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Final Report

AN ELECTROMAGNETIC ACCELERATOR UTILIZING SEQUENTIAL SWITCHING

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TABLE OF CONTENTS

TABLE OF CONTENTS
(Concluded)

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

The accelerator considered in this report consists of a single-layer electrically conducting solenoid and a metal projectile closely fitting inside or outside the coil. A series of switches delivers or removes current from each solenoid turn in synchronism with the traveling projectile. This produces within the projectile both steady-state and transient magnetic fields which are shown to increase the efficiency over a purely transient system.

Both mechanical and thermal limitations are studied. In the ten-meter accelerator treated as an example, mechanical destruction was the more severe limit; taking this into consideration, the analysis shows that, assuming zero friction, projectiles having several hundred grams mass can be accelerated to 10,000 meters per second in ten meters.

Both lateral displacement and tilting of the projectile axis with respect to the driving-coil axis are shown in certain instances to be unstable.

Parallel, series, and "traveling wave" circuits are applicable to this particular device.

I. INTRODUCTION AND SUMMARY

A. NEED FOR A HIGH-VELOCITY GUN

A device capable of accelerating an appreciable mass to high velocity has several applications of major importance. In the weapons field, anti-aircraft shells and missiles must have high velocity to be effective against the high-velocity missiles employed in modern warfare. Such an accelerator would also find use as a missile testing device when operated as an inverse wind tunnel and perhaps even as a missile launching mechanism. In the realm of component testing for modern high-thrust, high-velocity systems, high "g" impulse and impact tests also require a high-velocity accelerator. As a final example, a high-velocity gun can be used in some instances for its energy storage capabilities, as, for instance, in electrical impulse generators[1] and in impact heating (see Appendix II).

B. REVIEW OF PREVIOUS HIGH-VELOCITY-GUN RESEARCH

Chemically driven guns appear to be limited by the burning rates of fuels and explosives, although velocities as high as 7000 meters per second have been reached by shaped detonation waves.[2] A further limitation of an explosively accelerated system is the extreme difficulty of controlling such a reaction.

As an alternative to explosion-driven accelerators, electromagnetic schemes have been extensively studied. The Germans during World War II developed electrically driven guns which accelerated several gram pellets to over 1000 meters per second, and their work was analyzed and carried on after the war at Armour Research Foundation.[3] Other laboratories at which electrical gun development has been carried on include Air Force Cambridge Research Center,[4] University of Utah,[4] Zenith Radio Research Corporation,[4] The Rand Corporation,[5] and Holloman Air Research Center.[6]

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1. Conference on High Powered Electrical Impulse Techniques, Univ. of Mich. Eng. Res. Inst. Report No. 2522-4-T, Ann Arbor, May, 1957.
2. Cook, Univ. of Utah, as reported by W. S. Partridge, Conference on High Powered Electrical Impulse Techniques.
3. Electrical Gun and Power Source, ATI 90 744, Armour Research Foundation Tech. Report No. 3, Project No. 15-391E, May, 1947.
4. Conference on High Powered Electrical Impulse Techniques.
5. Huth, J. H., and Holbrook, R., "Linear Generator," Hypervelocity and Impact Effect Symposium, U. S. Naval Research Laboratory, 1957.
6. Millsaps, K., and Pohlhausen, K., The Acceleration of Large Masses by Electrical Means, Holloman Air Development Center, Operations Research Office, Summer Research Group, Tech. Memo. No. 3.

Two general types of systems have been considered in the above-mentioned studies. In one configuration, the projectile acts as a shorting bar across two parallel conducting rails. When a voltage is established across the rails causing current to flow through the projectile, a force on the projectile parallel to the rails results.

In the alternate scheme, acceleration of the pellet is derived from interaction between the magnetic field of a current-carrying coil and the magnetic field of currents in the projectile. The projectile current is most easily derived as current induced by the driving field, so this system is often referred to as an "induction" gun as contrasted with the "rail" gun described above.

C. BACKGROUND AND QUALITATIVE DESCRIPTION OF THE PARTICULAR SYSTEM TREATED IN THIS REPORT

The electromagnetic gun system being considered in this report was originally conceived and briefly reported by University of Michigan personnel in 1948. Although a proposal for further study and for construction of such a device was advanced at that time, no further work was done on the subject until the present contract was activated.

The University of Michigan accelerator is essentially an induction-type system mentioned earlier, but the unique feature of this particular system is a series of switches which control the currents to the individual turns of the driving coil. The current, thus controlled both in time and position in the coil, establishes a magnetic field configuration which has both a constant or bias component permeating the projectile and a transient component. As shown in Appendix I, this leads to greater efficiency than in the basic, or impulse, induction accelerator briefly described in the following paragraphs.

In the basic form of the induction accelerator, a cylindrical conducting pellet is placed coaxial with but centered slightly ahead of a cylindrical wire coil. When current is sent through the coil, the resulting magnetic field induces in the pellet circulating eddy currents which are normal to the radial components of the magnetic field, and which therefore result in axial forces being exerted within the pellet. Since the pellet is not centered, these forces will not be symmetric, and if the pellet is free to move, it will be accelerated out of the coil.

Now if a long solenoidal coil be considered as a series of coils such as described above, and if each coil section receives current as the projectile, already moving axially within the solenoid due to impulses from the previous coil sections, passes by, then a continued thrust is maintained on the pellet. This is a brief description of the accelerating system which is analyzed in this report.

There are two ways in which this sequentially switched accelerator may be operated. In both, each turn of a long solenoid constitutes a single-coil section as described in the previous paragraph, and the projectile is idealized as a conducting ring whose radius is only slightly less than the solenoid radius. In one instance, the operation is identical to the procedure outlined in the previous paragraph, while in the other, the solenoid initially carries current, and current is switched out of each section as the projectile passes by. Although the source of thrust is the same in either case, the particular geometry employed has important bearing on stability and heating of the projectile.

D. SUMMARY OF REPORTED RESULTS

Following a more thorough description of the accelerator, the projectile velocity is solved in the following sections in terms of the current in the driving coil and the dimensions of the device. One finds, of course, that velocity increases with current, but eventually one of two limits is reached; either the projectile becomes excessively hot or mechanically ruptures. Before making numerical estimates, therefore, calculations are made of maximum allowable currents considering heating and mechanical strength. For sample dimensions considered, allowable maximum currents are in the range of 10,000 to 100,000 amperes, and these result in exit velocities on the order of 10,000 meters/sec. In all cases acceleration is assumed to take place over a 10-meter length, and the accelerated masses range from 3 grams to 2 kilograms.

The report concludes with treatments of stability and circuit aspects of the device.

Two relevant topics are included as appended sections. In one, a more generalized approach to the theory of electromagnetic acceleration is considered, and it is shown that a constant-bias magnetic field increases the efficiency of the acceleration process. In the other, estimates are made of temperature which might be attainable when the kinetic energy of a moving mass is randomized to heat by collision. For a lithium projectile, it turns out that, assuming all kinetic energy is converted to heat in the original moving mass, a velocity of about 83,000 meters/sec is required to raise the mass beyond complete triple ionization temperature, perhaps 100,000°K.

II. DETAILED DESCRIPTION OF ACCELERATOR

When a conductor which is carrying an electric current is placed in a magnetic field having components normal to the current, a force is exerted on that conductor. This force is directly proportional to the magnetic field density, the magnitude of the current, the length of the conductor, and the angle between the current and the field density vectors. Symbolically this is represented as follows:

$$F = |\vec{B}||\vec{I}| \ \ell \sin \Theta \ldots\ldots \qquad (2\text{-}1)$$

where

F = force on the conductor, newtons,
$\vec{B}$ = magnetic flux density, webers per square meter,
$\vec{I}$ = current flowing in conductor, amperes,
l = length of conductor, meters, and
Θ = angle between $\vec{B}$ and $\vec{I}$.

If the magnetic field is due to a current flowing in another conductor, and if the two currents are parallel and in the same direction, an attractive force exists between the conductors, pulling them towards each other, but when the currents are in the opposite direction, a repulsive force exists, tending to separate the two conductors.

Since forces can be made to exist between current-carrying conductors, schemes can be developed by which electromagnetic accéleration of an object is achieved. Two such systems are analyzed here. The first utilizes attractive forces, the second, repulsive forces.

A. ATTRACTION CASE (See Fig. 1.)

A long stationary solenoid of N turns is energized with a current $+I_O$ flowing through each turn. The projectile, represented as a conducting ring, is brought to the starting end plane of the solenoid (z=0) in such a way that the net current in the ring at this position is zero. At this position the first turn of the solenoid is de-energized, thus inducing a current in the ring. Assuming perfect coupling between the ring and the solenoid turn, and taking into consideration the fact that induced current always flows in such a direction as to oppose the decrease in flux, a current $+I_O$ is established in the ring.

Since the currents in the ring and the solenoid are in the same direction, an attractive axial force is developed between the two, thus moving the ring axially in the +z direction towards the remaining energized portion of the solenoid.

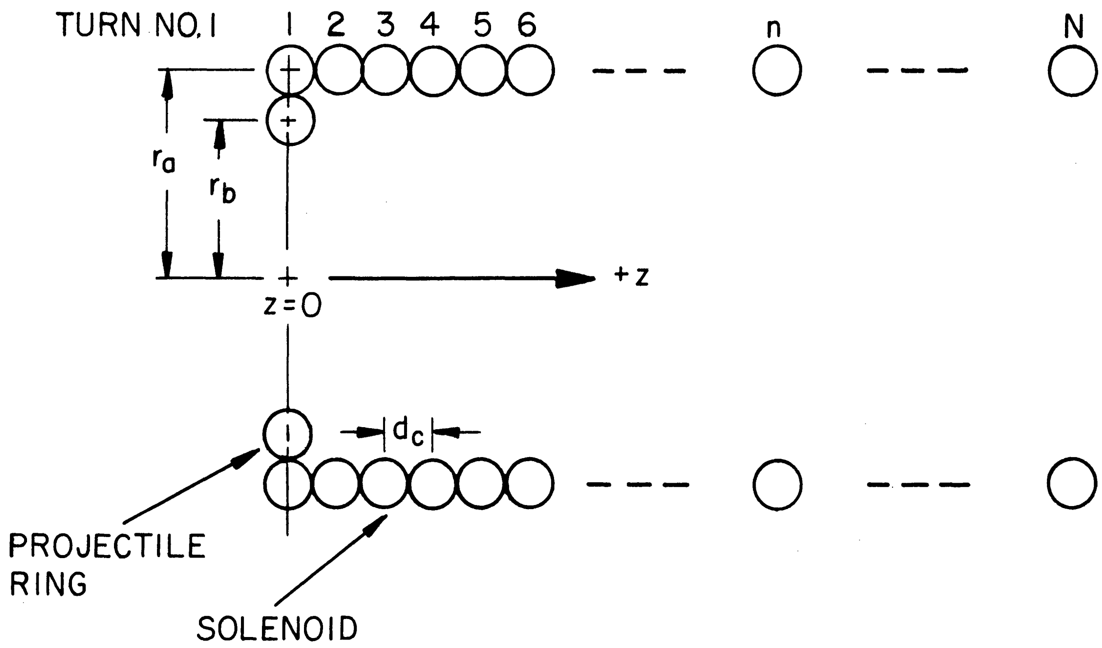

Fig. 1. Sequentially switched electromagnetic accelerator coil geometry.

As the ring moves into the magnetic field of the solenoid, a voltage is induced which opposes the current already flowing in the ring. It will be shown later that when the ring gets adjacent to the second turn, its net current will just be reduced to zero by this induced voltage.

Just as the ring reaches the second turn, this turn is caused to be deenergized, making current again flow in the ring and resulting in further forward thrust on the ring. The process described above is again repeated at the third and succeeding turns of the solenoid with the current in each turn being interrupted just as the projectile passes.

Subsequent analysis will reveal the following characteristics of the device:

1. There will be an average axial force on the ring through the whole length of the solenoid; thus continuous acceleration of the ring in the +z direction is achieved.

2. Energy originally stored in the magnetic field of the solenoid and in any external driving source which might be present is transferred into kinetic and magnetic energy in the projectile.

3. The current in the ring varies between zero and $+I_O$ through the major portion of travel through the solenoid. However, it builds up towards the end so that the ring emerges out of the solenoid with a net current which depends on dimensions and on the value of the driving current in the solenoid.

B. REPULSION CASE (See Fig. 1.)

In this case the solenoid does not have any initial current, and the projectile ring, also with zero initial current, is placed slightly to the right of the first turn. At this position the first turn of the solenoid is energized with a current of $+I_O$, thus inducing a current in the ring. Assuming perfect coupling between the ring and the solenoid turn, and again noting that induced current always flows in such a direction as to oppose the increase in flux, a current $(-I_O)$ is established in the ring.

Since the currents in the ring and the first turn of the solenoid are in opposite directions, a repulsive axial force is developed between the two, moving the ring axially in the +z direction, that is, away from the energized solenoid turn.

As the ring moves away from the first solenoid turn, a voltage is induced in the ring which again tends to cancel the current already present. It will be shown later that, as the ring moves through the first turn, the induced voltage will be very small, resulting in the ring reaching the second turn with an appreciable portion of its initial current remaining. As the ring just passes the second turn, this turn is energized, thus inducing an additional $(-I_O)$ in the ring. The process described above is repeated through the whole length of the solenoid, current being switched into each turn as the projectile passes. The results of this process will be shown to be:

1. There will be an average axial force on the ring through the whole length of the solenoid; thus continuous acceleration of the ring is achieved.

2. The energy from the driving generator is transferred into kinetic and magnetic energy in the ring plus magnetic energy in the solenoid.

3. As the ring travels through the first few turns of the solenoid, the ring current builds up to a certain value ($-I_{Rm}$), but through the main distance of travel, this current varies between $-I_{Rm}$ and $-(I_{Rm}-I_o)$. The ring will emerge out of the solenoid with a current equal to $-I_{Rm}$, but this current will be driven to zero by the opposing emf induced in the ring as it travels out of the solenoid magnetic field.

III. DERIVATION OF VELOCITY EXPRESSION

NOTATION

E_{ext} ≡ energy in an external storage device

E_{in} ≡ energy of the total system immediately after the n^{th} turn is de-energized (attraction) or energized (repulsion)

E_{fn} ≡ energy of the total system just before the $(n+1)^{th}$ turn is de-energized (attraction) or energized (repulsion)

δE ≡ energy extracted from the external storage device during the travel of the ring from the n^{th} to the $(n+1)^{th}$ turn

L_R ≡ ring self-inductance

L_n ≡ self-inductance of the n^{th} solenoid turn

L_S ≡ self-inductance of solenoid turns (n+1) through N (attraction) or turns 1 through n (repulsion)

M_{nS} ≡ mutual inductance between n^{th} turn and the remainder of the solenoid

M_{RS} ≡ mutual inductance between ring and energized solenoid turns

M_{Rd_cS} ≡ mutual inductance between ring and energized solenoid turns when the ring is in a plane d_c distance out from the end plane of the energized portion of the solenoid

M_{R_oS} ≡ mutual inductance between ring and energized solenoid turns when the ring is in the end plane of the energized portion of the solenoid

I_o ≡ solenoid current

I_R ≡ ring current

I_{Rna} ≡ current in ring just after n^{th} turn is de-energized (attraction) or energized (repulsion)

I_{Rnb} ≡ current in ring just before n^{th} turn is de-energized (attraction) or energized (repulsion)

I'_{Rn} ≡ current which would be induced in ring as it travels from the n^{th} to the $(n+1)^{th}$ turn

I_{Rm} ≡ maximum value to which the ring current builds

v_n, v_{n+1} ≡ ring velocity as it passes the n^{th} and $(n+1)^{th}$ solenoid turns, respectively

$K \triangleq I_{Rm}/I_o$

One could presumably analyze the kinetics of the system by applying Eq. (2-1) after first solving for the current and radial magnetic field at the projectile ring at each instant. It turns out, however, that in the case of an electromagnetic accelerator in which the radii of the ring and of the driving solenoid are approximately the same, B_R, the radial component of magnetic field at the ring, varies rapidly with position as the ring moves toward the end plane of the solenoid and is in fact infinite at the end plane when the radii are exactly the same. It is therefore very hard to determine the exact value of B_R on the ring at every instant.

Since B_r is hard to determine with any accuracy, it becomes awkward to obtain the velocity from Eq. (2-1). An energy balance may be resorted to, however, to determine the velocity of the projectile. That is, the stored energy from the solenoid plus the energy supplied by the source should be equal to the stored energy in the ring plus the kinetic energy of the ring. The analysis using energy considerations follows.

A. ENERGY BALANCE

1. Attraction Case

Assume that the ring reaches the n^{th} turn of the solenoid with a velocity v_n, and assume that we want to find the velocity when the ring reaches the $(n+1)^{th}$ turn.

When the n^{th} turn is de-energized and its energy is transferred to the projectile ring, the energy of the total system resides in the following places (using the symbols defined above):

a) ring motion ($1/2\ mv_n^2$) ,

b) ring self-magnetic field ($1/2\ L_R I_{Rna}^2$) ,

c) solenoid self-magnetic field ($1/2\ L_S I_o^2$) ,

d) ring-solenoid mutual field ($M_{R_{dc}S} I_{Rna} I_o$) , and

e) external storage device (E_{ext}) ;

that is,

$$E_{in} = \frac{1}{2} m v_n^2 + \frac{1}{2} L_R I_{Rna}^2 + \frac{1}{2} L_s I_o^2 + M_{RdcS} I_{Rna} I_o + E_{ext}. \qquad (3\text{-}1)$$

When the ring gets to the $(n+1)^{th}$ turn, but just before this turn is de-energized, the total energy will be (again using previously defined symbols):

$$E_{fn} = \frac{1}{2} m v_{n+1}^2 + \frac{1}{2} L_R I_{R(n+1)b}^2 + \frac{1}{2} L_s I_o^2 + M_{ROS} I_{R(n+1)b} I_o + E_{ext} - \delta E. \qquad (3\text{-}2)$$

δE is the energy from the external source required to keep the current in the solenoid I_0 constant; i.e., energy required to counter the induced emf in the solenoid due to the movement of the ring toward the solenoid, that is,

$$\delta E = \int_0^{\tau} e I_0 \, dt , \tag{3-3}$$

where

e = induced voltage in solenoid due to the movement of the ring

$$= -\frac{d\phi}{dt} = -\frac{d}{dt}\left(I_R M_{RS}\right)$$

∴

$$\delta E = -\int_0^{\tau} I_0 \, d\left(I_R M_{RS}\right). \tag{3-3'}$$

But
$$\bar{E}_{in} = \bar{E}_{fn} \tag{3-4}$$

∴
$$\frac{1}{2} m v_{n+1}^2 = \frac{1}{2} m v_n^2 + \frac{1}{2} L_R \left(I_{Rna}^2 - I_{R(n+1)b}^2\right)$$
$$+ I_0 \left(M_{Rd_cS} I_{Rna} - M_{R0S} I_{R(n+1)b}\right) + \delta E , \tag{3-5}$$

Substituting Eq.(3-3') in Eq. (3-5) and solving for v_{n+1}, we get:

$$v_{n+1} = \left\{ \frac{2}{m} \left[\frac{1}{2} m v_n^2 + \frac{1}{2} L_R \left(I_{Rna}^2 - I_{R(n+1)b}^2\right) + I_0 \left(M_{Rd_cS} I_{Rna} - M_{R0S} I_{R(n+1)b}\right) - \int_0^{\tau} I_0 d\left(I_R M_{RS}\right) \right] \right\}^{\frac{1}{2}} . \tag{3-6}$$

By a similar procedure, an equation for v_{n+1} for repulsion operation can be derived.

2. Repulsion Case

Just after the n^{th} turn is energized, assume that the velocity of the ring is v_n and the current is $I_{R_{na}}$. Then the total energy at this instant is (using symbols already defined):

$$E_{in} = \frac{1}{2} m v_n^2 + \frac{1}{2} L_R I_{Rna}^2 + \frac{1}{2} L_s I_o^2 + M_{Ros} I_{Rna} I_o + E_{ext} . \qquad (3\text{-}7)$$

When the ring reaches the $(n+1)^{th}$ turn, the total energy will be:

$$E_{fn} = \frac{1}{2} m v_{n+1}^2 + \frac{1}{2} L_R I_{R(n+1)b}^2 + \frac{1}{2} L_s I_o^2 + M_{Rd_cs} I_{R(n+1)b} I_o \qquad (3\text{-}8)$$
$$+ E_{ext} - \delta E .$$

Now

$$E_{in} = E_{fn} \qquad (3\text{-}4)$$

∴

$$\frac{1}{2} m v_{n+1}^2 = \frac{1}{2} m v_n^2 + \frac{1}{2} L_R \left(I_{Rna}^2 - I_{R(n+1)b}^2 \right)$$
$$+ I_o \left(M_{RoS} I_{Rna} - M_{Rd_cS} I_{R(n+1)b} \right) + \delta E . \qquad (3\text{-}9)$$

But

$$\delta E = - \int_0^\tau I_o \, d \left(I_R M_{RS} \right) \qquad (3\text{-}3')$$

Substituting in the above equation and solving for v_{n+1}, we get

$$v_{n+1} = \left\{ \frac{2}{m} \left[\frac{1}{2} m v_n^2 + \frac{1}{2} L_R \left(I_{Rna}^2 - I_{R(n+1)b}^2 \right) \right. \right. \qquad (3\text{-}10)$$
$$\left. \left. + I_o \left(M_{Ros} I_{Rna} - M_{Rd_cS} I_{R(n+1)b} \right) - \int_0^\tau I_o \, d \left(I_R M_{RS} \right) \right] \right\}^{\frac{1}{2}}$$

Evaluation of projectile velocity now reduces to the problems of determining projectile current and projectile-solenoid mutual inductance.

B. CURRENT IN THE RING AT EVERY INSTANT OF ITS TRAVEL THROUGH THE SOLENOID

1. Attraction Case (See Fig. 1.)

The entire coil is initially energized with a current $+I_o$ flowing through it. The ring is then given an initial velocity v_{RO} and an initial current $+I_{Ri}$

such that it arrives at the first turn of the coil with zero velocity and zero current.

If the ring is moved from $(-\infty)$ towards the energized solenoid, a current is induced in it such as to establish a flux ϕ_R just equal to and opposite in direction to the flux ϕ_C threading the ring from the solenoid. Symbolically, this is represented as follows:

$$\phi_R = -\phi_C . \tag{3-11}$$

$$I_R = \frac{\phi_R}{L_R} = -\frac{\phi_C}{L_R} , \tag{3-12}$$

where L_R is the self-inductance of the ring. However,

$$\phi_C = I_o M_{RS} \tag{3-13}$$

where M_{RS} is the mutual inductance between the ring and the coil, and is represented as Σm where (m) is the mutual inductance between the ring and each turn of the solenoid. It follows from Eq. (3-12) that:

$$I_R = -\frac{I_o}{L_R} \sum m , \tag{3-14}$$

where

$$m = f\sqrt{r_a r_b} \text{ .}^{7} \tag{3-15}$$

For $r_a \approx r_b$,

$$m = f\, r_b \; (\mu h) , \tag{3-15'}$$

where r_b is in cm, and f is tabulated with respect to the parameter

$$\delta = \frac{\text{distance}}{\text{diameter}} = \frac{z}{2r_b}$$

or

$$\Delta = \frac{\text{diameter}}{\text{distance}} ,$$

whichever is less than unity.[8]

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7. Grover, F. W., Inductance Calculations, McGraw-Hill Book Co., New York, 1946, p. 77.
8. Grover, p. 82.

Since the turns of the coil are separated by a distance d_c, m is represented as follows:

$$M_{RS} = r_b \sum_{j=0}^{N-1} f\left(\frac{z + jd_c}{2r_b}\right) . \qquad (3\text{-}16)$$

Also[9]

$$L_R = 4\pi r_b \left[\ln \frac{8r_b}{r} - 1.75\right] 10^{-7}$$

$$= 4\pi r_b \left[\ln \frac{r_b}{r} + 0.33\right] 10^{-7} \text{ (henrys)} , \qquad (3\text{-}17)$$

where

r = radius of ring wire ,

r_b = major radius of ring in meters .

Therefore, substituting into Eq. (3-14)

$$I_R = -\frac{I_o \times 10^3 \sum_{j=0}^{N-1} f\left(\frac{z+jd_c}{2r_b}\right)}{4\pi\left[\ln \frac{r_b}{r} + 0.33\right]} \qquad (3\text{-}18)$$

When the ring reaches the first turn of the coil, and just before this turn is de-energized, the induced current in the ring I_R would be

$$I_R = -I_o - \frac{I_o \times 10^3 \sum_{j=1}^{N-1} f\left(\frac{jd_c}{2r_b}\right)}{4\pi\left[\ln \frac{r_b}{r} + 0.33\right]} , \qquad (3\text{-}19)$$

and in order for the ring to have a zero net current at this instant, an initial current I_{Ri}, given by Eq. (3-20), must have been previously established in the projectile ring, since infinite ring conductivity is assumed.

$$I_{Ri} = +I_o + \frac{I_o \times 10^3 \sum_{j=1}^{N-1} f\left(\frac{jd_c}{2r_b}\right)}{4\pi\left[\ln \frac{r_b}{r} + 0.33\right]} \qquad (3\text{-}20)$$

9. American Institute of Physics Handbook, McGraw-Hill Book Co., New York, 1957, pp. 5-29.

At this instant, with a zero current in the ring, the first turn of the coil is de-energized, thus inducing a current $+I_0$ in the ring. Since the currents in the ring and the solenoid are in the same direction, a force is created on the ring which pulls it towards the energized portion of the solenoid. As the ring moves toward the energized portion, a voltage is induced in the ring which tends to cancel the already existing current in it.

Let I'_{R2} be defined as that current which is induced in the ring as it travels from the first to the second turn of the solenoid. This current is equal to the induced current in the ring as it travels from negative infinity to turn 2 minus the induced current from negative infinity to turn 1 with turn 1 de-energized. This is represented symbolically as follows:

$$I'_{R2} = I_{induced}\left[(-\infty \rightarrow 2) - (-\infty \rightarrow 1)\right] \quad \text{1 de-energized}, \tag{3-21}$$

where

$$I_{induced}\left[(-\infty \rightarrow 2)\right] = -I_0 - \frac{I_0 \times 10^3 \sum_{j=1}^{N-2} f\left(\frac{j d_c}{2r_b}\right)}{4\pi\left[\ln \frac{r_b}{r} + 0.33\right]} \tag{3-21'}$$

and

$$I_{induced}\left[(-\infty \rightarrow 1)\right] = -\frac{\sum_{j=1}^{N-1} f\left(\frac{j d_c}{2r_b}\right)}{4\pi\left[\ln \frac{r_b}{r} + 0.33\right]} \,. \tag{3-21''}$$

$\therefore$

$$I'_{R2} = -I_0 - \frac{I_0 \times 10^3}{4\pi\left[\ln \frac{r_b}{r} + 0.33\right]} f\left[\frac{(N-1)d_c}{2r_b}\right] \,. \tag{3-22}$$

If N is large enough, $f[(N-1)d_c/2r_b]$ will be very small,[10] and the second turn on the right in Eq. (3-22) can be neglected.

$$\therefore \quad I'_{R2} = -I_0 \,. \tag{3-22'}$$

The net current in the ring just before the second turn is de-energized will then be:

$$I_{R2b} = +I_0 - I_0 = 0 \,. \tag{3-23}$$

10. Grover, p. 82.

When N is small the second term on the right of Eq. (3-22) cannot be neglected and so:

$$I_{R2b} = \frac{I_o \times 10^3}{4\pi\left[\ln \frac{r_b}{r} + 0.33\right]} \; f\left[\frac{(N-1)\, d_c}{2 r_b}\right] \tag{3-24}$$

When the ring just reaches the second turn, this turn is de-energized, thus inducing an additional $+I_o$ in the ring, and so the net current in the ring just after the second turn is de-energized is:

$$I_{R2a} = I_o + I_{R2b} \,. \tag{3-25}$$

By a similar procedure it is shown that when the ring reaches the third turn, the net current in the ring will be:

$$I_{R3b} = \frac{I_o \times 10^3}{4\pi\left[\ln \frac{r_b}{r} + 0.33\right]} \left\{ f\left[\frac{(N-1)\, d_c}{2 r_b}\right] + f\left[\frac{(N-2)\, d_c}{2 r_b}\right] \right\} , \tag{3-26}$$

and

$$I_{R3a} = I_o + I_{R3b} \,. \tag{3-27}$$

Taking the above analysis into consideration, general equations may be written designating the net current in the ring at the n^{th} turn, just before and just after the turn is de-energized. These equations are:

$$I_{Rnb} = \frac{I_o \times 10^3}{4\pi\left[\ln \frac{r_b}{r} + 0.33\right]} \left\{ f\left[\frac{(N-1)\, d_c}{2 r_b}\right] + f\left[\frac{(N-2)\, d_c}{2 r_b}\right] + \cdots + f\left[\frac{(N-n+1)\, d_c}{2 r_b}\right] \right\} , \tag{3-28}$$

which expressed as a sum is shown below:

$$I_{Rnb} = \frac{I_o \times 10^3}{4\pi\left[\ln \frac{r_b}{r} + 0.33\right]} \sum_{j=N-1}^{N-n+1} f\left(\frac{j\, d_c}{2 r_b}\right) , \tag{3-29}$$

and

$$I_{Rna} = I_o + I_{Rnb} \,. \tag{3-30}$$

Equations (3-29) and (3-30) apply for any n except n=1. At n=1, $I_{R1b} = 0$ and $I_{R1a} = +I_o$.

It is seen from the above analysis that if the solenoid has a large number of turns, the current in the ring will oscillate between $+I_o$ and zero as it moves from one turn to the next until the projectile nears the end, in which region the ring current builds up to a certain value depending on the current in the solenoid and the dimensions. Therefore the ring will emerge out of the solenoid with a current flowing through it, and since the solenoid is left with no energy, its magnetic field is zero and has no effect on the current in the ring. The magnetic energy stored in the ring must therefore be eventually converted into heat in the projectile.

The current in the ring as it travels through the solenoid is plotted in Fig. 2 to help clarify the previous analysis. This plot is for specified dimensions, namely:

$$r_a \sim r_b = 10 \text{ cm}$$

$$d_c = 1 \text{ cm} .$$

It is worth noting here that the general shape of this curve is the same for any dimensions, but the current in the ring as it leaves the solenoid is a function of the dimensions.

Equations (3-29) and (3-30) show the magnitude of the current in the ring just before and just after each turn is de-energized, but they do not indicate how the current in the ring varies between one turn and the next. Since this is of importance, the current in the ring as it moves from the first to the second turn will be derived as a function of z, where $0 \leq z \leq d_c$.

It was shown previously that at z = 0, $I_R = +I_o$ and at $z = d_c$, $I_R = 0$.

The current induced in the ring as it moves from zero to z is, from Eqs. (3-16) and (3-18),

$$I' = -\frac{I_o \times 10^3}{4\pi\left[\ln \frac{r_b}{r} + 0.33\right]} \sum_{j=0}^{N-2} f\left[\frac{(d_c - z) + j d_c}{2 r_b}\right] . \qquad (3\text{-}31)$$

$\therefore$

$$I_{Rz} = I_o + I' . \qquad (3\text{-}31')$$

When I_R is calculated from Eq. (3-31') for different values of z between zero and d_c it is found that the current in the ring varies approximately linearly with position as it moves from one turn to the next.

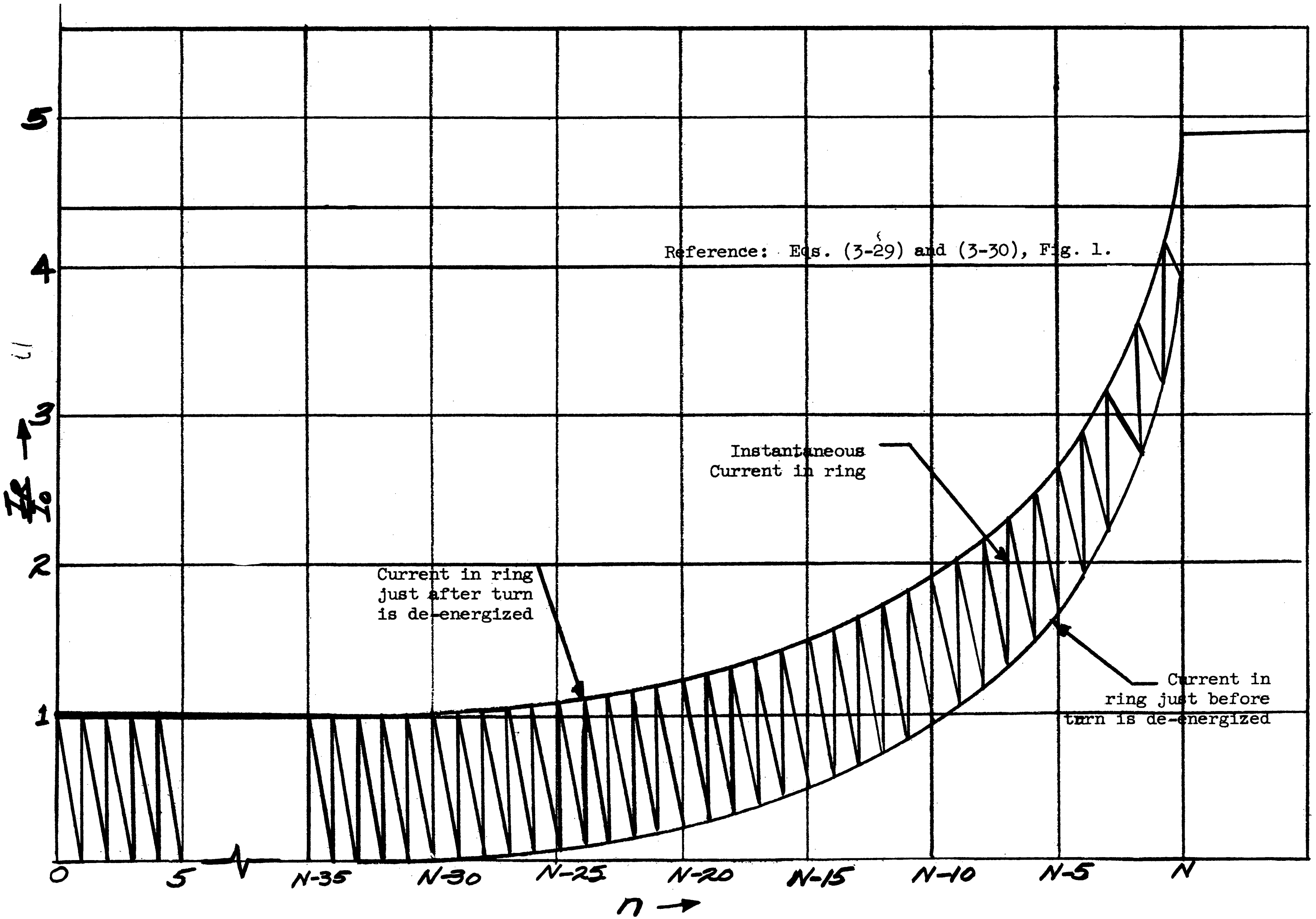

Fig. 2. Attraction case: current in projectile ring at n^{th} turn;
$r_a \approx r_b = 10$ cm, $d_c = 1$ cm.

2. Repulsion Case (See Fig. 1.)

This case is very similar in principle to the attraction one. In this case, however, the coil has no initial current, and the ring is placed slightly to the right of the first turn, and it also has no initial current or velocity.

At this position (z=0+) the first turn of the coil is energized with a current $+I_o$, thus inducing a current $(-I_o)$ in the ring, as explained previously. Since the currents in the ring and the coil are opposite in direction, a force is created on the ring which pushes it away from the energized turn of the coil in the +z direction.

When the ring moves away from the energized turn, a voltage is induced in it which tends to cancel the already existing current. However, the current in the ring will always be such as to cancel the flux threading the ring from the coil. Symbolically, this is represented as follows:

$$\varphi_R = -\varphi_C \tag{3-11}$$

$$I_R = \frac{\varphi_R}{L_R} = -\frac{\varphi_C}{L_R}, \tag{3-12}$$

where L_R is the self-inductance of the ring.

However,

$$\varphi_C = I_o M_{RS}, \tag{3-13}$$

where M_{RS} is the mutual inductance between the ring and the coil, and is represented as Σm, where m is the mutual inductance between the ring and each turn of the solenoid. Substitution in Eq. (3-12) gives:

$$I_R = -\frac{I_c}{L_R}\sum m \tag{3-14}$$

also

$$m = f\sqrt{r_a r_b}\,, \tag{3-15}$$ [11]

and for $r_a \cong r_b$,

$$m = f r_b \; (\mu h), \tag{3-15'}$$

where r_b is in centimeters, and f is tabulated with respect to the parameter

$$\delta = \frac{\text{distance}}{\text{diameter}} = \frac{z}{2r_a}$$

---

11. Grover, p. 77.

or
$$\Delta = \frac{\text{diameter}}{\text{distance}},$$

whichever is less than unity.[12]

Substituting in Eq. (3-14) we get

$$I_R = -\frac{I_0}{L_R} r_b \sum f\left(\frac{z}{2r_b}\right) . \tag{3-32}$$

But[13]

$$L_R = 4\pi r_b \left(\ln \frac{r_b}{r} + 0.33\right) 10^{-7} \text{ (henrys)}, \tag{3-19}$$

where r = radius of wire
r_a = radius of ring in meters.

This gives

$$I_R = -\frac{I_0 \times 10^3}{4\pi \left(\ln \frac{r_b}{r} + 0.33\right)} \sum f\left(\frac{z}{2r_b}\right) . \tag{3-33}$$

The current in the ring just after the first turn is energized is $(-I_0)$, and since the distance between each turn and the next is d_c, the current in the ring when it reaches the second turn and just before the second turn is energized is

$$I_{R2b} = -\frac{I_0 \times 10^3}{4\pi \left(\ln \frac{r_b}{r} + 0.33\right)} f\left(\frac{d_c}{2r_b}\right), \tag{3-34}$$

and I_{R2a} just after the second turn is energized is

$$I_{R2a} = -I_0 + I_{R2b} . \tag{3-35}$$

12. Grover, p. 82
13. A.I.P. Handbook, pp. 5-29.

From the above, general expressions showing the current in the ring just before and just after the n^{th} turn is energized are expressed as follows.

$$I_{Rnb} = -\frac{I_o \times 10^3}{4\pi\left(\ln\frac{r_b}{r}+0.33\right)} \sum f\left(\frac{i\,d_c}{2r_b}\right), \tag{3-36}$$

and

$$I_{Rna} = -I_o + I_{Rnb} \tag{3-37}$$

The current in the ring as it travels through the solenoid is plotted in Fig. 3 to help clarify the analysis. This plot is for specified dimensions, namely, $r_a \sim r_b = 10$ cm, $d_c = 1$ cm.

It is seen from the above plot that if the solenoid has a large number of turns, the current in the ring will build up to a certain value, $(-I_{Rm})$, depending on the current in the solenoid and the dimensions. This buildup of current occurs during the travel of the ring through the beginning few turns of the solenoid, and the current in the ring oscillates between $(-I_{Rm})$ and $-(I_{Rm}-I_o)$ through the major portion of travel.

The ring emerges out of the solenoid with a current flowing through it, but since the solenoid is left energized, the field of the solenoid tends to cancel the current in the ring as the ring moves out of the field of the solenoid. The current in the ring as a function of distance beyond the exit is plotted in Fig. 4 for both operating modes assuming infinite conductivity.

It is also noteworthy here that, like the attraction case, the current in the ring varies approximately linearly as the ring moves between one turn and the next.

C. MUTUAL INDUCTANCE BETWEEN THE RING AND THE SOLENOID

The mutual inductance between a solenoid and a circular current filament in its end plane is given in Eq. (3-38).[14] Dimensions are illustrated in Fig. 5.

$$M_{ROS} = 0.002\pi^2 r_b(\text{cm})\,\alpha\,\rho\,N\,Q_o \quad (\mu h) \tag{3-38}$$

14. Grover, p. 114.

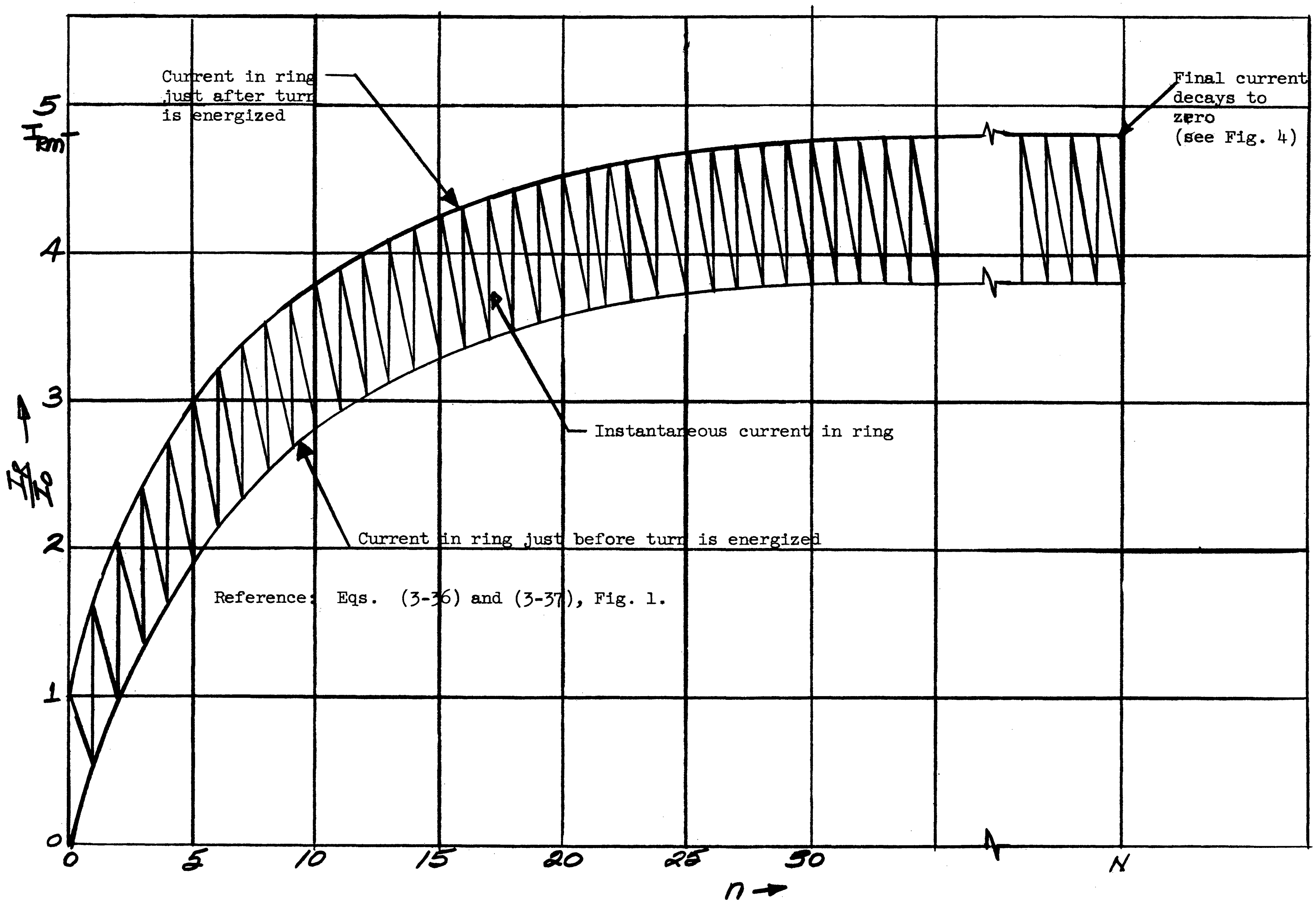

Fig. 3. Repulsion case: current in projectile ring at n^{th} turn; $r_a \sim r_b =$ 10 cm, $d_c = 1$ cm.

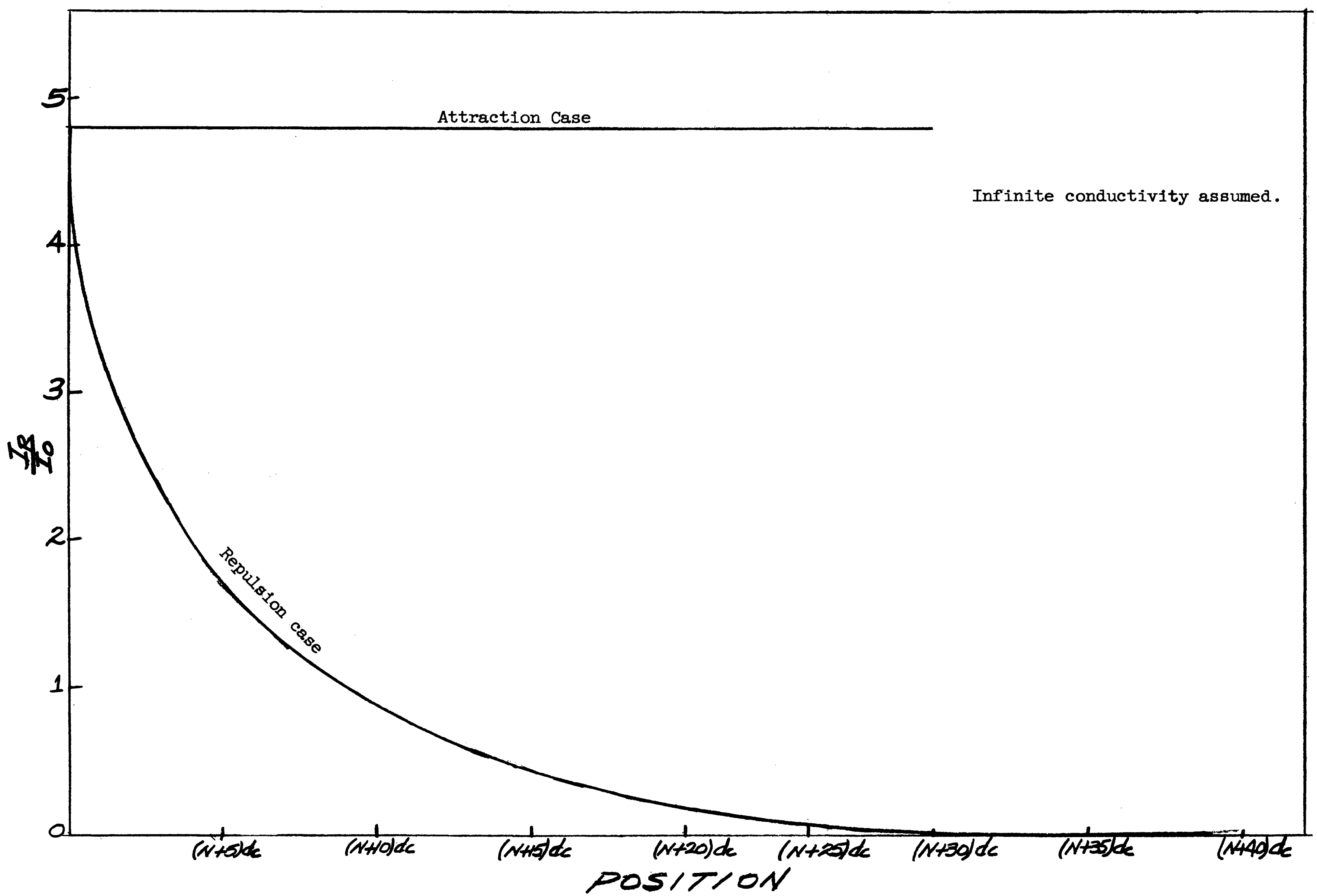

Fig. 4. Current in the projectile ring after it emerges from the solenoid: $r_a \sim r_b = 10$ cm, $d_c = 1$ cm.

where

$$\rho^2 = \frac{r_a^2}{r_a^2 + s^2} \tag{3-39}$$

$$\alpha = \frac{r_b}{r_a} \tag{3-40}$$

N = total number of turns, and
Q_o is tabulated as a function of ρ^2 and α.

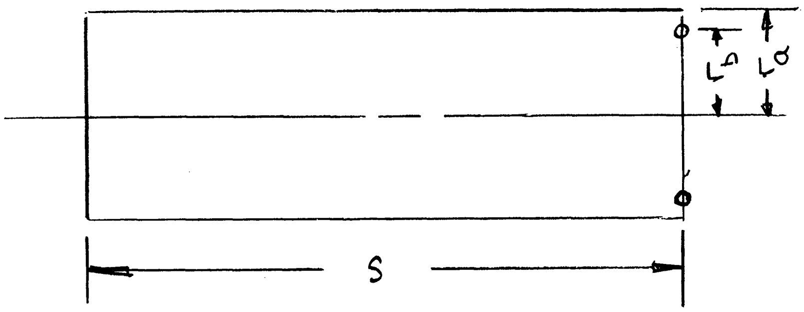

Fig. 5. Dimensions for mutual inductance calculations.

But for short solenoids and circles of the same radius (ρ^2 and α in the neighborhood of unity), the value of Q_o cannot be accurately interpolated from the table. In such cases this difficulty is avoided by the use of the following formula:[15]

$$M_{ROS} = 0.002\pi N A\sqrt{\alpha}\, R_o \left(\ln\frac{16}{\gamma^2+\zeta^2} - 2\frac{\gamma}{\zeta^2}\tan^{-1}\frac{\zeta}{\gamma} - 2\right) \tag{3-41}$$

where

$$\gamma^2 = \frac{(1-\alpha)^2}{4\alpha} \tag{3-42}$$

$$\zeta^2 = \frac{(1-\rho^2)}{\alpha\rho^2}\ , \tag{3-43}$$

and the value of R_o is tabulated as a function of γ^2 and ζ^2.

---

15. Grover, p. 116.

Now returning to Eq. (3-38), we see that, for $s \gg r_a$,

$$\rho^2 = \frac{r_a^2}{r_a^2 + s^2} \cong \frac{r_a^2}{s^2}$$

and

$$\rho = \frac{r_a}{s} \ ;$$

also

$$N = n_1 S \ ,$$

where

n_1 = turns/unit length.

Therefore, from Eq. (3-38) we get

$$M_{RS} = 0.002\pi^2 r_a r_b \alpha n_1 \varphi_0 \ . \qquad (3\text{-}38')$$

For the general case where the circular filament is not in the end plane but is at a distance d_c from the end plane,

$$M_{Rd_cS} = M_{Ro(s+d_c)} - M_{Rod_c} \qquad (3\text{-}44)$$

where

$M_{Ro(s+d_c)}$ ≡ mutual inductance of the solenoid of length $(s+d_c)$ and the circle in its end plane,

and

M_{Rod_c} ≡ mutual inductance between a solenoid of length d_c and the circle in its end plane.

But when $d_c \ll s$, $\quad M_{Rod_c} \ll M_{Ro(s+d_c)}$,

$$M_{Rd_cS} \cong M_{RoS} \triangleq M_o \ .$$

This shows that the mutual inductance between the ring and the solenoid in our case is almost constant through the major portion of travel, but starts to vary appreciably when only a few energized turns are present in the solenoid.

Therefore, in the attraction case, the mutual inductance between the ring and the solenoid is almost constant until the ring approaches the last few turns where the mutual inductance starts dropping.

In the repulsion case, however, the mutual inductance starts low, builds up to a certain value in the first few turns, and then stays almost constant at that value through the major portion of travel.

Graphically the mutual inductance will be as shown in Fig. 6.

Having now expressions for current and mutual inductance, we are now in a position to simplify the velocity equations.

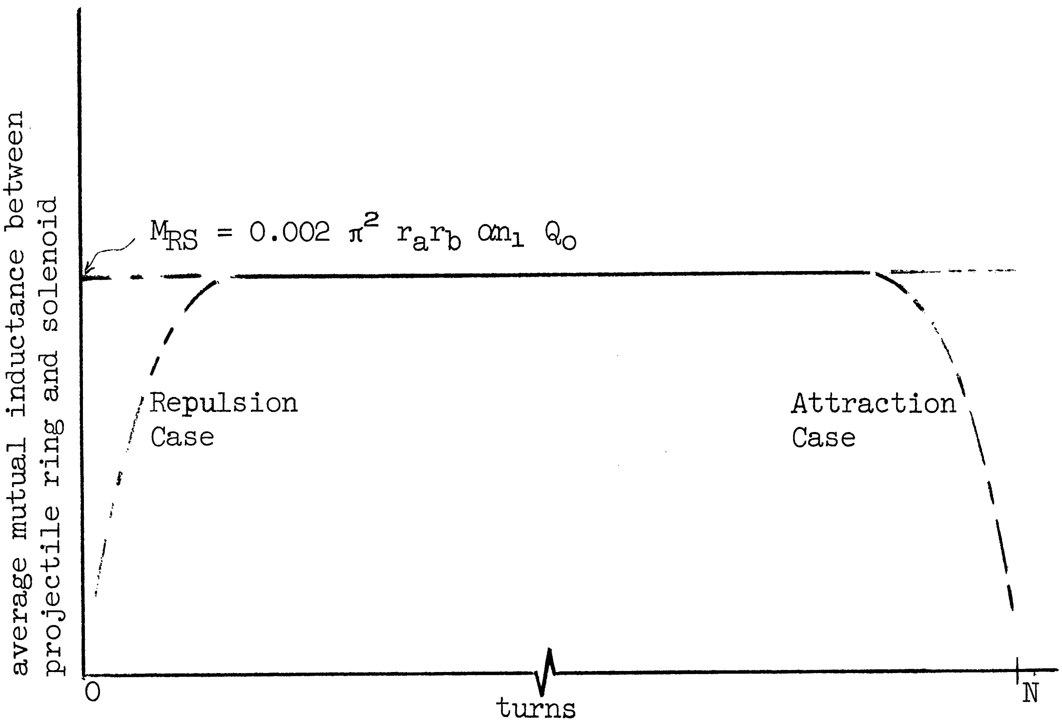

Fig. 6. Mutual inductance.

D. FINAL VELOCITY EXPRESSION

1. Attraction Case

Through the major portion of travel (except towards the end of the solenoid), we have:

a. Ring inductance (L_R) is approximately equal to the self-inductance of each solenoid turn (L_n).

b. Ring current just after a turn is de-energized is approximately equal to the solenoid current (I_o) and in the same direction as I_o.

c. Ring current just before a turn is de-energized is approximately zero.

d. The mutual inductance between ring and solenoid is approximately constant as the projectile moves between the n^{th} and $(n+1)^{th}$ turn; calling this M_o, it has been shown that

$$M_0 = 0.002\pi^2 r_a(\text{cm})\, r_b(\text{cm})\, \alpha n_1 \varphi_0 \quad (\mu h) \tag{3-38''}$$

where $n_1 \equiv$ turns per centimeter.

Since the current in the ring varies approximately linearly as the ring moves from the n^{th} turn to the $(n+1)^{th}$ turn, being I_0 at the n^{th} turn and zero at the $(n+1)^{th}$ turn, and since both M_{RS} ($\equiv M_0$) and I_0 are constant, we can evaluate the integral

$$\int_0^\tau I_0\, d(I_R M_{RS})$$

as follows:

$$\begin{aligned}
\int_0^\tau I_0\, d(I_R M_{RS}) &= \int_0^\tau I_0\, d(I_R M_0) \\
&= I_0 M_0 \int_0^\tau d I_R \\
&= I_0 M_0 \left[I_R \right]_0^\tau \\
&= I_0 M_0 (I_{R\tau} - I_{R0});
\end{aligned}$$

but

$$I_{R\tau} = 0, \quad I_{R0} \doteq I_0 .$$

Therefore

$$\int_0^\tau I_0\, d(I_R M_{RS}) = -I_0^2 M_0 . \tag{3-45}$$

Substituting back in the velocity equation (3-6), we get

$$v_{n+1} = \sqrt{\frac{2}{m}\left(\frac{1}{2} m v_n^2 + \frac{1}{2} I_0^2 L_R + I_0^2 M_0 + I_0^2 M_0\right)}$$

$$= \sqrt{v_n^2 + \frac{I_0^2}{m}(L_R + 4M_0)} \tag{3-46}$$

The kinetic energy gain in traveling N solenoid turns is then

$$K.E. = \frac{1}{2} m \left(v_f^2 - v_o^2 \right)$$

$$= \frac{1}{2} N I_o^2 \left(L_R + 4M_o \right), \tag{3-47}$$

and assuming zero initial velocity, the final velocity becomes

$$v_f = \sqrt{\frac{N I_o^2}{m} \left(L_R + 4M_o \right)} \tag{3-48}$$

2. Repulsion Case

Through the major portion of travel (except near the beginning of the solenoid), we have:

a. $L_R \cong L_n$ as in the attraction device.

b. Ring current just before a turn is energized (I_{Rnb}) is equal to some value $-(I_{Rm} - I_o)$ (see Fig. 4) dependent on dimensions and solenoid current I_o and is antiparallel to I_o.

c. Ring current just after a turn is energized (I_{Rna}) is $-I_{Rm}$, again directed oppositely to I_o.

d. Ring-coil mutual inductance is again approximately constant and is given by:

$$M_o = 0.002\pi^2 r_a(cm)\, r_b(cm)\, \alpha n_1 \phi_o \; (\mu h), \tag{3-38'}$$

where

$n_1 \equiv$ turns per centimeter.

As before, the back emf integral may now be evaluated:

$$\int_0^{\tau} I_o \, d(I_R M_{RS}) = I_o M_o \left(I_{R\tau} - I_{Ro} \right),$$

But $$I_{R\tau} = -(I_{RM} - I_0)\ , \quad I_{R0} = -I_{RM}$$

So, as in the attraction situation,

$$\int_0^\tau I_0\, d(I_R M_{RS}) = -\, I_0^2 M_0 \,. \tag{3-45}$$

Letting $I_{R_m} = KI_0$, and substituting back in the velocity equation (3-10), we get

$$v_{n+1} = \sqrt{\frac{2}{m}\left[\frac{1}{2} m v_n^2 + \frac{1}{2}(2k-1)L_R I_0^2 + I_0^2 M_0 + I_0^2 M_0\right]}$$

$$= \sqrt{v_n^2 + \frac{I_0^2}{m}\left[(2k-1)L_R + 4M_0\right]} \tag{3-49}$$

From the above analysis, then, the kinetic energy gained by the ring as it moves over N turns is:

$$K.E. = \frac{1}{2} m \left(v_f^2 - v_0^2\right)$$
$$= N I_0^2 \left[\frac{1}{2}(2k-1)L_1 + 2M_0\right] \tag{3-50}$$

Assuming that the initial kinetic energy of the ring is zero, we get an expression for the final velocity as follows:

$$v_f = \sqrt{\frac{N I_0^2}{m}\left[(2k-1)L_1 + 4M_0\right]} \tag{3-51}$$

Equations (3-48)and (3-51) derived above, apply through the major portion of travel, but they fail towards the end of the solenoid in the attraction case and at the beginning in the repulsion case.

The failure of these equations in the end regions will be due to what we shall call "end effects." These end effects impose certain limitations which will be treated in a later section.

For purposes of calculating the velocities attainable by the electromagnetic accelerator, however, Eqs. (3-48) and (3-51) will be quite adequate since the end regions constitute a negligible percentage of the total length of the device and therefore contribute little to the ultimate kinetic energy of the projectile.

IV. LIMITATIONS

The limitations on the electromagnetic accelerator are thermal and mechanical in nature. The thermal limitation results from the fact that the temperature of the ring cannot exceed a certain value which depends on the material used. The mechanical limitation arises from the interaction of the current in the ring and the magnetic field on it, which produces radial forces that tend to destroy the ring. The magnetic field on the ring is that of its self-field plus the field of the solenoid.

The mechanical limitation is assumed here to be an instantaneous (not time-dependent) factor. In most of the specific cases treated in the section of this report on numerical values (Section V), this turns out to be the dominating factor.

Heating of the projectile is of course a time-dependent function if there is no process for heat loss, as is approximately the case in the electromagnetic accelerator. Thus, heating will in all cases be the eventual controlling limitation if acceleration is carried on over a long enough time.

A. THERMAL LIMITATION

Heating will be analyzed in two parts. In the first part we will consider the ring as it travels from the beginning to the end of the solenoid. The second will take into consideration the magnetic energy stored in the ring as it emerges out of the solenoid. It is worthwhile to note here that, in the attraction case, this energy is converted into heat, while in the repulsion case, the current in the ring is wiped out by the solenoid field and does not contribute much to heating.

1. Attraction Case

The energy Q dissipated in the ring as heat is:

$$Q = I_R^2 \, R \, t \qquad (4\text{-}1)$$

where

R = a-c resistance in ohms

t = time in seconds.

The a-c resistance of a straight round wire conductor at very high frequencies is:[16]

$$R'_{h.f.} = \frac{1}{2\pi r_b \sigma \delta} \; (\text{ohm/unit length}) \qquad (4\text{-}2)$$

16. Ramo, S., and Whinnery, J. R., Fields and Waves in Modern Radio, J. Wiley and Sons, 1953, p. 245.

where

σ = conductivity, mho/meter
δ = skin depth, meters.

Applying this to our case, where we consider that the current in the ring flows in a skin depth δ which is distributed uniformly over approximately half of the periphery instead of all around it, we get

$$R'_{h.f.} = \frac{1}{\pi r_b \sigma \delta}, \tag{4-2'}$$

and

$$R_{h.f} = \frac{2\pi r_b}{\pi r_b \sigma \delta} = \frac{2}{\sigma \delta}, \tag{4-3}$$

where

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}, \quad f = \text{freq.}, (\text{c.p.s.}). \tag{4-4}$$

and for copper

$$\delta = \frac{0.066}{\sqrt{f}} \text{ meters}. \tag{4-4'}$$

Therefore

$$R_{h.f.} = \frac{30\sqrt{f}}{\sigma}. \tag{4-5}$$

Substituting back in the equation for Q, we get

$$Q = \frac{1}{\sigma} 30\sqrt{f}\, I_R^2\, t. \tag{4-6}$$

It should be noted that the above evaluation of the a-c resistance was based on sinusoidal variations. The current variation in our case does not quite meet this condition but it can be considered sinusoidal as an approximation. A more exact treatment will be found in Appendix I.

Now $f \triangleq v/\lambda$ (4-7)

where v = avg. velocity = $v_f/2$, and λ = wave length $\triangleq d_c$.

Therefore

$$f = \frac{v_f}{2d_c} \,. \tag{4-8}$$

Also

$$t \cong \frac{2S}{v_f} \tag{4-9}$$

where S = length of the solenoid.

Therefore

$$Q = \frac{30 I_R^2}{\sigma} \sqrt{\frac{v_f}{2d_c}} \left(\frac{2S}{v_f}\right) \quad \text{(joules)}$$

$$= \frac{10 I_R^2 S}{\sigma \sqrt{v_f d_c}} \quad \text{(calories)}. \tag{4-10}$$

The amount of heat required to give a temperature rise ΔT will be:

$$Q_{\Delta T} = m \times \Delta T \times c \quad \text{calories,} \tag{4-11}$$

where

m = mass of ring in gm
$= \pi r^2 \cdot 2\pi r_b \cdot \rho_m$, and
ρ_m = density in gm/m^3
c = specific heat in cal/gm°C.

Therefore

$$Q_{\Delta T} = 2\pi^2 r^2 r_b \rho_m \Delta T c \quad \text{(calories)} \tag{4-12}$$

and in order that the ring temperature does not go above a given temperature ($T_o + \Delta T$), where T_o is the ambient, the following should hold:

$$\frac{10 I_R^2 S}{\sigma \sqrt{v_f d_c}} \leqq 2\pi^2 r^2 r_b \rho_m \Delta T c \tag{4-13}$$

or

$$I_R^2 \leqq \frac{1}{S} 2 r^2 r_b \rho_m \Delta T c \sigma \sqrt{v_f d_c} \,. \tag{4-14}$$

But $I_R = K_a I_o$, where K_a is a constant determined from the current derivations developed previously; in this case (attraction) it is equal to 1/2. Therefore,

$$I_o^2 \leqq \frac{1}{S} 8 r^2 r_b \rho_m \Delta T c \sigma \sqrt{v_f d_c} . \tag{4-15}$$

We must also consider the effect of the magnetic energy stored in the ring as it emerges out of the solenoid. This magnetic energy all goes into heat and is:

$$Q_m = \frac{1}{2} I_{Rm}^2 L_R \text{ (joules)}, \tag{4-16}$$

where

$I_{Rm} = KI_o$ is the current in the ring as it emerges out of the solenoid;
K depends on the dimensions (as shown previously).

Therefore

$$Q_m = \frac{0.24}{2} (KI_o)^2 L_R \text{ (calories)}. \tag{4-17}$$

Now in order for the ring temperature not to exceed $(T_o + \Delta T)$, the following must hold [combining Eqs. (4-14) and (4-17)]:

$$0.12 K^2 I_o^2 L_R + \frac{10 K_a^2 I_o^2 S}{\sigma \sqrt{v_f d_c}} \leqq 2\pi^2 r^2 r_b \rho_m \Delta T c , \tag{4-18}$$

and

$$I_o^2 \leqq 20 r^2 r_b \rho_m \Delta T c \left[\frac{1}{0.12 K^2 L_R + \frac{10 K_a^2 S}{\sigma \sqrt{v_f d_c}}} \right]. \tag{4-19}$$

2. Repulsion Case

By a similar procedure the current limit in this case is as follows:

$$I_o^2 \leqq \frac{2 r^2 r_b \rho_m \Delta T c \sigma \sqrt{v_f d_c}}{K_r^2 S} . \tag{4-20}$$

The only difference in this equation from that of the attraction case is in the value for K_r. Here K_r depends on the dimensions and is different from K_a.

It was shown previously that the magnetic field of the solenoid wipes out the current in the ring as the ring moves away from the end of the solenoid. Therefore there is a negligible amount of heating contributed by the stored magnetic energy in the ring and Eq. (4-20) will be adequate for determination of maximum allowable current.

B. MECHANICAL LIMITATION

1. Attraction Case

The mechanical limitation arises from the fact that the mechanical stress on the projectile cannot exceed the tensile strength of the material involved. This mechanical stress appears as a radial pressure and will be due to two factors, the interaction of the projectile circumferential current with its own magnetic field, and with the axial component of the magnetic field from the external driving coil.

Consider a ring carrying current with major radius r_b and minor or cross-sectional radius r. If there is an outward directed radial force F' per unit peripheral length distributed uniformly around the ring, then the stress within the ring can be derived as follows (referring to Fig. 7).

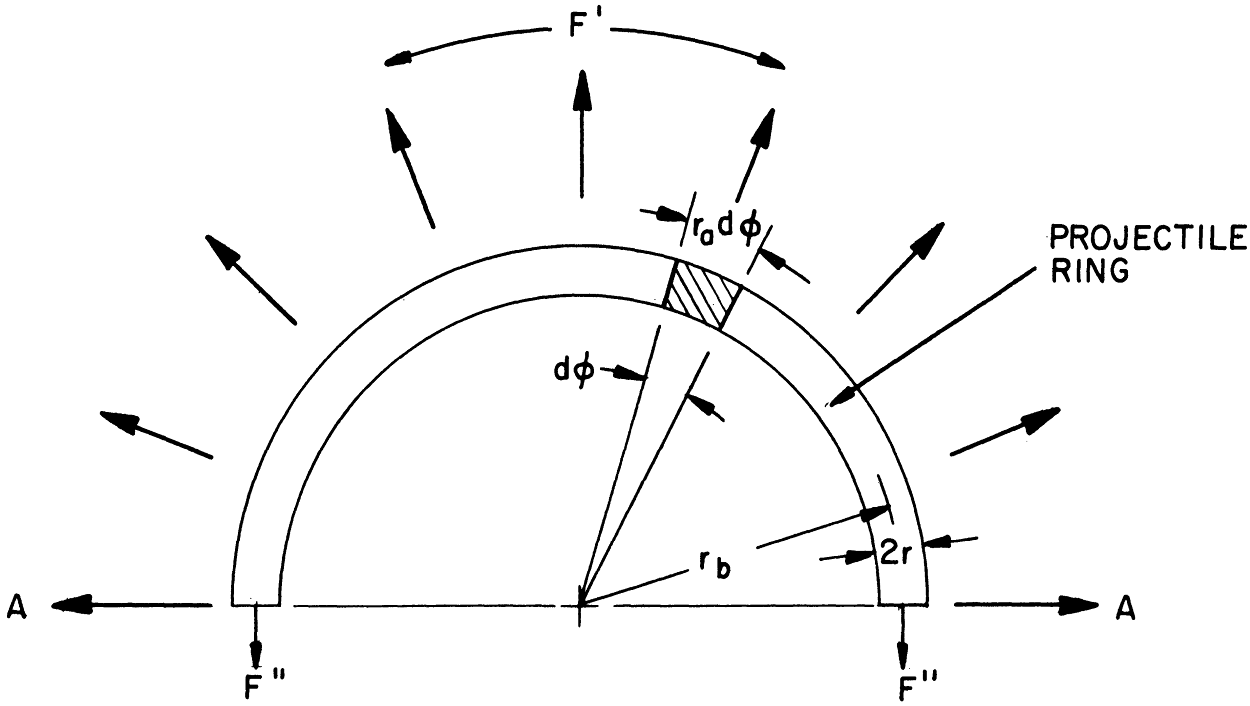

Fig. 7. Mechanical stress diagram.

2F" ≡ total downward force acting within the ring normal to the bisecting plane A-A* (newtons).

F' ≡ radial force per unit circumferential length acting on ring (newtons/meter).

*Plane A-A is horizontal and bisects the projectile ring along a major diameter.

In order that equilibrium exist, the sum of the vertical forces on the ring as shown in Fig. 7 must be zero; that is,

$$2F'' = \int_0^{\pi} F' \sin\phi \, r_b \, d\phi \tag{4-21}$$

which results in

$$F'' = F' r_b \quad . \tag{4-21'}$$

So that the ring is not destroyed, however, the following inequality must hold:

$$\frac{F''}{A} \leqq \sigma_t \quad , \tag{4-22}$$

where

$A \equiv$ ring cross-sectional area

$= \pi r^2$,

$\sigma_t \equiv$ tensile strength of the ring material,

or

$$\frac{F' r_b}{\pi r^2} \leq \sigma_t \quad . \tag{4-23}$$

The axial magnetic field on the ring due to the driving coil will be approximately:

$$B_{ext} = \frac{1}{2} \mu n_1 I_o \; , \tag{4-24}$$

where

I_o = coil current

n_1 = coil turns/unit length.

Therefore the external force F'_{ext} is given by

$$F'_{ext} = B_{ext} I_R = \frac{1}{2} \mu n_1 I_o I_R \; , \tag{4-25}$$

where

I_R = ring or projectile current.

The internal force due to the self-magnetic field of the ring current may be calculated by noting the change in stored energy caused by a small change in ring radius, keeping current constant.

$$W = \frac{1}{2} L_R I_R^2 \tag{4-26}$$

$$F'_{int} = \frac{1}{2\pi r_b} \frac{dW}{dr_b}$$

$$= \frac{I_R^2}{4\pi r_b} \frac{dL_R}{dr_b} . \qquad (4\text{-}27)$$

The inductance of a conducting loop is given approximately by[17]

$$L_R \cong \mu r_b \ln \frac{8r_b}{r}, \text{ where } \frac{r}{r_b} \ll 1. \qquad (4\text{-}28)$$

Therefore

$$\frac{dL_R}{dr_b} = \mu \ln \frac{8r_b}{r} + \mu$$

$$= \mu \left(\ln \frac{8r_b}{r} + 1\right) \qquad (4\text{-}29)$$

and

$$F'_{int.} = \frac{\mu I_R^2}{4\pi r_b}\left(\ln \frac{8r_b}{r} + 1\right) \qquad (4\text{-}30)$$

Now

$$F' = F'_{int.} + F'_{ext.} \qquad (4\text{-}31)$$

$$= \frac{\mu I_R^2}{4\pi r_b}\left(\ln \frac{8r_b}{r} + 1\right) + \frac{1}{2}\mu n_1 I_o I_R . \qquad (4\text{-}31')$$

Introducing Eq. (4-31') into Eq. (4-23) gives

$$\sigma_t \geq \frac{\frac{\mu I_R^2}{4\pi}\left(\ln \frac{8r_b}{r} + 1\right) + \frac{1}{2} r_b \mu n_1 I_o I_R}{\pi r^2} \qquad (4\text{-}32)$$

In attraction-type operation, the above analysis holds exactly, and through the major portion of travel $I_R \leqq I_o$. Although in the end region I_R rises considerably above I_o, the magnetic field strength from the solenoid at the same time falls off due to the shortness of the active coil, so we need not consider "end effects" in this analysis. Substituting I_o for I_R and $1/d_c$ for n_1 in Eq. (4-32), then, the limiting conditions are given by:

17. A.I.P. Handbook, pp. 5-29.

$$\sigma_t \geq \frac{\mu}{2\pi r}\left(\frac{r_b}{r}\right) I_o^2 \left[\frac{1}{d_c} + \frac{1}{2\pi r_b}\left(\ln \frac{8 r_b}{r} + 1\right)\right] . \qquad (4\text{-}33)$$

2. Repulsion Case

When the projectile is accelerated away from rather than toward the energized portion of the driving solenoid, as in the repulsion case, the radial force is inwardly directed, and the above derivation is therefore inapplicable. Mechanical limitation may in this case arise from two distinct types of failure. If the ratio r_b/r is small (say, less than 10), the ring is considered to be thick, and failure is reached when the axial stress around the ring exceeds the compressive strength (σ_c) of the ring material, that is, when $F' = (\pi r^2/r_b)\,\sigma_c$. The procedure followed for the attraction case is therefore also applicable with substitution of σ_c for σ_t, and the resulting maximum radial force in repulsion is higher than in attraction in the ratio of σ_c/σ_t.

When r_b/r becomes larger, however, as in general will be the case in the electromagnetic accelerator where velocity rather than mass is desired (leading to small r), deformation instability becomes the limiting factor. That is, beyond a certain radially inward loading (F' newton/meter of circumferential length), any slight elongation which appears in the nominally circular ring will grow so as to buckle the ring.

The analysis of the "slender" ring buckling limit is carried out in the literature[18,19] for a ring of rectangular cross section (radial width r, axial extent ℓ, major radius r_b, and the limiting condition is given by:

$$F'_{lim}(\text{repulsion}) = \frac{3EJ}{r_b^3} \qquad (4\text{-}34)$$

where

F'_{lim} ≡ maximum radially inward force per unit length around circumference of ring (newton/m)

E ≡ modulus of elasticity (newton/m^2)

J ≡ moment of inertia (m^4) $= \ell r^3/12$.[20] (4-35)

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18. Timoshenko, S., Theory of Elastic Stability, McGraw-Hill Book Co., New York, 1936, p. 216.
19. von Sanden, K. B., and Gunther, K., "The Strengths of Cylindrical Shells Stiffened by Frames and Bulkheads, Under Uniform External Pressure on All Sides," Werft and Reederei, 1, Nos. 8-10 (1920), 2, No. 17 (1921), translated by E. N. Labouvie, Translation No. 38, March, 1952, Department of the Navy, David W. Taylor Model Basin, Washington, D. C.
20. Timoshenko, S., and MacCullough, G. H., Elements of Strength of Materials, D. Van Nostrand Co., 1943, p. 350.

In the repulsion device, the maximum ring current I_{Rm} is given by

$$I_{Rm} = K I_o \tag{4-36}$$

where K is a constant (>1) depending on dimensions.

From Eq. (4-31'), then, (noting again $1/d_c = n_1$), the limiting conditions for a rectangular cross-sectional ring are given by:

$$3\frac{EJ}{r_b^3} \geq \frac{\mu K^2 I_o^2}{4\pi r_b}\left(\ln\frac{8r_b}{\ell}+1\right)+\frac{1}{2}\mu n_1 K I_o^2$$
$$\geq \frac{\mu K^2 I_o^2}{4\pi r_b}\left[\left(\ln\frac{8r_b}{\ell}+1\right)+\frac{2\pi r_b}{d_c K}\right], \tag{4-37}$$

or substituting in Eq. (4-35),

$$E \geq \frac{2\mu K}{\ell}\left(\frac{r_b}{r}\right)^3 I_o^2\left[\frac{K}{2\pi r_b}\left(\ln\frac{8r_b}{\ell}+1\right)+\frac{1}{d_c}\right] \tag{4-38}$$

Comparing Eqs. (4-38) and (4-33), one notes that the driving coil current I_o, and therefore the final velocity [see Eq. (3-49)] will be somewhat less in the repulsion device than in the alternate scheme. More quantitative estimates will be made in the later section covering numerical results.

Two methods exist for raising the mechanical limiting point of the repulsion-operated accelerator. In the first place, bracing or spokes could be used to oppose the collapse of the ring. The limiting Eq. (4-34) has been derived on the basis of buckling in the form of a single wavelength around the entire ring. A simple strut arrangement would necessitate at least a two-wavelength buckling configuration, which would not become unstable until a higher external load was reached. For further discussion of these matters, the reader may consult Refs. 18 and 19.

The other improvement of the mechanical strength situation may be brought about by having the projectile traveling coaxial with but outside of the accelerating solenoid. Referring to Fig. 1 this means that $r_b > r_a$ and the radial component of force on the ring is now outwardly directed. In calculating the velocity attainable by this geometry, the mutual inductance between ring and solenoid depends only on the ratios

$$\alpha = \frac{r_1}{r_2} \qquad (r_1 < r_2 \text{ always})$$

$$\rho = \frac{r_2^2}{r_2^2 + S^2}$$

regardless of whether the ring is larger or smaller than the solenoid (i.e., regardless of whether $r_a = r_1$, $r_b = r_2$, or $r_a = r_2$, $r_b = r_1$ in the above equations).* The current expression **already** derived (Fig. 3) is still applicable.

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*See Ref. 7 and the section in this report on mutual inductance, p.20.

V. NUMERICAL RESULTS

A. AVERAGE RING CURRENT IN TERMS OF SOLENOID CURRENT

It was found previously that the current in the ring depends on the dimensions of the system. Table I shows the current in the ring for several typical dimensions for both the attraction and repulsion accelerators. In this table $d_c = 2r$.

TABLE I

RING CURRENT

Attraction Case [Using Eqs. (3-29) and (3-30)]

r(cm)	r_b(cm)	Avg. I_R Through Majority of Travel*	I_{RM}(at end) = k I_o
0.1	2	0.5 I_o	3.6 I_o
0.5	10	0.5 I_o	4.8 I_o
0.5	20	0.5 I_o	8.0 I_o
0.5	30	0.5 I_o	10.5 I_o
0.5	40	0.5 I_o	13.0 I_o
0.5	50	0.5 I_o	15.5 I_o
0.5	20	0.5 I_o	8.0 I_o
0.4	20	0.5 I_o	9.5 I_o
0.3	20	0.5 I_o	11.5 I_o
0.2	20	0.5 I_o	15.5 I_o
0.1	20	0.5 I_o	26.5 I_o

Repulsion Case [Using Eqs. (3-36) and (3-37)]

r(cm)	r_b(cm)	I_{RM} = k I_o**
0.1	2	3.6 I_o
0.5	10	4.8 I_o
0.5	20	8.0 I_o
0.5	30	10.5 I_o
0.5	40	13.0 I_o
0.5	50	15.5 I_o
0.5	20	8.0 I_o
0.4	20	9.5 I_o
0.3	20	11.5 I_o
0.2	20	15.5 I_o
0.1	20	26.5 I_o

*I_R varies approximately linearly between I_o and 0.

**I_R varies between $-I_{RM}$ and $-(I_{RM} - I_o)$.

B. FINAL VELOCITY CALCULATION

With the current in the ring now determined for different dimensions, the mechanical limitation will be considered next. From this, a maximum allowable ring current is obtained, which in turn from the above tables determines the maximum solenoid current. Knowing the ring and solenoid currents, the final velocity can be obtained. This final velocity will then be substituted in the heating limitation to check if the inequalities (4-19) and (4-20) hold.

Copper will in all cases be used as the projectile material. Physical constants for copper are:[21]

tensile strength (σ_t) $\equiv$ 60,000 lb/in.^{2} = 5×10^8 newton/m^2
modulus of elasticity (E) $\equiv$ 15×10^6 lb/in.^{2}
density (ρ_m) $\equiv$ 8.9×10^6 gm/m^3
specific heat (c) $\equiv$ 0.094 cal/gm - °C
conductivity (σ) $\equiv$ 5.8×10^7 mhos/m

1. Attraction Case

a. Maximum Allowable Currents Considering the Mechanical Limitation.—From Eq. (4-33), the following inequality results:

$$I_o^2 \leq \frac{2.5 \times 10^{15}\, r^2}{r_b \left[\frac{1}{d_c} + \frac{1}{6.24 r_b}\left(\ln \frac{r_b}{r} + 3\right)\right]} \qquad (5\text{-}1)$$

But for the range of values involved here

$$\frac{1}{d_c} \gg \frac{1}{6.24 r_b}\left(\ln \frac{r_b}{r} + 3\right),$$

that is

$$F'_{ext} \gg F'_{int} \; ;$$

therefore,

$$I_o^2 \leq \frac{1}{r_b}\left(2.5 \times 10^{15}\, r^2 d_c\right), \qquad (5\text{-}1')$$

where r, r_b amd d_c are in meters, and I_o is in amperes.

Maximum allowable values for I_o from Eq. (5-1') are listed in Table II.

21. Handbook of Chemistry and Physics, 34th ed., Chemical Rubber Publishing Co., 1952.

TABLE II

MECHANICALLY ALLOWABLE CURRENT (ATTRACTION CASE)

[Using Eq. (5-1')]

$d_c \equiv 2r$

r(cm)	r_b(cm)	I_o(max)(amp)
0.1	2	1.58×10^4
0.5	10	7.9×10^4
0.5	20	5.6×10^4
0.5	50	3.54×10^4
0.5	20	5.6×10^4
0.3	20	2.6×10^4
0.1	20	$.5 \times 10^4$

b. Final Velocity as a Function of Dimensions.—From Eq. (3-48), using the dimensions of Tables I and II, final velocities have been calculated and are tabulated in Table III.

TABLE III

FINAL VELOCITY FOR N = 1000 TURNS (ATTRACTION CASE)

[Using Eq. (3-48)]

r(cm)	r_b(cm)	L_R(hys)	M_o(hys)	m(kg)	$v_f\ \frac{m}{sec}$
0.1	2	$.84 \times 10^{-7}$	3.95×10^{-7}	.0035	1.1×10^4
0.5	10	4.18×10^{-7}	16.2×10^{-7}	.445	1.0×10^4
0.5	20	$10. \times 10^{-7}$	$72. \times 10^{-7}$	.89	1.0×10^4
0.5	50	$31. \times 10^{-7}$	$480. \times 10^{-7}$	2.225	1.1×10^4
0.5	20	10×10^{-7}	72×10^{-7}	.89	1.0×10^4
0.3	20	11.3×10^{-7}	120×10^{-7}	.32	1.0×10^4
0.1	20	14×10^{-7}	360×10^{-7}	.035	1.0×10^4

The above table was calculated for a constant N, namely, one thousand turns in each case. If the table is calculated for a constant length of solenoid S = 10 m, we get the final velocities in Table IV.

TABLE IV
FINAL VELOCITY FOR S = 10 METERS (ATTRACTION CASE)
[Using Eq. (3-48)]

r(cm)	r_b(cm)	$v_f \frac{m}{sec}$	$v_f \frac{F+}{sec}$
0.1	2	2.5×10^4	8.3×10^4
0.5	10	1.0×10^4	3.3×10^4
0.5	20	1.0×10^4	3.3×10^4
0.5	50	1.1×10^4	3.5×10^4
0.5	20	1.0×10^4	3.3×10^4
0.3	20	1.3×10^4	4.3×10^4
0.1	20	2.3×10^4	7.6×10^4

c. Maximum Allowable Current Due to Heating Limitation.—The inequality to be satisfied is

$$I_o^2 \leq 20 r^2 r_b \rho_m \Delta T c \left[\frac{1}{0.12 k^2 L_\ell + \dfrac{10 k_a^2 S}{\sigma \sqrt{v_f d_c}}} \right] , \qquad (4\text{-}19)$$

where

$\rho_m \approx 8.9 \times 10^6 \frac{gm}{m^3}$.

$C = 0.094 \frac{cals}{gm°C}$,

ΔT will be taken as 500°C ,

$S = 10$ meters ,

$\sigma = 5.8 \times 10^7 \frac{mhos}{m}$,

$K_a = 1/2$, and

K depends on the dimensions.

Maximum allowable currents according to Eq. (4-19) are listed in Table V.

TABLE V

MAXIMUM THERMALLY ALLOWABLE CURRENT (ATTRACTION CASE)
[Using Eq. (4-19)]

r(cm)	r_b(cm)	Max. Allowable I_o
0.1	2	2.9 x 10^5
0.5	10	10.7 x 10^5
0.5	20	7.4 x 10^5
0.5	50	3.42 x 10^5
0.5	20	7.4 x 10^5
0.3	20	2.9 x 10^5
0.1	20	.375 x 10^5

It is seen from Table V that the maximum allowable current is limited mechanically for all dimensions given above. Therefore, the final velocities calculated previously are the maximum velocities obtainable.

2. Repulsion Case ($r_b < r_a$)

Again $1/d_c \ll (k/2\pi r_b)[\ell n(8r_b/\ell) + 1]$, so the maximum allowable current is given [from Eq. (4-38)] by:

$$I_o^2(\text{rep}) \leq \frac{E\ell}{2\mu k}\left(\frac{r}{r_b}\right)^3 \left[\frac{2\pi r_b}{k\left(\ln\frac{8r_b}{\ell}+1\right)}\right], \tag{5-2}$$

Comparing this with the equivalent attraction device equation [from Eq. (4-33), written for rectangular rather than round cross-sectioned ring],

$$I_o^2(\text{att}) \leq \frac{2\sigma_t \ell}{\mu}\left(\frac{r}{r_b}\right)\left[\frac{2\pi r_b}{\left(\ln\frac{8r_b}{\ell}+1\right)}\right], \tag{5-1"}$$

shows that

$$I_o^2(\text{rep})\Big|_{max} = \frac{E}{\sigma_t}\,\frac{1}{4k^2}\left(\frac{r}{r_b}\right)^2 I_o^2(\text{att})\Big|_{max}. \tag{5-3}$$

Typical results are shown in Table VI. From this table one concludes that inasmuch as velocity is proportional to I_o [Eq. (3-49)] the attraction device is far superior from the mechanical strength standpoint when $r_b < r_a$.

TABLE VI
COMPARISON OF MAXIMUM SOLENOID CURRENTS
IN ATTRACTION AND REPULSION CASES

r(cm)	r_b(cm)	$\frac{I_0^2(\text{rep})\|_{max}}{I_0^2(\text{att})\|_{max}}$
.5	10	6.9×10^{-3}
.5	20	6.1×10^{-4}
.5	50	2.5×10^{-5}
.1	20	2.25×10^{-8}

3. Repulsion Case ($r_b > r_a$)

The calculations here are straightforward following the procedures used in the attraction case above. Results are given in Tables VII and VIII.

TABLE VII
MECHANICALLY ALLOWABLE CURRENT (REPULSION CASE)
[Using Eq. (5-1')]

r(cm)	r_b(cm)	Max. I_{RM}(amp)	Max. I_0(amp)
0.1	2	1.58×10^4	$.44 \times 10^4$
0.5	10	8×10^4	$.44 \times 10^4$
0.5	20	5.6×10^4	1.66×10^4
0.5	50	3.54×10^4	$.23 \times 10^4$
0.5	20	5.6×10^4	$.53 \times 10^4$
0.3	20	2.6×10^4	$.228 \times 10^4$
0.1	20	$.5 \times 10^4$	$.02 \times 10^4$

TABLE VIII
FINAL VELOCITY FOR N = 1000 TURNS (REPULSION CASE)
[Using Eq. (3-51)]

r(cm)	r_b(cm)	$v_f \frac{m}{sec}$	$v_f \frac{ft}{sec}$
0.1	2	8.65×10^3	28×10^3
0.5	10	2.5×10^3	8.2×10^3
0.5	20	1.24×10^3	4.1×10^3
0.5	50	1.07×10^3	3.5×10^3
0.5	20	1.24×10^3	4.1×10^3
0.3	20	1.4×10^3	4.6×10^3
0.1	20	1.12×10^3	3.7×10^3

If Table VIII is calculated for a constant solenoid length, namely S = 10 m, we get:

TABLE IX
FINAL VELOCITY FOR S = 10 METERS (REPULSION CASE)
[Using Eq. (3-51)]

r(cm)	r_b(cm)	$v_f \frac{m}{sec}$	$v_f \frac{ft}{sec}$
0.1	2	19.3×10^3	63×10^3
0.5	10	2.5×10^3	8.2×10^3
0.5	20	1.24×10^3	4.1×10^3
0.5	50	1.07×10^3	3.5×10^3
0.5	20	1.24×10^3	4.1×10^3
0.3	20	1.8×10^3	5.9×10^3
0.1	20	2.62×10^3	8.65×10^3

TABLE X
MAXIMUM THERMALLY ALLOWABLE CURRENT (REPULSION CASE)
[Using Eq. (4-20)]

r(cm)	r_b(cm)	Max. I_o(amp)
0.1	2	18.5×10^4
0.5	10	17×10^5
0.5	20	12×10^5
0.5	50	9.5×10^5
0.5	20	12×10^5
0.3	20	4.63×10^5
0.1	20	$.73 \times 10^5$

It is seen from the above tables that the maximum allowable currents are limited mechanically in all cases considered, and therefore the final velocities obtained in Tables VIII and IX are the maximum obtainable velocities for this type of operation.

We conclude therefore that the attraction operation is superior to the repulsion one when final velocity is the goal.

VI. STABILITY

Two types of instability might arise in the sequentially switched accelerator. The axis of the projectile could suffer either a lateral displacement or a tilt with respect to its equilibrium position coaxial with the driving coil. If the forces on the projectile are such as to cause these displacements to grow, the system will be unstable.

A. LATERAL INSTABILITY

We are concerned here only with the radial variation of the axial component of magnetic field emerging from the end of a long solenoid. Over the end plane, this component is uniform, but at any small distance out, the axial component is maximum on the solenoid axis, and diminishes with radial distance away from the axis.

The projectile ring is shown in Fig. 8, displaced off axis in both attraction and repulsion cases.

In the repulsion case $|F_2|>|F_1|$, so the lateral perturbation will amplify and the device is inherently instable.

Under attraction operation, again $|F_2|>|F_1|$, but now the net radial force is in the direction to oppose the displacement, and the system is therefore stable to lateral disturbances. This would lead then to less frictional loss than in the repulsion device.

B. ROTATIONAL INSTABILITY

Now we must explore the behavior of the radial component of magnetic field issuing from a long solenoid since this will account for axial forces on the ring which will either oppose or enlarge any ring tilt which might appear. By considering Fig. 9, one can see that the radial field component, a distance r_b from the axis of a solenoid, must peak between $z = 0$ and $z = d_c$. Specifically, let all turns through the n^{th} be energized with current I_o; let the field at $z = 0$, $r=r_b$ be called B_{r_o}. Now, if the solenoid is long enough, and the $(n+1)^{th}$ turn is energized with I_o current, then the radial field at $z = d_c$, $r = r_o$ must also be B_{r_o}. But the $(n+1)^{th}$ turn can contribute no radial field in the $z = d_c$ plane, so returning to the original conditions we may say that $B_r = B_{r_o}$ at $z = 0$ and d_c, $r = r_b$ with turns only through the n^{th} energized. Knowing that the radial field approaches zero eventually in both the $+z$ and $-z$ directions, the radial magnetic field at $r = r_b$ as a function of z takes the form shown in Fig. 10, peaking at $z = \xi < d_c$.

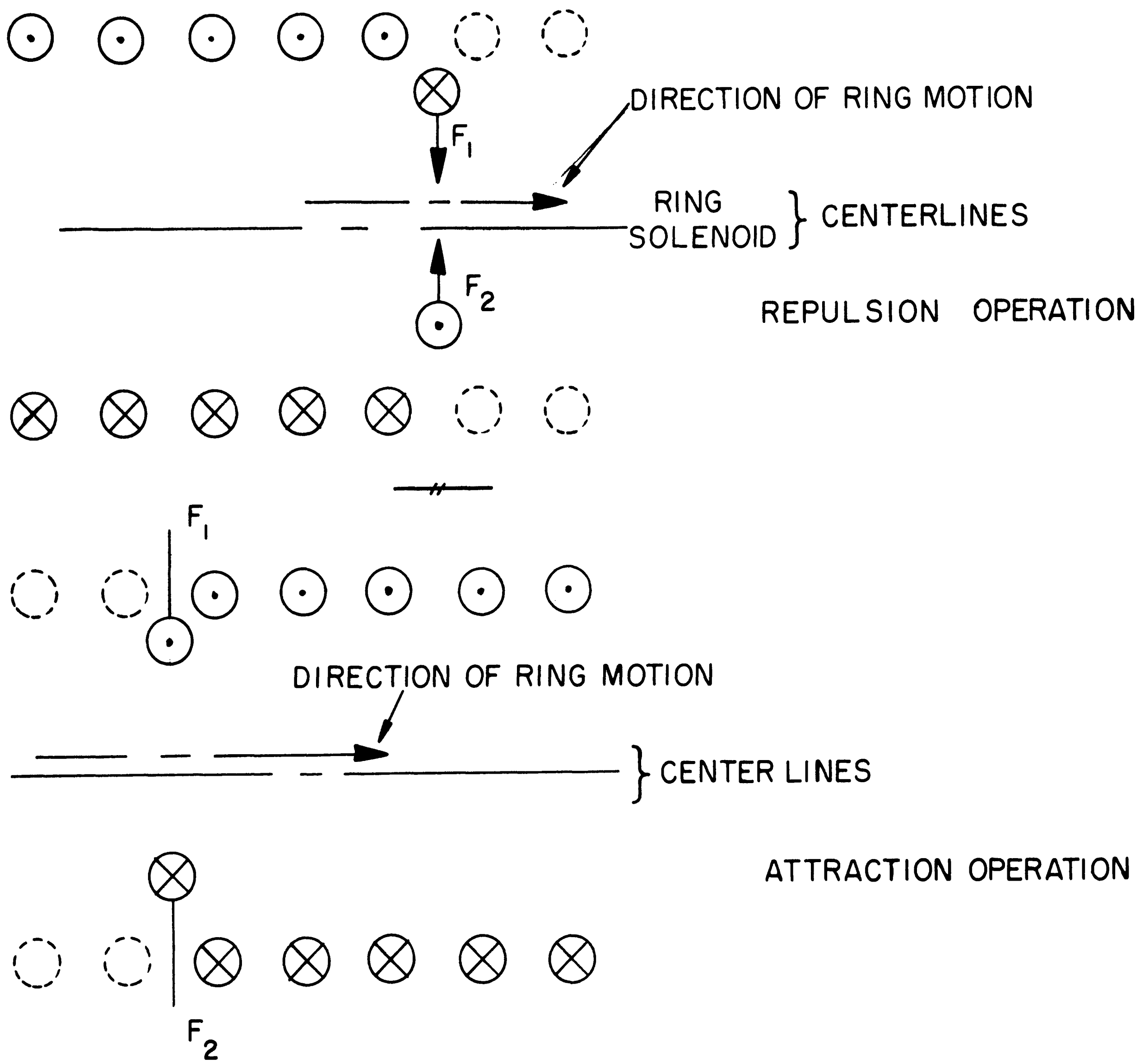

Fig. 8. Lateral Stability.

Currents are shown in the usual dot and cross notation. Un-energized coil turns are shown in dashed outline.

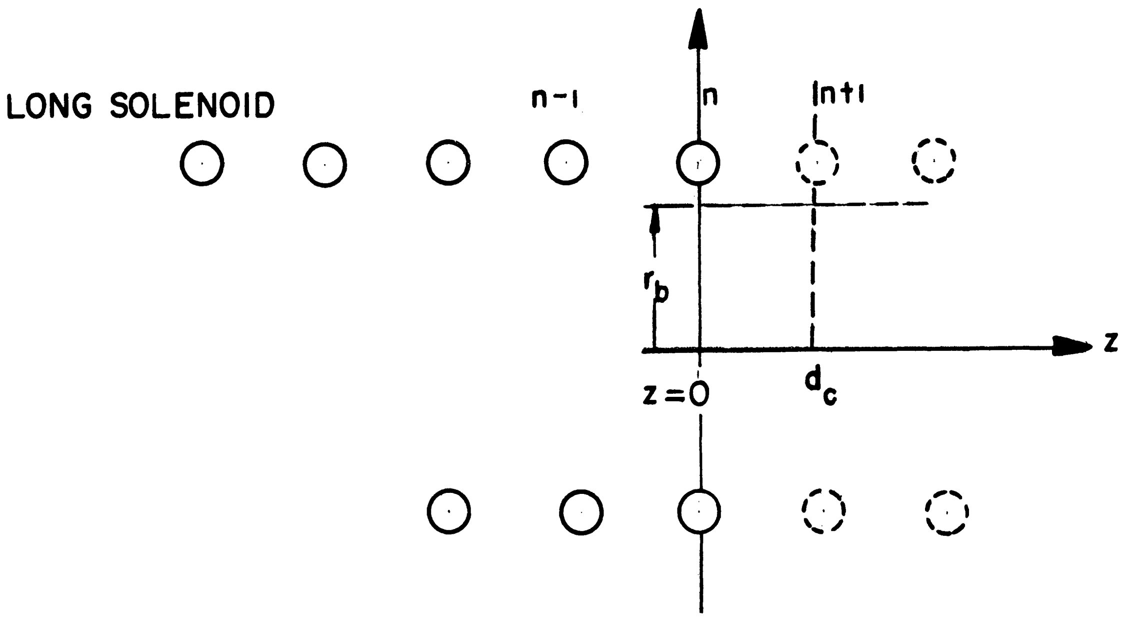

Fig. 9. Geometry for radial field evaluation.

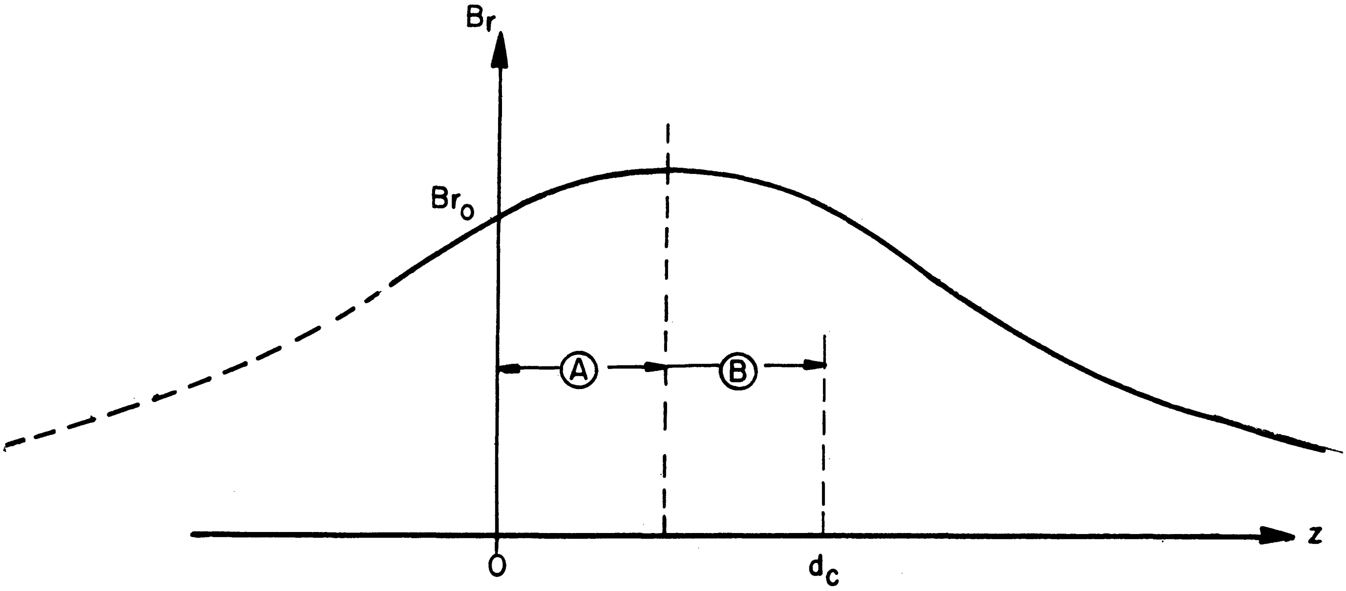

Fig. 10. Radial component of magnetic field off axis beyond end of a long solenoid.

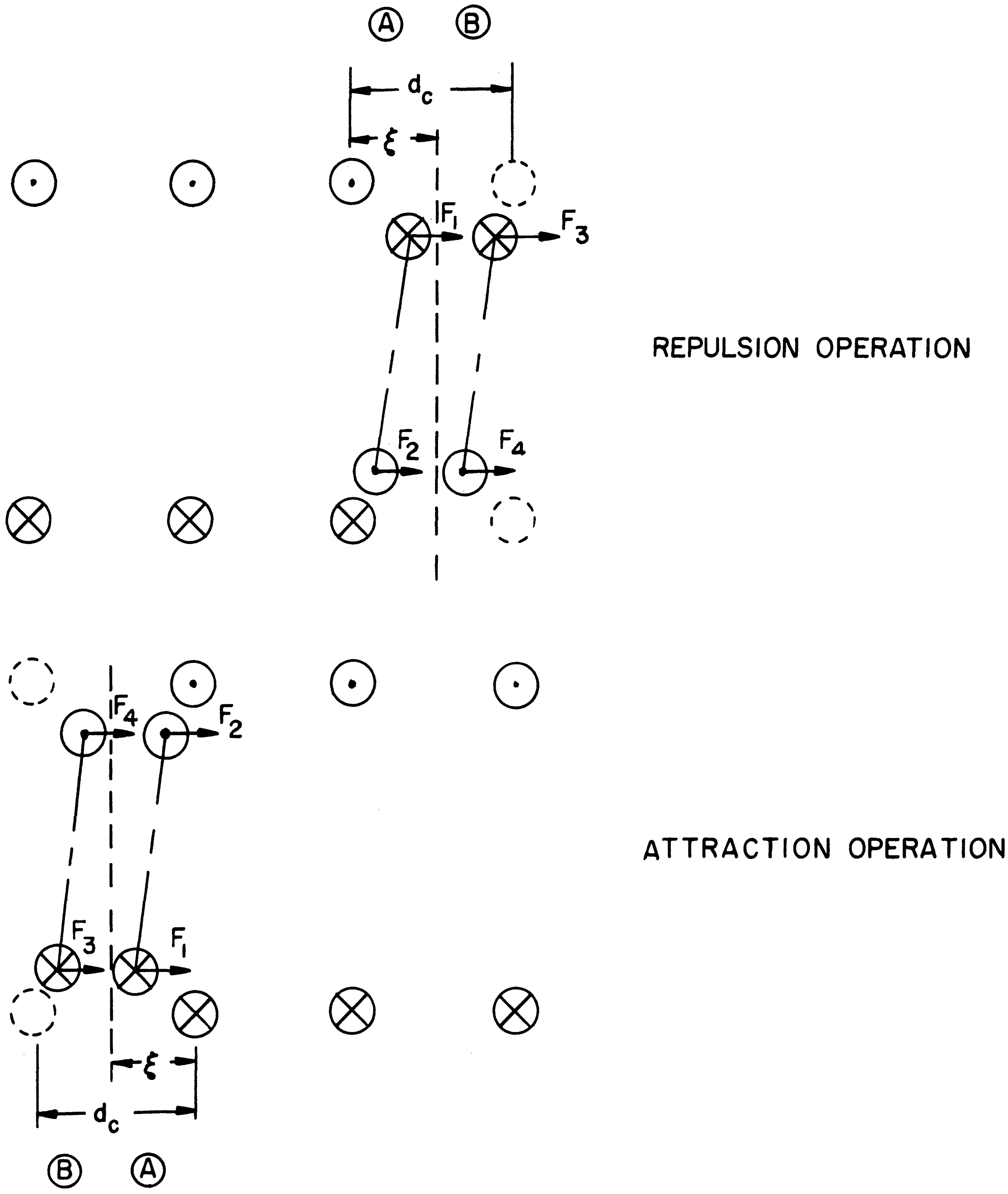

Fig. 11. Rotational stability.

Directions of ring and coil currents are shown by the usual dot cross notation.

Defining two regions A and B in Fig. 11, and referring to the forces as defined by Fig. 11, in each case $F_1 > F_2$ and $F_4 > F_3$. The result of this is that each operating mode will have both stable and unstable positions as shown in the chart below.

ROTATIONAL STABILITY

Region	Attraction	Repulsion
A	Stable	Unstable
B	Unstable	Stable

The ring current and resulting axial force are higher in region B and A , and from this one might be able to show that the rate of growth and decay of the disturbances differ in the two regions. This would then favor one system over the other.

VII. ENERGY STORAGE, CIRCUIT, AND SWITCHING REQUIREMENTS

The previous sections have derived expressions for velocity of the electromagnetic gun in terms of the gun geometry and driving current. It is also of interest to study the acceleration process in terms of energy storage and power since these will determine the form of the primary power source as well as reveal further the principles of operation of the device.

If one considers a mass m being uniformly accelerated to velocity v in a distance d, then the ultimate acceleration is given by

$$a = \frac{v_f^2}{2d} , \tag{7-1}$$

force on the mass is uniformly

$$F = ma , \tag{7-2}$$

and the transit time is

$$\tau = \frac{v_f}{a} = \frac{2d}{v_f} \tag{7-3}$$

The velocity at any instant t is given by

$$v = a\,(t - t_o) , \tag{7-4}$$

assuming initially $v_o = 0$ at time $t = t_o$, and at the same instant, the power being consumed is

$$P = Fv , \tag{7-5}$$

or

$$\begin{aligned} P' &= (ma)\,a(t - t_o) \\ &= m\,\frac{v_f^4}{4d^2}(t - t_o) . \end{aligned} \tag{7-6}$$

As an example, let 10 gm be accelerated to 3×10^5 cm/sec (~10,000 ft/sec) in a 10-meter length. Then

$$\tau = 6.7 \times 10^{-3} \text{ sec} ,$$

and

$$\begin{aligned} P(\tau) &= 136 \times 10^{12} \frac{\text{erg}}{\text{sec}} \\ &= 136 \text{ megawatts.} \end{aligned}$$

At these power levels, one is naturally led to think in terms of an intermediate electrical storage device supplying the driving power rather than considering withdrawal of this much power directly from the primary power lines.

In the case of the "attraction" type of system analyzed previously, this energy storage could conceivably be in the accelerator coil itself since it initially carries a current and therefore has energy stored in its magnetic field. This would result, of course, in current decay in the coil during the acceleration period due to back emf's induced in the coil by the moving projectile, and the constant current solution previously derived would not be strictly applicable.

Of course, in either operating mode, the accelerating current could come from a primary driver such as a series inductance or parallel capacitance. In this case, ultimate velocity for a given coil geometry and a given maximum coil current would be higher than in the self-driven situation.

The possible configurations for the switching type of electromagnetic accelerator can be categorized both by the manner in which the driving coil turns are connected and by the means used for initial electrical energy storage; the two divisions turn out not to be completely independent.

Accelerator turns may either be in series or in parallel, as shown in Fig. 12. Note also that either circuit is adaptable to both "attraction" and "repulsion" operation.

In the parallel circuit situation, assuming something approximating constant current operation is desired, the energy source must be essentially a constant voltage device able to provide up to NI_o current, where N is the total number of driving coil turns, and I_o is the current needed per turn. A capacitive circuit as shown in Fig. 12a would therefore perhaps be best.

The series arrangement, on the other hand, requires a driving supply which need handle at the most I_o current, although the supply voltage will vary with number of turns energized. This would suggest the inductive circuit of Fig. 12b. The series circuit is advantageous because in this case the accelerator coil itself can, in the "attraction" type of operation, easily be the energy source, as has been suggested earlier.

We will consider in detail the circuit of Fig. 12b. Initially, when used as an "attraction" device, all switches except Sw_N are open, and current I_o is established through the coil via Sw_N.

The projectile begins at the N^{th} turn and is given initial acceleration by closing Sw_{N-1} and then opening Sw_N. This procedure maintains a continuous path through Sw_{N-1} for the coil current as well as transferring to the projectile the energy of the N^{th} turn. As the projectile passes any turn (n), the switch Sw_{N-1} is closed and Sw_N is opened.

Although it might appear that opening the high current circuit would be a very difficult task, especially considering the speed with which this circuit must be broken, one must keep in mind that the projectile assumes most of the magnetic energy initially held by the turn, and only the leakage flux need be dissipated by the switch.

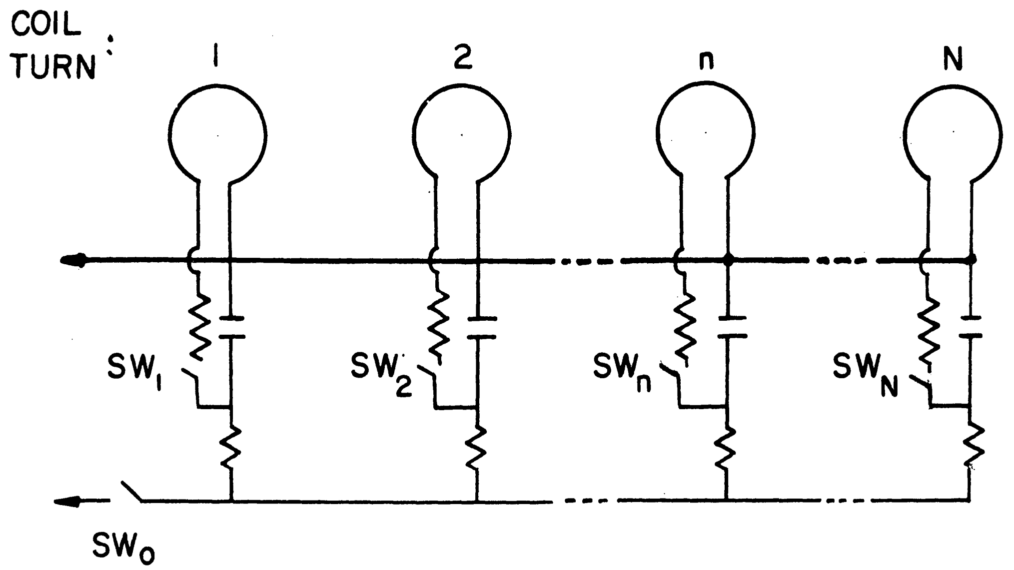

Fig. 12a. Parallel Circuit.

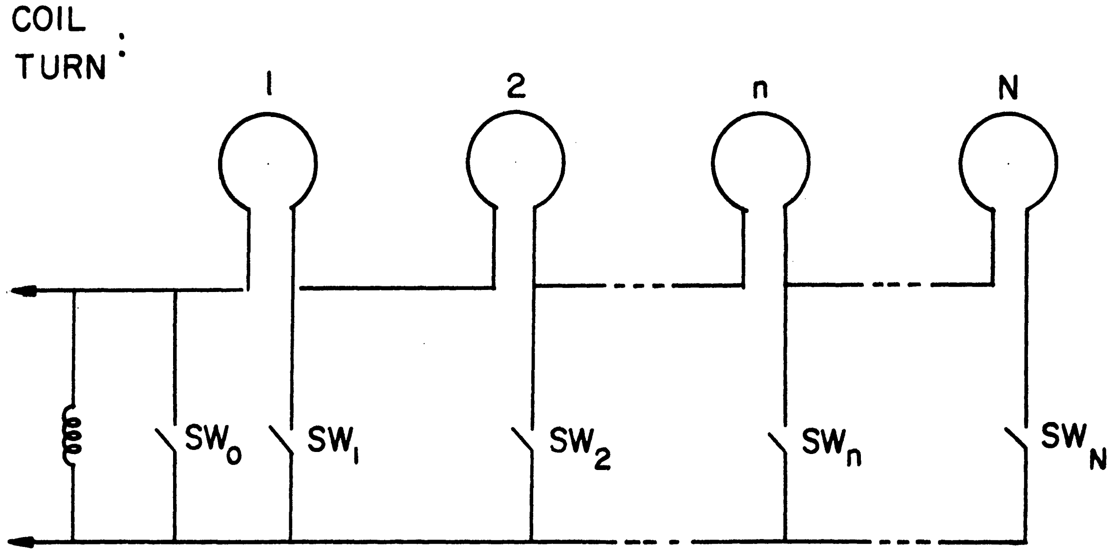

Fig. 12b. Series Circuit.

Fig. 12. Sequentially switched electromagnetic accelerator circuits.

The switch requirement then is not so much ability to handle large dissipated energy but rather ability to be timed accurately and to open very quickly. Reasonable criteria might be that the switch must be timed to open before the projectile passes beyond a turn by one tenth of the turn spacing and not to begin opening until the projectile is at least adjacent to a turn. This means that for the projectile which is traveling at 300,000 cm/sec with a turn spacing of one cm, the total time in which the switch must act, including both initiating time and opening duration randomness is only about 3 μsec.

In the repulsion situation using the circuit of Fig. 12b, the requirements are essentially the same since very little energy will be dissipated in any one switch, but timing is again critical.

The switch-timing problem becomes increasingly difficult as the ultimate desired projectile velocity is raised. At some point, however, one can begin to consider an automatically timed structure where the phase velocity of a pulse injected onto the accelerator coil would be synchronized with the projectile velocity.

In this "traveling wave" mode of operation, one might contemplate using a periodically loaded helical delay line which would simulate the action of the switched circuits described above, but would not require actually switching of currents into or out of coil turns.

Slow wave structures of this type would be feasible for velocities above perhaps 50,000 ft/sec, so an injection device giving the projectile this much initial velocity would be required.

A three-stage system might actually be considered, where a chemically driven gun would be used for initial acceleration. This would be followed by a section of the switch-operated electromagnetic accelerator leading into the self-time or "traveling wave" type of system.

VIII. REVIEW

The foregoing analysis has shown that acceleration of small masses to 10,000 meters per second is at least theoretically possible. The primary practical difficulty is the switching problem; consideration of the switch and timing circuit designs would perhaps be the next logical step to be taken in a development program of the sequentially switched accelerator.

Of the two operating modes considered, the "repulsion" device, having significantly higher projectile current than the "attraction" device, is more severely limited, both mechanically and thermally. Repulsion operation is also handicapped by "lateral" instability.

These considerations, plus the fact that the attraction accelerator may serve for its own energy storage, lead to the conclusion that attraction operation is the more applicable of the two operating modes. It will, however, require breaking rather than making a high-current circuit, and guides will be required to control the "rotational" instability.

For numerical values, copper has been the only projectile material considered in this report. Other metals such as aluminum, titanium, or beryllium might prove to have more advantageous combinations of electrical conductivity, specific heat, and mechanical properties.

APPENDIX I

MAGNETIC PRESSURE—A GENERALIZED APPROACH TO ELECTROMAGNETIC ACCELERATION

Electromagnetic acceleration of an electrically conducting mass may be analyzed on the basis of the Lorentz force given by

$$\vec{f} = \vec{j} \times \vec{B} , \tag{I-1}$$

where $\vec{j}$ is the current density field, $\vec{B}$ is an externally applied magnetic field, and $\vec{f}$ is the force per unit volume in the field.

The thrust on the projectile may also be thought to arise from a gradient of magnetic field pressure across the projectile, and either viewpoint may easily be derived from the other, as will now be shown. Consider a conducting sheet (Fig. 13) which is of finite thickness Δx in the x direction but infinite in the other two dimensions, and which carries uniform current density $\vec{j} \equiv \vec{a}_y j_y$. Let the whole system be immersed in a uniform external magnetic field $B_o \equiv \vec{a}_z B_{oz}$. Here, unit direction vectors are indicated by $\vec{a}_j$.

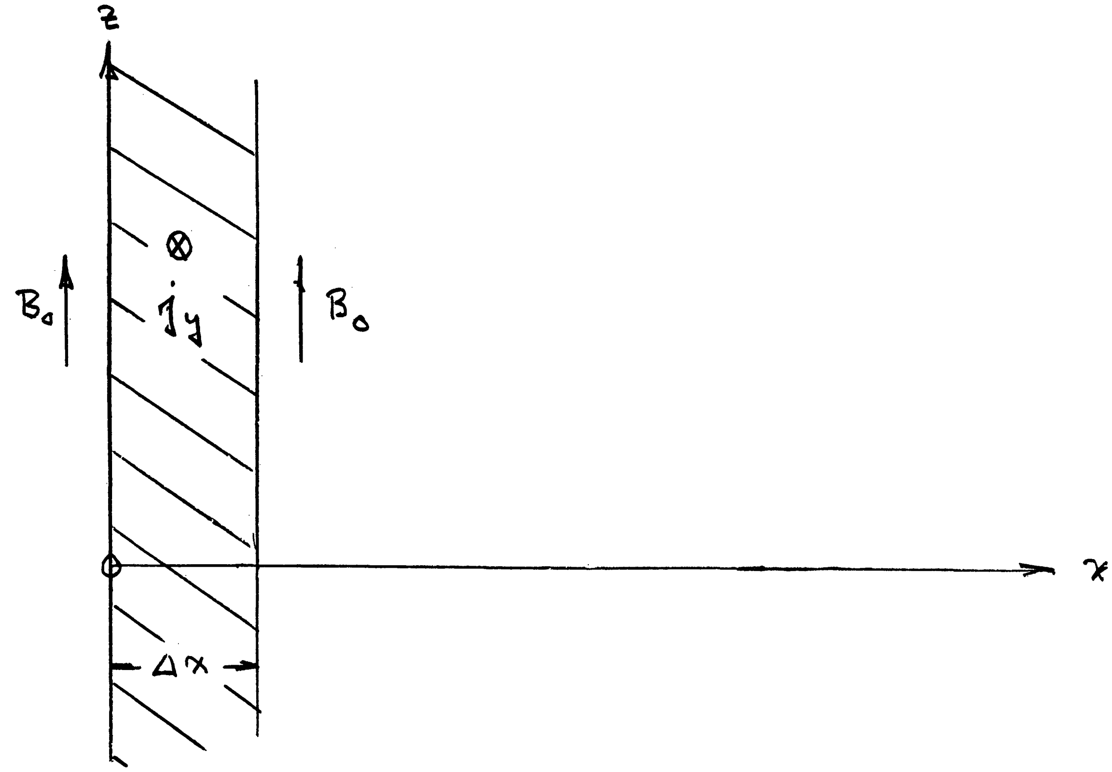

Fig. 13. Geometry for magnetic pressure derivation.

The force on a small area $\Delta y\ \Delta z$ of the side of this conducting sheet is, from the Lorentz expression:

$$F = (j_y \Delta x\ \Delta y)\ B_0 \Delta z\ , \tag{I-2}$$

so the pressure becomes:

$$p = j_y B_0 \Delta x\ . \tag{I-3}$$

In the absense of displacement current, the current density and magnetic intensity vector must be related by

$$\nabla \times \vec{H} = \vec{j}\ , \tag{I-4}$$

or in this case simply

$$j_y = \frac{\partial H_z}{\partial x}\ . \tag{I-4'}$$

Since j_y is constant and independent of x, the pressure then may be written:

$$\begin{aligned} p &= B_0 \frac{\partial H_z}{\partial x} \Delta x \\ &= B_0 \Delta H \qquad \text{(I-5)} \\ &= \frac{1}{\mu} B_0 \Delta B . \qquad \text{(I-5')} \end{aligned}$$

Now since the change in magnetic field is constant through Δx, and taking the magnetic field due just to the current j as $\pm\beta$ at the surfaces $x = 0,\ \Delta x$, the pressure becomes:

$$\begin{aligned} p &= \frac{2}{\mu} B_0 \beta \qquad \text{(I-6)} \\ &= \frac{1}{2\mu} \left(4 B_0 \beta + B_0^2 - B_0^2 + \beta^2 - \beta^2\right) \\ &= \frac{1}{2\mu} \left[(B_0 + \beta)^2 - (B_0 - \beta)^2\right] \\ &= \Delta \left(\frac{B_T^2}{2\mu}\right) , \qquad \text{(I-6')} \end{aligned}$$

where $B_T \equiv$ total magnetic field. Since $B_T^2/2\mu$ is just the density of energy stored in the magnetic field, the mechanical pressure now is shown to arise from a gradient of stored magnetic energy.

If j_y is a function of x, then the force expression must be written

$$\frac{\vec{F}}{v} = \vec{j}_y \times \vec{B} \tag{I-7}$$

where $v \equiv$ volume

$$\vec{F} = B_0 \Delta y \Delta z \int_0^{\Delta x} j_y\, dx , \tag{I-8}$$

and the pressure is:

$$P = B_0 \int_0^{\Delta x} J_y \, dx$$

$$= B_0 \int_0^{\Delta x} \frac{\partial H_z}{\partial x} \, dx \, , \qquad \text{(I-9)}$$

leading again to

$$P = B_0 \Delta H$$

$$= \Delta \left(\frac{B_T^2}{2\mu} \right) . \qquad \text{(I-6')}$$

All electromagnetic acceleration schemes may therefore be considered from the standpoint of thrust arising from gradients of $B_T^2/2\mu$. Two distinct cases are of interest:

1. Current with no external magnetic field.

2. Current in a uniform external magnetic field. (See discussion on page .)

Inasmuch as heating is one of the primary limiting factors, the ratio of kinetic to thermal energy delivered to the accelerated projectile will be derived in each of the above cases and will be taken as a measure of the efficiency of the acceleration process.

ACCELERATION PRODUCED BY A CURRENT WITH NO EXTERNAL MAGNETIC FIELD

There are two general ways in which a current can be produced in a conductor. Either an electric field may be applied directly across the conductor, or the electric field may be induced by a changing magnetic field.

1. Directly Applied Field

In the instance of a conductor subjected to an applied electric field in the absence of any external magnetic field, if the field is distributed symmetrically with respect to the conductor, as in the case of a voltage applied to a long straight wire, obviously there is no net lateral force experienced by the conductor regardless of its cross-sectional slope.

If one thinks in terms of a curved conductor, however, a net force does result. Consider a section of a cylindrical conductor having length L, inside radius R, and thickness ΔR, as in Fig. 14. Now if an electrical field is established across this semi-cylinder, as by the battery and rail arrangement shown in Fig. 14, causing circumferential current of density j_θ to flow, then since the path length and therefore resistance increases linearly with r, j_θ will decrease directly with r. This leads to a nonlinear variation of H with r:

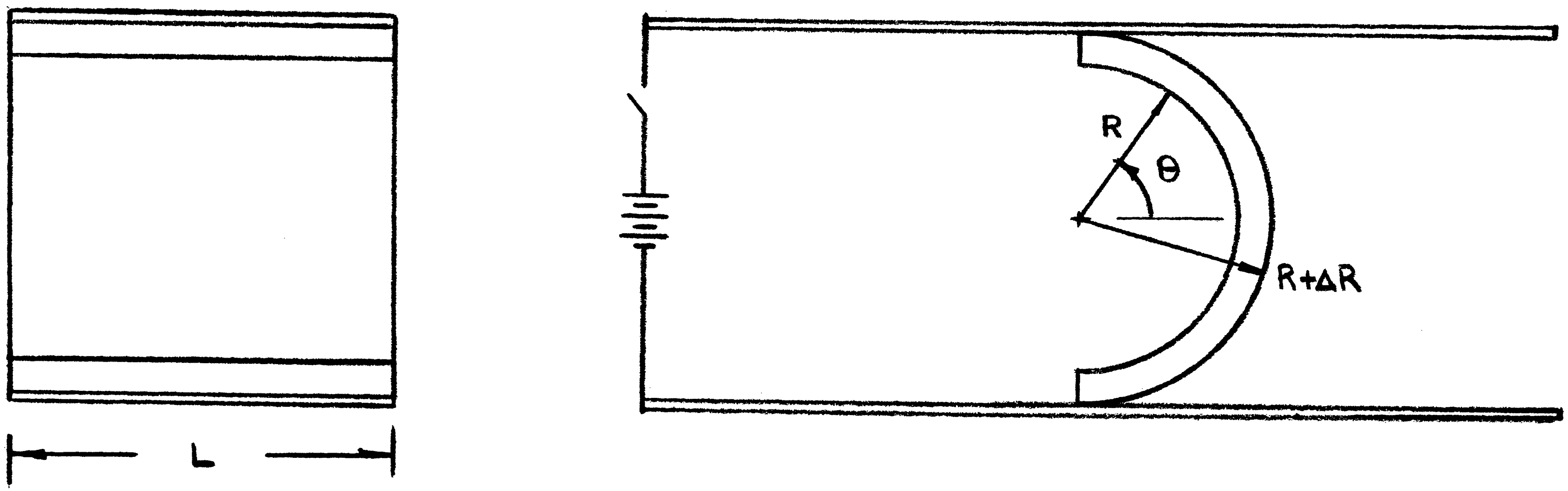

Fig. 14. Rail-type accelerator.

$$H_{R+\Delta R} = H_R - \int_R^{R+\Delta R} j_\theta \, dr \tag{I-10}$$

$$= H_R - \int_R^{R+\Delta R} \left[j_{\theta_R} - \frac{\partial j_\theta}{\partial r}(r-R) \right] dr$$

$$= H_R - \left[j_{\theta_R} r - \frac{1}{2} \frac{dj_\theta}{dr} r^2 \right]_R^{R+\Delta R}$$

$$= H_R - \delta H \tag{I-10'}$$

where

$$\delta H \triangleq j_{\theta_R} \Delta R + R \Delta j_\theta \tag{I-11}$$

$$j_{\theta_R} \triangleq j_\theta \quad \text{at} \quad r = R,$$

$$\Delta j_\theta \triangleq j_{\theta_R} - j_{\theta_{R+\Delta R}} . \tag{I-12}$$

The net outward pressure on the conducting surface at r is then:

$$P = \Delta\left(\frac{B^2}{2\mu}\right) \tag{I-6'}$$

$$= \frac{1}{2\mu}\left(B_R^2 - B_{R+\Delta R}^2\right)$$

$$= \frac{1}{2\mu}\left[2 B_R \delta B - (\delta B)^2\right]$$

$$\text{where } B_{R+\Delta R} = B_R + \delta B$$

$$= B_R \delta H - \frac{1}{2}\mu \delta H^2$$

$$= B_R\left(j_{\theta_R}\Delta R + R\Delta j_\theta\right) - \frac{1}{2}\mu\left(j_{\theta_R}\Delta R + R\Delta j_\theta\right)^2. \tag{I-13}$$

The usual rail-type electromagnetic gun can be analyzed along the lines of reasoning applied above as will be shown with reference to the somewhat idealized geometry of Fig. 14. The axial force on the incremental area LRdθ is

$$dF_z = PA$$

$$= PLR\cos\theta\, d\theta \qquad \text{(I-14)}$$

so the total axial force on the projectile is

$$F_z = 2\int_0^{\pi/2} dF$$

$$= 2PLR\sin\theta \Big|_0^{\pi/2} \qquad \text{(I-15)}$$

$$\doteq 2B_R\,\delta HLR \quad (\text{when } \delta B \ll B_R).$$

Kinetic energy at time t, being the integral of power over this time, is:

$$\mathcal{E}_K = \int_0^t F v\, dt$$

$$= \int_0^t F\left(\frac{F}{m}\right) t\, dt \qquad \text{(I-16)}$$

$$= \left(2B_R\,\delta HLR\right)^2 \frac{t^2}{2m} .$$

Energy per unit volume which goes into ohmic heating over this time is:

$$\mathcal{E}_{h_r} = \int \frac{j_\theta^2}{\sigma}\, dt = \frac{1}{\sigma}\int_0^t \left[j_{\theta R} - \frac{\partial j_\theta}{\partial r}(r-R)\right]^2 dt$$

$$= \frac{1}{\sigma}\left[j_{\theta R}^2 - j_{\theta R}\frac{\partial j_\theta}{\partial r}(r-R) + \left(\frac{\partial j_\theta}{\partial r}\right)^2 (r-R)^2\right] t \qquad \text{(I-17)}$$

The greatest heating will take place at r = R where

$$\mathcal{E}_{hR} = \frac{1}{\sigma} j_{\theta R}^2\, t . \qquad \text{(I-18)}$$

Combining Eqs. (I-16) and (I-18) shows the efficiency of the device to increase linearly with time.

2. Current Induced by a Changing Magnetic Field

This case will be studied with reference to the geometry of Fig. 15, consisting of two infinite parallel plane conductors of δx and Δx thickness. Let a constant current I_0 per unit z length be introduced in the y direction into the δx conductor at time t = 0. This will establish a uniform magnetic field

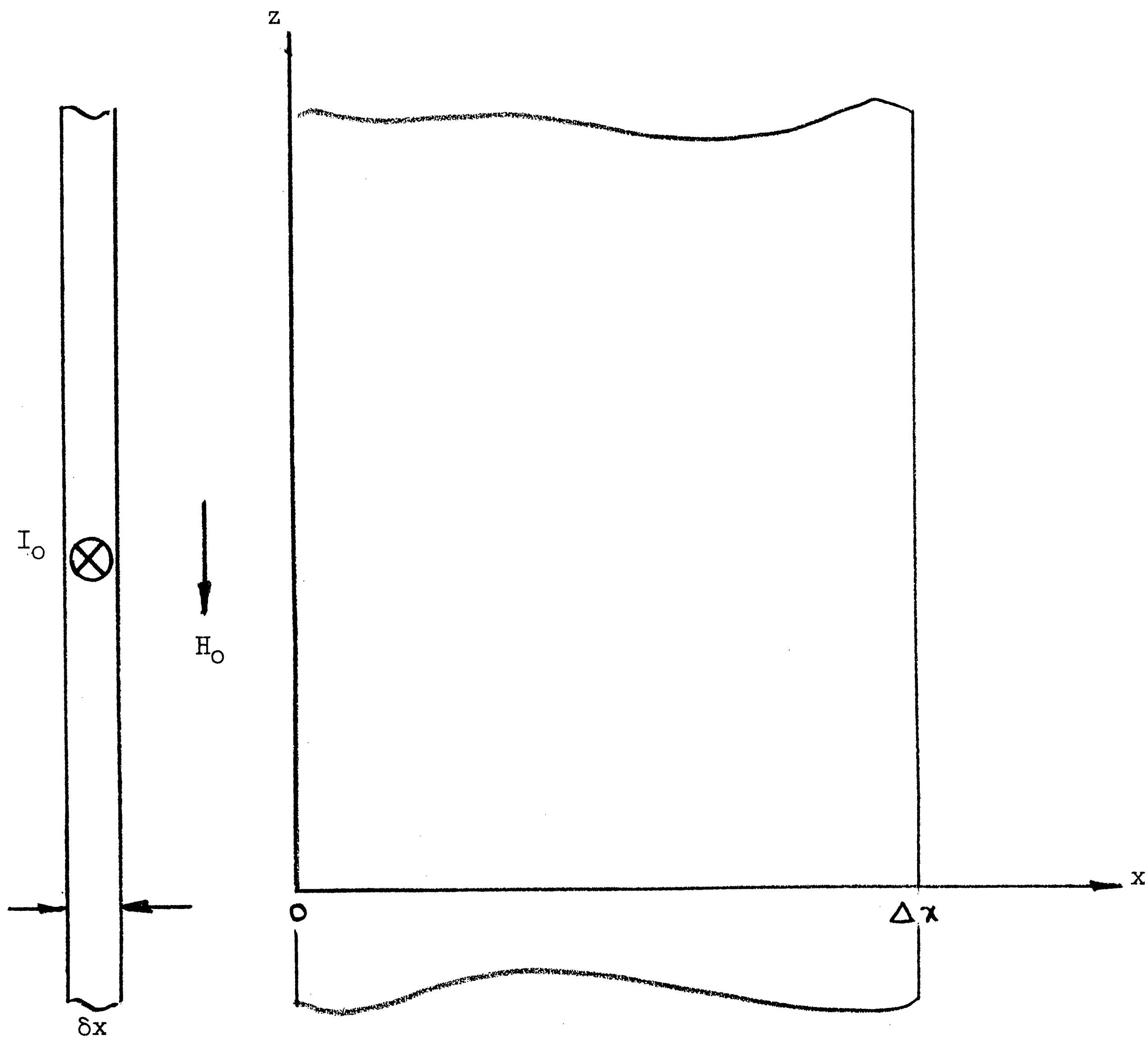

Fig. 15. Geometry for transient magnetic field derivation.

$B = \mu H_0$ in the space between planes and especially at the $x = 0$ surface of the Δx conductor. If this conductor has conductivity σ, the magnetic field will penetrate it governed by the following differential equation:

$$\frac{\partial H}{\partial t} = \frac{1}{\sigma\mu}\frac{\partial^2 H}{\partial x^2} . \tag{I-19}$$

For the transient type occurrence considered here, the solution to this equation is:[22]

$$H_z(x,t) = H_0\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{t/\sigma\mu}}} \exp(-\xi)^2 d\xi\right]$$

$$= H_0\left[1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t/\sigma\mu}}\right)\right] . \tag{I-20}$$

From this, the pressure on the Δx conductor is

$$P = \Delta\left(\frac{B}{2\mu}\right)^2 = \frac{1}{2}\mu\left(H_0^2 - H_{\Delta x}^2\right)$$

$$= \frac{1}{2}\mu H_0^2 \operatorname{erf}\left(\frac{\Delta x}{2\sqrt{t/\sigma\mu}}\right)\left[2 - \operatorname{erf}\left(\frac{\Delta x}{2\sqrt{t/\sigma\mu}}\right)\right] , \tag{I-21}$$

and if this plane is free to move, the kinetic energy per unit volume at time t becomes:

$$\mathcal{E}_K'(t) = \frac{1}{\Delta x}\int_0^t P v\, dt$$

$$= \frac{1}{\Delta x}\int_0^t \frac{P^2}{m} t\, dt \tag{I-22}$$

$$= \frac{1}{\rho(\Delta x)^2}\int_0^t \frac{\mu^2}{4} H_0^4 \left[\operatorname{erf}\left(\frac{\Delta x}{2\sqrt{\frac{t}{\sigma\mu}}}\right)\right]^2\left[2 - \operatorname{erf}\left(\frac{\Delta x}{2\sqrt{\frac{t}{\sigma\mu}}}\right)\right]^2 t\, dt .$$

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22. Churchill, R. V., Fourier Analysis and Boundary Value Problems, McGraw-Hill Book Co., New York, 1941, p. 123, problem 2; or see A.I.P. Handbook, pp. 5-93.

The current density within the moving plane is again given by

$$j_y = -\frac{\partial H_z}{\partial x}$$

$$j_y(x,t) = -\frac{\partial}{\partial x}\left\{H_o\left[1 - \mathrm{erf}\left(\frac{x}{2\sqrt{t/\sigma\mu}}\right)\right]\right\} \tag{I-23}$$

$$= H_o \frac{\partial}{\partial x}\,\mathrm{erf}\left(\frac{x}{2\sqrt{t/\sigma\mu}}\right).$$

The heat energy generated per unit volume in time t is given by the integral of j_y^2/σ over time t, so this becomes a function of x:

$$\mathcal{E}_h'(x,t) = \frac{H_o^2}{\sigma}\int_0^t \left[\frac{\partial}{\partial x}\,\mathrm{erf}\left(\frac{x}{2\sqrt{t/\sigma\mu}}\right)\right]^2 dt \tag{I-24}$$

The kinetic to thermal ratio in this case is then:

$$\frac{\mathcal{E}_K}{\mathcal{E}_h}(x,t) = \frac{\mu^2\sigma H_o^2 \int_0^t \left[\mathrm{erf}\left(\frac{\Delta x}{2\sqrt{\frac{t}{\sigma\mu}}}\right)\right]^2\left[2 - \mathrm{erf}\left(\frac{\Delta x}{2\sqrt{\frac{t}{\sigma\mu}}}\right)\right]^2 t\,dt}{4\rho(\Delta x)^2 \int_0^t \left[\frac{\partial}{\partial x}\,\mathrm{erf}\left(\frac{x}{2\sqrt{\frac{t}{\sigma\mu}}}\right)\right]^2 dt} \tag{I-25}$$

ACCELERATION ARISING FROM CURRENT IN THE PRESENCE OF A UNIFORM EXTERNAL MAGNETIC FIELD

As in the previous section, both induced currents and currents driven by a directly applied electric field may be produced. The former condition under proper circumstances will result in a steady or "d-c" force, while the latter again will be a transient phenomenon.

1. Directly Applied Field

A current flowing in a magnetic field will always have a force acting on it. Either curved or straight currents will result in net force in both transient and steady-state operation. For illustration we will analyze the case shown in Fig. 14, where an infinite plane of thickness Δx carrying uniform current density j_y is immersed in a uniform directed magnitude field B_o and is free to move.

The kinetic energy per unit volume at time t is

$$\mathcal{E}_k'(t) = \frac{1}{\Delta x}\int_0^t P v\, dt$$

$$= \frac{P^2}{\rho(\Delta x)^2}\int_0^t t\, dt \qquad \text{(I-26)}$$

$$= \frac{2 B_0^2 \beta^2}{\rho \mu^2 (\Delta x)^2} t^2 ,$$

using the notation of Eq. (I-6).

In the same time heat energy per unit volume generated by ohmic losses is given by:

$$\mathcal{E}_h'(t) = \int_0^t \frac{J_z^2}{\sigma} dt$$

$$= \frac{1}{\sigma}\int_0^t \left(\frac{\partial H_y}{\partial x}\right)^2 dt$$

$$= \frac{1}{\sigma}\left(\frac{\partial H_y}{\partial x}\right)^2 t \qquad \text{(I-27)}$$

$$= \frac{1}{\sigma}\left(\frac{2\beta}{\mu \Delta x}\right)^2 t$$

$$= \frac{4\beta^2}{\mu^2 \sigma (\Delta x)^2} t .$$

Finally:

$$\frac{\mathcal{E}_k}{\mathcal{E}_h}(t) = \frac{\sigma B_0^2}{2\rho} t . \qquad \text{(I-28)}$$

2. Current Induced by a Changing Magnetic Field

If a uniform magnetic field $\vec{B} = \vec{a}_z \vec{B}_1$ is included in Fig. 15, then the resulting configuration is applicable to the case considered in the following paragraphs.

Establishing a constant current I_0 per unit z length in the δx plane at $t = t_0$, the magnetic intensity within the Δx plane is given [from Eq. (I-20)] by:

$$H_z(x,t) = H_1 + H_0\left[1-\mathrm{erf}\left(\frac{x}{2\sqrt{t/\sigma\mu}}\right)\right] \qquad \text{(I-29)}$$

Net pressure across Δx therefore is:

$$P = \Delta\left(\frac{B_T^2}{2\mu}\right) = \frac{\mu}{2}\left[B_T^2(0,t) - B_T^2(\Delta x,t)\right]$$

$$= \frac{\mu}{2}\left\{(H_1+H_0)^2 - \left[H_1 + H_0\left(1-\mathrm{erf}\,\frac{\Delta x}{2\sqrt{t/\sigma\mu}}\right)\right]^2\right\}$$

$$= \frac{\mu}{2}\left\{2H_1H_0\,\mathrm{erf}\left(\frac{\Delta x}{2\sqrt{t/\sigma\mu}}\right) + H_0^2\,\mathrm{erf}\left(\frac{\Delta x}{2\sqrt{t/\sigma\mu}}\right)\left[2 - \mathrm{erf}\left(\frac{\Delta x}{2\sqrt{t/\sigma\mu}}\right)\right]\right\} \qquad \text{(I-30)}$$

$$= \frac{1}{2}\mu H_0^2\,\mathrm{erf}\left(\frac{\Delta x}{2\sqrt{t/\sigma\mu}}\right)\left[2\left(\frac{H_1}{H_0}+1\right) - \mathrm{erf}\left(\frac{\Delta x}{2\sqrt{t/\sigma\mu}}\right)\right]$$

This leads to the kinetic energy expression:

$$\mathcal{E}_k'(t) = \frac{1}{\rho(\Delta x)^2}\int_0^t P^2 t\,dt$$

$$= \frac{\mu^2 H_0^4}{4\rho(\Delta x)^2}\int_0^t\left[\mathrm{erf}\left(\frac{\Delta x}{2\sqrt{t/\sigma\mu}}\right)\right]^2\left[2\left(\frac{H_1}{H_0}+1\right) - \mathrm{erf}\left(\frac{\Delta x}{2\sqrt{t/\sigma\mu}}\right)\right]^2 t\,dt \qquad \text{(I-31)}$$

Since induced current density is a function only of $\partial H_z/\partial x$, the current and heat-energy equations will be identical to those derived in the induced current—no external magnetic field situation:

$$\mathcal{E}_H'(x,t) = \frac{H_0^2}{\sigma}\int_0^t\left[\frac{\partial}{\partial x}\mathrm{erf}\left(\frac{x}{2\sqrt{t/\sigma\mu}}\right)\right]^2 dt. \qquad \text{(I-24)}$$

The energy ratio now is given by

$$\frac{\mathcal{E}_k}{\mathcal{E}_h}(x,t) = \frac{\mu^2 \sigma H_0^2 \int_0^t \left[\operatorname{erf}\left(\frac{\Delta x}{2\sqrt{t/\sigma\mu}}\right)\right]^2 \left[2\left(\frac{H_1}{H_0}+1\right) - \operatorname{erf}\left(\frac{\Delta x}{2\sqrt{t/\sigma\mu}}\right)\right]^2 t\,dt}{4\rho(\Delta x)^2 \int_0^t \left[\frac{\partial}{\partial x}\operatorname{erf}\left(\frac{x}{2\sqrt{t/\sigma\mu}}\right)\right]^2 dt} \qquad \text{(I-32)}$$

Comparing this with Eq. (I-25) shows that the external field H_1 increases the efficiency of the accelerating process.

In the case of the switching-type induction accelerator analyzed in detail in this report, the action is similar to but not identical with the above situation. Somewhat simplified, the applicable geometry is shown in Fig. 16. Here, the solenoid and projectile turns are represented in cross section as large radius coaxial cylinders having ΔR and δR wall thickness, respectively, and extending over the axial length Δz. The radial component of the fringing field due to the excited solenoid turns is idealized as a uniform radial field, B_r. Introducing current per unit z length I_0' into the δR shell, a magnetic field H_{z_0} will be produced at the $R + \Delta R$ surface by I_0', and this will "diffuse" in the (-r) direction into the projectile inducing tangential currents in the projectile. But due to radial symmetry the only net force on the Δr cylinder is $\dot{z}$ directed, that is, in a direction normal to and therefore independent of the diffusing field H_z.

One must therefore follow a "quasi-steady-state" analysis since at any instant the current I_R in the projectile is uniform with z, and therefore the magnetic fields β due to I_R fore and aft of the projectile at an instant are equal in magnitude but oppositely directed.

This leads to a pressure expression identical with Eq. (I-13), but where in deriving the kinetic energy equation analogous to Eq. (I-31), one must remember that β is now time-dependent, as determined by the process of H_z diffusion radially through ΔR.

The important thing to note, however, is that, in the sequentially switched device, the projectile is permeated by a radial magnetic field which is essentially uniform in time and space and carries on top of this the transient field due to the decaying projectile current. One gains therefore the advantage of the bias field.

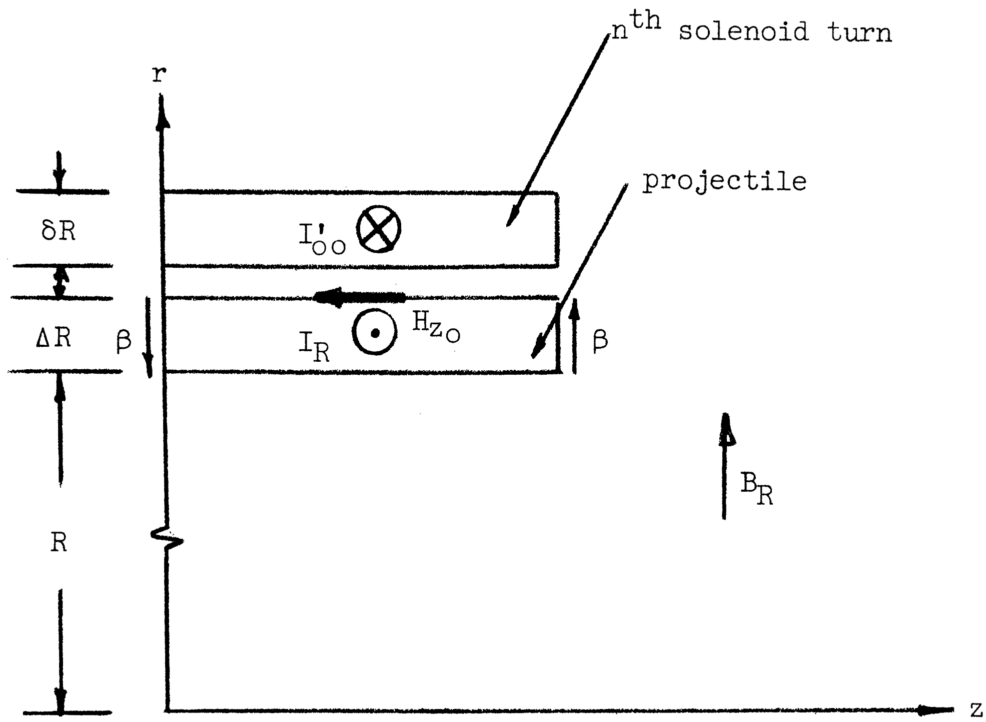

Fig. 16. Idealized sequentially switched accelerator field and conductor geometry.

B_R is the radial component of the magnetic field produced by all excited solenoid turns except the n^{th}; B_R is assumed uniform throughout the projectile cross section.

APPENDIX II

IMPACT HEATING

When a projectile having high kinetic energy suffers an inelastic collision with another mass, a portion of its energy is randomized into heat, and the possibility of attaining high temperature by a collision process therefore exists. Because of the present interest in high temperature systems and because of the obvious applicability of the electromagnetic accelerator in such a process, a brief analysis of this process will be presented here.

A pellet of mass m (kilogram) traveling at velocity v (meters/second) will have kinetic energy

$$E_K = 1/2\ mv^2 \text{ (joules)} \quad . \tag{II-1}$$

Assuming all this energy is converted to heat upon collision, and assuming this heat remains within the original mass, the temperature obtained is calculable from the following energy equality:

$$1/2\ mv^2 = cmJ\ (T_f - T_i) \tag{II-2}$$

where

T_f ≡ final temperature (°K),
T_i ≡ initial temperature (°K),
c ≡ specific heat (cal/kg-°K),
J ≡ mechanical equivalent of heat (4.2 joules/cal).

One must keep in mind when applying this equation that in practice it would be impossible to realize complete conversion of energy from kinetic to thermal, but results will at least be accurate to the right order of magnitude.

The specific heat used in Eq. (II-2) will vary with temperature and must include such things as phase changes and, in the higher temperature ranges, excitation and ionization. In higher atomic number materials having many orbital electrons, the latter two phenomena consume a large amount of energy and make attainment of high temperature very difficult; due to this, one is led to consideration of the lowest atomic number material constant with other requirements. In the case of the electromagnetic accelerator pellet where an electrical conducting solid is required, perhaps lithium metal is the most applicable. Physical constants for lithium are given below.

Thermodynamic Constants[23]

specific heat (solid) c_{ps} = 720 cal/kg-°C
specific heat (liquid) c_{pl} = 720 cal/kg-°C
specific heat (ideal monotomic gas) c_{vg} = 3/2 R cal/gm-atom°C = 430 cal/kg°C
latent heat of fusion $L_f = 10^5$ cal/kg
latent heat of vaporization $L_v = 4.64 \times 10^6$ cal/kg
melting point T_m = 180°C
boiling point T_b = 1331°C

Ionization Potentials[24]

VI = 5.36 ev/atom
VII = 75.26
VIII = 121.8
ΣV = 202.42 ev/atom

The velocity required to melt the projectile, assuming it is initially at 0°C, is given by:

$$1/2\ mv^2 = Jc_{ps}m\,T_m + JL_f m$$
$$v = \sqrt{2J(c_{ps}T_m + L_f)}$$
$$= 1390 \text{ meters/sec}$$

To vaporize lithium, again starting from 0°C, the required velocity is

$$\frac{1}{2}mv^2 = Jc_{ps}m\,T_m + JL_f m + Jc_{pl}m(T_b - T_m) + JL_v m$$
$$v = \sqrt{2J[c_{ps}T_m + L_f + c_{pl}(T_b - T_m) + L_v]}$$
$$= 6900 \text{ meters/sec.}$$

Up to about 10,000°C there will be little ionization or excitation, so this temperature is obtained by the following velocity:

$$v = \sqrt{2J[c_{ps}T_m + L_f + c_{pl}(T_b - T_m) + L_v + c_{vg}(10^4 - T_b)]}$$
$$= 8900 \text{ meters/sec.}$$

By the time 100,000°C is reached, approximately complete triple ionization of the lithium vapor has been achieved. To achieve complete ionization, each

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23. Stull, P. R., and Sinke, G. C., Thermodynamic Properties of the Elements, American Chemical Society, Washington, D. C., 1956.
24. Handbook of Chemistry and Physics, p. 2177.

atom must receive energy equal to the sum of the three ionization potentials, or per kilogram:

$$\mathcal{E}_{ion} = 202.4 \frac{ev}{atom} \times 6 \times 10^{23} \frac{atoms}{gm\text{-}atom} \times \frac{gm\text{-}atom}{6.94\ gm} \times 1.59 \times 10^{-19} \frac{joule}{ev} \times 10^{3} \frac{gm}{kg}$$

$$= 2.78 \times 10^{9}\ joules/kg.$$

In addition, assuming thermal equilibrium among electrons and ions, both the ion and electron gases must be raised to the high temperature. Here for simplicity we will assume up to 10,000°C, no ionization exists, and above this point, all atoms are completely stripped. Under these assumptions, the velocity equivalent to 100,000°C is given by:

$$v = \left\{ 2\mathcal{E}_{ion} + 2J \left[c_{ps} T_m + L_f + c_{pl} (T_b - T_m) + L_v + c_{vg} (10^4 - T_b) + 4 c_{vg} (10^5 - 10^4) \right] \right\}^{\frac{1}{2}}$$

$$= 83,500\ meters/sec.$$

Finally, the velocities necessary to reach one million and ten million degrees are:

$$v_{10^6} = 1.41 \times 10^5 \text{ meters/sec}$$

$$v_{10^7} = 3.9 \times 10^5 \text{ meters/sec.}$$

The above results are summarized in the following table.

TABLE XI
VELOCITIES NECESSARY TO ATTAIN REPRESENTATIVE TEMPERATURES

Temperature, °C	Velocity	Condition
180	1,390 meters/sec	melting
1,330	6,900	vaporization
10,000	8,900	no ionization
100,000	83,500	complete ionization
1,000,000	141,000	
10,000,000	390,000	