T H E U N I V E R S I T Y O F M I C H I G A N
COLLEGE OF ENGINEERING
Department of Engineering Mechanics

Technical Report

THE STABILITY OF PARALLEL FLOWS OF FLUIDS WITH MEMORIES

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ORA Project 06505

under contract with:

DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH
CONTRACT NO. Nonr-1224(49), NR NO. 062-342
CHICAGO, ILLINOIS

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

September 1967

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The Stability of Parallel Flows
of Fluids with Memories

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Abstract

The equations governing the stability of plane parallel flows are developed for three models of fluids with memories. Asymptotic solutions valid for large Reynolds numbers are obtained and the effect of the memory are shown to be destabilizing. The approach to the problem allows evaluation of how fast a memory must fade to allow evaluation of the stresses in power series in the time interval. An alternate approach to inverting convected derivatives is also presented.

I. Introduction

A recent paper by Chan Man Fong and Walters (1965) considered the stability of parallel flows of two visco-elastic fluids with very short memories. The present work extends their analysis to such fluids with long but still fading memory and also extends the analysis to a newer model which has been proposed by Goddard and Miller (1966). A discussion of the various convected derivatives is also presented in a manner which allows more ready physical interpretation as well as a quicker way of obtaining forms for convected integrals.

Quasi-linear models of visco-elastic fluids are usually written in the form

$$\underline{\underline{p}} + \sum_{n=1}^{N} \lambda_n L_t^{(n)} \underline{\underline{p}} = 2\mu_0 \left(\underline{\underline{d}} + \sum_{m=1}^{M} \tau_m L_t^{(m)} \underline{\underline{d}} \right) \quad (1)$$

where $\underline{\underline{p}}$ is related to the total stress $\underline{\underline{t}}$ by

$$\underline{\underline{t}} = -p\underline{\underline{I}} + \underline{\underline{p}}$$

$\underline{\underline{d}}$ is the rate of deformation tensor, and L_t is a time derivative operator satisfying the principal of material indifference. The constants λ_n and τ_m are related to the stress relaxation times and rate of deformation relaxation times, respectively. Several forms for these operators have been proposed in the past (See Oldroyd (1968) for a review); we present here briefly three of these definitions in a somewhat different manner which facilitates their physical interpretation.

If θ is a convected material reference frame, and $\underline{\gamma}^{\alpha}$ is a set of covariant base vectors defined by $\frac{\partial \underline{r}}{\partial \theta^{\alpha}}$ ($\underline{r}$ a position vector) so that they are tangent to the frame (see for example Sokolnikoff (1951), Chapter 3), then a second order tensor $\underline{\underline{T}}$ can be written as

$$\underline{\underline{T}} = T^{\alpha\beta}\, \underline{\gamma}_{\alpha}\, \underline{\gamma}_{\beta}$$

or

$$\underline{\underline{T}} = T_{\alpha\beta}\, \underline{\gamma}^{\alpha}\, \underline{\gamma}^{\beta},$$

$\underline{\gamma}^{\alpha}$ being the contravariant base vectors defined by

$$\underline{\gamma}^{\alpha} = e^{\delta\beta\alpha}\, \underline{\gamma}_{\delta} \times \underline{\gamma}_{\beta}\,.$$

The absence of a dot or cross between two vectors indicates the indefinite, or dyadic, product. Oldroyd (1950) proposed two separate definitions for L_t; denoting time differentiation with material coordinates held fixed by D/Dt, they are

$$L_{tA}\, \underline{\underline{T}} \equiv \frac{DT_{\alpha\beta}}{Dt}\, \underline{\gamma}^{\alpha}\, \underline{\gamma}^{\beta} \equiv \frac{d_A T_{ij}}{dt}\, \underline{g}^{i}\, \underline{g}^{j}$$

for the model he called type A, and

$$L_{tB}\, \underline{\underline{T}} \equiv \frac{DT^{\alpha\beta}}{Dt}\, \underline{\gamma}_{\alpha}\, \underline{\gamma}_{\beta} \equiv \frac{d_B T^{ij}}{dt}\, \underline{g}_{i}\, \underline{g}_{j}$$

for the model he called type B. Latin indices and base vectors here refer to a space fixed reference frame. It is readily found by the normal tensor transformation laws that

$$\frac{d_A T_{ij}}{dt} = \frac{DT_{ij}}{Dt} + (d^m_{\cdot i} + \omega^m_{\cdot i}) T_{mj} + (d^n_{\cdot j} + \omega^n_{\cdot j}) T_{in},$$

$$\frac{d_B T^{ij}}{dt} = \frac{DT^{ij}}{Dt} - (d^i_{\cdot m} + \omega^i_{\cdot m}) T^{mj} - (d^j_{\cdot n} + \omega^j_{\cdot n}) T^{in},$$

and that

$$\frac{D\underline{\gamma}_\alpha}{Dt} = (d^\beta_{\cdot \alpha} + \omega^\beta_{\cdot \alpha}) \underline{\gamma}_\beta,$$

$$\frac{D\underline{\gamma}^\alpha}{Dt} = -(d^\alpha_{\cdot \beta} + \omega^\alpha_{\cdot \beta}) \underline{\gamma}^\beta,$$

where $\underline{\omega}$ is the vorticity tensor. Thus Oldroyd's definition of the convected rate is the material rate of those tensor components which an observer would measure with respect to a coordinate system both rotating and deforming with the material; for type A the components are measured with respect to the contravariant base vectors, while for type B the covariant base vectors are used, the base vectors in both cases being both stretched and rotated with the material.

The present form of writing the convected derivative allows ready inversion, for letting

$$W_{ij} = \frac{d_A T_{ij}}{dt},$$

then since

$$W_{\alpha\beta} = \frac{DT_{\alpha\beta}}{Dt},$$

and since the θ^α are constant in time for a material particle,

$$T_{\alpha\beta} = \int^t W_{\alpha\beta}(\theta, t')\, dt',$$

or

$$T_{ij} = \frac{\partial\theta^\alpha}{\partial x^i}\frac{\partial\theta^\beta}{\partial x^j}\int^t \frac{\partial x'^m}{\partial\theta^\alpha}\frac{\partial x'^n}{\partial\theta^\beta} W_{mn}(x', t')\,dt'$$

$$= \int^t \frac{\partial x'^m}{\partial x^i}\frac{\partial x'^n}{\partial x^j} W_{mn}(x', t')\,dt',$$

and similarly for the type B derivative. These integrals were presented first by Oldroyd (1950); they were used by Walters (1962) in models designated as A' and B' by writing

$$p_{ij} = 2\int_{-\infty}^{t} \psi(t-t')\frac{\partial x'^m}{\partial x^i}\frac{\partial x'^n}{\partial x^j} d_{mn}(x', t')\,dt' \qquad (2)$$

and

$$p^{ij} = 2\int_{-\infty}^{t} \psi(t-t')\frac{\partial x^i}{\partial x'^m}\frac{\partial x^j}{\partial x'^n} d^{mn}(x', t')\,dt', \qquad (3)$$

respectively, where $\psi(t)$ is a material relaxation function.* When $\psi(t)$ consists of a combination of exponentials and Dirac delta functions,

*(It is frequently more convenient to work with N, the distribution of relaxation times, defined by $\psi(t) = \int_0^\infty N(p)\exp(-t/p)\,dp/p$.)

equations (2) and (3) are exactly equivalent to equation (1); otherwise they are generalizations of equation (1). (For example, when $N = M = 1$, equation (1) is obtained by taking $N(p) = \mu_o\left[\frac{\tau_1}{\lambda_1}\delta(p) + \left(1 - \frac{\tau_1}{\lambda_1}\right)\delta(p-\lambda_1)\right]$.

Jaumann (1911) proposed a different definition of L_t which we shall designate as L_{tC}. Introducing new base vectors $\underline{\Gamma}_\alpha$ by

$$\underline{\Gamma}_\alpha = S_{\alpha}^{\cdot\beta}\,\underline{\gamma}_\beta$$

where $\underline{\underline{S}}$ satisfies the equation

$$\frac{DS_{\alpha}^{\cdot\beta}}{Dt} = - S_{\alpha}^{\cdot\delta}\,d_{\delta}^{\cdot\beta}$$

and reduces to the identity matrix as an initial condition, then, denoting the inverse tensor with a minus unity superscript,

$$\underline{\underline{T}} = \tilde{T}^{\alpha\beta}\,\underline{\Gamma}_\alpha\underline{\Gamma}_\beta\,, \quad \tilde{T}^{\alpha\beta} = T^{\delta\epsilon}\,\overset{-1}{S}_{\delta\alpha}\,\overset{-1}{S}_{\epsilon\beta}\,,$$

and

$$L_{tC}\,\underline{\underline{T}} \equiv \frac{D\tilde{T}_{\alpha\beta}}{Dt}\,\underline{\Gamma}_\alpha\underline{\Gamma}_\beta \equiv \frac{d_c T_{ij}}{dt}\,\underline{g}_i\,\underline{g}_j,$$

as before it is readily found from the transformation laws that

$$\frac{d_c T_{ij}}{dt} = \frac{DT_{ij}}{Dt} + \omega^{m}_{\cdot i}\,T_{mj} + \omega^{n}_{\cdot j}\,T_{in}$$

and that

$$\frac{D\underline{\Gamma}_\alpha}{Dt} = \omega^{\beta}_{\cdot\delta}\, S_{\alpha}^{\cdot\delta}\, \underline{\gamma}_\beta .$$

Since by Ricci's theorem and the above $d_t g_{ij}/dt = 0$, raising and lowering of indices commutes with the operation of Jaumann differentiation.

The above results can be put in a simpler appearing form by introducing a further tensor $\underline{\underline{R}}$, defined by

$$R_i^{\cdot\alpha} \equiv S^{-1\cdot\alpha}_{\beta}\, \frac{\partial\theta^\beta}{\partial x_i} ;$$

then

$$\frac{DR_i^{\cdot\alpha}}{Dt} = \omega_{ij}\, R^{j\alpha} \tag{4}$$

and

$$R_i^{\cdot\alpha}\, \underline{\Gamma}_\alpha = \underline{g}_i , \text{ or } \underline{\Gamma}_\alpha = \overset{-1}{R}{}_{\alpha}^{\cdot i}\, \underline{g}_i$$

where $\underline{\underline{R}}$ is equal initially to the identity matrix. Thus $\underline{\underline{S}}$ is the tensor rotating the material base vectors $\underline{\gamma}_\alpha$ into the material base vectors $\underline{\Gamma}_\alpha$, and $\underline{\underline{R}}$ is the tensor rotating the material base vectors $\underline{\Gamma}_\alpha$ into the fixed base vectors $\underline{g}_i$. As has been shown by Goddard and Miller (1965), $\underline{\underline{R}}$ corresponds to an orthogonal transformation, and hence its inverse and transpose are equivalent. Thus the Jaumann derivative is the material time rate of those tensor components which an observer would measure with respect to the base vectors $\underline{\Gamma}_\alpha$ rotating locally with the same rate as the vorticity, i.e., moving with the principal axes of $\underline{\underline{d}}$. The length of these base vectors changes also, but not directly with the material. Inversion of the derivative again follows readily from the definition; if now

$$Y_{ij} = \frac{d_c T_{ij}}{dt},$$

then

$$\tilde{Y}_{\alpha\beta} = Y^{\delta\epsilon} S^{-1}_{\delta\alpha} S^{-1}_{\epsilon\beta} = \frac{D\tilde{T}_{\alpha\beta}}{Dt} = \frac{D}{Dt}\left(T^{\mu\nu} S^{-1}_{\mu\alpha} S^{-1}_{\nu\beta}\right)$$

and

$$\tilde{T}_{\alpha\beta} = \int^t \tilde{Y}_{\alpha\beta}(\theta, t')\,dt',$$

$$T_{ij} = \frac{\partial\theta^\delta}{\partial x^i}\frac{\partial\theta^\epsilon}{\partial x^j} S^{\alpha}_{\cdot\delta} S^{\beta}_{\cdot\epsilon} \int^t \frac{\partial\theta^\mu}{\partial x^m}\frac{\partial\theta^\nu}{\partial x^n} Y^{mn}(x', t')$$

$$\cdot \left(S^{-1\,\mu}_{\cdot\,\alpha} S^{-1\,\nu}_{\cdot\,\beta}\right)_{t'} dt'$$

$$= R_i^{\cdot\alpha} R_j^{\cdot\beta} \int^t (R_{m\alpha} R_{n\beta})_{t'} Y^{mn}(x', t')\,dt'$$

$$= \int^t R_i^{\cdot m} R_j^{\cdot n} Y_{mn}(x', t')\,dt',$$

the initial conditions on R_{ij} being imposed at time t'. A material of type C' could now be defined by

$$p_{ij} = 2\int_{-\infty}^{t} \psi(t-t') R_i^{\,m} R_j^{\,n} d_{mn}(x',t')\,dt'. \quad (5)$$

We note that equation (5) is the constitutive equation presented by Goddard and Miller (1966), their integration being presented by other arguments. No simple relation has so far been found relating the various Oldroyd and Jaumann integrals.

II. Governing equations

The solution for steady flow between stationary parallel plates is next presented for materials of type A', B', and C'. The stability problem for parallel flows is then formulated for each by superimposing a wavy infinitesimal disturbance on the primary flow and then determining under what conditions this disturbance will grow.

Cartesian coordinates are used, the z-axis and the y-axis being chosen parallel and perpendicular to the plates, respectively. For the steady flow the velocity components are assumed in the form

$$U = V = 0, \quad W = W(y),$$

with $W = 0$ at $y = \pm h$. By inspection the motion is

$$x' = x, \quad y' = y, \text{ and } z' = z - W(y)(t-t'),$$

primed coordinates denoting the position of a particle at time t'. The only non-zero component of the rate of deformation is

$$d_{23} = \tfrac{1}{2}\,DW,$$

where D represents differentiation with respect to y. The non-zero displacement gradients and rotation components are

$$\frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y} = \frac{\partial z'}{\partial z} = 1, \quad \frac{\partial z'}{\partial y} = -(t-t')DW,$$

$$R_{11} = 1, \quad R_{22} = R_{33} = \cos\left[\tfrac{1}{2}(t-t')DW\right],$$

$$\text{and} \quad R_{23} = -R_{32} = -\sin\left[\tfrac{1}{2}(t-t')DW\right].$$

Substitution of these into equations (2), (3) and (5) and the equations of motion yields the non-zero partial stress components and velocity as shown in Table I, where

$$B_n = \int_0^\infty \tau^n N(\tau)\,d\tau = J_n^0, \tag{6}$$

and

$$J_n^m = \int_0^\infty \tau^n N(\tau)\left[1+(\tau DW)^2\right]^{-m} d\tau. \tag{7}$$

Hence, fluids A', B', and C' all predict different normal stresses. Normal stress differences are consistent with the sudden expansion or contraction of the stream when some non-Navier-Stokes liquids suddenly emerge from a tube into the atmosphere (the Merrington effect, which would occur in A' and C') as well as with the differences in the shape of the free surface for such different liquids undergoing

	A'	B'	C'
p_{xx}	0	0	0
p_{yy}	$-2B_1(DW)^2$	0	$-2J_1^1\,(DW)^2$
p_{zz}	0	$2B_1(DW)^2$	$2J_1^1\,(DW)^2$
p_{yz}	B_oDW	B_oDW	$J_o^1\,DW$
p	$\frac{\partial p}{\partial z}\,z - 2B_1(DW)^2$	$z\,\frac{\partial p}{\partial z}$	$\frac{\partial p}{\partial z}\,z - 2J_1^1\,(DW)^2$
W	$W_o(1 - \frac{y^2}{h^2})$	$W_o\,(\frac{1}{2} - \frac{y^2}{h^2})$	Satisfies $J_o^1\,DW = \frac{\partial p}{\partial z}\,y$
$\frac{\partial p}{\partial z}$	B_oD^2W	B_oD^2W	$(J_o^1 - 2J_2^2DW)\,D^2W$

TABLE I
Velocity and Stresses for primary flow of materials A', B', and C'.

Couette flow (the Weissenberg effect, which would also occur in A' and C'). Only C' shows a variable effective viscosity.

In the development of the Orr-Sommerfeld equation (the stability equation for Navier-Stokes liquids), consideration is limited to a disturbance that corresponds to a velocity field which is both temporally and spatially (in z) periodic. Subject to this limitation, Squire (1933) has shown that it is sufficient to consider a disturbance that corresponds to a two-dimensional velocity field. In the visco-elastic case, only such disturbances will be considered also, although no proof of the sufficiency of this exists for these fluids. (In fact, Listrov (1966) has shown that at least for a Stokesian fluid three dimensional disturbances are less stable than two dimensional disturbances.) Accordingly, the disturbance velocity components are taken in the form

$$u^* = 0, \quad v^* = v(y)E, \quad \text{and} \quad w^* = \omega(y)E,$$

where E = exp iA(z - Ct); A is the (real) wave number, and c the (complex) wave speed.

We first derive the stability equation for materials A' and B'. The total displacement is assumed to be the sum of the primary flow displacement and the displacement resulting from the disturbance, namely,

$$x' = x,$$

$$y' = y + f(y,z,t) - f(y',z',t'),$$

and

$$z' = z - [W(y) + k(y,z,t)](t - t') + g(y,z,t) - g(y',z',t').$$

Taking

$$f = \sum_{n=1}^{\infty} f_n(y)\, E^n$$

and similar expressions for g and k, since

$$\frac{Dx'}{Dt} = \frac{Dy'}{Dt} = \frac{Dz'}{Dt} = 0,$$

the solutions are readily obtained as

$$f_1 = -v / iA(W - C),$$

$$f_{n+1} = (f_n Dv - v Df_n) / iA(W - C),$$

$$g_1 = -(k_1 + wr) / iA(W - C),$$

$$g_{n+1} = (-k_n + g_n Dv - v Dg_n) / iA(W - C),$$

$$k_1 = v DW / iA(W - C),$$

$$k_{n+1} = (k_n Dv - v Dk_n) / iA(W - C).$$

Expanding all quantities about values at time t, it follows that to the lowest order

$$y' = y - \frac{v E (1-F)}{i A (W-C)} \tag{8}$$

and

$$z' = z - (t-t')W + \frac{v D W E F (t-t')}{i A (W-C)} + \left[\frac{v D W}{A^2 (W-C)^2} + \frac{D v}{A^2 (W-C)} \right] E (1-F), \tag{9}$$

where F = exp iA(C-W) (t-t').

If consideration is restricted to a short time interval, then equations (8) and (9) can be approximated by expanding F in a power series in (t-t') and retaining only the first order terms. Hence,

$$y' = y - v (t-t') E$$

and

$$z' = z - \left(W - \frac{D v}{i A} E\right)(t-t').$$

These ... the expressions used by Walters (1962). The more general forms given by equations (8) and (9) will, however, be used here.

We note also that equations (8) and (9) are well-behaved at the critical point where W and C have the same value. Applying the limiting process W→C results in

and

$$\lim_{W\to C} (y'-y) = -vE(t-t')$$

$$\lim_{W\to C} [z'-z+W(t-t')] = \frac{Dv}{iA}E(t-t') - \tfrac{1}{2}vEDW(t-t')^2,$$

in agreement with the limit as t approaches t' to the order of the linear terms in t-t'.

The non-zero rate of deformation components at time t are

$$d_{yy} = -d_{zz} = EDv$$

and

$$d_{yz} = \tfrac{1}{2}\left[DW + \left(iAv - \frac{D^2v}{iA}\right)E\right].$$

To the same approximation used for the displacements, the non-zero rate of deformation components at time t' are

$$d_{yy} = -d_{zz} = Dv\,EF$$

and

$$d_{yz} = \tfrac{1}{2}\left[DW + \left(iAv - \frac{D^2v}{iA}\right)EF - \frac{vD^2W\,E(1-F)}{iA(W-C)}\right].$$

Making use of the continuity equation, the non-zero derivatives of the displacements can thus be written as

$$\frac{\partial y'}{\partial y} = 1 - \left(Dv - \frac{vDW}{W-C}\right)\frac{E(1-F)}{iA(W-C)},$$

$$\frac{\partial y'}{\partial z} = -\frac{vE(1-F)}{W-C},$$

$$\frac{\partial z'}{\partial z} = 1 + \frac{vDWE(t-t')}{W-C} + \left(Dv - \frac{vDW}{(W-C)}\right)(t-t'),$$

and

$$\frac{\partial z'}{\partial y} = -(t-t')DW + (t-t')\frac{DWEF}{iA(W-C)}\left(Dv - \frac{vDW}{W-C}\right) + \frac{(t-t')E}{iA(W-C)}\left(vD^2W + DvDW - \frac{v(DW)^2}{W-C}\right) - \frac{E(1-F)}{A^2(W-C)}\left[D^2v + \frac{2v(DW)^2}{(W-C)^2} - \frac{2DvDW + vD^2W}{W-C}\right].$$

Combining these results with the constitutive equations for A' (equation (2)) leads to the disturbance stresses

$$\bar{p}_{zz} = -2K_0^1 Dv - 2iADWK_1^1 v,$$

$$\bar{p}_{yz} = -\frac{K_0^1}{iA}D^2v + 2K_1^2 DWDv + \left[iAK_0^1 - K_1^1 D^2W + 2iA(K_2^2 + K_2^1)\right]v,$$

and

$$\bar{p}_{yy} = \frac{2DW}{iA}(K_1^1 + K_1^2)D^2v + 2[K_0^1$$
$$+ 2(DW)^2 iA(W-C)(K_3^3 + K_3^2)]Dv$$
$$+2[DWD^2W(2K_2^1 + K_2^2)$$
$$-2(DW)^3 iA(K_3^2 + K_3^3) - iADWK_1^2]v.$$

The functions $K_n^m(y)$, are generalizations of Walters' constants and are defined by.

$$K_n^m = \int_0^\infty \tau^n N(\tau)[1 + iA(W-C)\tau]^{-m} d\tau. \quad (10)$$

Upon considering the equations of motion, one obtains, after subtracting the equations satisfied by the velocity components of the primary flow and eliminating the pressure by cross-differentiation,

$$\rho[(W-C)(D^2-A^2) - D^2W]v$$
$$= iAD(\bar{p}_{yy} - \bar{p}_{zz}) - (D^2+A^2)\bar{p}_{yz}. \quad (11)$$

In order to write equation (11) in terms of non-dimensional variables, the following dimensionless functions of y are introduced:

$$P_0 = K_0^1/\rho W_0 h, \quad P_1 = K_1^1/\rho h^2,$$
$$P_2 = K_2^2 W_0/\rho h^3, \quad P_3 = K_4^3 W_0^3/\rho h^5.$$

Putting these and the previous expressions for $\overline{P}_{yy}$, $\overline{P}_{zz}$ and $\overline{P}_{yz}$ into equation (11) results in

$$
\begin{aligned}
i\left[(U-c)(D^2-\alpha^2)v - v D^2 U\right] \\
&= \frac{P_0}{\alpha}(D^2-\alpha^2)^2 v - \alpha^2 P_2 (U \\
&- c)(2DU)D^3 v - \alpha(U-c)[3P_2 D^2 U \\
&- 2\alpha^2(U-c)P_3] D^2 v - \alpha(U \\
&- c)\{4\alpha^2 DU[(DU)^2 - D^2U(U \\
&- c)]P_3 - (2\alpha^2 DU - D^3U)P_2\} Dv \\
&+ \{[4DU(\alpha^2 DU + D^3U) + \alpha^2(U-c)D^2U \\
&+ 3(D^2U)^2]\alpha P_2 - [\alpha^2(U-c)^2 \\
&+ 3(U-c)D^2U - 2(DU)^2]2\alpha^3(DU)^2 P_3 \\
&+ i D^4 U P_1\} v, \qquad (12)
\end{aligned}
$$

where α is the non-dimensional wave number Ah, c is the non-dimensional wave speed C/W_0, U is now the non-dimensional primary flow velocity, W/W_0, and D represents differentiation with respect to the non-dimensional y. For a Navier-Stokes liquid all the P_n are zero except P_0, and equation (12) would then reduce to the Orr-Sommerfeld equation.

Walters (1962) has suggested that for some viscoelastic liquids, called "slightly viscoelastic," it is reasonalbe to expect the $\psi(\tau)$ to vanish rapidly as τ increases. This is the justification he gave for using the approximate forms for equations (8) and (9). This is equivalent to replacing the upper limit in the expressions for the P_n by a finite limit (say T) which may even be quite small. Assuming $|iA(W-C)| \ll T^{-1}$ everywhere in the flow field and neglecting all B_n for $n \geq 2$, equation (12) reduces to

$$
\begin{aligned}
i\left[(U-c)(D^2-\alpha^2) - D^2U\right] v \\
= \left[1 - i\alpha R_1 (U-c)\right](\alpha R)^{-1}(D^2-\alpha^2)^2 v \\
+ i R_1 R^{-1} v D^4 U
\end{aligned}
$$

where

$$R = \rho W_0 h / B_0$$

and

$$R_n = B_n (W_0)^n / B_0 h^n$$

which is the equation given by Walters.

Two remarks are appropriate. Equation (12) was derived without specifying the form of W(y) and hence is not restricted to the primary flow here considered. Also, if equation (3) (the constitutive equation for material B') had been used in place of equation (2) (the constitutive equation for material A'), the resulting equation for v would have been exactly equation (12) - a perhaps surprising result, considering the primary flows involving the two constitutive equations give quite different normal stress results.

The stability equation for the C' material is developed in much the same way. For the perturbation velocities, the solution to equation (4) is found to be, to the order of the linearization,

$$R_{yy} = R_{zz} = \cos\left[\tfrac{1}{2}(t-t')DW\right] + EH\sin\left[\tfrac{1}{2}(t-t')DW\right],$$

$$R_{yz} = -R_{zy} = -\sin\left[\tfrac{1}{2}(t-t')DW\right] + EH\cos\left[\tfrac{1}{2}(t-t')DW\right],$$

where

$$H = \frac{(1-F)}{2iA(W-C)}\left[v\left(iA - \frac{D^2W}{iA(W-C)}\right) + \frac{D^2v}{iA}\right] + \frac{(t-t')vD^2W}{2iA(W-C)}.$$

From equation (5), the disturbance stresses are

$$\bar{p}_{22} = -\bar{p}_{33} = vDW\Big\{-2iAL_1^1 + \Big[\Big(iA - \frac{D^2W}{iA(W-C)}\Big)\big(J_0^1 - L_0^1\big) + D^2W\big(J_1^2 + \tfrac{1}{2}J_1^1\big)\Big]\big[iA(W-C)\big]^{-1}\Big\} + 2Dv\big[L_0^1 + iA(W-C)L_1^1\big] + D^2v\frac{DW}{A^2(W-C)}\big(L_0^1 - J_0^1\big), \qquad (13)$$

$$\bar{p}_{23} = v\Big\{iAL_0^1 + \big[-A^2(W-C) + D^2W\big]L_1^1 + \big[iA(W-C)\big]^{-1}\Big[\big(L_0^1 - J_0^1\big)D^2W + \Big(iA - \frac{D^2W}{iA(W-C)}\Big)\big(J_1^1 - L_1^1\big)(DW)^2 + 2J_2^2(DW)^2D^2W\Big]\Big\}$$

$$+ 2DWL_1^1 Dv + \frac{D^2 v}{\iota A}\Big\{-\big[L_0^1 + \iota A(W-C)L_1^1\big] + \frac{(DW)^2}{\iota A(W-C)}\big(J_1^1 - L_1^1\big)\Big\}, \quad (14)$$

where

$$L_n^m = \int_0^\infty \tau^n N(\tau)\Big\{\big[1 + \iota A(W-C)\tau\big]^2 + (\tau DW)^2\Big\}^{-m} d\tau. \quad (15)$$

Because of the complexity of these stress terms we do not write out the stability equation here, but consider the appropriate approximation in the next section.

III. Asymptotic Solutions

An approximate solution to the stability equation (12) for the primary flow $U = 1 - y^2$ is next obtained. The determination of the characteristics equation for a Navier-Stokes liquid is first briefly discussed in Part A, the procedure used being that presented by Graebel (1966) with only slight modifications. The counterpart characteristics equation for viscoelastic liquids A' and B' is next presented in Part B, and the procedure used to actually solve this characteristic equation and to determine the points on the neutral stability curve is presented in Part C. Viscoelastic liquid C' is discussed in Part D.

A. The Determination of the Characteristic Equation for a Navier-Stokes Liquid

It is anticipated that both c and $1/\alpha R$ will be small for the case of interest, which suggests a solution in terms of matched asymptotic expansions. The flow region is first divided into an inner and an outer region. The inner region is a strip that includes the rigid bottom boundary at $y = -1$ and the "critical point" at $y = y_c$, where $U(y_c) = c$. The outer region extends from the inner region to the centerline between the plates at $y = 0$ which, as a result of the symmetry of the geometry and equations, serves as the other boundary. It is assumed that the two regions overlap.

The procedure is to obtain a solution valid in the inner region and a solution valid in the outer region and then to merge the two solutions. The inner solution is made to satisfy the boundary conditions at $y = -1$ and the outer solution the conditions at $y = 0$. The merging then produces a characteristic equation which gives α as a function of R and c, the plot of α versus R for $c_i = 0$ being the neutral stability curve.

In the inner region v is obtained by introducing the change in variable (coordinate stretching)

$$\eta = (y - y_c)/\mu = z/\mu$$

and by putting

$$v(y) = \chi(\mu, \eta) = \chi^{(0)}(\eta) + \mu\chi^{(1)}(\eta) + \cdots;$$

where μ is a function of αR, expected to be small, but unknown at this point.

Substitution of the above into equation (11) with all of the R_n for $n \geq 1$ set equal to zero suggests that the proper choice for μ is $\mu = (\alpha R)^{-1/3}$, and equation (12) becomes

$$\frac{d^4\chi^{(0)}}{d\eta^4} - i\eta DU_c \frac{d^2\chi^{(0)}}{d\eta^2} + \mu\Big[2i\chi^{(0)} + i\eta^2 \frac{d^2\chi^{(0)}}{d\eta^2} + \frac{d^4\chi^{(1)}}{d\eta^4} - i\eta DU_c \frac{d^2\chi^{(1)}}{d\eta^2}\Big] + \cdots = 0.$$

$\chi^{(0)}$ and $\chi^{(1)}$ are the solutions of

$$\frac{d^4\chi^{(0)}}{d\eta^4} - i\eta DU_c \frac{d^2\chi^{(0)}}{d\eta^2} = 0$$

and

$$\frac{d^4\chi^{(1)}}{d\eta^4} - i\eta DU_c \frac{d^2\chi^{(1)}}{d\eta^2} = i\left(\eta^2 \frac{d^2\chi^{(0)}}{d\eta^2} + 2\chi^{(0)}\right).$$

Hence,

$$\chi_1^{(0)} = \eta,$$

$$\chi_2^{(0)} = 1,$$

$$\chi_3^{(0)} = \int_{-\infty}^{\eta} d\eta \int_{-\infty}^{\eta} d\eta \; h_1[i\eta(DU_c)^{1/3}],$$

and

$$\chi_4^{(0)} = \int_{\infty}^{\eta} d\eta \int_{\infty}^{\eta} d\eta \; h_2[i\eta(DU_c)^{1/3}],$$

where

$$h_1[x] = \frac{2}{3} x^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3} x^{3/2}\right],$$

$$h_2[x] = \frac{2}{3} x^{1/2} H_{1/3}^{(2)}\left[\frac{2}{3} x^{3/2}\right];$$

$H^{(1)}$ and $H^{(2)}$ are Hankel functions of order one-third. Since h_2 increases exponentially with large positive η at a much faster rate than does the outer solution, $\chi_4^{(0)}$ cannot be merged with the outer solution and is therefore discarded. For $\chi^{(1)}$ we have

$$\chi_1^{(1)} = -\eta^2/DU_c\,.$$

Graebel (1966) gives the solutions for $\chi_2^{(1)}$ and $\chi_3^{(1)}$ as

$$\chi_2^{(1)} = 2(iDU_c)^{-2/3}\int_0^\eta d\eta\int_0^\eta d\eta\int_0^\infty ds\,\cdot$$
$$\cdot\exp\{-i[(iDU_c)^{1/3}\eta s + \tfrac{1}{3}s^3]\}$$

and

$$\chi_3^{(1)} = 2i\left[\frac{7}{10(DU_c)^2}\frac{d^3\chi_3^{(0)}}{d\eta^3} - \frac{9}{10(DU_c)^2}\frac{d^2\chi_3^{(0)}}{d\eta^2}\right.$$
$$\left.+\frac{i\eta}{2DU_c}\chi_3^{(0)}\right] + \gamma\,\chi_2^{(1)},$$

where

$$\gamma = 0.67830\,(DU_c)^{-1} - 0.39099$$
$$+\,i(0.39160\,(DU_c)^{-1} - 0.67896).$$

For large η,

$$\chi_2^{(1)} \sim \frac{-2\eta}{DU_c}\ln\eta + \frac{2i}{3\eta^2(DU_c)^2},$$

where

$$-\tfrac{7}{6}\pi < \arg\eta < \frac{\pi}{6}\,.$$

In the outer region v is obtained by introducing the expansion

$$v(\mu, y) = \epsilon_0(\mu)\, v^{(0)}(y) + \epsilon_1(\mu)\, v^{(1)}(y) + \cdots.$$

Substituting this into equation (12) gives

$$(U-c)(D^2-\alpha^2)v^{(0)} - v^{(0)} D^2 U = 0,$$

and $v^{(0)} = v^{(n)}$ for all n such that $O(\epsilon_n(\mu)) > O(\mu^3)$. The Tollmien solutions (1936) are

$$v_1^{(0)}(y) = F_1(z) = \sum_{n=0}^{\infty} A_n z^{n+1},$$

and

$$v_2^{(0)}(y) = F_2(z) = F_1 \ln z + \sum_{n=0}^{\infty} B_n z^n,$$

where

$$A_0 = 1,\quad A_1 = -(DU_c)^{-1},\quad A_2 = \tfrac{1}{6}\alpha^2,$$

$$A_{n+3} = \frac{(n+1)(n+4)A_{n+2} + \alpha^2 DU_c A_{n+1} - \alpha^2 A_n}{(n+3)(n+4)\, DU_c},$$

and

$$B_0 = -\tfrac{1}{2} DU_c,\quad B_1 = 0,\quad B_2 = \frac{2}{DU_c} - \tfrac{1}{4}\alpha^2 DU_c, \text{ etc.}$$

In the inner region Graebel used

$$v(y) \approx \chi^{(0)}(\eta) = C_1 \eta + C_2 + C_3 \chi_3^{(0)},$$

where the C_i are arbitrary constants. For the inner and outer solutions to merge for large positive η the proper choice is $1/\mu$ for ϵ_0, $\ln\mu$ for ϵ_1, and 1 for ϵ_2. In the outer region v is given by

$$v(y) = \frac{1}{\mu} C_1 F_1 + \ln\mu \left(\frac{2 C_2 F_1}{DU_c} \right) - \frac{2C_2}{DU_c} F_2 + A F_1 + O(\mu \ln\mu),$$

where A is a constant that cannot be determined until higher order terms are considered. $\chi_3^{(0)}$ decreases exponentially with large positive η and, hence, no terms are needed to merge with it.

The boundary conditions are

$$v = Dv = 0 \quad at \quad y = -1$$

and

$$Dv = D^3 v = 0 \quad at \quad y = 0$$

if v is an even function of y, or

$$v = D^2 v = 0 \quad at \quad y = 0$$

if v is an odd function of y. The Orr-Sommerfeld equation allows separation into even and odd parts when U is an even function of y; since v_0 in the outer region satisfies a second order equation which is even in y, satisfaction of only one boundary condition at $y = 0$ is sufficient. Using symmetrical disturbances since they are less stable than asymmetrical disturbances, the boundary conditions give

$$\frac{1}{\mu} C_1 DF_1(z_0) - \frac{2C_2}{DU_c} DF_2(z_0) = 0,$$

$$C_1 \eta + C_2 + C_3 \chi_3^{(0)}(\eta_1) = 0,$$

and

$$C_1 + C_3 \frac{d\chi_3^{(0)}(\eta_1)}{d\eta} = 0$$

to the lowest non-trivial order in μ, where $\eta_1 = -(1 + y_c)/\mu = z_1/\mu$ and $z_0 = -y_c$. For the existence of a non-trivial solution,

$$\frac{1}{z_1}\left[\frac{DU_c DF_1(z_0)}{2DF_2(z_0)} + 1\right] = \frac{\chi_3^{(0)}(\eta_1)}{\eta_1 \dfrac{d\chi_3^{(0)}(\eta_1)}{d\eta}}$$

which is the characteristic equation obtained by Graebel. The plot of α vs R for $c_i = 0$ obtained from this, however, does not contain the characteristic loop of a neutral stability curve, but gives only the lower branch of the curve. This shortcoming is explained by considering the magnitudes of the $\chi^{(1)}$ terms in the inner expansion, which in turn requires some knowledge of η_1. With Lin's (1945) results ($\eta_1 \approx 2.5$, $\mu \approx 1/20$) as a guide it appears that $\chi^{(0)}$ does not adequately describe the solution near $y = -1$, for $\mathrm{Im}(\mu\chi_2^{(1)}) \approx 0.3$ as compared with the $\mathrm{Im}\chi_2^{(0)} = 0$. The real parts of $\mu\chi_2^{(1)}$ and

$\mu\chi_2^{(1)}$ are however small compared to $\chi_1^{(0)}$ and $\chi_2^{(0)}$ respectively, and $\mu\chi_3^{(1)}$ is small compared to $\chi_3^{(0)}$. Thus by including the $\chi^{(1)}$ terms the imaginary part of χ_2 is altered significantly, this additional term apparently being responsible for generating the loop in the stability curve. It seems then that a better choice for the solution in the inner region is

$$v(y) = C_1\left(\eta - \frac{\mu}{DU_c}\eta^2\right) + C_2\left(1 + \mu\chi_2^{(1)}\right) + C_3\chi_3^{(0)}. \qquad (16)$$

For merging equation (16) with the outer solution, again the proper choice is $1/\mu$ for ϵ_0, $\ln\mu$ for ϵ_1, and 1 for ϵ_2, as before. Then v(y) in the outer region is given by

$$v(y) = \frac{C_1F_1}{\mu} + \ln\mu\left(\frac{2C_2F_1}{DU_c}\right) - \frac{2C_2}{DU_c}F_2 + O(\mu).$$

Substituting the appropriate boundary conditions leads to

$$C_1\left(\eta_1 - \frac{\mu\eta_1^2}{DU_c}\right) + C_2\left(1 + \mu\chi_2^{(1)}(\eta_1)\right) + C_3\chi_3^{(0)}(\eta_1) = 0,$$

$$C_1\left(1 - \frac{2\mu\eta_1}{DU_c}\right) + C_2\mu\frac{d\chi_2^{(1)}(\eta_1)}{d\eta} + C_3\frac{d\chi_3^{(0)}(\eta_1)}{d\eta} = 0$$

$$\frac{1}{\mu}C_1DF_1(z_0) - \frac{2C_2}{DU_c}DF_2(z_0) = 0,$$

$$P_0 = R^{-1}\{1 - i\alpha R_1 z DU_c - [\tfrac{1}{2} i\alpha R_1 D^2U_c + \alpha^2 R_2 (DU_c)^2] z^2 + \cdots\},$$

$$P_1 = R^{-1}\{R_1 - i\alpha R_2 z DU_c - [\tfrac{1}{2} i\alpha R_2 D^2U_c + \alpha^2 R_3 (DU_c)^2] z^2 + \cdots\},$$

$$P_2 = R^{-1}\{R_2 - 2i\alpha R_3 z DU_c - [i R_3 D^2U_c + 3\alpha R_4 (DU_c)^2]\alpha z^2 + \cdots\},$$

and

$$P_3 = R^{-1}\{R_4 - 3i\alpha R_5 z DU_c - [\tfrac{3}{2} i R_5 D^2U_c + 6\alpha R_6 (DU_c)^2]\alpha z^2 + \cdots\},$$

where $D^nU_c = D^nU$ evaluated at $y = y_c$. These are valid in the neighborhood of the critical point for any distribution function that vanishes as τ becomes large, specifically for $N(\tau)$ negligible when $\tau > T$, where $|iA(W-C)T| < 1$.

A change to a stretched variable and inner and outer expansions are introduced as in part A. When all these are substituted into equation (12), the stability equation in the inner region is now modified to and thus to the characteristics equation

$$\frac{\chi_3^{(0)}(\eta_1)}{\eta_1 \dfrac{d\chi_3^{(0)}(\eta_1)}{d\eta}} = \frac{\frac{1}{2} D U_c \left[1 + \mu \dfrac{d\chi_2^{(1)}(\eta_1)}{d\eta}\right] D F_1(z_0) + z_1 \left(1 - \dfrac{z_1}{D U_c}\right) D F_2(z_0)}{z_1 \left\{\left[1 - \dfrac{2\mu\eta_1}{D U_c}\right] D F_2(z_0) + \frac{1}{2} D U_c D F_1(z_0) \dfrac{d\chi_2^{(1)}(\eta_1)}{d\eta}\right\}} . \quad (17)$$

The plot of α versus R resulting from this does have a loop and the equation is essentially that used by Lin, although he elected to express the outer solution as an expansion in powers of α^2. With present computers the expansion in terms of the coordinate rather than α^2 seems to be much simpler and more accurate.

B. The Determination of the Characteristic Equation for a Viscoelastic Liquid of Type A' or B'

For a viscoelastic liquid which does not depart too drastically from a Navier-Stokes liquid the solution of equation (11) can be carried out in a manner analogous to the solution presented in part A. Specifically, it is anticipated that in equation (12) both $|c|$ and $|P_n/\alpha|$ will be small. Hence, the flow region can be divided into an inner and an outer region as previously done.

In the inner region the P_n can be expressed as series in z by expanding the denominators and then integrating term by term. The results are

$$\frac{d^4\chi^{(0)}}{d\eta^4} - i\eta DU_c \frac{d^2\chi^{(0)}}{d\eta^4}$$

$$-\mu[i\alpha R_1 \eta DU_c \frac{d^4\chi^{(0)}}{d\eta^4}$$

$$-2i\chi^{(0)} + i\eta^2 \frac{d^2\chi^{(0)}}{d\eta^2} - \frac{d^4\chi^{(1)}}{d\eta^4}$$

$$+ i\eta DU_c \frac{d^2\chi^{(1)}}{d\eta^2}] + \cdots = 0,$$

and thus

$$(1 - i\epsilon\eta DU_c)\frac{d^4\chi^{(0)}}{d\eta^4} - i\eta DU_c \frac{d^2\chi^{(0)}}{d\eta^2} = 0, \quad (18)$$

where $\epsilon = \mu\alpha R_1$. The outer equation is unchanged. Since the term containing R_1 multiplies the fourth derivative, χ_1 and χ_2 remain as in part A. Taking ϵ to be small but still of larger order than μ, χ_3 is approximated by

$$\chi_3^{(0)}(\eta) \approx \phi_0(\eta) + \epsilon\phi_1(\eta), \quad (19)$$

where

$$\frac{d^4\phi_0}{d\eta^4} - i\eta DU_c \frac{d^2\phi_0}{d\eta^4} = 0$$

and

$$\frac{d^4\phi_1}{d\eta^4} - i\eta DU_c \frac{d^2\phi_1}{d\eta^2} = i\eta DU_c \frac{d^4\phi_0}{d\eta^4}.$$

Then ϕ_o is the same as $\chi_3^{(0)}$ in Part A, and

$$\phi_1 = \frac{1}{5}\left(\eta \frac{d^3\phi_0}{d\eta^3} - 4 \frac{d^2\phi_0}{d\eta^2}\right).$$

Thus for the solution in the inner region, χ_1 and χ_2 as given in Part A are used, and $\phi_o + \epsilon\phi_1$ is used for χ_3.

Because of the additional linearization introduced by equation (19), the present results are limited to small ϵ and serve mainly to indicate a trend. For larger values of ϵ the perturbation scheme used to solve equation (18) may not be adequate. In this case, one would have to resort to an exact solution of equation (18) as presented in the Appendix.

Since the outer solution remains unchanged and χ_3 is not directly involved in the merging, the characteristic equation retains the same form as equation (17), the only difference being χ_3 is now given by equation (19).

C. Solution of the Characteristics Equation

For calculation pruposes, it is convenient to introduce the change in variable

$$\zeta = \beta\eta$$

where

$$\beta = (DU_c)^{1/3}.$$

Then the left hand side of equation (17) becomes

$$\frac{\chi_3^{(0)}(\eta_1)}{\eta_1 \dfrac{d\chi_3^{(0)}(\eta_1)}{d\eta_1}} \tag{20}$$

$$= \frac{\int_\infty^{\varsigma_1} d\varsigma \int_\infty^{\varsigma} d\varsigma\, h_1(i\varsigma) + \lambda\left[\varsigma_1 \dfrac{dh_1(i\varsigma_1)}{d\varsigma} - 4h_1(i\varsigma_1)\right]}{\varsigma_1\left\{\int_\infty^{\varsigma_1} d\varsigma\, h_1(i\varsigma) + \lambda\left[i\varsigma_1^2 h_1(i\varsigma_1) - 3\dfrac{dh_1(i\varsigma_1)}{d\varsigma}\right]\right\}}$$

where

$$\lambda = \frac{1}{5}\,\epsilon\,\beta^2$$

$$= \frac{(\alpha W_0 D W_c)^{2/3} \int_0^\infty \tau N(\tau)\, d\tau}{5\,(\rho h^4)^{1/3} \left(\int_0^\infty N(\tau)\, d\tau\right)^{2/3}} \,. \tag{21}$$

The functions h_1 and h_2 are discussed, and tables of h_1 and h_2 and their derivatives are given in Annals (1945). Putting $x = i\varsigma$, it follows that

$$\mathrm{Re}\ h_1(i\varsigma) = \sum_{m=0}^{\infty}\left\{ \frac{(-1)^{m+1} B_{2m+1}\,\varsigma^3}{(200)^{2m+1}} \right.$$

$$\frac{\sqrt{3}}{3}\left[\frac{2(-1)^m A_{2m+1}\,\varsigma^2}{(200)^{2m+1}} + \right.$$

$$\left. + \frac{(-1)^{m+1} B_{2m}}{(200)^{2m}}\right] \varsigma^{6m+1} \Big\} ,$$

and

$$\mathcal{I}m\; h_1(\iota\zeta) = \sum_{m=0}^{\infty}\left\{ \frac{(-1)^m \zeta B_{2m}}{(200)^{2m}} + \frac{\sqrt{3}}{3}\left[\frac{(-1)^{m+1}\zeta^4 B_{2m+1}}{(200)^{2m+1}} + \frac{2(-1)^{m+1}A_{2m}}{(200)^{2m}}\right]\zeta^{6m}\right\}$$

where

$$A_0 = 2^{2/3}/\Gamma(2/3)\,,\quad A_m = 200A_{m-1}/3m(3m-1),$$

$$B_0 = 2^{1/3}/3^{2/3}\,\Gamma(\tfrac{4}{3}),\; B_m = 200\,B_{m-1}/3m(3m+1)$$

are also tabulated.

For the integration indicated on the left hand side of equation (20) the above series were integrated term by term for $\zeta \leq 5$. For $\zeta > 5$, the integration was continued numerically using Simpson's rule and the asymptotic expansion of h_1 until there were no noticeable changes in the value of the integral in the seventh significant figure. These asymptotic expansions are

$$Re\; h_1(\iota\zeta) \sim S\,\zeta^{-1/4} e^{-\frac{\sqrt{2}}{3}\zeta^{3/2}} \cos\left(\frac{\sqrt{2}}{3}\zeta^{3/2} + \frac{13\pi}{24}\right),$$

and

$$\operatorname{Im} h_1(i\zeta) \sim -S\,\zeta^{-1/4} e^{-\frac{\sqrt{3}}{3}\zeta^{3/2}} \sin\left(\frac{\sqrt{3}}{3}\zeta^{3/2} + \frac{13\pi}{24}\right)$$

where $S = (2\sqrt{3})^{1/3}/\sqrt{\pi}$ and $-\frac{7\pi}{6} < \arg\zeta < \frac{5\pi}{6}$.

The results of the integration are

$$\int_\infty^\zeta d\zeta\,\operatorname{Re} h_1(i\zeta) = 0.504361 + \sum_{m=0}^{\infty}\left\{\frac{(-1)^{m+1}\zeta^3 B_{2m+1}}{(6m+5)(200)^{2m+1}} + \frac{\sqrt{3}}{3}\left[\frac{2(-1)^m A_{2m+1}\zeta^2}{(6m+4)(200)^{2m+1}} + \frac{(-1)^{m+1}B_{2m}}{(6m+2)(200)^{2m}}\right]\right\}\zeta^{6m+2},$$

$$\int_\infty^\zeta d\zeta\int_\infty^\zeta d\zeta\,\operatorname{Re} h_1(i\zeta) = -0.390992 + 0.504361\,\zeta + \sum_{m=0}^{\infty}\left\{\frac{(-1)^{m+1}\zeta^3 B_{2m+1}}{(6m+5)(6m+6)(200)^{2m+1}} + \frac{\sqrt{3}}{3}\left[\frac{2(-1)^m\zeta^2 A_{2m+1}}{(6m+4)(6m+5)(200)^{2m+1}} + \frac{(-1)^{m+1}B_{2m}}{(6m+2)(6m+3)(200)^{2m}}\right]\right\}\zeta^{6m+3},$$

$$\int_{\infty}^{\varsigma} d\varsigma \, \mathrm{Im}\, h_1(i\varsigma) = 0.873588$$
$$+ \sum_{m=0}^{\infty} \left\{ \frac{(-1)^m B_{2m}}{(6m+2)\,(200)^{2m}} \right.$$
$$+ \frac{\sqrt{3}}{3} \left[\frac{(-1)^{m+1} \varsigma^4 B_{2m+1}}{(6m+5)\,(200)^{2m+1}} \right.$$
$$\left.\left. + \frac{2(-1)^{m+1} A_{2m}}{(6m+1)\,(200)^{2m}} \right] \right\} \varsigma^{6m+1},$$

and

$$\int_{\infty}^{\varsigma} d\varsigma \int_{\infty}^{\varsigma} d\varsigma \, \mathrm{Im}\, h_1(i\varsigma) = -0.678640$$
$$+ 0.873588\, \varsigma$$
$$+ \sum_{m=0}^{\infty} \left\{ \frac{(-1)^m B_{2m}}{(6m+2)(6m+3)(200)^{2m}} \right.$$
$$+ \frac{\sqrt{3}}{3} \left[\frac{(-1)^{m+1} \varsigma^4 B_{2m+1}}{(6m+5)(6m+6)(200)^{2m+1}} \right.$$
$$\left.\left. + \frac{2(-1)^{m+1} A_{2m}}{(6m+1)(6m+2)(200)^{2m}} \right] \right\} \varsigma^{6m+2}.$$

For given values of λ and c the solution of equation (20) was obtained by plotting the real part against the imaginary part for each side. The intersections of these two families of curves gave α and ζ_1. R was computed from

$$R = \frac{DU_c}{\alpha}\left(-\frac{1+y_c}{\zeta_1}\right)^3 .$$

The results are shown in Figure 1. The graph for $\lambda = 0$, which corresponds to a Navier-Stokes liquid, is seen to be in close agreement with Lin's results, and the preceeding statements regarding the anticipated size of the various parameters are seemingly consistent with the final results. The results are qualitatively in agreement with those of Chan Man Fong and Walters, shown also in Figure 1. (The λ in their paper is defined as five times the value of the present one.) The quantitative disagreement of the two results is not understood; it is noted, however, that Chan Man Fong and Walters results for $\lambda = 0$, departing considerably from the results of Lin, do agree with the results of Stuart (1954). On this basis, it is believed that the present results are the more accurate ones.

D. <u>The Stability of a Viscoelastic Liquid of Type C'</u>

Introducing stretched coordinates again as in part B, near the critical point the stresses become

$$\bar{p}_{yy} = -\bar{p}_{zz} = \frac{2DW_c}{iA\tilde{\mu}^2} J^2_{1c} \frac{d^2 v}{d\eta^2} + O\left(\frac{v}{\tilde{\mu}}\right),$$

$$\bar{p}_{yz} = \frac{1}{iA\tilde{\mu}^2}\left[J^1_{0c} - 2(DW_c)^2 J^2_{2c}\right]\frac{d^2 v}{d\eta^2}$$

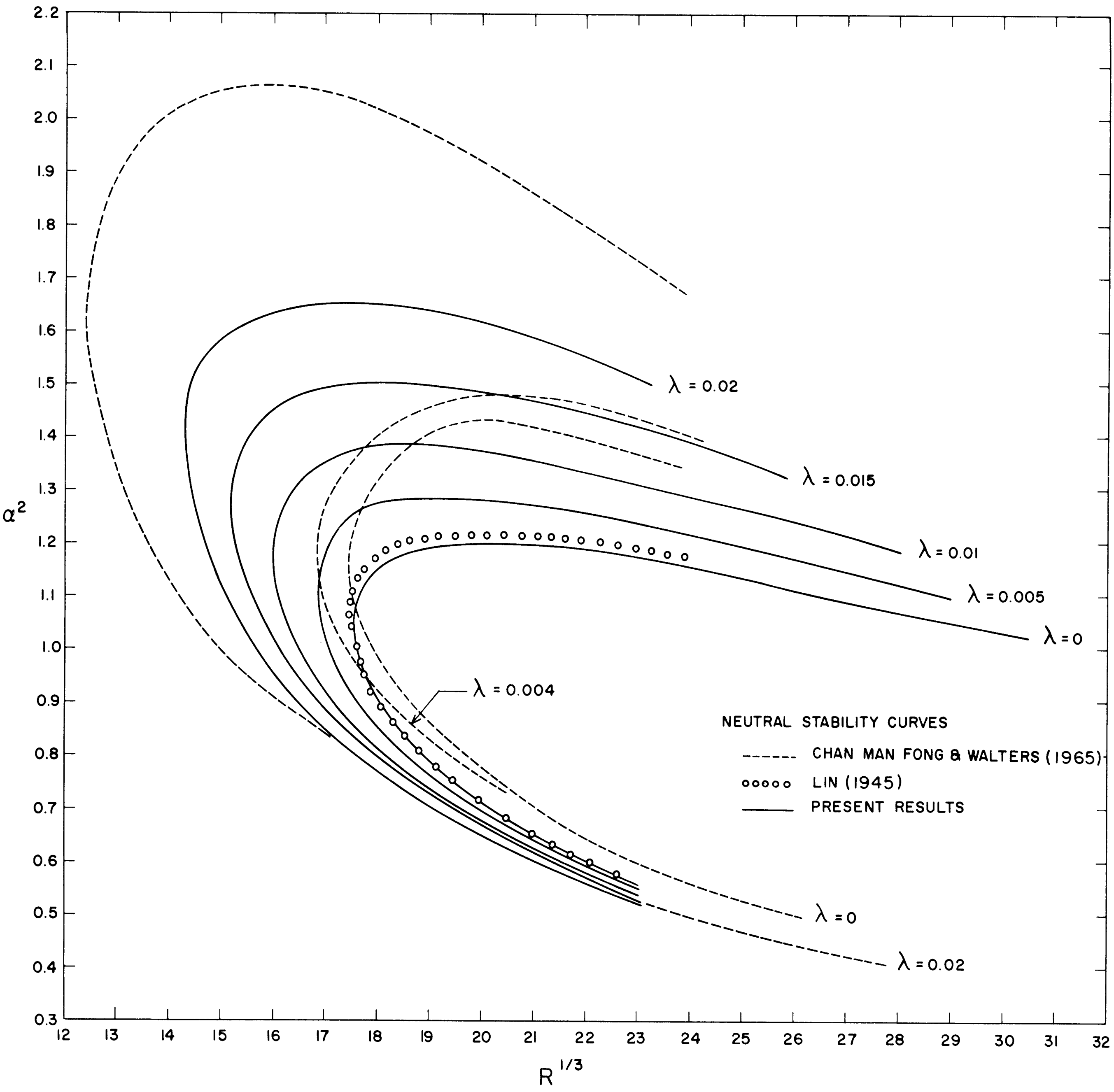

Figure 1. Neutral stability curves for various values of λ.

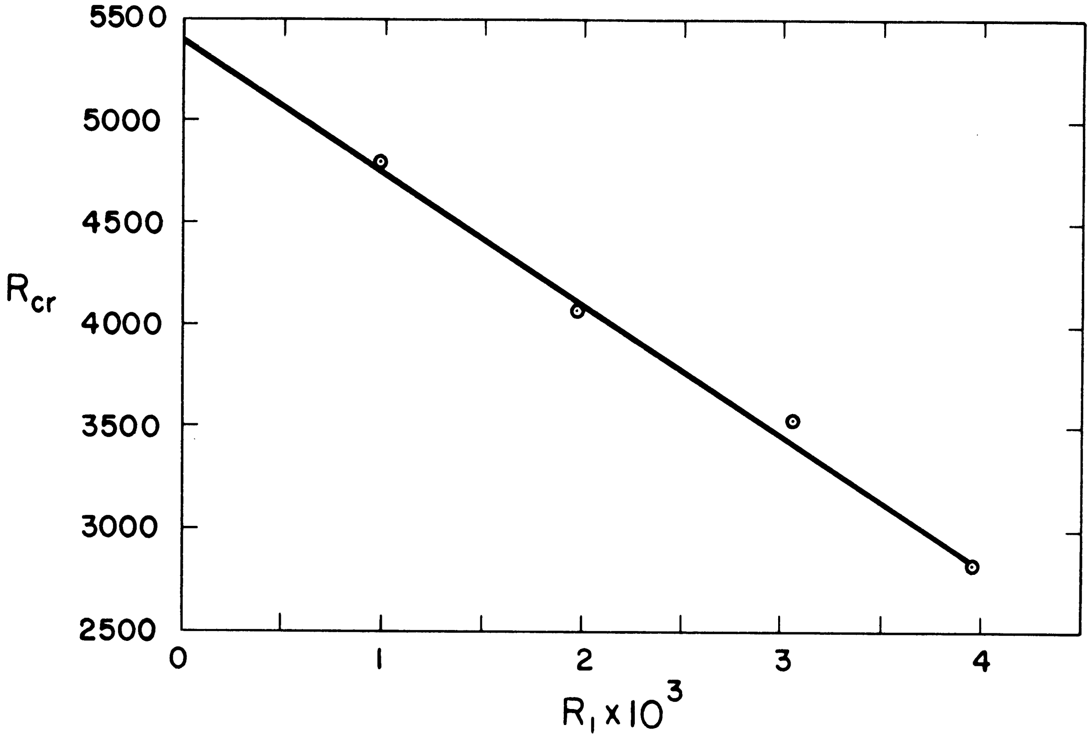

Figure 2. Critical Reynolds number versus the parameter R_1.

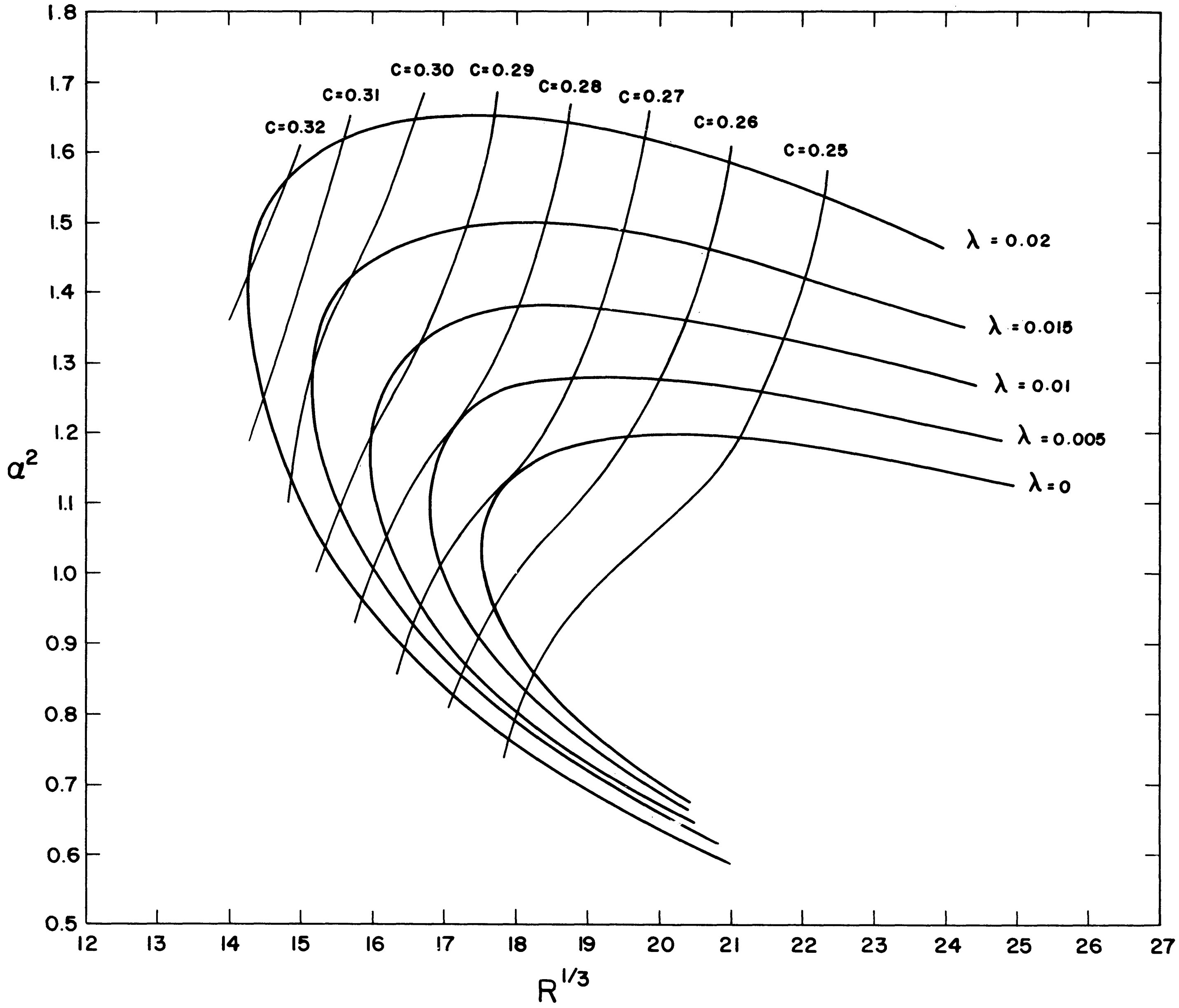

Figure 3. Neutral stability curves and curves of constant c for various values of λ.

$$-\frac{1}{iA\hat{\mu}}\Big\{\eta\frac{d^2v}{d\eta^2}\Big[6DW_cD^2W_cJ^2_{2c} - 8(DW_c)^3D^2W_cJ^3_{4c}+iADW_c\big(2J^2_{1c} - J^1_{1c}+(DW_c)^2(J^2_{3c}-4J^3_{3c})\big)\Big] + 2iADW_cJ^1_{1c}\frac{dv}{d\eta}\Big\} + O(v)$$

where the subscript appended to the J^m_n's indicates that they are evaluated at y_c; the J^m_n's and K^m_n's have been assumed to be all of the same order of magnitude. The parameter $\tilde{\mu}$ is defined by $\tilde{\mu} = (\alpha\tilde{R})^{-1/3}$, where $\tilde{R} = \rho h W_0 D^2W_c(\partial p/\partial z)^{-1} = \dfrac{\rho h y_c W_0 D^2W_c}{J^1_{0c}DW_c}$.

Making the distances and velocities dimensionless as before, and defining

$$\tilde{R}_1 = (\tilde{R}/\rho h^2)\Big[J^2_{1c} - 4J^3_{3c}\Big(\frac{W_0DU_c}{h}\Big)^2\Big],$$

$$\tilde{R}_2 = (\tilde{R}/\rho h^2)\Big[J^1_{1c} + J^2_{1c} + 8J^3_{3c}\Big(\frac{W_0DU_c}{h}\Big)^2\Big],$$

substitution of these stresses into equation (11) along with use of the equation for the primary flow results in

$$\frac{d^4v}{d\eta^4} - i\eta DU_c\frac{d^2v}{d\eta^2} + \tilde{\mu}\Big\{-\frac{1}{2}i\eta^2D^2U_c\frac{d^2v}{d\eta^2} - \eta\Big(\frac{D^3U_c}{D^2U_c} + i\alpha DU_c\tilde{R}_1\Big)\frac{d^4v}{d\eta^4} + 2\Big(-\frac{D^3U_c}{D^2U_c} + i\alpha DU_c\tilde{R}_2\Big)\frac{d^3v}{d\eta^3}\Big\} + O(\tilde{\mu}^2) = 0.$$

Proceeding further as in case B, with $\epsilon_1 = \alpha \tilde{\mu} \tilde{R}_1, \epsilon_2 = \alpha \tilde{\mu} \tilde{R}_2$, we have

$$(1 - i\epsilon_1 \eta DU_c) \frac{d^4\chi^{(0)}}{d\eta^4} + 2i\epsilon_2 DU_c \frac{d^3\chi^{(0)}}{d\eta^3} - i\eta DU_c \frac{d^2\chi^{(0)}}{d\eta^2} = 0 \tag{22}$$

as the governing equation in the inner region. It is seen then that if ϵ_1 and ϵ_2 are both small compared to unity, but larger that $\tilde{\mu}$, again $\chi_1^{(0)}$ and $\chi_2^{(0)}$ remain unchanged, but a first approximation to $\chi_3^{(0)}$ is now

$$\chi_3^{(0)} \approx (1 - \tfrac{1}{2} i\epsilon_2 \eta DU_c)\phi_0 + \epsilon_1 \phi_1$$

where ϕ_0, ϕ_1 are as given in part B.

Away from the critical region, the inviscid equation will again hold, but the primary velocity profile is of course different from the parabolic one. (For the model typified by equation (1) with $M = N = 1$ and $L_t = L_{tC}$, for instance, DW satisfies a cubic equation where the coefficients are linear functions of y.) Comparing a fluid of type A' (or B') with type C', if N and $\partial p/\partial z$ are the same for both cases, the fluid of type C' will have a steeper velocity profile than the parabolic one. To carry out the details of the solution, it is necessary to specify N. When this is known, W can readily be found by numerical methods, and equation (17) then used to determine the neutral stability curve.

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APPENDIX

An exact solution of equation (18) is possible and has, in fact, been given by Chan Man Fong and Walters (1965). A modified and more complete version of their results is presented here to show its use in the present method.

Equation (18) is of the form

$$(1 - i\lambda\eta)\frac{d^2\Phi}{d\eta^2} - i\eta\,\Phi = 0.$$

Writing

$$z = 2(i\lambda\eta - 1)\lambda^{-3/2}, \quad \delta = \tfrac{1}{2}\lambda^{-3/2},$$

and

$$\Phi = e^{-z/2} f(z),$$

the equation for f is the confluent hypergeometric form

$$z f'' - z f' - \delta f = 0$$

with solutions

$$U(z,\delta) = \frac{1}{\Gamma(\delta)}\int_0^{\infty e^{i\beta}} e^{-zv}\, v^{\delta-1}(v+1)^{-\delta-1}\,dv, \quad (23)$$

$$\left(-\pi < \beta < \pi;\ -\tfrac{\pi}{2} < \beta + \arg z < \pi/2\right),$$

and

$$V(z,\delta) = \frac{e^{z}\,\Gamma(\delta)}{2\pi i}\int_{-\infty,0+}^{-\infty} e^{vz}\, v^{-\delta}(v+1)^{\delta}\, dv.$$

As z approaches $i\infty$, U(z) approaches $z^{-\delta}$ and V(z) approaches $z^{\delta}e^{\delta}$. This suggested that U must be the solution corresponding to χ_3 and V to χ_4. To verify that this is indeed the case, replace v in equation (23) by $1/2(-1 + s\sqrt{\lambda})$. Then in the limit as λ approaches zero with arg $z = -\pi$ and $\beta = 2\pi/3$,

$$\lambda^{-1/2}\,\Gamma(\delta)\,(-1)^{\delta}\, e^{-z/2}\, U(z)$$
$$\rightarrow -2\int_{\infty}^{\infty e^{i\beta}} \exp\left[-i\eta s - \tfrac{1}{3}s^{3}\right]\left[1 + \lambda\left(s^{2} - \tfrac{1}{5}s^{5}\right) + \cdots\right] ds,$$

or, upon expressing the integral in terms of Hankel functions,

$$\lambda^{-1/2}\,\Gamma(\delta)\,(-1)^{\delta}\, e^{-z/2}\, U(z)$$
$$\rightarrow -2\left[I - \lambda\left(\frac{d^{2}I}{d\eta^{2}} + \frac{i}{5}\frac{d^{5}I}{d\eta^{5}}\right)\right]$$
$$= -2\left[I + \tfrac{1}{5}\lambda\frac{d^{2}}{d\eta^{2}}\left(\eta\frac{dI}{d\eta} - 4I\right)\right],$$

where

$$I = \int_{\infty}^{\infty e^{2\pi i/3}} \exp\left[-is\eta - \tfrac{1}{3}s^{3}\right] ds$$
$$= -\pi\sqrt{\eta/3}\; e^{-i\pi/12}\, H^{(1)}_{1/3}\left[\tfrac{2}{3}(i\eta)^{3/2}\right].$$

This is the desired result apart from a multiplicative constant. Thus the general form of $\chi_3^{(0)}$ for arbitrary λ is given by integrating

$$\frac{d^2\chi_3^{(0)}}{d\eta^2} = \int_0^{\infty e^{2\pi i/3}} \exp\left[-z(v+\tfrac{1}{2})\right](v+1)^{-\delta-1} v^{\delta-1}\, dv$$

and using this $\chi_3^{(0)}$ in the characteristics equation (17).

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3. REPORT TITLE

The Stability of Parallel Flows of Fluids with Memories

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (Last name, first name, initial)

Mook, Dean T.

Graebel, W. P., Project Director

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1967	51	--

8a. CONTRACT OR GRANT NO. Nonr-1224(49) b. PROJECT NO. NR 062-342 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 06505-3-T 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

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