable us to determine a_1 and $A(\nu)$ from the equations

$$\Psi'(0,\mu) - f(0,\mu) = \frac{a_1}{2} + \int_0^1 A(\nu)\,\varphi_\nu(\mu)\,d\nu = \begin{cases} \dfrac{\mu}{2}\left(\dfrac{3}{3-b_1}\right),\ \mu > 0 & \text{a)} \\[2ex] \delta(\mu - \mu_0)\ ,\ \mu > 0 & \text{b).} \end{cases} \tag{6.30}$$

When comparing the work of this chapter with that concerning linearly anisotropic scattering in Chapters II and III, one observes that the polynomial $f(\mu)$ replaces the polynomials $cd(\nu\mu)$ and $cd(\mu\mu)$ in the prior work. Therefore, in analogy to (2.67) to (2.69),

the half-space X-function of Case (which has the properties attributed

to all X-functions in Chapter II and vanishes like z^{-1} as $z \to \infty$)

satisfies

$$X(z) = \int_0^1 \frac{\gamma(\mu) f(\mu) d\mu}{\mu - z} \tag{6.31}$$

$$\gamma(\nu) \lambda(\nu) = \frac{\nu}{2} P \int_0^1 \frac{\gamma(\mu) f(\mu) d\mu}{\mu - \nu} \ , \ 0 < \nu < 1 \tag{6.32}$$

$$1 = \int_0^1 \gamma(\mu) f(\mu) d\mu \tag{6.33}$$

where

$$\gamma(\nu) = \frac{\nu}{2} \frac{X^-(\nu)}{\Lambda^-(\nu)} \ , \ 0 < \nu < 1 \ . \tag{6.34}$$

Equation (6.9) may be used to rewrite (6.32) as

$$\int_0^1 \varphi_\nu(\mu) \gamma(\mu) d\mu = 0 \ . \tag{6.35}$$

Also by analogy, Equation (3.65) becomes

$$\varphi_\nu(\mu) \varphi_{\nu'}(\mu) = \frac{1}{\nu - \nu'} \left[\frac{\nu}{2} f(\mu) \varphi_{\nu'}(\mu) - \frac{\nu'}{2} f(\mu) \varphi_\nu(\mu) \right]$$
$$+ \Lambda^+(\mu) \Lambda^-(\mu) \delta(\mu - \nu) \delta(\mu - \nu') \ . \tag{6.36}$$

Since $f(\mu)$ has no zeros in $[0,1]$, Equation (6.36) may be multiplied

by $\gamma(\mu) d\mu / f(\mu)$ and integrated over μ to get

$$\int_0^1 \phi_\nu(\mu)\, \phi_{\nu'}(\mu)\, \frac{\gamma(\mu)}{f(\mu)}\, d\mu = \frac{\gamma(\nu)\, \Lambda^+(\nu)\Lambda^-(\nu)\, \delta(\nu-\nu')}{f(\nu)} \tag{6.37}$$

where (6.35) has been used. This equation is the normalization rela-tion, with weight function $\gamma(\mu)/f(\mu)$ between the continuum modes. This is no guarantee, however, that the continuum eigenfunction $\phi_\nu(\mu)$ will be orthogonal to the eigenfunction $\psi_1(0,\mu) = \frac{1}{2}$ with weight function $\gamma(\mu)/f(\mu)$. In fact, by use of the identity

$$\frac{f(\nu)}{\mu-\nu} = \frac{f(\mu)}{\mu-\nu} + \frac{3b_2}{4}(\mu+\nu), \tag{6.38}$$

it can be shown that

$$\int_0^1 \frac{1}{2}\, \phi_\nu(\mu)\, \frac{\gamma(\mu)}{f(\mu)}\, d\mu = -\left(\frac{1}{2}\right)\frac{3b_2\,\nu(\nu+\bar{\nu})\,\gamma^{(0)}}{8\,f(\nu)} . \tag{6.39}$$

In (6.39), we have again used the definitions

$$\gamma^{(i)} = \int_0^1 \mu^i\, \gamma(\mu)\, d\mu \quad , \quad i = 0,1 \tag{6.40}$$

and

$$\bar{\nu} = \frac{\gamma^{(1)}}{\gamma^{(0)}} . \tag{6.41}$$

Equation (6.39) may be rearranged to give

$$\int_0^1 \frac{1}{2}\left[\mathcal{G}_\nu(\mu) + \frac{3b_2\,\nu\,f(\mu)(\nu+\bar{\nu})}{8\,f(\nu)}\right]\frac{\gamma(\mu)}{f(\mu)}\,d\mu = 0 \,. \tag{6.42}$$

Use of (6.35) and (6.37) shows that

$$\int_0^1 \mathcal{G}_\nu(\mu)\left[\mathcal{G}_{\nu'}(\mu) + \frac{3b_2\,\nu'f(\mu)(\nu'+\bar{\nu})}{8\,f(\nu')}\right]\frac{\gamma(\mu)}{f(\mu)}\,d\mu =$$
$$\frac{\gamma(\nu)\,\Lambda^+(\nu)\,\Lambda^-(\nu)\,\delta(\nu-\nu')}{f(\nu)} \,. \tag{6.43}$$

In order to completely determine the continuum mode coefficient by use of the above relations, one also needs the equation

$$\int_0^1 \left(-\frac{\mu}{2}\right)\left[\mathcal{G}_\nu(\mu) + \frac{3b_2\,\nu\,f(\mu)(\nu+\bar{\nu})}{8\,f(\nu)}\right]\frac{\gamma(\mu)}{f(\mu)}\,d\mu =$$
$$\frac{\nu}{4\,\gamma^{(0)}f(\nu)}\left[\left[\gamma^{(0)}\right]^2\left(1+\frac{b_2}{4}\right) - \frac{3b_2}{4}\left[\gamma^{(1)}\right]^2\right] \,. \tag{6.44}$$

Following the approach of Shure and Natelson,[22] the right hand side of (6.44) may be simplified. The X-function under consideration obeys the equation

$$X(z)X(-z) = \Lambda(z)X^2(0) \tag{6.45}$$

where

$$\chi^2(0) = \lim_{z \to \infty} \frac{1}{-z^2 \Lambda(z)} = \frac{3}{1 - \frac{b_2}{5}} \tag{6.46}$$

is obtained using (6.14). The equation $f(\mu) = 0$ has roots

$$\mu = \pm \left[\frac{1 + \frac{b_2}{4}}{\frac{3b_2}{4}} \right]^{1/2} = \pm \sigma .$$

From (6.31),

$$\chi(\pm\sigma) = -\frac{3b_2}{4} \int_0^1 \gamma(\mu)(\mu \pm \sigma) d\mu = -\frac{3b_2}{4} \left[\gamma^{(1)} \pm \sigma \gamma^{(0)} \right] . \tag{6.47}$$

Using (6.45) through (6.47), we find that

$$\frac{3\Lambda(\sigma)}{1 - \frac{b_2}{5}} = \left(\frac{3b_2}{4} \right)^2 \left[[\gamma^{(1)}]^2 - \sigma^2 [\gamma^{(0)}]^2 \right] , \tag{6.48}$$

Setting $z = \sigma$ in (6.13) gives

$$\Lambda(\sigma) = 1 - \frac{3b_2 \sigma}{8} \int_{-1}^1 (\sigma + \mu) d\mu = -\frac{b_2}{4} \tag{6.49}$$

so that (6.48) becomes

$$\frac{1}{1 - \frac{b_2}{5}} = [\gamma^{(0)}]^2 \left(1 + \frac{b_2}{4} \right) - \frac{3b_2}{4} [\gamma^{(1)}]^2 = [\gamma^{(0)}]^2 f(\bar{\nu}) , \tag{6.50}$$

Thus, Equation (6.44) becomes

$$\int_0^1 \left(-\frac{\mu}{2}\right)\left[\varphi_\nu(\mu) + \frac{3b_2\,\nu\,f(\mu)\,(\nu+\bar\nu)}{8f(\nu)}\right]\frac{\gamma(\mu)}{f(\mu)}\,d\mu =$$

$$\frac{\nu}{4\,\gamma^{(0)}\,f(\nu)\,(1-b_2/5)} \tag{6.51}$$

and the four equations necessary to determine the expansion coefficients in the half-space Milne and albedo problems are (6.35), (6.42), (6.43), and (6.51). These expansion coefficients will be found in terms of b_2 , $X(-\mu)$ or $\gamma(\mu)$, and $\gamma^{(i)}$ for $i = 0,1$. The function $\gamma(\mu)$ can be related to $X(-\mu)$ by use of (6.34), (6.45), and (6.46); the result is:

$$\gamma(\mu) = \frac{3\mu}{2\,X(-\mu)\,(1-b_2/5)} \,,\quad 0 \le \mu < 1 \,. \tag{6.52}$$

We now apply the orthogonality relations to Equation (6.30) to obtain a_1 and $A(\nu)$. Using (6.35), (6.30) reduces to

$$a_1 = \begin{cases} \dfrac{3\,\bar\nu}{3-b_1} & \text{a)} \\[3mm] \dfrac{2\,\gamma(\mu_0)}{\gamma^{(0)}} & \text{b)}. \end{cases} \tag{6.53}$$

Equations (6.42), (6.43), and (6.51) reduce (6.30) to

$$A(\nu) = \begin{cases} -\dfrac{X(-\nu)}{2(3-b_1)\,\gamma^{(0)}\,\Lambda^+(\nu)\,\Lambda^-(\nu)} & \text{a)} \\[6mm] \dfrac{\left[\mathscr{P}_\nu(\mu_0) + \dfrac{3\nu\,b_2\,f(\mu_0)(\nu+\bar\nu)}{8\,f(\nu)}\right]\dfrac{f(\nu)}{f(\mu_0)}\,\gamma(\mu_0)}{\gamma(\nu)\,\Lambda^+(\nu)\,\Lambda^-(\nu)} & \text{b)} \end{cases} \qquad (6.54)$$

so the angular density $\Psi'(x,\mu)$ for quadratically anisotropic scatter-
ing with $c = 1$ and $b_1 = 0$ is now completely specified. Equations
(6.28), (6.29), (6.53), and (6.54) give the solutions to the half-
space a) Milne and b) albedo problems for quadratically anisotropic
scattering with $c = 1$.

As was the case in Chapter IV, the integrals for the emergent
angular distribution from the half-space may be done explicitly. From
(6.28) and (6.29),

$$\Psi'(0,-\mu) = \Psi(0,-\mu) = f(0,-\mu) + \frac{a_1}{2}$$

$$+ \int_0^1 A(\nu)\,\mathscr{P}_{-\nu}(\mu)\,d\nu \quad , \mu > 0 , \qquad (6.55)$$

where, for

$$\Psi(\mu) = \Psi(0,\mu) - f(0,\mu) , \qquad (6.56)$$

the expansion coefficients are

$$\frac{a_1}{2} = \frac{1}{\gamma^{(0)}} \int_0^1 \psi(\mu') \gamma(\mu') d\mu'$$

$$A(\nu) = \frac{f(\nu)}{\gamma(\nu) \Lambda^+(\nu) \Lambda^-(\nu)} \int_0^1 \psi(\mu') \left[\varphi_\nu(\mu') + \frac{3\nu b_2 f(\mu')(\nu + \bar{\nu})}{8 f(\nu)} \right] \frac{\gamma(\mu')}{f(\mu')} d\mu'. \quad (6.57)$$

Using (6.57) in (6.55) gives

$$\psi(0, -\mu) = f(0, -\mu) + \frac{1}{\gamma^{(0)}} \int_0^1 \psi(\mu') \gamma(\mu') d\mu'$$

$$+ \int_0^1 \psi(\mu') \frac{\gamma(\mu')}{f(\mu')} \left[\int_0^1 \frac{\varphi_{-\nu}(\mu) f(\nu)}{\gamma(\nu) \Lambda^+(\nu) \Lambda^-(\nu)} \right. \quad (x)$$

$$(x) \left\{ \varphi_\nu(\mu') + \frac{3\nu b_2 f(\mu')(\nu + \bar{\nu})}{8 f(\nu)} \right\} d\nu \left. \right] d\mu', \mu > 0. \quad (6.58)$$

From Appendix G,

$$\int_0^1 \frac{\varphi_{-\nu}(\mu) f(\nu) \left[\varphi_\nu(\mu') + \frac{3\nu b_2 f(\mu')(\nu+\bar{\nu})}{8 f(\nu)} \right] d\nu}{\gamma(\nu) \Lambda^+(\nu) \Lambda^-(\nu)} \Bigg|_{\substack{\mu > 0 \\ \mu' > 0}} =$$

$$- f(\mu') \left[\frac{1}{\gamma(0)} + \frac{1}{X(-\mu)} \left[\frac{3 b_2}{4} (\bar{\nu} - \mu) - \frac{f(\mu)}{\mu + \mu'} \right] \right], \qquad (6.59)$$

and so (6.59) simplifies to

$$\Psi(0, -\mu) = f(0, -\mu)$$

$$- \frac{1}{X(-\mu)} \int_0^1 \Psi(\mu') \gamma(\mu') \left[\frac{3 b_2}{4} (\bar{\nu} - \mu) - \frac{f(\mu)}{\mu + \mu'} \right] d\mu', \ \mu > 0. \ (6.60)$$

Since

$$\Psi(\mu') = \begin{cases} \dfrac{3\mu'}{2(3 - b_1)} & \text{a)} \\[2mm] \delta(\mu' - \mu_0) & \text{b)}, \end{cases} \qquad (6.61)$$

the results of (6.60) can finally be reduced to

$$\Psi(0,-\mu) = \begin{cases} \dfrac{3}{2(3-b_1)\,\gamma^{(0)}\,X(-\mu)(1-b_2/5)} \;, \mu > 0 & \text{a)} \\[3em] \dfrac{3\mu_0\left[1 + \dfrac{b_2}{4} + \dfrac{3b_2}{4}\mu\mu_0 - \dfrac{3b_2}{4}\bar{\nu}(\mu_0+\mu)\right]}{2\,(\mu_0+\mu)\,X(-\mu)\,(1-b_2/5)\,X(-\mu_0)} \;, \mu > 0 & \text{b)} \end{cases} \qquad (6.62)$$

where (6.50) was used in the derivation of (6.62a) and (6.52) was used to obtain (6.62b).

Equations (6.53a), (6.54a) and (6.62a) were obtained in a different manner by Shure and Natelson.* Equation (6.62b) agrees with the result of Chandrasekhar[2] when the conversion equations of Appendix H are used.

To find the surface densities and net currents, we can integrate over (6.60) or (6.62). The net currents are already known by (6.27), however. We will find the surface densities by integration of (6.62). From Appendix G, the integrals needed are

*Shure and Natelson[22] normalized to $J = -1/3$ instead of $J = -1/(3-b_1)$.

$$\int_0^1 \frac{d\mu}{X(-\mu)} = \frac{2(1-b_2/5)}{3(1+b_2/4)}\left[\sqrt{\frac{3}{1-b_2/5}} + \frac{3b_2\,\gamma^{(1)}}{4}\right]$$

$$\int_0^1 \frac{\left[1 + \frac{b_2}{4} + \frac{3b_2}{4}\mu\mu_0 - \frac{3b_2}{4}\bar{\nu}(\mu_0+\mu)\right]d\mu}{X(-\mu)(\mu+\mu_0)} =$$

$$\frac{2(1-b_2/5)}{3}\left[-\frac{X(-\mu_0)}{\mu_0} + \frac{\frac{b_2}{4}X^2(0)}{(1+b_2/4)\,\gamma^{(0)}}\right.$$

$$+ \left.\frac{\left[1 + \frac{b_2}{4} - \frac{3b_2}{4}\bar{\nu}\mu_0\right]X(0)}{\mu_0(1+b_2/4)}\right] \, . \tag{6.63}$$

The surface densities are given by

$$\rho(0) = \int_0^1 \Psi(0,\mu)\,d\mu + \int_0^1 \Psi(0,-\mu)\,d\mu \tag{6.64}$$

and, using (6.23) and (6.62), are found to be

$$\rho(0) = \begin{cases} \dfrac{\sqrt{\dfrac{3}{1-b_2/5}} + \dfrac{3b_2}{4}\gamma^{(1)}}{(3-b_1)\,\gamma^{(0)}(1+b_2/4)} & \text{a)} \\[4ex] \dfrac{X(0)}{X(-\mu_0)}\left[1 + \dfrac{-\dfrac{b_2}{4}\mu_0\left[X(0)-3\gamma^{(1)}\right]}{\gamma^{(0)}(1+b_2/4)}\right] & \text{b).} \end{cases} \tag{6.65}$$

For $x \neq 0$, the density may be found by use of Equations (6.8), (6.28), and (6.29). The result is

$$p(x) = \int_{-1}^{1} f(x, \mu) \, d\mu \; + \; a_1 \; + \int_{0}^{1} A(\nu) e^{-x/\nu} \, d\nu$$

$$- \begin{cases} \dfrac{b_1 \, x}{3 - b_1} & \text{a)} \\[4mm] 0 & \text{b).} \end{cases} \tag{6.66}$$

CHAPTER VII

GENERAL ANISOTROPIC SCATTERING

The transport problem for a medium which scatters neutrons in an arbitrary manner was considered by Mika,[21] who proved the completeness theorem and derived full-range orthogonality relations. For this case where the scattering is N'th-order, Equation (1.4) is to be solved. In this chapter, we shall proceed in a manner similar to that of Chapter III and attempt to determine a bi-orthogonal form which makes an eigenfunction of the homogeneous Boltzmann operator orthogonal to an adjoint function with respect to an appropriate weight function. No conclusive results, however, are obtained. The calculations are presented mainly to suggest a reasonable approach and to illustrate the manner in which difficulties arise.

Using the _ansatz_ of (2.2) in (1.4) and defining

$$\xi_k(\nu) = \int_{-1}^{1} P_k(\mu) \, \varphi_\nu(\mu) \, d\mu ,$$ (7.1)

one obtains

$$(\nu - \mu) \, \varphi_\nu(\mu) = \frac{c\nu}{2} \sum_{k=0}^{N} b_k \, P_k(\mu) \, \xi_k(\nu)$$ (7.2)

Multiplying both sides of (7.2) by $(2\ell+1) \, P_\ell(\mu)$, integrating over μ , and using the orthogonality and recurrence relations for Legendre polynomials, we find that the $\xi_k(\nu)$ are themselves polynomials with the recurrance relation

$$(\ell+1)\,\xi_{\ell+1}\,(\nu) - \nu\,h_\ell\,\xi_\ell\,(\nu) + \ell\,\xi_{\ell-1}\,(\nu) = 0, \quad \ell \geqslant 1, \tag{7.3}$$

where

$$h_\ell = 2\ell + 1 - c\,b_\ell \quad . \tag{7.4}$$

Again we choose the normalization

$$\xi_0(\nu) = \int_{-1}^{1} \mathscr{P}_\nu(\mu)\,d\mu = 1 \quad . \tag{7.5}$$

One then obtains

$$\xi_1(\nu) = h_0\,\nu = (1-c)\,\nu$$

$$\xi_2(\nu) = \frac{1}{2}\left(h_0\,h_1\,\nu^2 - 1\right)$$

$$\xi_\ell(\nu) = \frac{h_0 h_1 \cdots h_{\ell-1}}{\ell!}\left[\nu^\ell - \left\{\frac{1^2}{h_0 h_1} + \frac{2^2}{h_1 h_2} + \cdots + \frac{(\ell-1)^2}{h_{\ell-2} h_{\ell-1}}\right\}\nu^{\ell-2}\right.$$

$$\left. + \left\{\frac{1^2 \cdot 3^2}{h_0 h_1 h_2 h_3} + \frac{1^2 \cdot 4^2}{h_0 h_1 h_3 h_4} + \cdots + \frac{(\ell-3)^2(\ell-1)^2}{h_{\ell-4} h_{\ell-3} h_{\ell-2} h_{\ell-1}}\right\}\nu^{\ell-4} - \cdots\right], \ell \geqslant 1, \tag{7.6}$$

where the general term has been given by Kuščer.[34] The general term
may also be written as

$$\xi_\ell(\nu) = \frac{h_0 h_1 \cdots h_{\ell-1}}{\ell!} \sum_{i=0}^{[\ell/2]} \left[(-1)^i \nu^{\ell-2i} \prod_{j=0}^{i-1} \left\{ \sum_{k=2j}^{\ell-2i+2j} \frac{(k+1)^2}{h_k h_{k+1}} \right\} \right] , \quad \ell \geq 1 , \qquad (7.7)$$

where $\prod^{-1} = 1$ and where $[\ell/2]$ is defined as the greatest integer in
$\ell/2$. The function $\xi_\ell(\nu)$ is a polynomial of order ℓ and is a
generalization of the Legendre polynomial $P_\ell(\mu)$ to which it degenerates
if $c = 0$. Using (7.6), one has

$$\xi_\ell(\nu) = \sum_{j=0}^{[\ell/2]} k_{j\ell}\, \nu^{\ell-2j} \qquad (7.8)$$

where

$$k_{00} = 1 \qquad (7.9)$$

and where, for $\ell \geq 1$,

$$k_{0\ell} = \frac{h_0 h_1 \cdots h_{\ell-1}}{\ell!}$$

$$k_{1\ell} = -\left[\frac{1^2}{h_0 h_1} + \frac{2^2}{h_1 h_2} + \cdots + \frac{(\ell-1)^2}{h_{\ell-2} h_{\ell-1}} \right] k_{0\ell} \qquad (7.10)$$

and so on.

For the general-order anisotropic scattering, define

$$D(\nu,\mu) = \sum_{\ell=0}^{N}{}' b_\ell \, \xi_\ell(\nu) \, P_\ell(\mu) \; . \tag{7.11}$$

The eigenfunctions for $-1 < \nu < 1$ are

$$\varphi_\nu(\mu) = \varphi_{-\nu}(-\mu) = P \frac{\frac{c\nu}{2} D(\nu,\mu)}{\nu-\mu} + \lambda(\nu) \, \delta(\nu-\mu) \tag{7.12}$$

where

$$\lambda(\nu) = 1 - \frac{c\nu}{2} P \int_{-1}^{1} \frac{D(\nu,\mu)\,d\mu}{\nu-\mu} \; , \quad -1 < \nu < 1 \; . \tag{7.13}$$

The discrete eigenfunctions are

$$\varphi_{\pm j}(\mu) = \varphi_{\mp j}(-\mu) = \frac{\frac{c\nu_j}{2} D(\pm\nu_j,\mu)}{\nu_j \mp \mu} \; . \tag{7.14}$$

In Equation (7.14), the eigenvalues $\pm \nu_j$ for $j = 0,1,\ldots M$ are the roots of

$$\Lambda(\pm\nu_j) = 0 \tag{7.15}$$

where

$$\Lambda(z) = 1 - \frac{cz}{2} \int_{-1}^{1} \frac{D(z,\mu)\,d\mu}{z-\mu} = 1 - \frac{cz}{2} \int_{-1}^{1} \frac{D(\mu,\mu)\,d\mu}{z-\mu} \tag{7.16}$$

and where $M \leq N$.[8] Equation (7.16) is proved by showing that the integral

$$\int_{-1}^{1} \frac{\xi_\ell(z) - \xi_\ell(\mu)}{z - \mu} P_\ell(\mu) \, d\mu \quad , \ell \leq N$$

vanishes. Since $\xi_\ell(z)$ is a polynomial of order $\leq \ell$ in z , the integral reduces to a sum of integrals of the form

$$\int_{-1}^{1} \mu^j P_\ell(\mu) \, d\mu \quad , \; j \leq \ell - 1$$

which indeed vanish. The values of $\Lambda(z)$ as z approaches the cut from above/below are

$$\Lambda^{\pm}(\nu) = \lambda(\nu) \pm \frac{i \pi c \nu \, D(\nu, \nu)}{2} \quad , -1 < \nu < 1 . \tag{7.17}$$

The behavior of $\Lambda(z)$ at infinity can be found by considering

$$\Lambda(\infty) = 1 - \frac{c}{2} \sum_{\ell=0}^{N} b_\ell \lim_{z \to \infty} \int_{-1}^{1} \xi_\ell(\mu) P_\ell(\mu) \left[1 + \frac{\mu}{z} + \frac{\mu^2}{z^2} + \cdots \right] d\mu . \tag{7.18}$$

The value of $\Lambda(\infty)$ is finite and is given by

$$\Lambda(\infty) = h_0 - \frac{h_0 \, c b_1}{3} - \frac{h_0 h_1 \, c b_2}{3 \cdot 5} - \frac{h_0 h_1 h_2 \, c b_3}{3 \cdot 5 \cdot 7} - \cdots \left(N \ \text{TERMS} \right) . \tag{7.19}$$

By induction, it may be proved that[35]

$$\Lambda(\infty) = \prod_{i=0}^{N} \left[\frac{h_i}{2i+1} \right] .$$ (7.20)

For the case of general anisotropic scattering where N is finite, the solution to the Boltzmann equation becomes, by virtue of the completeness theorem of Mika:[21]

$$\Psi(x,\mu) = \sum_{j=0}^{M} a_{+j} \varphi_{+j}(\mu) e^{-x/\nu_j} + \sum_{j=0}^{M} a_{-j} \varphi_{-j}(\mu) e^{x/\nu_j}$$

$$+ \int_{-1}^{1} A(\nu) \varphi_{\nu}(\mu) e^{-x/\nu} d\nu .$$ (7.21)

The function $X_g(z)$ is again given by a linear combination of functions analogous to the forms in Equations (2.26) and (2.62):

$$X_g(z) = X_{g0}(z) \left[w_0 + \frac{w_1 h(-\alpha)}{(z-\alpha)^{p_1}} + \frac{w_2 h(\beta)}{(\beta-z)^{p_2}} \right] , \quad \begin{matrix} w_1 \neq 0 \\ w_2 \neq 0 \end{matrix}$$ (7.22)

where $X_{g0}(z)$ has the form of (2.20) and where the constants w_i, $i = 0,1,2$, depend upon c and b_i, $i = 0$ to N. The integer constants p_1 and p_2 are found from the relations

$$0 \leq \frac{\ln \Lambda^+(\alpha) - \ln \Lambda^-(\alpha)}{2\pi i} + p_1 < 1$$

$$0 \leq -\frac{\ln \Lambda^+(\beta) - \ln \Lambda^-(\beta)}{2\pi i} + p_2 < 1 \tag{7.23}$$

since

$$X_{g0}(z) \approx (z - \alpha)^{-\left[\frac{\ln \Lambda^+(\alpha) - \ln \Lambda^-(\alpha)}{2\pi i}\right]} \quad \text{AS} \quad z \to \alpha$$

$$X_{g0}(z) \approx (\beta - z)^{\left[\frac{\ln \Lambda^+(\beta) - \ln \Lambda^-(\beta)}{2\pi i}\right]} \quad \text{AS} \quad z \to \beta . \tag{7.24}$$

These value of p_1 and p_2 , for the special cases of interest, are given by:

α	β	p_1	p_2
-1	1	M+1	M+1
0	1	0	M+1
-1	0	M+1	0

Two equations of interest for the N'th-order anisotropic scattering are

$$X_g(z) - X_g(\infty) = \int_\alpha^\beta \frac{\gamma_g(\mu) D(\mu, \mu) d\mu}{\mu - z} \tag{7.25}$$

$$\gamma_g(\nu)\lambda(\nu) - \frac{c\nu}{2} X_g(\infty) = \frac{c\nu}{2} P\int_\alpha^\beta \frac{\gamma_g(\mu) D(\mu,\mu)d\mu}{\mu-\nu} \quad , \alpha<\nu<\beta \; . \tag{7.26}$$

The particular relations in the half-range case, where $w_0 = 0$ and $w_2 = 1$, are:

$$X(z) = \int_0^1 \frac{\gamma(\mu) D(\mu,\mu) d\mu}{\mu-z} \tag{7.27}$$

$$\gamma(\nu)\lambda(\nu) = \frac{c\nu}{2} P\int_0^1 \frac{\gamma(\mu) D(\mu,\mu) d\mu}{\mu-\nu} \quad , 0<\nu<1 \tag{7.28}$$

$$\int_0^1 \gamma(\mu) D(\mu,\mu) d\mu = 1 \; . \tag{7.29}$$

The half-range X-function satisfies the identity

$$X(z) X(-z) = \frac{\Lambda(z)}{\Lambda(\infty)\prod_{j=0}^M (\nu_j^2 - z^2)} \tag{7.30}$$

which means that

$$\gamma(\nu) = \frac{\frac{c\nu}{2}}{X(-\nu)\Lambda(\infty)\prod_{j=0}^M (\nu_j^2-\nu^2)} \quad , 0<\nu<1 \; . \tag{7.31}$$

For $\alpha = -1$ and $\beta = 1$ the full-range X-function may be written as

$$X(z) = \frac{\Lambda(z)}{\Lambda(\infty)\prod_{j=0}^M (\nu_j^2-z^2)} \tag{7.32}$$

so that

$$\gamma(\nu) = \frac{\dfrac{c\nu}{2}}{\Lambda(\infty) \displaystyle\prod_{j=0}^{M} (\nu_j^2 - \nu^2)}, \quad -1 < \nu < 1 .\tag{7.33}$$

Other full-range relations corresponding to $w_0 = 0$ and $w_1 = w_2 = \dfrac{1}{2}$ are:

$$X(z) = \int_{-1}^{1} \frac{\gamma(\mu) D(\mu,\mu) \, d\mu}{\mu - z}\tag{7.34}$$

$$\gamma(\nu) \lambda(\nu) = \frac{c\nu}{2} P \int_{-1}^{1} \frac{\gamma(\mu) D(\mu,\mu) \, d\mu}{\mu - \nu}, \quad -1 < \nu < 1\tag{7.35}$$

$$\int_{-1}^{1} \gamma(\mu) D(\mu,\mu) \, d\mu = 0\tag{7.36}$$

$$\int_{-1}^{1} \mu \, \gamma(\mu) D(\mu,\mu) \, d\mu = 1 .\tag{7.37}$$

In the present form, the preliminary analysis indeed appears formidable. It can be greatly simplified, however, by assuming that there is only one pair of roots to the dispersion relation (7.15). The question of the number of discrete modes has received attention by Chandrasekhar,[2] Davison,[6] and Kuščer.[36] Although it is tempting to argue that there are only two discrete roots for general anisotropic scattering, Kuščer[36] has shown that this is not necessarily true. However, for scattering which is not too highly anisotropic, the assumption of only one pair of roots should be permissible and we will therefore make this simplification. This assumption

is reasonable since, according to Davison,* it is true in "all practical problems hitherto considered."

In the full-range case, one verifies[21] directly from (7.2) that the orthogonality relations (3.1) to (3.3) for isotropic scattering remain valid. We proceed directly to the problem of finding orthogonality relations for the half-space problems. From our experience in Chapter III, we shall assume that the weight function for the half-range case is still $\gamma_g(\mu) = (\nu_0-\mu)\,\gamma(\mu)$. This assumption, however, is only a guess.

Following the approach used in the case of linearly anisotropic scattering, one can prove that

$$(\nu-\nu')\int_0^1 \frac{\varphi_\nu(\mu)}{D(\nu,\mu)}\,\frac{\varphi_{\nu'}(\mu)}{D(\nu',\mu)}\,D(\mu,\mu)(\nu_0-\mu)\,\gamma(\mu)\,d\mu = 0 , \quad \begin{array}{c} \alpha < \nu < \beta \\ \alpha < \nu' < \beta \end{array} \qquad (7.38)$$

provided $X_g(\infty) \neq 0$. The rearrangement of this equation into a generalization of the useful bi-orthogonality relation developed before is not as straight-forward since $D(\nu,\mu)$ is not a symmetric function except for the two cases previously considered ($N = 1$ or $N = 2$ with $c = 1$). Thus it is much more difficult to remove the polynomials $D(\nu,\mu)$ and $D(\nu',\mu)$ from the denominator of (7.38). Instead, we will approach the problem in a slightly different manner and examine a "recipe" for finding a bi-orthogonal form for half-space problems.

From (7.12) or (7.14),

$$(\nu-\mu)\,\varphi_\nu(\mu) = \frac{c\nu}{2}\,D(\nu,\mu) \qquad (7.39)$$

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* Reference 6, p. 242

and writing the equation again for v' and operating on the two equations
and subtracting gives

$$(v-v')\int_0^1 \varphi_v(\mu)\,\varphi_{v'}(\mu)\,(v_0-\mu)\,\gamma(\mu)\,d\mu =$$

$$\int_0^1 \left[\frac{cv}{2}\,D(v,\mu)\,\varphi_{v'}(\mu) - \frac{cv'}{2}\,D(v',\mu)\,\varphi_v(\mu)\right](v_0-\mu)\gamma(\mu)\,d\mu \quad . \qquad (7.40)$$

In order to obtain the desired bi-orthogonal form which will make the
right hand side of (7.40) vanish, add the term

$$(v-v')\int_0^1 \frac{cv'}{2}\,B(mn;v',\mu)\,\varphi_v(\mu)\,(v_0-\mu)\,\gamma(\mu)\,d\mu$$

to both sides. Here, the m in $B(mn;v',\mu)$ denotes the mode of v in
(7.40): $m = c$ for $0 < v < 1$ and $m = d$ for $v = v_0$. The mode of
v' is denoted by n and $n = c$ for $0 < v' < 1$ and $n = d$ for $v' = v_0$.
Note that $B(cn;v',\mu)$ can not contain the variable of integration in the
eigenfunction expansion (7.21). However, $B(mn; v',\mu)$ can be an explicit
function of v' and of μ , as is denoted by the notation selected. One
now has

$$(v-v')\int_0^1 \varphi_v(\mu)\left[\varphi_{v'}(\mu) + B(mn;v',\mu)\,\frac{cv'}{2}\right](v_0-\mu)\gamma(\mu)\,d\mu = 0 \qquad (7.41)$$

provided $B(mn;v',\mu)$ is constructed such that

$$\sum_{i=0}^{N} b_i \left[\frac{c\nu}{2} \, \xi_i(\nu) \int_0^1 P_i(\mu) \, \varphi_{\nu'}(\mu)(\nu_0 - \mu) \, \gamma(\mu) \, d\mu \right.$$

$$\left. - \frac{c\nu'}{2} \, \xi_i(\nu') \int_0^1 P_i(\mu) \, \varphi_{\nu}(\mu)(\nu_0 - \mu) \, \gamma(\mu) \, d\mu \right]$$

$$+ (\nu - \nu') \frac{c\nu'}{2} \int_0^1 B(mn; \nu', \mu) \, \varphi_\nu(\mu)(\nu_0 - \mu) \gamma(\mu) \, d\mu = 0 \, . \qquad (7.42)$$

Equation (7.11) has been used in obtaining the above equation.

To simplify (7.42), define

$$\int_0^1 P_i(\mu) \, \varphi_\nu(\mu)(\nu_0 - \mu) \, \gamma(\mu) \, d\mu = \frac{c\nu}{2} \, K_{i,N}(\nu) \, , \quad i \le N \qquad (7.43)$$

where $K_{i,N}(\nu)$ is a polynomial in ν of order $\le N$ (ν may be either a continuum or discrete mode). Equations (7.42) and (7.43) combine to give

$$\frac{c\nu}{2} \frac{c\nu'}{2} \sum_{i=0}^{N} b_i \left[\xi_i(\nu) K_{i,N}(\nu') - \xi_i(\nu') K_{i,N}(\nu) \right]$$

$$+ (\nu - \nu') \frac{c\nu'}{2} \int_0^1 B(mn; \nu', \mu) \, \varphi_\nu(\mu)(\nu_0 - \mu) \gamma(\mu) \, d\mu = 0 \, . \qquad (7.44)$$

After dividing by $v-v'$, this may be written in the form

$$\int_0^1 B(mn; v',\mu)\, \mathcal{P}_v(\mu)(\mu_0-\mu)\, \mathcal{T}(\mu)\, d\mu = \frac{c\nu}{2} L_N(v, v')$$

(7.45)

where

$$L_N(v, v') = \frac{\sum_{i=0}^{N} b_i \left[\xi_i(v') K_{i,N}(v) - \xi_i(v) K_{i,N}(v') \right]}{v - v'}$$

(7.46)

The function $L_N(v, v')$ is a symmetric polynomial of order $\leq$ (N-1) in v and v' so therefore $B(mn; v', \mu)$ must be a polynomial of order $\leq$ (N-1) in v' if (7.45) is to be valid. In order to simplify the solution of (7.45), we will try to expand $B(mn; v', \mu)$ in the form

$$B(mn; v',\mu) = \sum_{i=0}^{N-1} Q_i(mn; v') P_i(\mu)$$

(7.47)

where $\sum^{-1} = 0$ and where

$$Q_i(mn; v) = \sum_{j=0}^{N-1} q_{ij}(mn)\, v^j$$

(7.48)

Use of (7.47) reduces (7.45) to the equation

$$\sum_{i=0}^{N-1} Q_i(mn; v') K_{i,N}(v) = L_N(v, v')$$

(7.49)

after (7.43) is used. From (7.47) and (7.48), the final form of

$B(mn;v',\mu)$ (such that (7.41) is valid) may be written as

$$B(mn; v', \mu) = \sum_{i=0}^{N-1} P_i(\mu) \sum_{j=0}^{N-1} q_{ij}(mn)(v')^j \tag{7.50}$$

where the values of $q_{ij}(mn)$ may be determined from the equation

$$\sum_{i=0}^{N-1} K_{i,N}(v) \sum_{j=0}^{N-1} q_{ij}(mn)(v')^j = L_N(v,v') \tag{7.51}$$

provided they exist.

Notice that no where have we proved the existence of the

$q_{ij}(mn)$ which may depend only upon m and n , c , v_0 , and $\gamma^{(i)}$

and b_i , i = 0 to N . We will merely assume that they may be found

from the above speculative analysis. In the solution for $q_{ij}(mn)$, the

left and right hand sides of (7.51) may be grouped in terms multiplied

by v^i and $(v')^j$, i, j = 0 to N-1 . Equation (7.51) then dissolves

itself into N^2 equations for $q_{ij}(mn)$. Because of the symmetry on the

right hand side, however, we may judiciously group the equations into

N systems of N equations each. Thus the values of $q_{ij}(mn)$ may be

found after evaluating N'th-order determinants. The need for evaluating

N'th-order determinants when considering the transport equation with

N'th-order anisotropic scattering and arbitrary c is not unique to

this particular method of solution; it was also encountered by Kuscer

in his work on the Milne problem for anisotropic scattering.[30]

Using the bi-orthogonality relation formulated above, one can now calculate the normalization condition for the continuum eigenfunctions where $0 < \nu < 1$ and $0 < \nu' < 1$. An equation analogous to (3.65) is

$$\mathcal{P}_{\nu}(\mu)\,\mathcal{P}_{\nu'}(\mu) = \frac{1}{\nu-\nu'}\left[\frac{c\nu}{2}\,D(\nu,\mu)\,\mathcal{P}_{\nu'}(\mu) - \frac{c\nu'}{2}\,D(\nu',\mu)\,\mathcal{P}_{\nu}(\mu)\right]$$

$$+ \Lambda^{+}(\mu)\,\Lambda^{-}(\mu)\,\delta(\mu-\nu)\,\delta(\mu-\nu') \ . \tag{7.52}$$

Using (7.43) and (7.52),

$$\int_{0}^{1}\mathcal{P}_{\nu}(\mu)\left[\mathcal{P}_{\nu'}(\mu) + B(cc;\nu',\mu)\frac{c\nu'}{2}\right](\nu_{0}-\mu)\,\gamma(\mu)\,d\mu =$$

$$(\nu_{0}-\nu)\,\gamma(\nu)\,\Lambda^{+}(\nu)\,\Lambda^{-}(\nu)\,\delta(\nu-\nu') \tag{7.53}$$

provided (7.50) is valid for $m = c$ and $n = c$.

As can be seen from the above analysis, a first step in the solution for $q_{ij}(mn)$ is to determine a general expression for $K_{i,N}(\nu)$. Using (7.11), (7.12), and (7.43),

$$\frac{c\nu}{2}\,K_{i,N}(\nu) = P_{i}(\nu)\,(\nu_{0}-\nu)\,\gamma(\nu)\,\lambda(\nu)$$

$$+ \frac{c\nu}{2}\sum_{\ell=0}^{N}b_{\ell}\,P\!\int_{0}^{1}\frac{P_{i}(\mu)\,\zeta_{\ell}(\nu)\,P_{\ell}(\mu)(\nu_{0}-\mu)\,\gamma(\mu)}{\nu-\mu}\,d\mu, \quad \begin{array}{c}0<\nu<1 \\ i\leq N\end{array}. \tag{7.54}$$

In order to use (7.28) to eliminate the first term on the right hand side of the above equation, we utilize (7.8) and also the expansion[37]

$$P_i(\mu) = \sum_{k=0}^{[i/2]} \frac{(-1)^k (2i-2k)! \, \mu^{i-2k}}{2^i \, k! \, (i-k)! \, (i-2k)!} \qquad (7.55)$$

to obtain

$$\frac{\xi_\ell(\nu) P_i(\mu)}{\nu - \mu} = \frac{\xi_\ell(\mu) P_i(\nu)}{\nu - \mu} + P_i(\nu) \sum_{j=0}^{[\ell/2]} k_{j\ell} \sum_{s=0}^{\ell-2j-1} \nu^{\ell-2j-1-s} \mu^s$$

$$- \xi_\ell(\nu) \sum_{k=0}^{[i/2]} \frac{(-1)^k (2i-2k)!}{2^i \, k! \, (i-k)! \, (i-2k)!} \sum_{r=0}^{i-2k-1} \nu^{i-2k-1-r} \mu^r \qquad (7.56)$$

where $\overset{-1}{\sum} = 0$. Use of the equation

$$\frac{\nu_0 - \mu}{\nu - \mu} = \frac{\nu_0 - \nu}{\nu - \mu} + 1$$

and (7.54) and (7.56) gives the general form for $K_{i,N}(\nu)$,
$0 < \nu < 1$:

$$K_{i,N}(v) = \int_0^i P_i(\mu) D(v,\mu) \gamma(\mu) d\mu$$

$$+ (v_0 - v)\left[P_i(v) \sum_{l=0}^{N} b_l \sum_{j=0}^{[l/2]} k_{jl} \sum_{s=0}^{l-2j-1} v^{l-2j-1-s} \int_0^l P_l(\mu) \mu^s \gamma(\mu) d\mu \right.$$

$$\left. - \sum_{l=0}^{N} b_l \, \xi_l(v) \sum_{k=0}^{[i/2]} \frac{(-1)^k (2i-2k)!}{2^i \, k! \, (i-k)! \, (i-2k)!} \sum_{r=0}^{i-2k-1} v^{i-2k-1-r} \int_0^l P_l(\mu) \mu^r \gamma(\mu) d\mu \right], \quad (7.57)$$

$$0 < v < 1$$
$$i \leq N.$$

Notice that (7.57) is also valid for $v = v_0$ since the result

$$K_{i,N}(v_0) = \int_0^1 P_i(\mu) D(v_0,\mu) \gamma(\mu) d\mu \qquad (7.58)$$

is also obtained by use of (7.14) and (7.43). This is a consequence of the selection of the half-range weight function as $(v_0 - \mu) \, \gamma(\mu)$.

Specific values of $K_{i,N}(v)$ may be obtained from (7.57) and are:

$$K_{0,0}(\nu) = \gamma^{(0)}$$

$$K_{0,1}(\nu) = K_{0,0}(\nu) + b_1 h_0 \nu_0 \gamma^{(1)}$$

$$K_{0,2}(\nu) = K_{0,1}(\nu) - \frac{b_2}{4}\left[3\gamma^{(2)} - \gamma^{(0)}\right]\left[1 - h_0 h_1 \nu_0 \nu\right]$$

$$+ \frac{b_2 h_0 h_1}{4}(\nu_0 - \nu)\left[3\gamma^{(3)} - \gamma^{(1)}\right]$$

$$K_{1,1}(\nu) = \gamma^{(1)} - (\nu_0 - \nu)\gamma^{(0)} + b_1 h_0 \nu \gamma^{(2)}$$

$$K_{1,2}(\nu) = K_{1,1}(\nu) - \frac{b_2}{4}\left[3\gamma^{(3)} - \gamma^{(1)}\right]\left[1 - h_0 h_1 \nu_0 \nu\right]$$

$$+ \frac{b_2}{4}(\nu_0 - \nu)\left[3\gamma^{(2)} - \gamma^{(0)}\right]$$

$$K_{2,2}(\nu) = \frac{1}{2}\left[3\gamma^{(2)} - \gamma^{(0)}\right] + \frac{3b_1 h_0 \nu}{2}\gamma^{(3)} - \frac{b_1 h_0 \nu_0 \gamma^{(1)}}{2}$$

$$+ \frac{b_2}{8}\left[h_0 h_1 \nu^2 - 1\right]\left[9\gamma^{(4)} - 6\gamma^{(2)} + \gamma^{(0)}\right]$$

$$- \frac{3}{2}(\nu_0 - \nu)\left[\nu\gamma^{(0)} + \gamma^{(1)} + b_1 h_0 \nu \gamma^{(2)}\right]$$

$$- \frac{b_2}{8}(\nu_0 - \nu)\left[h_0 h_1 - 3\right]\left[\nu\left(3\gamma^{(2)} - \gamma^{(0)}\right) + \left(3\gamma^{(3)} - \gamma^{(1)}\right)\right] \tag{7.59}$$

Because of the complexity of the $K_{i,N}(\nu)$, it is desirable

to determine a recursion relation to check the polynomials obtained by

use of (7.57). Using the recurrence relation for Legendre polynomials,

multiplying by $\phi_\nu(\mu)(\nu - \mu)\gamma(\mu)$, and integrating from 0 to 1 gives

$$(n+1)\frac{c\nu}{2} K_{n+1,N}(\nu) - (2n+1)\int_0^1 \mu P_n(\mu) \phi_\nu(\mu) (\nu_0 - \mu) \gamma(\mu) d\mu$$

$$+ n \frac{c\nu}{2} K_{n-1,N}(\nu) = 0 , \quad 1 \le n \le N-1 \tag{7.60}$$

Using partial fractions, the middle term may be reduced to

$$-(2n+1)\left[\nu \frac{c\nu}{2} K_{n,N}(\nu) - \frac{c\nu}{2}\int_0^1 P_n(\mu) D(\nu,\mu)(\nu_0 - \mu) \gamma(\mu) d\mu\right] ,$$

so the desired recurrence relation is

$$(n+1)K_{n+1,N}(\nu) - (2n+1)\nu K_{n,N}(\nu) + n K_{n-1,N}(\nu) =$$

$$-(2n+1)\int_0^1 P_n(\mu) D(\nu,\mu)(\nu_0 - \mu) \gamma(\mu) d\mu , \quad 1 \le n \le N-1 . \tag{7.61}$$

Equation (7.61) is satisfied by the $K_{i,N}(\nu)$ of (7.59).

Up to now, we have only shown how to construct the $q_{ij}(mn)$.
We have not yet said anything about which orthogonality relations must

be constructed. From (7.41), we see that for $0 < \nu < 1$ and $0 < \nu' < 1$,

then

$$(\nu-\nu')\int_0^1 \varphi_\nu(\mu)\left[\varphi_{\nu'}(\mu) + B(cc; \nu',\mu)\frac{c\nu'}{2}\right](\nu_0-\mu)\gamma(\mu)d\mu = 0 \qquad (7.62)$$

and the normalization condition is given by (7.53). In (7.41), if $\nu = \nu_0$ and $0 < \nu' < 1$, then

$$\int_0^1 \varphi_+(\mu)\left[\varphi_{\nu'}(\mu) + B(dc; \nu',\mu)\frac{c\nu'}{2}\right](\nu_0-\mu)\gamma(\mu)d\mu = 0 \ . \qquad (7.63)$$

We hope that

$$B(cc; \nu',\mu) = B(dc; \nu',\mu) \qquad (7.64)$$

so that both the continuum and discrete eigenfunctions are orthogonal to $[\phi_{\nu'}(\mu) + B(mc; \nu',\mu)\frac{c\nu'}{2}]$, $m = c = d$. If the weight function (which we assumed was given by $(\nu_0-\mu)\gamma(\mu)$) is correct, then Equation (7.64) will be valid. This would make the determination of the continuum mode expansion coefficient for half-space problems trivial after the other necessary relations were found. Even if (7.64) is not true, however, the present analysis could be used to find $B(cc; \nu', \mu)$ so that the determination of the continuum mode coefficient using (7.53) might still be possible after the integral

$$\int_0^1 \varphi_+(\mu)\left[\varphi_{\nu'}(\mu) + B(cc; \nu',\mu)\frac{c\nu'}{2}\right](\nu_0-\mu)\gamma(\mu)d\mu$$

were evaluated. It must be noted that orthogonal forms are sometimes useful even if the discrete eigenfunction is not orthogonal to an eigenfunction of the form $\phi_{\nu'}(\mu) + B(cc; \nu', \mu)\frac{c\nu'}{2}$. For an example of this, see Appendix D.

Another useful bi-orthogonality equation could be obtained from (7.41) by setting $0 < \nu < 1$ and $\nu' = \nu_0$ to obtain the result

$$\int_0^1 \phi_\nu(\mu)\left[\phi_+(\mu) + B(cd; \nu_0,\mu)\frac{c\nu_0}{2}\right](\nu_0-\mu)\gamma(\mu)d\mu = 0 , \qquad (7.65)$$

The construction of $B(cd; \nu_0, \mu)$ would permit the discrete expansion coefficient in half-space problems to be determined after the other necessary relations were derived. In general, since $K_{i,N}(\nu)$ for $\nu = \nu_0$ is an easier expression to work with than if $0 < \nu < 1$, $B(cd; \nu_0, \mu)$ will be more easily found than $B(cc; \nu, \mu)$. The asymptotic distribution in any half-space transport problem with general anisotropic scattering, which is obtained from the discrete mode expansion coefficients, is therefore more accessible to evaluation than the non-asymptotic distribution.

One advantage of the present method of construction of the bi-orthogonality relations (which could have been used in Part B of Chapter III) over the construction which would have to be followed using (7.38) and proceeding in a manner analogous to that of Chapter III is the following result:

Theorem. If $B(dc; \nu, \mu)$ is constructed such that

$$\int_0^1 \phi_+(\mu)\left[\phi_\nu(\mu) + B(dc; \nu,\mu)\frac{c\nu}{2}\right](\nu_0-\mu)\gamma(\mu)d\mu = 0,$$ (7.63)

then

$$\int_0^1 \phi_\nu(\mu)\left[\phi_+(\mu) + B(cd; \nu_0,\mu)\frac{c\nu_0}{2}\right](\nu_0-\mu)\gamma(\mu)d\mu = 0.$$ (7.65)

The converse is also true. The proof is trivial and consists of showing that

$$\frac{c\nu}{2}\int_0^1 B(dc; \nu,\mu)\,\phi_+(\mu)(\nu_0-\mu)\gamma(\mu)d\mu =$$

$$\frac{c\nu_0}{2}\int_0^1 B(cd; \nu_0,\mu)\,\phi_\nu(\mu)(\nu_0-\mu)\gamma(\mu)d\mu$$ (7.66)

Use of (7.50) and (7.43) reduces (7.66) to

$$\sum_{i=0}^{N-1} K_{i,N}(\nu_0)\sum_{j=0}^{N-1} q_{ij}(dc)\nu^j =$$

$$\sum_{i=0}^{N-1} K_{i,N}(\nu)\sum_{j=0}^{N-1} q_{ij}(cd)\nu_0^j.$$ (7.67)

Equation (7.51) can be used in (7.67) to give the equation

$$L_N(\nu_0,\nu) = L_N(\nu,\nu_0).$$ (7.68)

But from its definition in (7.46), $L_N(\nu,\nu')$ is a symmetric function

so the theorem is proved. An example of the theorem is seen in

Equations (3.80) and (3.81).

We have now formulated a "recipe" for finding bi-orthogonalit

relations of the form of Equation (7.41). In the analysis, however,

we have assumed that the $q_{ij}(mn)$, $i,j = 0$ to $N - 1$, exist and can

be found. Rather than proceeding to the existence proof, let us examin

the results for the special cases $N = 0,1,2$. For $N = 0$, (7.50)

becomes $B(mn; \nu' , \mu) = 0$. For $N = 1$, the following relations

are true:

$$B(cc; \nu'_1\mu) = q_{00}(cc) = B$$

$$B(dc; \nu'_1\mu) = q_{00}(dc) = B$$

$$B(cd; \nu_0\mu) = q_{00}(cd) = B \tag{7.69}$$

where B is given in Equation (3.78). Note that it is only coinci-

dental that $q_{00}(dc) = q_{00}(cd)$ for linearly anisotropic scattering

since the theorem in no way guarantees that $B(dc; \nu' , \mu) = B(cd; \nu_0,\mu)$.

For $N = 2$, the analysis becomes quite involved and explicit answers

have not been obtained. Thus, even if the constants $q_{ij}(mn)$ do exist

for general N , the computation required in order to find them is

indeed formidable.

It should also be remembered that only if (7.64) is valid

did we assume the proper weight function such that the eigenfunctions

$\phi_\nu(\mu)$, $0 < \nu < 1$, and $\phi_+(\mu)$ are necessarily both orthogonal to

$\phi_{\nu'}(\mu) + B(cc; \nu', \mu)\dfrac{c\nu'}{2}$. It thus appears pointless to attempt

to prove the existence of the $q_{ij}(mn)$ as presently defined if we

want both the continuum and discrete eigenfunctions orthogonal to the

same function.

With the ideas of the previous paragraph in mind, however,

it is easy to use the present analysis to formulate the proper func-

tion $\phi_{\nu'}(\mu) + \bar{B}(cc; \nu', \mu)\dfrac{c\nu'}{2}$ such that

$$\int_0^1 \phi_\nu(\mu)\left[\phi_{\nu'}(\mu) + \bar{B}(cc; \nu', \mu)\frac{c\nu'}{2}\right]\gamma_g(\mu)\, d\mu =$$

$$\gamma_g(\nu)\Lambda^+(\nu)\Lambda^-(\nu)\,\delta(\nu-\nu') \tag{7.70}$$

$$\int_0^1 \phi_+(\mu)\left[\phi_{\nu'}(\mu) + \bar{B}(dc; \nu', \mu)\frac{c\nu'}{2}\right]\gamma_g(\mu)\, d\mu = 0 \tag{7.71}$$

$$\int_0^1 \phi_\nu(\mu)\left[\phi_+(\mu) + \bar{B}(cd; \nu_0, \mu)\frac{c\nu_0}{2}\right]\gamma_g(\mu)\, d\mu = 0 . \tag{7.72}$$

One simply proceeds from Equation (7.39) writing everything in terms

of the unknown weight function $\gamma_g(\mu)$. Since $\gamma_g(\mu)$ must reduce

to $(\nu_0 - \mu)\gamma(\mu)$ when $N = 0,1$, Equation (7.39) can be used to write

the general form of $\gamma_g(\mu)$ as

$$\gamma_g(\mu) = \gamma(\mu)\left[w_0(1-\mu) + w_2\right], \quad \begin{array}{l} w_0 \neq 0 \\ w_2 \neq 0 \end{array} . \tag{7.73}$$

The existence of all the constants $\bar{q}_{ij}(cc)$ and $\bar{q}_{ij}(dc)$ for $i,j = 0$ to $N - 1$ must be verified and the values must be obtained. Then the values of w_O and w_2 are determined by requiring that

$$\bar{B}(cc; \nu', \mu) = \bar{B}(dc; \nu', \mu) \ . \tag{7.74}$$

The existence of $\bar{B}(cd; \nu_O, \mu)$ necessary to make (7.72) valid is guaranteed by the theorem.

To summarize, we have attempted to determine a bi-orthogonal form which makes an eigenfunction of the homogeneous Boltzmann operator orthogonal to an adjoint function with respect to an appropriate weight function. We have observed that by proceeding in a manner similar to that of Chapter III, the determination of the bi-orthogonal form (if it exists) is quite involved. Perhaps this problem could better be attacked from a more abstract point of view using the properties of the adjoint Boltzmann operator.

CHAPTER VIII

SUMMARY

The method developed by Case was used to consider several time-independent problems of neutron transport theory for a homogeneous medium in plane geometry. This method utilizes a direct expansion of the neutron angular density in terms of the eigenfunctions of the homogeneous transport equation. The solutions to these problems were greatly facilitated by the use of half-range orthogonality relations (for $0 < \mu < 1$) between the eigenfunctions of the transport equation. Both the half-range orthogonality equations and the previously known full-range equations (for $-1 < \mu < 1$) were shown to be special cases of a more general partial-range orthogonality equation.

These orthogonality relations were derived and were used to obtain solutions for the following problems:

1. The half-space Milne, albedo, constant isotropic source, and Green's function problems for linearly anisotropic scattering (i.e., for a half-space medium which scatters neutrons in such a manner that the scattering function is linear in the cosine of the scattering angle).

2. The half-space Milne and albedo problems for quadratically anisotropic scattering when there is no absorption.

3. The albedo problem for isotropic scattering in a slab of finite thickness.

In each case, simplified expressions were derived for the emerging angular density and the density and net current on the surface(s).

A speculative approach for finding the half-range orthogonality relations for general-order anisotropic scattering was also discussed.

Some of the results from the slab albedo problem were used to determine new integral expressions for the X- and Y-functions of Chandrasekhar. These yielded convenient asymptotic values valid for large slab thicknesses. Also included was a generalization of the slab albedo boundary conditions to the case of a slab bounded on one side by a vacuum and on the other side by a specularly and diffusely reflecting face.

APPENDICES

APPENDIX A

DERIVATION OF EQUATIONS (2.30) AND (2.31)

Extensive use will be made of the Cauchy integral formula,[27] which states that if $f(z)$ is analytic on an open region S in the complex plane and if C is a closed curve such that C and its interior region lie within S, then for every point z inside of C,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{z'-z} \qquad (A.1)$$

where $\oint$ denotes the positive orientation of C.

Derivation of Equation (2.30)

The derivation of (2.30) for the function $X_g(z)$ is similar to the derivation for the half-space $X(z)$.[7] The function $X_g(z) - X_g(\infty)$ is analytic everywhere in the entire complex plane cut from α to β on the real axis and vanishes at infinity. One can therefore use (A.1) to write

$$X_g(z) - X_g(\infty) = \frac{1}{2\pi i} \oint_C \frac{\left[X_g(z') - X_g(\infty)\right]dz'}{z'-z} \qquad (A.2)$$

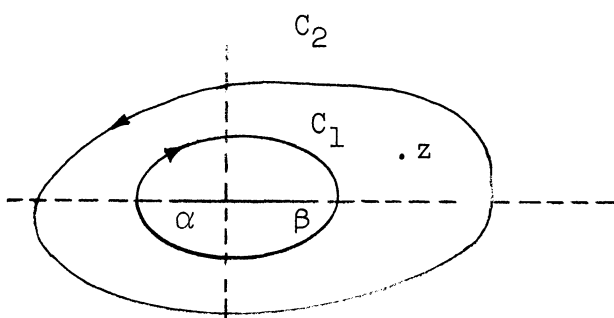where $C = C_1 + C_2$ is defined as the contour shown in Figure A.1:

Figure A.1. The Path C for Equation (A.2).

If we deform the contour C_2 to a very large circle and observe that the integrand vanishes faster than $1/z$, we find that there is no contribution from the integral over C_2 . (This reasoning will be tacitly used throughout the remaining appendices.) If we also allow C_1 to shrink and enclose the cut, we then have

$$X_g(z) - X_g(\infty) = \frac{1}{2\pi i} \int_\alpha^\beta \frac{\left[X_g(\mu) - X_g(\infty)\right]^+ - \left[X_g(\mu) - X_g(\infty)\right]^-}{\mu - z} \, d\mu$$

$$= \frac{1}{2\pi i} \int_\alpha^\beta \frac{X_g^+(\mu) - X_g^-(\mu)}{\mu - z} \, d\mu \qquad (A.3)$$

since there is no special contribution from the contour in the neighbor-hood of the endpoints (in view of the fact that $X_g(\mu)$ possesses at

worst a weak singularity as $\mu \to \alpha$, β). From (2.28),

$$\gamma_g(\mu) = \frac{c\mu}{2} \frac{X_g^-(\mu)}{\Lambda^-(\mu)} \quad , \alpha \leq \mu < \beta$$

and this can be rewritten, using (2.15) and (2.19), as

$$\gamma_g(\mu) = \frac{\Lambda^+(\mu) - \Lambda^-(\mu)}{2\pi i} \frac{X_g^-(\mu)}{\Lambda^-(\mu)}$$

$$= \frac{X_g^+(\mu) - X_g^-(\mu)}{2\pi i} \quad , \alpha \leq \mu < \beta \quad . \tag{A.4}$$

Equations (A.3) and (A.4) combine to yield

$$X_g(z) - X_g(\infty) = \int_\alpha^\beta \frac{\gamma_g(\mu)\,d\mu}{\mu - z} \quad . \tag{2.30}$$

Derivation of Equation (2.31)

For $z = \nu$, $\alpha < \nu < \beta$, the use of (2.13) reduces (2.30) to

$$\frac{1}{2}\left[\left[X_g(\nu) - X_g(\infty)\right]^+ + \left[X_g(\nu) - X_g(\infty)\right]^-\right] = P\int_\alpha^\beta \frac{\gamma_g(\mu)\,d\mu}{\mu - \nu} \quad . \tag{A.5}$$

Multiplication by $\frac{c\nu}{2}$ gives

$$\frac{c\nu}{4}\left[X_g^+(\nu) + X_g^-(\nu)\right] - \frac{c\nu}{2}X_g(\infty) = \frac{c\nu}{2}P\int_\alpha^\beta \frac{\gamma_g(\mu)\,d\mu}{\mu - \nu} \quad . \tag{A.6}$$

Use of (2.15) changes the first term on the left hand side of (A.6) to

$$\frac{c\nu}{4}\left[X_g^+(\nu) + X_g^-(\nu)\right] = \left[\frac{\Lambda^+(\nu) - \Lambda^-(\nu)}{4\pi i}\right]\left[X_g^+(\nu) + X_g^-(\nu)\right] \qquad (A.7)$$

and use of (2.19) gives

$$\frac{c\nu}{4}\left[X_g^+(\nu) + X_g^-(\nu)\right] = \frac{\Lambda^+(\nu)X_g^+(\nu) - \Lambda^-(\nu)X_g^-(\nu)}{4\pi i} \qquad (A.8)$$

This may be rewritten as

$$\frac{c\nu}{4}\left[X_g^+(\nu) + X_g^-(\nu)\right] = \left[\frac{\Lambda^+(\nu) + \Lambda^-(\nu)}{2}\right]\left[\frac{X_g^+(\nu) - X_g^-(\nu)}{2\pi i}\right] \qquad (A.9)$$

Equations (2.15) and (A.4) reduce (A.9) to

$$\frac{c\nu}{4}\left[X_g^+(\nu) + X_g^-(\nu)\right] = \lambda(\nu)\Upsilon_g(\nu) \qquad (A.10)$$

and (A.6) and (A.10) yield the result

$$\lambda(\nu)\Upsilon_g(\nu) - \frac{c\nu}{2}X_g(\infty) = \frac{c\nu}{2}P\int_\alpha^\beta \frac{\Upsilon_g(\mu)\,d\mu}{\mu-\nu}, \quad \alpha<\nu<\beta \,. \qquad (2.31)$$

APPENDIX B

DETERMINATION OF THE GENERAL-RANGE, HALF-RANGE, AND FULL-RANGE WEIGHT FUNCTIONS

Determination of the General-Range Weight Function

In the constructive proof of the general-range completeness theorem,[7] the function $\Psi(\mu)$ was expanded as

$$\Psi(\mu) = \int_\alpha^\beta A(\nu)\, \varphi_\nu(\mu)\, d\nu \tag{B.1}$$

where the value of the expansion coefficient, $A(\nu)$, was obtained by using the set of equations[7]

$$\frac{c\nu}{2} A(\nu) = N^+(\nu) - N^-(\nu)$$

$$N(z) = \frac{1}{2\pi i\, X_g(z)} \int_\alpha^\beta \frac{\gamma_g(\mu)\, \Psi(\mu)\, d\mu}{\mu - z} \qquad . \tag{B.2}$$

Using the Plemelj formulas (2.13) and (2.14) and (B.2) gives

$$\frac{c\nu}{2} A(\nu) = \frac{1}{2\pi i} \left[\frac{1}{X_g^+(\nu)} - \frac{1}{X_g^-(\nu)} \right] P\int_\alpha^\beta \frac{\gamma_g(\mu)\, \Psi(\mu)\, d\mu}{\mu - \nu}$$

$$+ \frac{1}{2} \gamma_g(\nu)\, \Psi(\nu) \left[\frac{1}{X_g^+(\nu)} + \frac{1}{X_g^-(\nu)} \right] \qquad . \tag{B.3}$$

Use of (2.19) shows that the function $(X(z))^{-1}$ obeys the following relations along the cut $\alpha < \nu < \beta$ in the complex plane:

$$\frac{1}{2\pi i}\left[\frac{1}{X_g^+(\nu)} - \frac{1}{X_g^-(\nu)}\right] = -\frac{\left(\frac{c\nu}{2}\right)^2}{\gamma_g(\nu)\Lambda^+(\nu)\Lambda^-(\nu)} \quad , \quad \alpha < \nu < \beta$$

$$\frac{1}{2}\left[\frac{1}{X_g^+(\nu)} + \frac{1}{X_g^-(\nu)}\right] = \frac{\frac{c\nu}{2}\lambda(\nu)}{\gamma_g(\nu)\Lambda^+(\nu)\Lambda^-(\nu)} \quad , \quad \alpha < \nu < \beta \quad . \quad (B.4)$$

These equations may be verified by using (2.19) to obtain

$$\left[\frac{1}{X^+(\nu)} \mp \frac{1}{X^-(\nu)}\right] = \mp \frac{\Lambda^-(\nu)}{X^-(\nu)}\frac{\left[\Lambda^+(\nu) \mp \Lambda^-(\nu)\right]}{\Lambda^+(\nu)\Lambda^-(\nu)} \quad (B.5)$$

and then using (2.15) and (2.28). Substituting the results of (B.4) into (B.3) yields the result

$$A(\nu)\gamma_g(\nu)\Lambda^+(\nu)\Lambda^-(\nu) =$$

$$\int_\alpha^\beta \gamma_g(\mu)\,\psi(\mu)\left[-P\,\frac{\frac{c\nu}{2}}{\mu-\nu} + \lambda(\mu)\,\delta(\mu-\nu)\right]d\mu \quad (B.6)$$

and this can be rewritten as

$$A(\nu)\gamma_g(\nu)\Lambda^+(\nu)\Lambda^-(\nu) = \int_\alpha^\beta \gamma_g(\mu)\,\varphi_\nu(\mu)\,\psi(\mu)\,d\mu \quad . \quad (B.7)$$

Use of (B.1) in (B.7) gives

$$A(\nu)\, \gamma_g(\nu) \Lambda^+(\nu) \Lambda^-(\nu) \;=\;$$

$$\int_\alpha^\beta \gamma_g(\mu)\, \phi_\nu(\mu) \left[\int_\alpha^\beta A(\nu')\, \phi_{\nu'}(\mu)\, d\nu' \right] d\mu \; . \tag{B.8}$$

Equation (B.8) can also be written in the abbreviated form

$$\int_\alpha^\beta \phi_\nu(\mu)\, \phi_{\nu'}(\mu)\, \gamma_g(\mu)\, d\mu \;=\; \gamma_g(\nu)\, \Lambda^+(\nu) \Lambda^-(\nu)\, \delta(\nu - \nu') \tag{B.9}$$

to parallel the form of (3.1). This is exactly the equation we are looking for since it shows that $\gamma_g(\mu)$ is the general-range weight function which makes $\phi_\nu(\mu)$ and $\phi_{\nu'}(\mu)$ orthogonal over the range $\alpha < \mu < \beta$.

Determination of the Half-Range Weight Function

When we are considering $\alpha = 0$ and $\beta = 1$, we know that

$$\gamma(\nu)\, \lambda(\nu) \;=\; \frac{c\nu}{2}\, P\!\int_0^1 \frac{\gamma(\mu)\, d\mu}{\mu - \nu} \;,\; 0 < \nu < 1 \;, \tag{2.35}$$

This may be rewritten as

$$\int_0^1 \phi_\nu(\mu)\, \gamma(\mu)\, d\mu \;=\; 0 \;,\; 0 < \nu < 1 \tag{B.10}$$

and (B.10) may be written as

$$\int_0^1 \left[\frac{\frac{c\nu_0}{2}}{\nu_0 \mp \mu} \right] \phi_\nu(\mu)\,(\nu_0 \mp \mu)\,\gamma(\mu)\,d\mu =$$

$$\int_0^1 \phi_\pm(\mu)\,\phi_\nu(\mu)\,(\nu_0 \mp \mu)\,\gamma(\mu)\,d\mu = 0 \ . \qquad \text{(B.11)}$$

The eigenfunction $\phi_\nu(\mu)$ is already orthogonal to $\phi_{\nu'}(\mu)$ by (B.9). To make $\phi_+(\mu)$ also orthogonal to $\phi_\nu(\mu)$, $0 < \nu < 1$, the weight function for the half-range case must be $\gamma_g(\mu) = (\nu_0 - \mu)\gamma(\mu)$.* (This weight function is the desired one for problems were the region under consideration is in the right half-space because the discrete coefficient a_+ is unknown. The weight function should be $(\nu_0 + \mu)\gamma(\mu)$ if the region is located in the left half-space.)

Determination of the Full-Range Weight Function

For $\alpha = -1$ and $\beta = 1$, we have the equation

$$\gamma(\nu)\lambda(\nu) = \frac{c\nu}{2}\,P\int_{-1}^1 \frac{\gamma(\mu)\,d\mu}{\mu - \nu} \ , \quad -1 < \nu < 1 \qquad \text{(2.43)}$$

which may be rewritten as

$$\int_{-1}^1 \phi_\nu(\mu)\,\gamma(\mu)\,d\mu = 0 \qquad \text{(B.12)}$$

or as

$$\int_{-1}^1 \left[\frac{\frac{c\nu_0}{2}}{\nu_0 \mp \mu} \right] \phi_\nu(\mu)\,(\nu_0 \mp \mu)\,\gamma(\mu)\,d\mu = 0 \ . \qquad \text{(B.13)}$$

---

*Of course $\gamma_g(\mu) = K(\nu_0 - \mu)\gamma(\mu)$ where K is any constant is also a satisfactory weight function.

In order to make both the $\phi_+(\mu)$ and $\phi_-(\mu)$ orthogonal to $\phi_\nu(\mu)$, $-1 < \nu < 1$, we must use (2.44). Multiplying (2.43) by ν and using partial fractions and (2.44) gives

$$\int_{-1}^{1} \mu \, \phi_\nu(\mu) \, \gamma(\mu) \, d\mu = 0 \, . \tag{B.14}$$

Therefore we also have the equation

$$\int_{-1}^{1} \mu \left[\frac{\frac{c\nu_0}{2}}{\nu_0 \mp \mu} \right] \phi_\nu(\mu) \, (\nu_0 \mp \mu) \, \gamma(\mu) \, d\mu = 0 \, , \tag{B.15}$$

Multiplying (B.13) by ν_0 and adding/subtracting (B.15) gives

$$\int_{-1}^{1} \phi_\pm(\mu) \, \phi_\nu(\mu) \, (\nu_0^2 - \mu^2) \, \gamma(\mu) \, d\mu = 0 \, . \tag{B.16}$$

Therefore the full-range weight function is $(\nu_0^2 - \mu^2)\gamma(\mu)$.

This weight function may be reduced to that obtained by Case.[7] Multiplication of (B.16) by $2(1-c)/c$ and use of (2.9) and (2.47) gives the result

$$\int_{-1}^{1} \mu \, \phi_\pm(\mu) \, \phi_\nu(\mu) \, d\mu = 0 \tag{B.17}$$

so the function μ is also a weight function for the full-range case-- in agreement with Equations (3.1) to (3.3).

EVALUATION OF INTEGRALS NEEDED TO FIND THE SURFACE DISTRIBUTIONS OF CHAPTER IV

Derivation of Equation (4.17)

The integral which must be evaluated in order to simplify the emergent angular density is

$$I_1 = \int_0^1 \frac{\varphi_{-\nu}(\mu)\left[\varphi_\nu(\mu') + B \frac{c\nu}{2}\right] d\nu}{(\nu_0 - \nu)\, \gamma(\nu)\, \Lambda^+(\nu)\, \Lambda^-(\nu)} \Bigg|_{\substack{\mu > 0 \\ \mu' > 0}} \quad , \tag{C.1}$$

Use of the identity

$$d(-\nu\mu)\, d(\nu\mu') = d(-\mu\mu')\, d(\nu\nu) - b_1(1-c)(\nu+\mu)(\nu-\mu') \tag{C.2}$$

and (2.52) gives

$$I_1 = d(-\mu\mu')\left[\frac{\frac{c\mu'}{2}\, \lambda(\mu')}{(\mu'+\mu)(\nu_0 - \mu')\, \gamma(\mu')\, \Lambda^+(\mu')\, \Lambda^-(\mu')} \right.$$

$$\left. + P\int_0^1 \frac{\left(\frac{c\nu}{2}\right)^2 d(\nu\nu)\, d\nu}{(\nu+\mu)(\nu_0 - \nu)(\nu-\mu')\, \gamma(\nu)\, \Lambda^+(\nu)\, \Lambda^-(\nu)} \right]$$

$$- \int_0^1 \frac{\left(\frac{c\nu}{2}\right)^2 \left[B \frac{d(-\nu\mu)}{\nu+\mu} - b_1(1-c)\right] d\nu}{(\nu-\nu_0)\, \gamma(\nu)\, \Lambda^+(\nu)\, \Lambda^-(\nu)} \quad , \tag{C.3}$$

For linearly anisotropic scattering, the function $(X(z))^{-1}$ has the following behavior on the cut $0 \leq \nu \leq 1$ in the complex plane:

$$\frac{1}{2\pi i}\left[\frac{1}{X^+(\nu)} - \frac{1}{X^-(\nu)}\right] = -\frac{\left(\frac{c\nu}{2}\right)^2 d(\nu\nu)}{\gamma(\nu)\,\Lambda^+(\nu)\,\Lambda^-(\nu)} \, , \, 0 < \nu < 1$$

$$\frac{1}{2}\left[\frac{1}{X^+(\nu)} + \frac{1}{X^-(\nu)}\right] = \frac{\left(\frac{c\nu}{2}\right)\lambda(\nu)}{\gamma(\nu)\,\Lambda^+(\nu)\,\Lambda^-(\nu)} \, , \, 0 < \nu < 1 \, . \quad (C.4)$$

The derivation of (C.4) is analogous to that used to obtain (B.4).

Using the results of (C.4) as a guide, consider the function $[(z-\mu')(\nu_0-z)X(z)]^{-1}$ which is analytic in the complex plane except for a simple pole at $z = \nu_0$ and a simple pole at $z = \mu'$ imbedded in the cut from 0 to 1. The function vanishes as $z \to \infty$. Letting C_3 go to infinity in the contour shown,
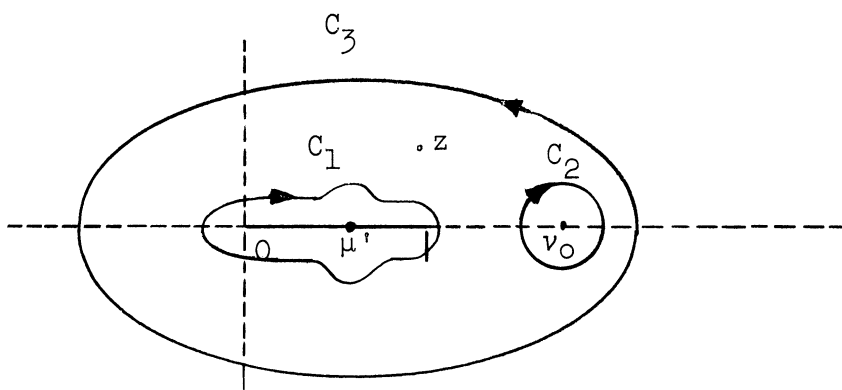

Figure C.1. The Path C for Equation (C.5).

the use of the Cauchy integral formula gives

$$\frac{1}{(z-\mu')(\nu_0-z)X(z)} = \frac{1}{2\pi i}\oint_{C_1+C_2} \frac{dz'}{(z'-\mu')(\nu_0-z')X(z')(z'-z)} \quad . \tag{C.5}$$

Equations (2.12) and (C.5) can be used to obtain the result

$$\frac{1}{(z-\mu')(\nu_0-z)X(z)} = P\int_0^1 \frac{\frac{1}{2\pi i}\left[\frac{1}{X^+(\nu)} - \frac{1}{X^-(\nu)}\right]d\nu}{(\nu-\mu')(\nu_0-\nu)(\nu-z)}$$

$$- \frac{\frac{1}{2}\left[\frac{1}{X^+(\mu')} + \frac{1}{X^-(\mu')}\right]}{(\mu'-z)(\nu_0-\mu')}$$

$$+ \frac{1}{(\nu_0-\mu')(\nu_0-z)X(\nu_0)} \quad . \tag{C.6}$$

Here, the sign of the second term on the right hand side is negative because the contour is clockwise around the point $z = \mu'$. The third term comes from the residue at $z = \nu_0$. Use of (C.4) in (C.6) shows that, for $z = -\mu$,

$$\frac{1}{(\mu+\mu')(\nu_0+\mu)X(-\mu)} + \frac{1}{(\nu_0-\mu')(\nu_0+\mu)X(\nu_0)} =$$

$$P\int_0^1 \frac{\left(\frac{c\nu}{2}\right)^2 d(\nu\nu)\,d\nu}{(\nu-\mu')(\nu_0-\nu)(\nu+\mu)\gamma(\nu)\Lambda^+(\nu)\Lambda^-(\nu)} + \frac{\frac{c\mu'}{2}\lambda(\mu')}{(\mu'+\mu)(\nu_0-\mu')\gamma(\mu')\Lambda^+(\mu')\Lambda^-(\mu')} \quad . \tag{C.7}$$

Use of the identity

$$\frac{d(-\nu\mu)}{\nu+\mu} = \frac{d(\nu\nu)}{\nu+\mu} - b_1(1-c)\nu \qquad (C.8)$$

and Equation (C.7) reduces (C.3) to

$$I_1 = \frac{d(-\mu\mu')}{(\mu+\mu')(\nu_0+\mu)X(-\mu)} + \frac{d(-\mu\mu')}{(\nu_0-\mu')(\nu_0+\mu)X(\nu_0)}$$

$$+ B\int_0^1 \frac{\left(\frac{c\nu}{2}\right)^2 d(\nu\nu)\, d\nu}{(\nu_0-\nu)(\nu+\mu)\, \gamma(\nu)\, \Lambda^+(\nu)\, \Lambda^-(\nu)}$$

$$+ \frac{b_1(1-c)}{d(\nu_0\bar\nu)}\int_0^1 \frac{\left(\frac{c\nu}{2}\right)^2\left[b_1(1-c)(\nu_0-\bar\nu)\nu + d(\nu_0\bar\nu)\right]}{(\nu-\nu_0)\,\gamma(\nu)\,\Lambda^+(\nu)\Lambda^-(\nu)}\, d\nu \quad , \qquad (C.9)$$

Now consider the function $[(\nu_0-z)X(z)]^{-1}$. It is analytic in the complex plane except for a simple pole at $z = \nu_0$ and the cut from 0 to 1 and varies like $+1$ as $z \to \infty$. The contour for the Cauchy integral formula is shown in Figure C.2.
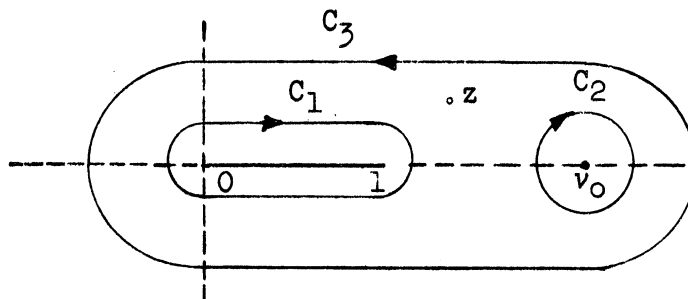

Figure C.2. The Path C Used to Obtain Equation (C.10).

Again, the curve C_3 is allowed to become large. The result obtained when $z = -\mu$ is

$$-\frac{1}{(\nu_0+\mu)\, X(-\mu)} + 1 + \frac{1}{(\nu_0+\mu)\, X(\nu_0)} = \int_0^1 \frac{(\frac{c\nu}{2})^2\, d(\nu\nu)\, d\nu}{(\nu_0-\nu)(\nu+\mu)\,\gamma(\nu)\,\Lambda^+(\nu)\Lambda^-(\nu)} , \qquad (C.10)$$

The last function to be considered is

$$\frac{b_1(1-c)(\nu_0-\bar\nu)\, z + d(\nu_0\bar\nu)}{d(zz)\, X(z)} .$$

It is analytic in the complex plane except for the simple poles at $\pm\sqrt{-1/[b_1(1-c)]} \equiv \pm\sigma$ * and the cut from 0 to 1 and it behaves like $-(\nu_0 - \bar\nu)$ as $z \to \infty$. The Cauchy formula then yields the following result if $z = \nu_0$:

$$-\frac{b_1(1-c)(\nu_0-\bar\nu)\nu_0 + d(\nu_0\bar\nu)}{d(\nu_0\nu_0)\, X(\nu_0)} - (\nu_0 - \bar\nu) =$$

$$\int_0^1 \frac{(\frac{c\nu}{2})^2\left[b_1(1-c)(\nu_0-\bar\nu)\nu + d(\nu_0\bar\nu)\right]}{(\nu-\nu_0)\,\gamma(\nu)\,\Lambda^+(\nu)\Lambda^-(\nu)}\, d\nu$$

$$+\frac{1}{2\sigma b_1(1-c)}\left[\frac{\left[b_1(1-c)(\nu_0-\bar\nu)\sigma + d(\nu_0\bar\nu)\right]}{(\sigma-\nu_0)\, X(\sigma)} - \frac{\left[b_1(1-c)(\nu_0-\bar\nu)\sigma - d(\nu_0\bar\nu)\right]}{(\sigma+\nu_0)\, X(-\sigma)}\right] \quad (C.11)$$

---

*No pole is imbedded in the cut since $b_1(1-c)$ is always ≥ -1. If b_1 were less than -1, the scattering function would be negative for some angles and this is not physically possible.

The first term on the left hand side of (C.11) reduces to $-[X(\nu_0)]^{-1}$. To simplify the right hand side, one needs to know the values of $X(\pm\,\sigma)$. Following the approach of Shure and Natelson,[22] one may write (2.67) as

$$X(\pm\sigma) = b_1 (1-c)\int_0^1 \gamma(\mu)\, (\mu \pm \sigma)\, d\mu = b_1\,(1-c)\gamma^{(0)}\,(\bar{\nu} \pm \sigma) \ . \qquad (C.12)$$

Using (2.37) and (C.12), it is evident that

$$X(\sigma)X(-\sigma) = \frac{\Lambda(\sigma)}{(\nu_0^2 - \sigma^2)\,\Lambda(\infty)} = b_1^2(1-c)^2\left[\gamma^{(0)}\right]^2(\bar{\nu}^2 - \sigma^2) \qquad (C.13)$$

and using the definition of $\pm\,\sigma$ gives the identity

$$\left[\gamma^{(0)}\right]^2 d(\bar{\nu}\,\bar{\nu})\, d(\nu_0\nu_0) = \frac{\Lambda(\sigma)}{\Lambda(\infty)} \ . \qquad (C.14)$$

From (2.56) it may be shown that $\Lambda(\sigma) = 1-c$ so we have the identity

$$\left[\gamma^{(0)}\right]^2 d(\bar{\nu}\,\bar{\nu})\, d(\nu_0\nu_0) = \frac{1}{\left(1 - \frac{cb_1}{3}\right)} \qquad (C.15)$$

where (2.59) has been used. After considerable algebraic manipulation, one finds that the contribution in (C.11) from the residues at $+\,\sigma$ and $-\,\sigma$ cancel. Thus, Equation (C.11) reduces to

$$\int_0^1 \frac{\left(\frac{c\nu}{2}\right)^2\left[b_1(1-c)(\nu_0 - \bar{\nu})\nu + d\,(\nu_0\nu)\right]}{(\nu - \nu_0)\,\gamma(\nu)\,\Lambda^+(\nu)\Lambda^-(\nu)}\, d\nu = -\frac{1}{X(\nu_0)} - (\nu_0 - \bar{\nu}) \ . \qquad (C.16)$$

Putting (C.10) and (C.16) into (C.9) eventually leads to the result

$$
I_1 = \frac{1}{(\nu_0 + \mu)\, X(-\mu)} \left[\frac{d(-\mu\mu')}{\mu + \mu'} - B \right]
$$

$$
+ \frac{d(-\nu_0\mu)\, d(\mu'\bar{\nu})}{(\nu_0 + \mu)(\nu_0 - \mu')\, d(\nu_0\bar{\nu})\, X(\nu_0)} \qquad \qquad (C.17)
$$

Derivation of Equation (4.24)

In order to derive the equations in (4.24), we need the following generalizations of Equations (A.4) and (A.10) to the case of linearly anisotropic scattering:

$$
\frac{1}{2\pi i} \left[X^+(\nu) - X^-(\nu) \right] = \frac{\frac{c\nu}{2}\, d(\nu\nu)}{(\nu_0^2 - \nu^2)(1-c)\left(1 - \frac{cb_1}{3}\right) X(-\nu)}
$$

$$
= \gamma(\nu)\, d(\nu\nu) \,, \quad 0 < \nu < 1
$$

$$
\frac{1}{2} \left[X^+(\nu) + X^-(\nu) \right] = \frac{2}{c\nu}\, \gamma(\nu)\, \lambda(\nu), \quad 0 < \nu < 1 \,. \qquad (C.18)
$$

We consider the function

$$
\frac{(\nu_0^2 - z^2)\, X(z)}{d(zz)(z - \epsilon)}
$$

where ϵ is a small negative number. The function vanishes as

$z \to \infty$ and is analytic in the entire complex plane except for simple

poles at $z = \epsilon$ and $z = \pm \sigma$. It also has a cut from 0 to 1

because of the X-function. Using the Cauchy integral formula for
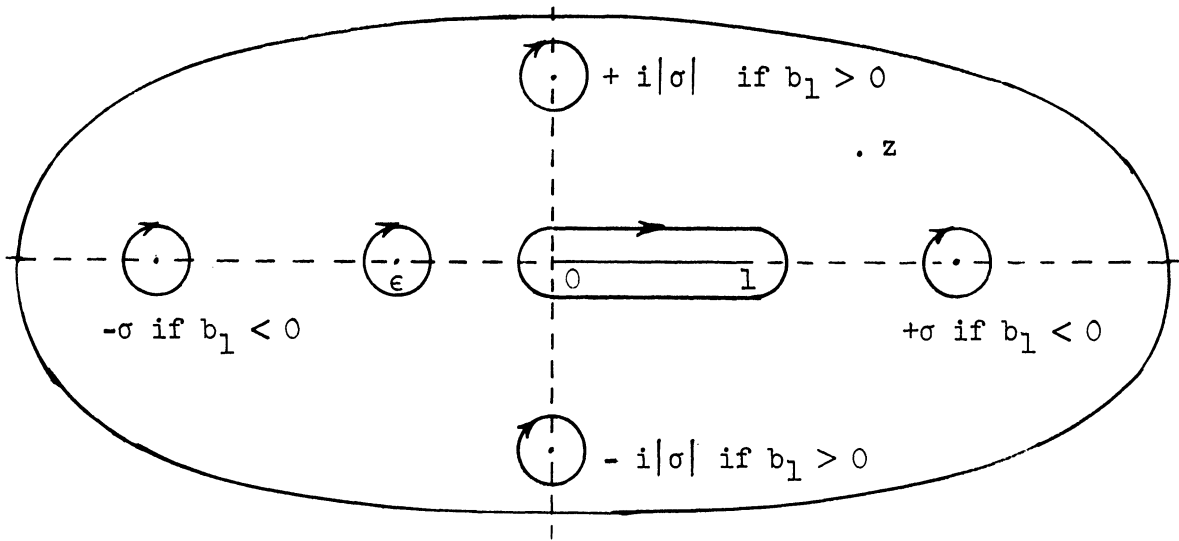
the region of Figure C.3 shows that

Figure C.3. The Path C Used to Obtain Equation (C.19).

$$\frac{(\nu_o^2 - z^2) X(z)}{d(zz)(z - \epsilon)} = \int_0^1 \frac{(\nu_o^2 - \nu^2) \frac{1}{2\pi i} \left[X^+(\nu) - X^-(\nu) \right] d\nu}{d(\nu\nu)(\nu - \epsilon)(\nu - z)}$$

$$- \frac{(\nu_o^2 - \sigma^2) X(\sigma)}{b_1 (1-c) 2\sigma (\sigma - \epsilon)(\sigma - z)} + \frac{(\nu_o^2 - \sigma^2) X(-\sigma)}{b_1 (1-c) 2\sigma (\sigma + \epsilon)(\sigma + z)}$$

$$- \frac{(\nu_o^2 - \epsilon^2) X(\epsilon)}{d(\epsilon\epsilon)(\epsilon - z)} \qquad (C.19)$$

Taking the limit as $\epsilon \to 0$ and using (C.12) and (C.18) reduces (C.19) to

$$\int_0^1 \frac{d\mu}{X(-\mu)(\mu-z)} = \frac{2(1-c)(1-\frac{cb_1}{3})}{c}\left[\frac{(\nu_0^2-z^2)X(z)}{z\,d(zz)}\right.$$

$$\left. - \frac{\nu_0^2 X(0)}{z} - \frac{\gamma^{(0)} d(\nu_0\nu_0) d(-z\bar{\nu})}{d(zz)}\right] \quad , \qquad (C.20)$$

Letting $z = -\mu'$ and $z = \overline{+}\,\nu_0$ gives the first two equations of (4.24).

To evaluate the third integral in (4.24), consider the function

$$\frac{(\nu_0^2 - z^2)X(z)}{d(zz)}$$

and follow through a similar analysis to obtain

$$\int_0^1 \frac{\mu\,d\mu}{X(-\mu)(\mu-z)} = \frac{2(1-c)(1-\frac{cb_1}{3})}{c}\left[(\nu_0^2 - z^2)X(z)\right.$$

$$\left. - \gamma^{(0)} d(\nu_0\nu_0)(\bar{\nu}+z)\right] \quad , \qquad (C.21)$$

Setting $z = 0$ in (C.21) gives the last equation of (4.24).

APPENDIX D

MODIFIED ORTHOGONALITY RELATIONS FOR ISOTROPIC SCATTERING

In the derivation of relations (3.16) to (3.25), the weight function $(\nu_0-\mu)\gamma(\mu)$ was used in order to make $\phi_+(\mu)$ and $\phi_\nu(\mu)$, $0 < \nu < 1$, orthogonal. If the weight function $(\nu_0+\mu)\gamma(\mu)$ were used, then $\phi_-(\mu)$ and $\phi_\nu(\mu)$ would have been orthogonal over the half-range of μ . With this in mind, the following generalities of (3.16) through (3.21) may be obtained for $0 < \nu < 1$ and $0 < \nu' < 1$:

$$\int_0^1 \phi_\nu(\mu)\,\phi_{\nu'}(\mu)(\nu_0 \mp \mu)\gamma(\mu)\,d\mu = (\nu_0 \mp \nu)\gamma(\nu)\,\Lambda^+(\nu)\Lambda^-(\nu)\,\delta(\nu-\nu') \quad \text{(D.1)}$$

$$\int_0^1 \phi_\pm(\mu)\,\phi_\nu(\mu)(\nu_0 \mp \mu)\gamma(\mu)\,d\mu = 0 \quad \text{(D.2)}$$

$$\int_0^1 \phi_\pm(\mu)\,\phi_\nu(\mu)(\nu_0 \pm \mu)\gamma(\mu)\,d\mu = c\nu_0\nu\,X(\pm\nu_0)\phi_\pm(\nu) \quad \text{(D.3)}$$

$$\int_0^1 \phi_\pm(\mu)\,\phi_\pm(\mu)(\nu_0 \mp \mu)\gamma(\mu)\,d\mu = \mp\left(\frac{c\nu_0}{2}\right)^2 X(\pm\nu_0) \quad \text{(D.4)}$$

$$\int_0^1 \phi_\pm(\mu)\,\phi_\mp(\mu)(\nu_0 \mp \mu)\gamma(\mu)\,d\mu = \frac{c^2\nu\nu_0}{4}\,X(-\nu) \quad \text{(D.5)}$$

$$\int_0^1 \phi_{-\nu}(\mu)\,\phi_{\nu'}(\mu)(\nu_0 \mp \mu)\gamma(\mu)\,d\mu = \frac{c\nu'}{2}\,\phi_{-\nu'}(\nu')(\nu_0 \pm \nu)X(-\nu) \, , \quad \text{(D.6)}$$

Equations (D.2), (D.4), and (D.5) simplify to

$$\int_0^1 \phi_\nu(\mu)\,\gamma(\mu)\,d\mu = 0 \quad \text{(D.7)}$$

$$\int_0^1 \phi_\pm(\mu)\,\gamma(\mu)\,d\mu = \mp\frac{c\nu_0}{2}\,X(\pm\nu_0) \quad \text{(D.8)}$$

$$\int_0^1 \varphi_{-\nu}(\mu)\, \gamma(\mu)\, d\mu = \frac{c\nu}{2}\, X(-\nu) \, . \tag{D.9}$$

Adding (D.2) and (D.3) gives

$$\int_0^1 \varphi_{\pm}(\mu)\, \varphi_\nu(\mu)\, \gamma(\mu)\, d\mu = \frac{c\nu}{2}\, X(\pm\nu_0)\, \varphi_{\pm}(\nu) \, . \tag{D.10}$$

Adding both equations of (D.1) results in the answer

$$\int_0^1 \varphi_\nu(\mu)\, \varphi_{\nu'}(\mu)\, \gamma(\mu)\, d\mu = \gamma(\nu)\, \Lambda^+(\nu)\, \Lambda^-(\nu)\, \delta(\nu-\nu') \tag{D.11}$$

and adding both equations of (D.6) yields

$$\int_0^1 \varphi_{-\nu}(\mu)\, \varphi_{\nu'}(\mu)\, \gamma(\mu)\, d\mu = \frac{c\nu'}{2}\, X(-\nu)\, \varphi_{-\nu}(\nu') \, . \tag{D.12}$$

From (D.10) it is seen that neither $\phi_+(\mu)$ nor $\phi_-(\mu)$ is orthogonal to $\phi_\nu(\mu)$, $0 < \nu < 1$, with respect to the weight function $\gamma(\mu)$. This is of no concern, however, in problems which do not have solutions in closed form such as the slab problem. In fact, Equations (5.11) and (5.16) are most easily obtained using the above relations. Multiplying (5.7) by $\gamma(\mu)d\mu$ and integrating over the half-range of μ gives (5.11) after Equations (D.7) through (D.9) are used. Use of (D.10) through (D.12) gives (5.16) directly--without the partial fraction decomposition which is needed if (3.16), (3.17), (3.18), and (3.21) are used on Equation (5.7).

EVALUATION OF INTEGRALS NEEDED TO FIND THE
SURFACE DISTRIBUTIONS OF CHAPTER V

Derivation of Equation (5.33)

The reflected and transmitted angular densities at the sur-
faces may be written, for $\mu \geq 0$, as

$$\Psi(0,-\mu) = a_+ \, \varphi_-(\mu) + a_- \, \varphi_+(\mu) + \int_0^1 A(-\nu) \, \varphi_\nu(\mu) \, d\nu$$

$$+ \int_0^1 \left[\frac{B_+(\nu) + B_-(\nu)}{2} \right] \varphi_{-\nu}(\mu) \, d\nu \qquad (E.1)$$

$$\Psi(d,\mu) = a_+ \, \varphi_+(\mu) e^{-d/\nu_0} + a_- \, \varphi_-(\mu) e^{d/\nu_0} + \int_0^1 A(\nu) \varphi_\nu(\mu) e^{-d/\nu} \, d\nu$$

$$+ \int_0^1 \left[\frac{B_+(\nu) - B_-(\nu)}{2} \right] \varphi_{-\nu}(\mu) \, d\nu \qquad (E.2)$$

where (5.3) and (5.8) have been used. Use of (5.16) in the above
equations gives

$$\Psi(0,-\mu) = a_+ \left[\varphi_-(\mu) - X(\nu_0) \int_0^1 \frac{\varphi_{-\nu}(\mu) \, \frac{c\nu}{2} \, \varphi_+(\nu) d\nu}{\gamma(\nu) \, \Lambda^+(\nu) \, \Lambda^-(\nu)} \right]$$

$$+ a_- \left[\varphi_+(\mu) - X(-\nu_0) \int_0^1 \frac{\varphi_{-\nu}(\mu) \, \frac{c\nu}{2} \, \varphi_-(\nu) d\nu}{\gamma(\nu) \, \Lambda^+(\nu) \, \Lambda^-(\nu)} \right]$$

$$+ \int_0^1 A(-\nu) \left[\varphi_\nu(\mu) - X(-\nu) \int_0^1 \frac{\varphi_{-\nu'}(\mu) \, \frac{c\nu'}{2} \, \varphi_{-\nu}(\nu') d\nu'}{\gamma(\nu') \, \Lambda^+(\nu') \, \Lambda^-(\nu')} \right] d\nu$$

$$+ \gamma(\mu_0) \int_0^1 \frac{\varphi_{-\nu}(\mu) \, \varphi_\nu(\mu_0) \, d\nu}{\gamma(\nu) \, \Lambda^+(\nu) \, \Lambda^-(\nu)} \tag{E.3}$$

and

$$\Psi(d,\mu) = a_+ e^{-d/\nu_0} \left[\varphi_+(\mu) - X(-\nu_0) \int_0^1 \frac{\varphi_{-\nu}(\mu) \, \frac{c\nu}{2} \, \varphi_-(\nu) d\nu}{\gamma(\nu) \, \Lambda^+(\nu) \, \Lambda^-(\nu)} \right]$$

$$+ a_- e^{d/\nu_0} \left[\varphi_-(\mu) - X(\nu_0) \int_0^1 \frac{\varphi_{-\nu}(\mu) \, \frac{c\nu}{2} \, \varphi_+(\nu) d\nu}{\gamma(\nu) \, \Lambda^+(\nu) \, \Lambda^-(\nu)} \right]$$

$$+ \int_0^1 A(\nu) e^{-d/\nu} \left[\varphi_\nu(\mu) - X(-\nu) \int_0^1 \frac{\varphi_{-\nu'}(\mu) \, \frac{c\nu'}{2} \, \varphi_{-\nu}(\nu') d\nu'}{\gamma(\nu') \, \Lambda^+(\nu') \, \Lambda^-(\nu')} \right] d\nu . \tag{E.4}$$

Consider first the function $[(z-\nu_0)X(z)]^{-1}$ which is analytic everywhere in the complex plane except for a simple pole at ν_0 and the cut from 0 to 1 . It behaves like -1 as $z \to \infty$. Considering the contour in Figure E.1 and using the Cauchy integral formula yields

$$\frac{1}{(z-\nu_0)X(z)} + 1 = \frac{1}{2\pi i} \oint_{C_1+C_2} \frac{dz'}{(z'-\nu_0) X(z')(z'-z)} \tag{E.5}$$
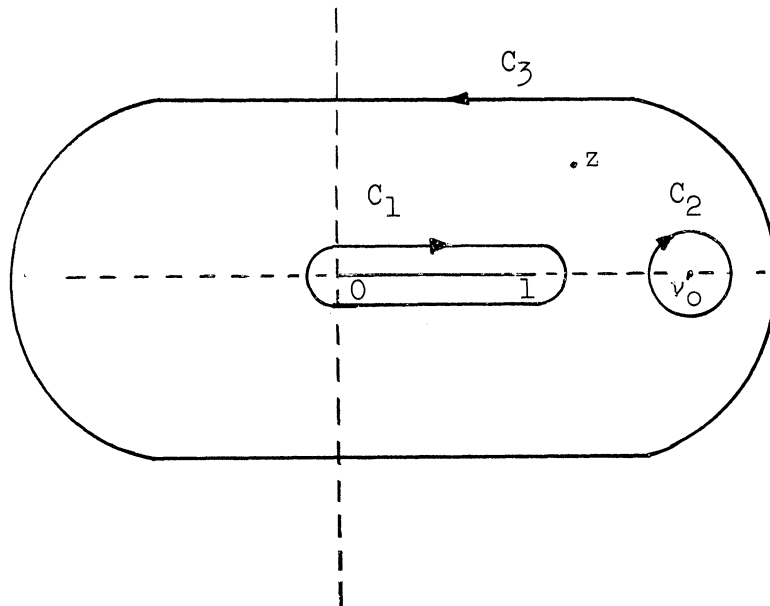
Figure E.1. The Path C for Equation (E.5).

after letting C_3 go to infinity. Use of (B.4) with $z = -\mu$, $0 < \mu < 1$, gives

$$-\frac{1}{(\nu_0 + \mu)\, X(-\mu)} + 1 = -\frac{1}{(\nu_0 + \mu)\, X(\nu_0)} + \int_0^1 \frac{\left(\frac{c\nu}{2}\right)^2 d\nu}{(\nu_0 - \nu)(\nu + \mu)\, \gamma(\nu)\, \Lambda^+(\nu)\, \Lambda^-(\nu)}, \quad \text{(E.6)}$$

A similar equation may be obtained by letting $\nu_0 \to -\nu_0$. Multiplying the equations by $\mp \frac{c\nu_0}{2} X(\pm \nu_0)$ gives the result

$$\varphi_\mp(\mu) - X(\pm\nu_0) \int_0^1 \frac{\varphi_{-\nu}(\mu)\, \frac{c\nu}{2}\, \varphi_\pm(\nu)\, d\nu}{\gamma(\nu)\, \Lambda^+(\nu)\, \Lambda^-(\nu)} =$$

$$\frac{\varphi_\mp(\mu)\, X(\pm\nu_0)}{X(-\mu)} \mp \frac{c\nu_0}{2}\, X(\pm\nu_0) \qquad \text{(E.7)}$$

The second function to be considered is $[(\nu+z)X(z)]^{-1}$.

It is analytic in the complex plane except for a simple pole at $-\nu$

and the cut from 0 to 1 and it behaves like -1 as $z \to \infty$.

Proceeding in the same manner as above, one obtains

$$\frac{1}{(\nu-\mu)X(-\mu)} + 1 = \frac{1}{(\nu-\mu)X(-\nu)} - \int_0^1 \frac{\left(\frac{c\nu'}{2}\right)^2 d\nu'}{(\nu+\nu')(\nu'+\mu)\gamma(\nu')\Lambda^+(\nu')\Lambda^-(\nu')} \cdot \quad (E.8)$$

Multiplying by $\frac{c\nu}{2} X(-\nu)$ and then adding the term $\frac{X(-\nu)}{X(-\mu)} \lambda(\mu)\delta(\mu-\nu)$
to each side allows us to write (E.8) as

$$\varphi_\nu(\mu) - X(-\nu)\int_0^1 \frac{\varphi_{-\nu'}(\mu) \frac{c\nu'}{2} \varphi_{-\nu}(\nu') d\nu'}{\gamma(\nu')\Lambda^+(\nu')\Lambda^-(\nu')} =$$

$$\frac{\varphi_\nu(\mu) X(-\nu)}{X(-\mu)} + \frac{c\nu}{2} X(-\nu) \ , \qquad (E.9)$$

The last function to be considered is $[(z-\mu_0)X(z)]^{-1}$. It

is analytic everywhere except for a pole at μ_0 which is imbedded in

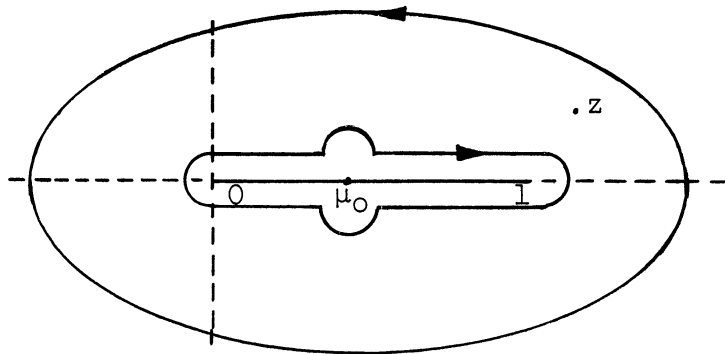the cut from 0 to 1 . It also behaves like -1 as $z \to \infty$.

Figure E.2. The Path C Used to Obtain Equation (E.10).

The Cauchy integral formula and Equation (2.12) give

$$- \frac{1}{(\mu + \mu_0) X(-\mu)} + 1 = \frac{1}{2\pi i} P \int_0^1 \frac{\left[\frac{1}{X^+(\nu)} - \frac{1}{X^-(\nu)} \right] d\nu}{(\nu - \mu_0)(\nu + \mu)}$$

$$- \frac{\frac{1}{2}}{\mu_0 + \mu} \left[\frac{1}{X^+(\mu_0)} + \frac{1}{X^-(\mu_0)} \right] \tag{E.10}$$

after letting the outer contour go to infinity and setting $z = -\mu$.

Use of (B.4) and multiplication by $-\gamma(\mu_0)$ reduces (E.10) to the form

$$\gamma(\mu_0) \int_0^1 \frac{\varphi_{-\nu}(\mu) \varphi_\nu(\mu_0) d\nu}{\gamma(\nu) \Lambda^+(\nu) \Lambda^-(\nu)} = \frac{\gamma(\mu_0)}{(\mu + \mu_0) X(-\mu)} - \gamma(\mu_0) \quad , \tag{E.11}$$

Equations (E.7), (E.9), and (E.11) reduce (E.3) to

$$\Psi(0, -\mu) = \frac{1}{X(-\mu)} \left[\frac{\gamma(\mu_0)}{\mu + \mu_0} + a_+ \varphi_-(\mu) X(\nu_0) + a_- \varphi_+(\mu) X(-\nu_0) \right.$$

$$\left. + \int_0^1 \Lambda(-\nu) \varphi_\nu(\mu) X(-\nu) d\nu \right] - \gamma(\mu_0) - a_+ \frac{c\nu_0}{2} X(\nu_0)$$

$$+ a_- \frac{c\nu_0}{2} X(-\nu_0) + \int_0^1 \Lambda(-\nu) \frac{c\nu}{2} X(-\nu) d\nu \tag{E.12}$$

and (E.7) and (E.9) in (E.4) gives

$$\psi(d,\mu) = \frac{1}{X(-\mu)}\left[a_+ e^{-d/\nu_0}\,\varphi_+(\mu)X(-\nu_0) + a_- e^{d/\nu_0}\,\varphi_-(\mu)X(\nu_0)\right.$$

$$\left.+\int_0^1 A(\nu)\,e^{-d/\nu}\,\varphi_\nu(\mu)X(-\nu)d\nu\right] + a_+ e^{-d/\nu_0}\,\frac{c\nu_0}{2}X(-\nu_0)$$

$$- a_- e^{d/\nu_0}\,\frac{c\nu_0}{2}X(\nu_0) + \int_0^1 A(\nu)e^{-d/\nu}\,\frac{c\nu}{2}X(-\nu)d\nu \ , \qquad (E.13)$$

Now examine Equations (5.4) and (5.6). Multiplication of both equations by $\gamma(\mu)d\mu$ and integration over μ from 0 to 1 gives

$$\gamma(\mu_0) + a_+ \frac{c\nu_0}{2}X(\nu_0) - a_- \frac{c\nu_0}{2}X(-\nu_0)$$

$$- \int_0^1 A(-\nu)\frac{c\nu}{2}X(-\nu)d\nu = 0$$

$$- a_+ e^{-d/\nu_0}\frac{c\nu_0}{2}X(-\nu_0) + a_- e^{d/\nu_0}\frac{c\nu_0}{2}X(\nu_0)$$

$$- \int_0^1 A(\nu)e^{-d/\nu}\frac{c\nu}{2}X(-\nu)d\nu = 0 \qquad (E.14)$$

where (D.7), (D.8), and (D.9) have been used. The reflected and trans-mitted angular densities, for $\mu \geq 0$, are therefore given by

$$\psi(0,-\mu) = \frac{1}{X(-\mu)}\left[\frac{\gamma(\mu_0)}{\mu + \mu_0} + a_+\,\varphi_-(\mu)X(\nu_0) + a_-\,\varphi_+(\mu)X(-\nu_0)\right.$$

$$\left.+\int_0^1 A(-\nu)\,\varphi_\nu(\mu)X(-\nu)d\nu\right]$$

$$\Psi(d,\mu) = \frac{1}{X(-\mu)}\left[a_+ e^{-d/\nu_0} \varphi_+(\mu)X(-\nu_0) + a_- e^{d/\nu_0} \varphi_-(\mu)X(\nu_0)\right.$$

$$\left. + \int_0^1 A(\nu) e^{-d/\nu} \varphi_\nu(\mu) X(-\nu)d\nu \right]. \tag{5.33}$$

The same equations can be derived in another manner[33] by use of results obtained in the constructive derivation of (5.11) and (5.16).

Derivation of Equation (5.36)

Using (5.1), (5.2), and (5.33), the surface densities and net currents are

$$\rho(0) = 1 + \gamma(\mu_0)\int_0^1 \frac{d\mu}{X(-\mu)(\mu+\mu_0)} + a_+ \frac{c\nu_0}{2} X(\nu_0)\int_0^1 \frac{d\mu}{X(-\mu)(\mu+\nu_0)}$$

$$- a_- \frac{c\nu_0}{2} X(-\nu_0)\int_0^1 \frac{d\mu}{X(-\mu)(\mu-\nu_0)} + \int_0^1 A(-\nu)X(-\nu)\left[\int_0^1 \frac{\varphi_\nu(\mu)d\mu}{X(-\mu)}\right]d\nu$$

$$\rho(d) = -a_+ e^{-d/\nu_0}\frac{c\nu_0}{2} X(-\nu_0)\int_0^1 \frac{d\mu}{X(-\mu)(\mu-\nu_0)} + a_- e^{d/\nu_0}\frac{c\nu_0}{2} X(\nu_0)\int_0^1 \frac{d\mu}{X(-\mu)(\mu+\nu_0)}$$

$$+ \int_0^1 A(\nu) e^{-d/\nu} X(-\nu)\left[\int_0^1 \frac{\varphi_\nu(\mu)d\mu}{X(-\mu)}\right]d\nu \tag{E.15}$$

and

$$j(0) = \mu_0 - \gamma(\mu_0) \int_0^1 \frac{\mu \, d\mu}{X(-\mu)(\mu + \mu_0)} - a_+ \frac{c\nu_0}{2} X(\nu_0) \int_0^1 \frac{\mu \, d\mu}{X(-\mu)(\mu + \nu_0)}$$

$$+ a_- \frac{c\nu_0}{2} X(-\nu_0) \int_0^1 \frac{\mu \, d\mu}{X(-\mu)(\mu - \nu_0)} - \int_0^1 A(-\nu) X(-\nu) \left[\int_0^1 \frac{\mu \, \varphi_\nu(\mu) \, d\mu}{X(-\mu)} \right] d\nu$$

$$j(d) = -a_+ e^{-d/\nu_0} \frac{c\nu_0}{2} X(-\nu_0) \int_0^1 \frac{\mu \, d\mu}{X(-\mu)(\mu - \nu_0)} + a_- e^{d/\nu_0} \frac{c\nu_0}{2} X(\nu_0) \int_0^1 \frac{\mu \, d\mu}{X(-\mu)(\mu + \nu_0)}$$

$$+ \int_0^1 A(\nu) e^{-d/\nu} X(-\nu) \left[\int_0^1 \frac{\mu \, \varphi_\nu(\mu) \, d\mu}{X(-\mu)} \right] d\nu \quad . \tag{E.16}$$

Using partial fractions, one can reduce the integrals in (E.16) to

$$\int_0^1 \frac{\mu \, d\mu}{X(-\mu)(\mu + \mu_0)} = \int_0^1 \frac{d\mu}{X(-\mu)} - \mu_0 \int_0^1 \frac{d\mu}{X(-\mu)(\mu + \mu_0)}$$

$$\int_0^1 \frac{\mu \, d\mu}{X(-\mu)(\mu \pm \nu_0)} = \int_0^1 \frac{d\mu}{X(-\mu)} \mp \nu_0 \int_0^1 \frac{d\mu}{X(-\mu)(\mu \pm \nu_0)}$$

$$\int_0^1 \frac{\mu \, \varphi_\nu(\mu) \, d\mu}{X(-\mu)} = -\frac{c\nu}{2} \int_0^1 \frac{d\mu}{X(-\mu)} + \nu \int_0^1 \frac{\varphi_\nu(\mu) \, d\mu}{X(-\mu)} \quad , \tag{E.17}$$

In order to evaluate the above integrals, consider the

function $(v_o^2 - z^2)X(z)(z-\epsilon)^{-1}$ where ϵ is a small negative number.

The function behaves like $+1$ as $z \to \infty$ and is analytic in the cut

plane except for a simple pole at $z = \epsilon$. Applying the Cauchy inte-

gral formula to the analytic region enclosed by the contour shown in

Figure E.3 gives

$$\frac{(v_o^2 - z^2)X(z)}{z-\epsilon} - 1 = \int_0^1 \frac{(v_o^2 - \mu^2)\frac{1}{2\pi i}\left[X^+(\mu) - X^-(\mu)\right]d\mu}{(\mu-\epsilon)(\mu-z)} - \frac{v_o^2 X(\epsilon)}{\epsilon - z} \quad \text{(E.18)}$$
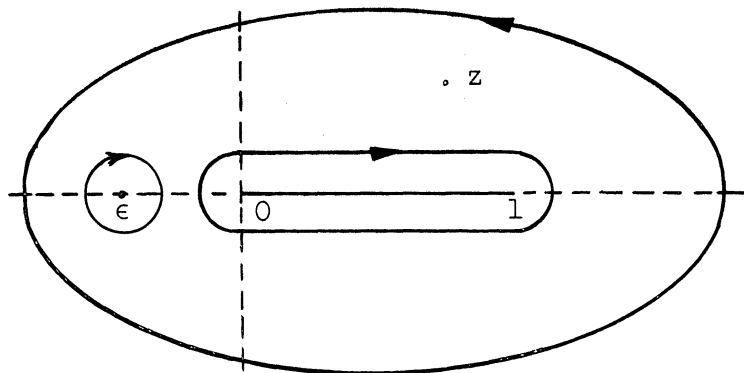

Figure E.3. The Path C Used to Obtain Equation (E.18).

From (2.38) and (A.4),

$$\frac{1}{2\pi i}\left[X^+(\mu) - X^-(\mu)\right] = \frac{\frac{c\mu}{2}}{(v_o^2 - \mu^2)(1-c)X(-\mu)}, \quad 0 \le \mu < 1, \quad \text{(E.19)}$$

Using this and taking the limit as $\epsilon \to 0$ in (E.18) yields

$$\int_0^1 \frac{d\mu}{X(-\mu)(\mu - z)} = \frac{2(1-c)}{c}\left[\frac{(\nu_0^2 - z^2)X(z)}{z} - 1 - \frac{\nu_0^2 X(0)}{z}\right] \qquad (E.20)$$

As $z = -\mu_0$, $\mp \nu_0$, and $+\mu$, one obtains

$$\int_0^1 \frac{d\mu}{X(-\mu)(\mu + \mu_0)} = \frac{2(1-c)}{c}\left[-1 + \frac{\nu_0^2 X(0)}{\mu_0}\right] - \frac{1}{\gamma(\mu_0)}$$

$$\int_0^1 \frac{d\mu}{X(-\mu)(\mu \pm \nu_0)} = \frac{2(1-c)}{c}\left[-1 \pm \nu_0 X(0)\right]$$

$$\int_0^1 \frac{\varphi_\nu(\mu)\,d\mu}{X(-\mu)} = \frac{2(1-c)}{c}\left[\frac{c\nu}{2} + \frac{c\nu_0^2 X(0)}{2}\right]. \qquad (E.21)$$

Equations (E.15), (E.16), (E.17), and (E.21) combine to give

$$\rho(0) = \frac{1}{X(0)}\left[\frac{2\,\gamma(\mu_0)}{c\mu_0} + a_+ X(\nu_0) + a_- X(-\nu_0) + \int_0^1 A(-\nu)X(-\nu)\,d\nu\right]$$

$$- \frac{2(1-c)}{c}\left[\gamma(\mu_0) + a_+ \frac{c\nu_0}{2} X(\nu_0) - a_- \frac{c\nu_0}{2} X(-\nu_0) - \int_0^1 A(-\nu)X(-\nu)\frac{c\nu}{2}\,d\nu\right]$$

$$\rho(d) = \frac{1}{X(0)}\left[a_+ e^{-d/\nu_0} X(-\nu_0) + a_- e^{d/\nu_0} X(\nu_0) + \int_0^1 A(\nu)e^{-d/\nu} X(-\nu)\,d\nu\right]$$

$$- \frac{2(1-c)}{c}\left[-a_+ e^{-d/\nu_0}\frac{c\nu_0}{2} X(-\nu_0) + a_- e^{d/\nu_0}\frac{c\nu_0}{2} X(\nu_0) - \int_0^1 A(\nu)e^{-d/\nu}\frac{c\nu}{2} X(-\nu)\,d\nu\right]$$

$$\dot{j}(0) = -(1-c)\left[\frac{2\mu_0^2\,\gamma(\mu_0)}{c\mu_0} + a_+\nu_0^2 X(\nu_0) + a_-\nu_0^2 X(-\nu_0) + \int_0^1 A(-\nu)\nu^2 X(-\nu)\,d\nu\right]$$

$$+\left[\frac{2}{cX(0)} - \int_0^1 \frac{d\mu}{X(-\mu)}\right](x)$$

$$(x)\left[\gamma(\mu_0) + a_+\frac{c\nu_0}{2}X(\nu_0) - a_-\frac{c\nu_0}{2}X(-\nu_0) - \int_0^1 A(-\nu)\frac{c\nu}{2}X(-\nu)\,d\nu\right]$$

$$\dot{j}(d) = (1-c)\left[a_+ e^{-d/\nu_0}\nu_0^2 X(-\nu_0) + a_- e^{d/\nu_0}\nu_0^2 X(\nu_0) + \int_0^1 A(\nu)e^{-d/\nu}\nu^2 X(-\nu)\,d\nu\right]$$

$$-\left[\frac{2}{cX(0)} - \int_0^1 \frac{d\mu}{X(-\mu)}\right](x)$$

$$(x)\left[-a_+ e^{-d/\nu_0}\frac{c\nu_0}{2}X(-\nu_0) + a_- e^{d/\nu_0}\frac{c\nu_0}{2}X(\nu_0) - \int_0^1 A(\nu)e^{-d/\nu}\frac{c\nu}{2}X(-\nu)\,d\nu\right] \quad (E.22)$$

and these results reduce to (5.36) when the equations in (E.14) are used.

APPENDIX F

GENERALIZATION OF SLAB BOUNDARY CONDITIONS

Using the solution of the slab albedo problem obtained in Chapter V, it is possible to write the solutions to related slab problems. For example, suppose we wish the solution to the albedo problem for the slab which specularly reflects neutrons (or photons) at the surface located at $x = d$. The boundary conditions on the angular density, $\psi_r(x,\mu)$, are:

$$\psi_r(0,\mu) = \delta(\mu - \mu_0) \;, \mu > 0$$

$$\psi_r(d,-\mu) = \psi_r(d,\mu), \; -1 \leq \mu \leq 1 \;, \tag{F.1}$$

Using an approach which is essentially the same as the method of images in electrostatic theory, the result may be written down immediately. The angular density is given by

$$\psi_r(x,\mu) = \psi_a(x,\mu) + \psi_a'(x,\mu) \;, \; 0 \leq x \leq d \;, \tag{F.2}$$

after the slab thickness in the expansion coefficients is doubled to account for the image slab. Here, $\psi_a(x,\mu)$ is the solution of Chapter V and $\psi_a'(x,\mu) = \psi_a(d-x,-\mu)$.

As another example, suppose that one desires to solve the slab albedo problem under the condition that the face at $x = d$ reflects neutrons (photons) in a diffuse manner. The boundary conditions on the angular density, $\psi_d(x,\mu)$, are:

$$\Psi_d\,(0,\mu) = \delta(\mu - \mu_0)\,,\ \mu > 0$$

$$\Psi_d\,(d,-\mu) = S\int_0^1 f(\mu_0)\,\Psi_a'(d,-\mu)\,d\mu_0\,,\ \mu > 0\ .\tag{F.3}$$

The function $f(\mu_0)$ is a function which prescribes the angular distribution of the diffusely reflected neutrons. The constant S is to be determined by the requirement that the number of neutrons leaving the slab at $x = d$ be equal to the number entering, i.e.,

$$-\int_{-1}^0 \mu\,\Psi_d\,(d,\mu)\,d\mu = \int_0^1 \mu\,\Psi_d\,(d,\mu)\,d\mu$$

or

$$S = \int_0^1 \mu\,\Psi_d\,(d,\mu)\,d\mu\ .\tag{F.4}$$

In (F.4), we have chosen the normalization $\int_0^1 \mu_0 f(\mu_0)\,d\mu_0 = 1$. The diffuse angular density is given by

$$\Psi_d\,(x,\mu) = \Psi_a\,(x,\mu) + S\int_0^1 f(\mu_0)\,\Psi_a'(x,\mu)\,d\mu_0\tag{F.5}$$

and (F.4) and (F.5) combine to determine the value of S as

$$S = \frac{\int_0^1 \mu\,\Psi_a(d,\mu)\,d\mu}{1-\int_0^1 \mu\,d\mu\int_0^1 f(\mu_0)\,\Psi_a'(d,\mu)\,d\mu_0}\ .\tag{F.6}$$

Equation (F.6) may also be written as

$$S = \frac{\mu_0 T}{1 - \int_0^1 \mu_0 R \, f(\mu_0) \, d\mu_0}$$ (F.7)

where T and R are given by (5.43), (5.47), (5.60), and (5.61).

As a guide to the study of photon transport in the atmosphere, the solutions to the slabs with specularly and diffusely reflecting faces may be used. Assume that one desires to know the one-speed steady-state photon distribution (ignoring polarization effects) in a homogeneous atmosphere illuminated by an incident distribution from the sun. Scattering in the atmosphere is assumed to be isotropic. Photons striking the earth's surface, however, are permitted to be either specularly or diffusely reflected back into the atmosphere or absorbed. The angular density in the atmosphere, $\Psi(x,\mu)$, is then given by

$$\Psi(x,\mu) = M \, \Psi_r(x,\mu) + N \, \Psi_d(x,\mu)$$ (F.8)

where M and N are the coefficients of specular and diffuse reflection of the earth's surface, respectively. Clearly, $M + N \leq 1$. If the diffuse reflection obeys Lambert's law, i.e., if the diffuse reflection is isotropic, then $f(\mu_0) = 2$ and the problem is solved once M and N are known.

A problem arises when one attempts to apply the above theory to a model of the earth's atmosphere, however. When the scattering in

the atmosphere is most nearly isotropic, the atmosphere is clear and the mean free path thickness (dependent upon the photon energy) is of the order of unity. For a slab thickness this small, a high order of approximation is required for the solution of the Fredholm integral equations containing the continuum expansion coefficients. This becomes quite unwieldly since even the first-order approximation involves the numerical evaluation of the complicated integrals of (5.30). In attempting to apply the zeroth- or first-order approximation to the model of the earth's atmosphere for thicker slabs, one must be concerned with the transfer of monoenergetic radiation through a cloudy or foggy homogeneous atmosphere. This has the advantage that inside the atmosphere, polarization effects may be ignored which make the model more realistic. The primary failure of the above theory in this case, however, is that a foggy medium scatters radiation in a highly anisotropic manner.

Another important drawback to the present formulation of the problem of photon scattering in an earth's atmosphere is that azimuthal symmetry was assumed. If the angle of the sun's monodirectional beam of incident radiation is such that $\mu_0 \approx 1$, the above model is satisfactory. For other values of μ_0, however, the transport equation without azimuthal symmetry would have to be considered since the boundary condition for the sun's radiation is of the form

$$\Psi(0, \mu, \phi) = \delta(\mu - \mu_0)\, \delta(\phi - \phi_0) , \quad \mu > 0, \tag{F.9}$$

Equation (1.3) would then appear as

$$\left[\mu \frac{\partial}{\partial x} + 1\right] \Psi(x, \mu, \phi) = \frac{c}{4\pi} \sum_{k=0}^{N} b_k \int_{-1}^{1} d\mu' \int_{0}^{2\pi} d\phi' \left[P_k(\mu) P_k(\mu') \right.$$

$$\left. + 2 \sum_{\ell=1}^{k} \frac{(k-\ell)!}{(k+\ell)!} P_k^{\ell}(\mu) P_k^{\ell}(\mu') \cos\left[\ell(\phi - \phi')\right] \right] \Psi(x, \mu', \phi') \qquad \text{(F.10)}$$

which can be written more concisely as

$$\left[\mu \frac{\partial}{\partial x} + 1\right] \Psi(x, \mu, \phi) = \frac{c}{4\pi} \sum_{k=0}^{N} b_k \sum_{\ell=0}^{k} (2 - \delta_{\ell 0}) \frac{(k-\ell)!}{(k+\ell)!} (x)$$

$$(x) P_k^{\ell}(\mu) \int_{-1}^{1} P_k^{\ell}(\mu') d\mu' \int_{0}^{2\pi} \cos\left[\ell(\phi - \phi')\right] \Psi(x, \mu', \phi') d\phi' \qquad \text{(F.11)}$$

With the method of invariant imbedding, Chandrasekhar[2] was able to solve (F.11) with the boundary condition (F.9). After separating out the uncollided beam from the angular density, he transformed (F.11) into an inhomogeneous equation for the angular density of the radiation field which had suffered at least one scattering interaction. He then expanded the angular density of the transport equation in a finite Fourier cosine series in terms of the angle $(\phi - \phi_0)$.

Because he had separated out the uncollided beam of particles, the solution eventually reduced to the solution of N azimuthally-independent transport equations.

An identical approach could be followed using Case's method. The solution to each azimuthally-independent, non-homogeneous transport equation is found by solving the Green's function problem.

APPENDIX G

EVALUATION OF INTEGRALS NEEDED TO FIND THE
SURFACE DISTRIBUTIONS OF CHAPTER VI

Derivation of Equation (6.59)

The integral which must be evaluated is

$$I_2 = \int_0^1 \frac{\varphi_{-\nu}(\mu)\, f(\nu)}{\gamma(\nu)\, \Lambda^+(\nu)\, \Lambda^-(\nu)} \left[\varphi_{\nu}(\mu') \right.$$

$$\left. + \frac{3\nu\, b_2\, f(\mu')(\nu+\bar{\nu})}{8\, f(\nu)} \right] d\nu \Bigg|_{\substack{\mu>0 \\ \mu'>0}} \tag{G.1}$$

Use of (6.9) gives

$$I_2 = \frac{\mu'\, f(-\mu)\, f(\mu')\, X(\mu')}{2\, \gamma(\mu')\, \Lambda^+(\mu')\Lambda^-(\mu')\, (\mu'+\mu)}$$

$$+ \frac{f(-\mu)\, f(\mu')}{4}\, P\!\int_0^1 \frac{\nu^2\, f(\nu)\, d\nu}{\gamma(\nu)\, \Lambda^+(\nu)\Lambda^-(\nu)(\nu+\mu)(\nu-\mu')}$$

$$+ \frac{3\, b_2\, f(-\mu)\, f(\mu')}{16}\int_0^1 \frac{\nu^2\, (\nu+\bar{\nu})\, d\nu}{\gamma(\nu)\Lambda^+(\nu)\Lambda^-(\nu)(\nu+\mu)}\ , \tag{G.2}$$

The behavior of $(X(z))^{-1}$ on the cut $0 \le \nu \le 1$ is given

by

$$\frac{1}{2\pi i}\left[\frac{1}{X^+(\nu)} - \frac{1}{X^-(\nu)}\right] = - \frac{\left(\frac{\nu}{2}\right)^2 f(\nu)}{\gamma(\nu)\,\Lambda^+(\nu)\,\Lambda^-(\nu)} \quad, \quad 0 < \nu < 1$$

$$\frac{1}{2}\left[\frac{1}{X^+(\nu)} + \frac{1}{X^-(\nu)}\right] = \frac{\frac{\nu}{2}\,\lambda(\nu)}{\gamma(\nu)\,\Lambda^+(\nu)\,\Lambda^-(\nu)} \quad, \quad 0 < \nu < 1 \quad. \qquad (G.3)$$

Now consider the function $[(z-\mu')X(z)]^{-1}$. It is analytic in the complex plane except for a pole at μ' imbedded in the cut from 0 to 1. The function behaves like -1 as $z \to \infty$. The contour to be used when applying the Cauchy integral formula is that of Figure G.1. Upon letting the contour C_1 become very large,

$$\frac{1}{(z-\mu')X(z)} + 1 = \frac{1}{2\pi i}\oint_{C_2} \frac{dz'}{(z'-\mu')\,X(z')\,(z'-z)} \quad. \qquad (G.4)$$
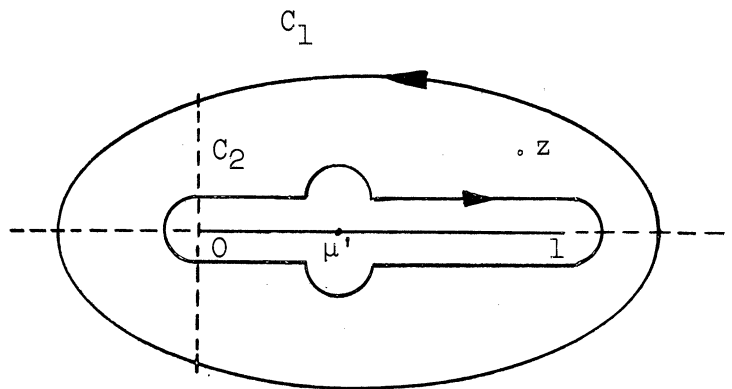

Figure G.1. The Path C Used to Obtain Equation (G.4).

Using Equations (2.12), (G.3), and (G.4) gives, for $z = -\mu$,

$$\frac{1}{(\mu + \mu') X(-\mu)} - 1 = P \int_0^1 \frac{\left(\frac{\nu}{2}\right)^2 f(\nu) d\nu}{(\nu - \mu')(\nu + \mu) \gamma(\nu) \Lambda^+(\nu) \Lambda^-(\nu)} + \frac{\frac{\mu'}{2} \lambda(\mu')}{\gamma(\mu') \Lambda^+(\mu') \Lambda^-(\mu')(\mu + \mu')} , \qquad (G.5)$$

We now consider the function $(z+\bar{\nu})/[f(z)X(z)]$. It behaves like $4/(3b_2)$ as $z \to \infty$ and is analytic in the complex plane except for simple poles at $z = \pm \sqrt{(1 + \frac{b_2}{4})/\frac{3b_2}{4}} = \pm \sigma$ and the cut from 0 to 1 . Applying the Cauchy integral formula gives the equation

$$\frac{z + \bar{\nu}}{f(z) X(z)} - \frac{4}{3b_2} = \frac{1}{2\pi i} \int_0^1 \frac{\left[\frac{1}{X^+(\nu)} - \frac{1}{X^-(\nu)}\right](\nu + \bar{\nu}) d\nu}{f(\nu)(\nu - z)}$$

$$+ \frac{4(\sigma + \bar{\nu})}{3b_2 (2\sigma)(\sigma - z) X(\sigma)}$$

$$+ \frac{4(-\sigma + \bar{\nu})}{3b_2 (2\sigma)(\sigma + z) X(-\sigma)} , \qquad (G.6)$$

After setting $z = -\mu$ and using (6.47), (6.50), and (G.3), one finds that

$$\int_0^1 \frac{\left(\frac{\nu}{2}\right)^2 f(\nu)(\nu + \bar{\nu}) d\nu}{f(\nu) \gamma(\nu) \Lambda^+(\nu) \Lambda^-(\nu)(\nu + \mu)} = \frac{\mu - \bar{\nu}}{f(-\mu) X(-\mu)} + \frac{4}{3b_2}$$

$$- \frac{4}{3b_2 f(-\mu) \gamma^{(0)}} , \qquad (G.7)$$

Equations (G.5) and (G.7) reduce (G.2) to

$$I_z = -f(\mu') \left[\frac{1}{\gamma(0)} \right.$$

$$\left. + \frac{1}{X(-\mu)} \left\{ \frac{3b_2(\bar{\nu} - \mu)}{4} - \frac{f(\mu)}{\mu + \mu'} \right\} \right] \qquad (6.59)$$

since $f(-\mu) = f(\mu)$.

Derivation of Equation (6.63)

The behavior of the X-function on the cut is given by

$$\frac{1}{2\pi i} \left[X^+(\nu) - X^-(\nu) \right] = \gamma(\nu) f(\nu) = \frac{3\nu f(\nu)}{2\left(1 - \frac{b_2}{5}\right) X(-\nu)} , \quad 0 < \nu < 1 . \qquad (G.8)$$

Equation (G.8) and the Cauchy integral formula can be used to consider

the function $X(z)/f(z)$. This function is analytic everywhere except

at the poles $z = \pm \sigma$ and the cut from 0 to 1 . It vanishes for

large z . Using the Cauchy formula, one therefore obtains

$$\frac{X(z)}{f(z)} = \int_0^1 \frac{3\mu \, d\mu}{2\left(1 - \frac{b_2}{5}\right)(\mu - z) X(-\mu)}$$

$$+ \frac{1}{2\sigma\left(\frac{3b_2}{4}\right)} \left[\frac{X(\sigma)}{\sigma - z} + \frac{X(-\sigma)}{\sigma + z} \right] \qquad (G.9)$$

Letting $z = 0$ and using (6.47) reduces (G.9) to

$$\int_0^1 \frac{d\mu}{X(-\mu)} = \frac{2\left[1 - \frac{b_2}{5}\right]}{3\left[1 + \frac{b_2}{4}\right]} \left[-\sqrt{\frac{3}{1 - b_2/4}} + \frac{3b_2 \gamma^{(1)}}{4} \right] \qquad (6.63)$$

To evaluate the second integral of (6.63), we consider the
function

$$Q(z) = \frac{\left[1 + \frac{b_2}{4} + \frac{3b_2}{4}\mu_0 z - \frac{3b_2}{4}\bar{\nu}(\mu_0 + z)\right] X(z)}{f(z)(z - \epsilon)}$$

and use the Cauchy formula to find that

$$Q(z) = \frac{3}{2(1 - b_2/5)} \int_0^1 \frac{\mu\left[1 + \frac{b_2}{4} + \frac{3b_2}{4}\mu_0\mu - \frac{3b_2}{4}\bar{\nu}(\mu_0 + \mu)\right] d\mu}{X(-\mu)(\mu - \epsilon)(\mu - z)}$$

$$+ \frac{\left[1 + \frac{b_2}{4} + \frac{3b_2}{4}\mu_0\sigma - \frac{3b_2}{4}\bar{\nu}(\mu_0 + \sigma)\right] X(\sigma)}{2\sigma\left(\frac{3b_2}{4}\right)(\sigma - z)(\sigma - \epsilon)}$$

$$- \frac{\left[1 + \frac{b_2}{4} - \frac{3b_2}{4}\mu_0\sigma - \frac{3b_2}{4}\bar{\nu}(\mu_0 - \sigma)\right] X(-\sigma)}{2\sigma\left(\frac{3b_2}{4}\right)(\sigma + z)(\sigma + \epsilon)}$$

$$+ Q(\epsilon) \tag{G.10}$$

Letting ϵ go to zero and setting $z = -\mu_0$ gives

$$\int_0^1 \frac{\left[1 + \frac{b_2}{4} + \frac{3b_2}{4}\mu_0\mu - \frac{3b_2}{4}\bar{\nu}(\mu_0 + \mu)\right] d\mu}{X(-\mu)(\mu + \mu_0)} =$$

$$\frac{2(1 - b_2/5)}{3}\left[-\frac{X(-\mu_0)}{\mu_0} + \frac{\frac{b_2}{4}X(0)}{(1 + b_2/4)\gamma^{(0)}}\right.$$

$$\left. + \frac{\left[1 + \frac{b_2}{4} - \frac{3b_2}{4}\bar{\nu}\mu_0\right]X(0)}{(1 + b_2/4)\mu_0}\right] \tag{6.63}$$

after Equations (6.47) and (6.50) are used.

APPENDIX H

RELATIONSHIP BETWEEN CHANDRASEKHAR AND CASE QUANTITIES AND NEW INTEGRAL EXPRESSIONS FOR CHANDRASEKHAR'S X- AND Y-FUNCTIONS

Some of the emergent angular distributions obtained in Chapters IV and VI were also obtained by Chandrasekhar[2] using the method of invariant imbedding. This appendix contains the relations needed to check the agreement with these results. These half-space relations correlate the Chandrasekhar H-function with the Case X-function and also the moments of the H-function with the moments of the Case γ-function. Also included in this appendix is a derivation of new integral expressions for Chandrasekhar's X- and Y-functions for slab problems using the results of the slab albedo problem of Chapter V.

Correlation Between Chandrasekhar and Case Formulations

One of the primary notational differences between results obtained by using Case's method and those of Chandrasekhar arises because of the differences in the coordinate systems used:
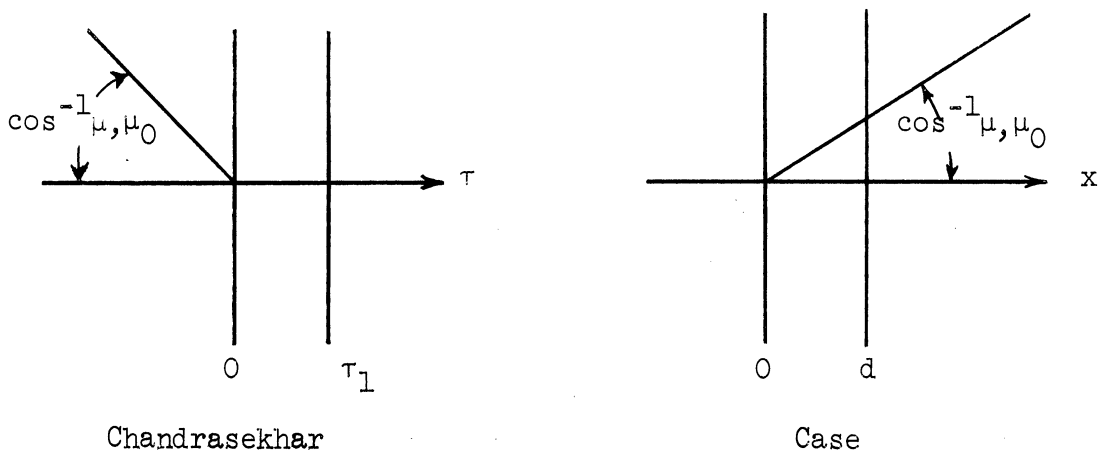

Chandrasekhar   Case

Figure H.1. Coordinate Systems Used in the Chandrasekhar and Case Formulations.

Another important difference is the fact that Chandrasekhar's

work evaluates only the emergent angular distributions $I(0,\mu)$ and

$I(d,-\mu)$, of particles which have been scattered at least once whereas

the approach of Case evaluates the entire angular distribution, $\Psi(x,\mu)$,

regardless of the past history of any particle. This means that

Chandrasekhar's law of diffuse reflection and transmission and the

slab albedo problem of our Chapter V, to which it corresponds, are

related by the equations

$$\frac{2\,I(0,\mu)}{F} = \Psi(0,-\mu) \qquad\qquad , \mu \geq 0, \mu_0 \geq 0 \qquad (H.1)$$

$$\frac{2\,I(\tau_1,-\mu)}{F} = \Psi(d,\mu) - \delta(\mu-\mu_0)\,e^{-d/\mu} \quad , \mu \geq 0, \mu_0 \geq 0 \qquad (H.2)$$

provided the scattering is isotropic.* If the scattering is aniso-

tropic, (H.1) and (H.2) relate the Case angular density to the azimuthally-

symmetric part of the Chandrasekhar intensity. This distinction arises

since the boundary conditions of Chapters IV, V, and VI assumed azimuthal

symmetry whereas Chandrasekhar solved the problem without making this

assumption. In the limit as the slab thickness becomes infinite, (H.1)

can be used to correlate the emergent angular density for the half-space

albedo problem with the azimuthally symmetric part of Chandrasekhar's

solution to the problem of diffuse reflection.

*The quantity $2/F$ is a normalization factor.

When attempting to correlate the results from Chandrasekhar's

constant net flux problem, which is the half-space Milne problem for

c = 1 ,* Equation (H.1) again is valid. In fact, (H.1) is valid for a

medium which scatters particles in an anisotropic manner since the

constant net flux problem has azimuthal symmetry.

The solutions to Chandrasekhar's half-space problems are

found in terms of the H-function, which is related to the Case X-

function by[11]

$$\frac{1}{H(\mu)} = (\nu_0 + \mu)\sqrt{1-c}\ X(-\mu)$$ (H.3)

in the case of isotropic scattering. By the use of (2.37) with z = 0 ,

this may be written as

$$\frac{1}{H(\mu)} = \left(1 + \mu/\nu_0\right) \frac{X(-\mu)}{X(0)}$$ (H.4)

This latter form is valid for arbitrary scattering law and conveniently

illustrates the behavior as c → 1 $\left(\nu_0 \to \infty\right)$:

$$\frac{1}{H(\mu)} = \frac{X(-\mu)}{X(0)}, \quad c = 1 .$$ (H.5)

---

*Note that the Milne problem answers of Chapter IV can not be used if
c = 1 because the normalization was selected such that

$$\rho(0) = \int_{-1}^{1} \Psi(0,\mu)\,d\mu = 1$$

and this normalization is not correct for the discrete eigenfunction
carrying the current, $\Psi_0(x,\mu)$. In Chapter VI, the normalization for
the Milne problem is given by Equation (6.27a). The normalization

$$j(0) = \int_{-1}^{1} \mu\ \Psi(0,\mu)\,d\mu = -1$$

must be selected for the solution by Case's method in order for it
to agree exactly with the results of Chandrasekhar.

From (2.37) and (3.34), we note that $X(0) = 3^{1/2}$ for isotropic

scattering and $c = 1$. For the linearly anisotropic scattering

problems of Chapter IV, we have the result

$$X(0) = \frac{1}{\nu_0 \sqrt{\Lambda(\infty)}} = \frac{1}{\nu_0 \sqrt{(1-c)(1-cb_1/3)}} \tag{H.6}$$

where we have used (2.37) and (2.59). For the quadratically aniso-

tropic scattering problems of Chapter VI where $c = 1$, Equation (6.46)

shows that

$$X(0) = \left[\frac{3}{1 - b_2/5} \right]^{1/2} . \tag{H.7}$$

Thus, the relation between the H-function and the X-functions of

Chapters IV and VI are

$$\frac{1}{H(\mu)} = \begin{cases} (\nu_0 + \mu)\left[(1-c)(1 - \frac{cb_1}{3}) \right]^{1/2} X(-\mu) & \text{IV)} \\ \left[(1 - \frac{b_2}{5})/3 \right]^{1/2} X(-\mu) & \text{VI).} \end{cases} \tag{H.8}$$

The solutions of Chandrasekhar are in terms of the H-functions

and the moments of the H-function, defined by

$$\alpha_i = \int_0^1 \mu^i H(\mu) d\mu \qquad i = 0, 1, 2 \cdots . \tag{H.9}$$

Thus in order to complete the correspondence between the notations, we must relate α_i to the $\gamma^{(i)}$ of Case's formalism. For the γ-function of Chapter IV, Equations (2.38), (2.59), and (H.8) give

$$\int_0^1 \mu^i (\nu_0 - \mu) \gamma(\mu) \, d\mu = \frac{c}{2 \sqrt{(1-c)(1- \frac{cb_1}{3})}} \int_0^1 \mu^{i+1} H(\mu) \, d\mu \qquad \text{(H.10)}$$

so

$$\bar{\omega}_0 \, \alpha_i = 2 \sqrt{(1-c)(1- \frac{cb_1}{3})} \left[\nu_0 \, \gamma^{(i-1)} - \gamma^{(i)} \right] , \quad i = 0, 1 . \qquad \text{(H.11)}$$

The function $\gamma^{(-1)}$ is obtained by setting $z = 0$ in (2.67). In particular, the specific relation needed to correlate the emergent angular distributions is

$$\frac{\bar{\omega}_0 \, \alpha_1}{2 - \bar{\omega}_0 \, \alpha_0} = \frac{\nu_0 - \bar{\nu}}{d(\nu_0 \bar{\nu})} , \qquad \text{(H.12)}$$

In Chapter VI, Equations (6.52), (H.8), and (H.9) can be combined to give the relation

$$\alpha_i = \left[\frac{4(1 - b_2/5)}{3} \right] \gamma^{(i-1)} , \quad i = 1, 2 . \qquad \text{(H.13)}$$

The only other relation needed is given by

$$\frac{3 b_2 \alpha_2^2}{\alpha_1^2 (4 + b_2)} + \frac{16}{3 \alpha_1^2 (4 + b_2)} = 1 \qquad \text{(H.14)}$$

and this is proved using (6.50). It should again be noted that the
source strength in the solution to the Milne problem is normalized by
Chandrasekhar to a net current of -1 instead of $-1/(3-b_1)$ as was
done in (6.27a).

New Integral Expressions for Chandrasekhar's X- and Y-Functions

The X- and Y-functions are defined by the coupled equations

$$X(\mu) = 1 + \frac{\mu \bar{\omega}_0}{2} \int_0^1 \frac{\left[X(\mu)X(\mu') - Y(\mu)Y(\mu') \right] d\mu'}{\mu + \mu'}$$

$$Y(\mu) = e^{-\tau_1/\mu} + \frac{\mu \bar{\omega}_0}{2} \int_0^1 \frac{\left[Y(\mu)X(\mu') - X(\mu)Y(\mu') \right] d\mu'}{\mu - \mu'} \qquad (H.15)$$

for isotropic scattering. From Section 62.1 of Chandrasekhar's book,[2]
it is seen that his functions are related to the reflected and trans-
mitted angular intensities at the surfaces of a slab illuminated by
incident radiation from all directions $\mu_0 > 0$ with magnitude propor-
tional to $1/\mu_0$. Using (H.1) and (H.2), the relations for $X(\mu)$
and $Y(\mu)$ are

$$X(\mu) = 1 + \mu \int_0^1 \frac{\Psi(0, -\mu) d\mu_0}{\mu_0}$$

$$Y(\mu) = e^{-d/\mu} + \mu \int_0^1 \frac{\left[\Psi(d, \mu) - e^{-d/\mu} \delta(\mu - \mu_0) \right] d\mu_0}{\mu_0} \qquad (H.16)$$

where $\psi(0,-\mu)$ and $\psi(d,\mu)$ are given by (5.33). Adding and sub-tracting the above equations and using (5.8) and (5.33) gives

$$X(\mu) \pm Y(\mu) = 1 + \frac{\mu}{X(-\mu)} \int_0^1 \frac{1}{\mu_0} \left[\frac{\gamma(\mu_0)}{\mu_0 + \mu} \right. $$

$$+ b_\pm \left[\varphi_-(\mu) X(\nu_0) \pm e^{-d/\nu_0} \varphi_+(\mu) X(-\nu_0) \right]$$

$$\left. \pm \int_0^1 B_\pm(\nu) e^{-d/\nu} \varphi_\nu(\mu) X(-\nu) d\nu \right] d\mu_0 \quad . \tag{H.17}$$

From (2.34),

$$X(0) = \int_0^1 \frac{\gamma(\mu_0) d\mu_0}{\mu_0} \quad . \tag{H.18}$$

Using partial fractions, (H.18), and (2.43), one can show that

$$1 + \frac{\mu}{X(-\mu)} \int_0^1 \frac{\gamma(\mu_0) d\mu_0}{\mu_0(\mu_0 + \mu)} = \frac{X(0)}{X(-\mu)} = \frac{1}{\nu_0 \sqrt{1-c} \; X(-\mu)}$$

$$\int_0^1 \frac{\gamma(\mu_0) \varphi_\mu(\mu_0) d\mu_0}{\mu_0} = \frac{c}{2} X(0) = \frac{c}{2\nu_0 \sqrt{1-c} \; X(-\mu)} \quad . \tag{H.19}$$

Using the above equations and (5.12), Equation (H.17) becomes

$$X(\mu) \pm Y(\mu) = \frac{X(0)}{X(-\mu)} - \mu \frac{X(-\nu_0)}{X(-\mu)} \left[e^{2z_0/\nu_0} \varphi_-(\mu) \mp e^{-d/\nu_0} \varphi_+(\mu) \right] g_\pm$$

$$\pm \frac{\mu}{X(-\mu)} \int_0^1 \varphi_\nu(\mu) e^{-d/\nu} X(-\nu) Q_\pm(\nu) d\nu \tag{H.20}$$

where, using (5.13) and (5.16),

$$
q_{\pm} = \int_0^1 \frac{b_{\pm}\, d\mu_0}{\mu_0}
$$

$$
= \frac{\frac{2}{c} X(0) \mp \int_0^1 \nu\, Q_{\pm}(\nu)\, e^{-d/\nu}\, X(-\nu)\, d\nu}{\nu_0\, X(-\nu_0)\left[e^{2z_0/\nu_0} \pm e^{-d/\nu_0}\right]}
\tag{H.21}
$$

and

$$
Q_{\pm}(\nu) = \int_0^1 \frac{B_{\pm}(\nu)\, d\mu_0}{\mu_0}
$$

$$
= \frac{X(-\nu)(\nu_0^2 - \nu^2)(1-c)}{\Lambda^+(\nu)\, \Lambda^-(\nu)}\left[\frac{X(0)}{\nu} + X(-\nu_0)\left[\varphi_+(\nu)\, e^{2z_0/\nu_0}\right.\right.
$$

$$
\left.\left. \mp \varphi_-(\nu)\, e^{-d/\nu_0}\right] q_{\pm} \mp \int_0^1 e^{-d/\nu'}\, \varphi_{-\nu'}(\nu)\, X(-\nu')\, Q_{\pm}(\nu')\, d\nu'\right] .
\tag{H.22}
$$

The following identity may be proved:

$$
\frac{e^{2z_0/\nu_0} \mp e^{d/\nu_0}}{e^{2z_0/\nu_0} \pm e^{-d/\nu_0}} = \coth\left(\frac{2z_0 + d}{\nu_0}\right) \mp \operatorname{csch}\left(\frac{2z_0 + d}{\nu_0}\right) .
\tag{H.23}
$$

Use of (H.21) and (H.23) to eliminate $q_{\pm}$ in Equations (H.20) and (H.22) gives two equations of the form

$$X(\mu) \pm Y(\mu) = \frac{1}{X(-\mu)(\nu_0{}^2 - \mu^2)} \left[\frac{1}{\sqrt{1-c}} \left\{ \nu_0 - \mu \coth\left(\frac{2z_0+d}{\nu_0}\right) \right. \right.$$

$$\left. \pm \mu \operatorname{csch}\left(\frac{2z_0+d}{\nu_0}\right) \right\} \pm \mu \int_0^1 Q_\pm(\nu) e^{-d/\nu} X(-\nu) \; (x)$$

$$(x) \left\{ (\nu_0{}^2 - \mu^2) \varphi_\nu(\mu) + \frac{c\nu}{2} \left[-\mu + \nu_0 \coth\left(\frac{2z_0+d}{\nu_0}\right) \right. \right.$$

$$\left. \left. \left. \mp \nu_0 \operatorname{csch}\left(\frac{2z_0+d}{\nu_0}\right) \right] \right\} \, d\nu \right] \qquad \text{(H.24)}$$

and

$$Q_\pm(\nu) = \frac{X(-\nu)(1-c)}{\nu \Lambda^+(\nu)\Lambda^-(\nu)} \left[\frac{1}{\sqrt{1-c}} \left\{ \nu_0 + \nu \coth\left(\frac{2z_0+d}{\nu_0}\right) \right. \right.$$

$$\left. \mp \nu \operatorname{csch}\left(\frac{2z_0+d}{\nu_0}\right) \right\} \mp \nu \int_0^1 Q_\pm(\nu') e^{-d/\nu'} X(-\nu') \; (x)$$

$$(x) \left\{ (\nu_0{}^2 - \nu^2) \varphi_{-\nu'}(\nu) + \frac{c\nu'}{2} \left[\nu + \nu_0 \coth\left(\frac{2z_0+d}{\nu_0}\right) \right. \right.$$

$$\left. \left. \left. \mp \nu_0 \operatorname{csch}\left(\frac{2z_0+d}{\nu_0}\right) \right] \right\} \, d\nu' \right] \, . \qquad \text{(H.25)}$$

where, using (5.13) and (5.16),

$$
q_\pm = \int_0^1 \frac{b_\pm \, d\mu_0}{\mu_0}
$$

$$
= \frac{\frac{2}{c} X(0) \mp \int_0^1 \nu \, Q_\pm(\nu) e^{-d/\nu} X(-\nu) d\nu}{\nu_0 \, X(-\nu_0)\left[e^{2z_0/\nu_0} \pm e^{-d/\nu_0}\right]}
\tag{H.21}
$$

and

$$
Q_\pm(\nu) = \int_0^1 \frac{B_\pm(\nu) \, d\mu_0}{\mu_0}
$$

$$
= \frac{X(-\nu)(\nu_0^2 - \nu^2)(1-c)}{\Lambda^+(\nu) \, \Lambda^-(\nu)} \left[\frac{X(0)}{\nu} + X(-\nu_0)\left[\varphi_+(\nu) e^{2z_0/\nu_0} \right. \right.
$$

$$
\left. \left. \mp \varphi_-(\nu) e^{-d/\nu_0} \right] q_\pm \mp \int_0^1 e^{-d/\nu'} \varphi_{-\nu'}(\nu) X(-\nu') Q_\pm(\nu') d\nu' \right] .
\tag{H.22}
$$

The following identity may be proved:

$$
\frac{e^{2z_0/\nu_0} \mp e^{d/\nu_0}}{e^{2z_0/\nu_0} \pm e^{-d/\nu_0}} = \coth\left(\frac{2z_0+d}{\nu_0}\right) \mp \operatorname{csch}\left(\frac{2z_0+d}{\nu_0}\right) .
\tag{H.23}
$$

Use of (H.21) and (H.23) to eliminate $q_\pm$ in Equations (H.20) and (H.22) gives two equations of the form

$$X(\mu) \pm Y(\mu) = \frac{1}{X(-\mu)(\nu_0^2 - \mu^2)}\left[\frac{1}{\sqrt{1-c}}\left\{\nu_0 - \mu \coth\left(\frac{2z_0 + d}{\nu_0}\right)\right.\right.$$

$$\left.\pm \mu \operatorname{csch}\left(\frac{2z_0 + d}{\nu_0}\right)\right\} \pm \mu \int_0^1 Q_\pm(\nu) e^{-d/\nu} X(-\nu) \,(x)$$

$$(x)\left\{(\nu_0^2 - \mu^2)\varphi_\nu(\mu) + \frac{c\nu}{2}\left[-\mu + \nu_0 \coth\left(\frac{2z_0 + d}{\nu_0}\right)\right.\right.$$

$$\left.\left.\left.\mp \nu_0 \operatorname{csch}\left(\frac{2z_0 + d}{\nu_0}\right)\right]\right\}\,d\nu\right] \qquad\qquad \text{(H.24)}$$

and

$$Q_\pm(\nu) = \frac{X(-\nu)(1-c)}{\nu \Lambda^+(\nu)\Lambda^-(\nu)}\left[\frac{1}{\sqrt{1-c}}\left\{\nu_0 + \nu \coth\left(\frac{2z_0 + d}{\nu_0}\right)\right.\right.$$

$$\left.\mp \nu \operatorname{csch}\left(\frac{2z_0 + d}{\nu_0}\right)\right\} \mp \nu \int_0^1 Q_\pm(\nu') e^{-d/\nu'} X(-\nu') \,(x)$$

$$(x)\left\{(\nu_0^2 - \nu^2)\varphi_{-\nu'}(\nu) + \frac{c\nu'}{2}\left[\nu + \nu_0 \coth\left(\frac{2z_0 + d}{\nu_0}\right)\right.\right.$$

$$\left.\left.\left.\mp \nu_0 \operatorname{csch}\left(\frac{2z_0 + d}{\nu_0}\right)\right]\right\}\,d\nu'\right] \;. \qquad\qquad \text{(H.25)}$$

Equations (H.24) and (H.25) correlate the X-function of Case with the X- and Y-functions of Chandrasekhar. For large slab thicknesses, integral terms with $e^{-d/\nu}$ vanish more rapidly than terms proportional to $\text{csch}\,(2z_0 + d)/\nu_0$ and can be ignored in the lowest-order approximation. Ignoring the integral terms corresponds to obtaining the effect from the discrete modes, i.e., the asymptotic variation. Equation (H.24) then gives

$$X(\mu) \pm Y(\mu) = \frac{\nu_0 - \mu\,\coth\left(\frac{2z_0+d}{\nu_0}\right) \pm \mu\,\text{csch}\left(\frac{2z_0+d}{\nu_0}\right)}{\sqrt{1-c}\;\;X(-\mu)\,(\nu_0^2 - \mu^2)} \tag{H.26}$$

so, for large d ,

$$X(\mu) = \frac{\nu_0 - \mu\,\coth\left(\frac{2z_0+d}{\nu_0}\right)}{\sqrt{1-c}\;\;X(-\mu)\,(\nu_0^2 - \mu^2)}$$

$$Y(\mu) = \frac{\mu\,\text{csch}\left(\frac{2z_0+d}{\nu_0}\right)}{\sqrt{1-c}\;\;X(-\mu)\,(\nu_0^2 - \mu^2)} \;\;, \tag{H.27}$$

The equations of (H.27) can be obtained more directly by using the asymptotic forms of (H.20) and (H.21).

It will be noticed that the forms of (H.27) are familiar-- they are just the asymptotic emergent surface densities, $\rho(0)$ and $\rho(d)$, obtained from (5.37) with μ_0 replaced by μ . This corresponds to the alternative interpretation of the X- and Y-functions given by Chandrasekhar.

The results in (H.27), although only approximate, are useful since the numerical work on the Chandrasekhar functions[38-40] considers d to be small. The integral equations of (H.16) are best for smaller thicknesses and require more iterations as the thickness increases.

In the limit of infinite thickness, the equations in (H.27) yield

$$X(\mu) = \frac{1}{\sqrt{1-c}\ X(-\mu)(\nu_0+\mu)} = H(\mu)$$

$$Y(\mu) = 0 \tag{H.28}$$

where, in the first equation, (H.3) was used. This behavior was first noted by Chandrasekhar.

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