"On Relationships Between L_1, L_2 and L_∞ Minima"

K.G. Murty and K.S. Al-Sultan
Department of Industrial and Operations Engineering
University of Michigan
Ann Arbor, MI 48109-2117, USA

and

Systems Engineering Department
King Fahd University of Petroleum and Minerals
Dhahran-31261, Saudi Arabia

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Abstract

Let K be a nonempty convex polyhedron { x : Ax $\geqq$ b } in R^n. We investigate the problems of finding the nearest point in K to the origin by the L_2 (Euclidean), L_1 (Rectilinear), and the L_∞ (Tschebychev) distances. Problems of this type occur very often in curve fitting and parameter estimation applications, and there has been a long standing controversy on which distance measure to use in such applications, for which there is as yet no mathematically satisfactory resolution. We establish relationships between the sets of optimum solutions of the nearest point problems under these three distance measures. We provide results comparing the distances of the three minima when measured using the same measure (any one of L_1, L_2, or L_∞). Finally, we discuss ways in which an optimum solution of one of these problems might be used as an initial point in algorithms for solving the other problems.

Keywords: Convex polyhedron; Euclidean, rectilinear and Tschebychev distances; nearest points.

1. INTRODUCTION

Given a point $x = (x_1, \ldots, x_n)^T \in R^n$ the distance between the origin 0 and x, by three commonly used distance measures is defined to be:

L_2, or Euclidean distance between 0 and x, $||x||_2 = \sqrt{x_1^2 + \ldots + x_n^2} = \sqrt{x^T x}$

L_1, or rectilinear distance between 0 and x, $||x||_1 = |x_1| + \ldots + |x_n|$

L_∞, or Tschebychev distance between 0 and x, $||x||_\infty = \max\{|x_1|, \ldots, |x_n|\}$

Given a $\delta \in R^1$, $\delta > 0$, we define $S_r(\delta) = \{ x : ||x||_r \leqq \delta \}$, the L_r- sphere with 0 as center and radius δ, for $r = 2, 1, \infty$. These spheres are drawn in Figure 1 for n = 2.

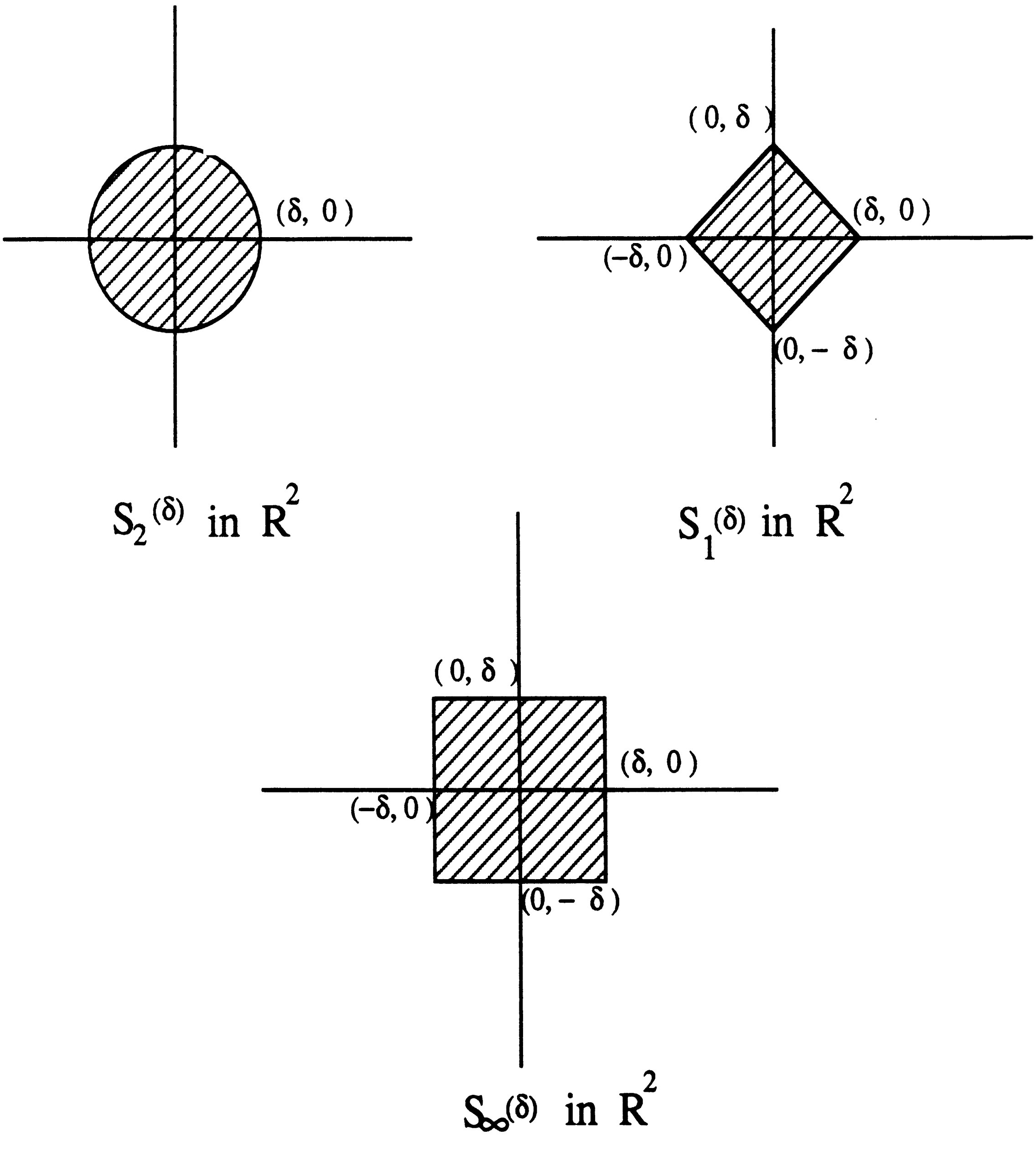

Figure 1: The L_2-, L_1- and L_∞- spheres in R^2

In R^n, $S_1(\delta)$ has 2^n facets (one corresponding to each orthant), but only 2n extreme points (one on each half-axis); whereas $S_\infty(\delta)$ has 2n facets (one corresponding to each half-axis), and 2^n extreme points (one in each orthant).

Let $K = \{x : Ax \geqq b\}$ be a given nonempty convex polyhedron in R^n. Finding the nearest point in K to the origin by the three distance measures leads to the following problems:

$$\text{minimize } \sum_{i=1}^{n} |x_i| \qquad (P1)$$
$$\text{subject to } Ax \geqq b$$

$$\text{minimize } x^T x \qquad (P_2)$$
$$\text{subject to } Ax \geqq b$$

$$\text{minimize } (\max\{|x_1|, \ldots, |x_n|\}) \qquad (P_\infty)$$
$$\text{subject to } Ax \geqq b$$

(P_2) is a strict convex quadratic program (QP).

It is well known that (P_1) is equivalent to the following linear program (LP) (P'_1), and similarly (P_∞) is equivalent to the LP (P'_∞)

$$\text{minimize} \sum_{j=1}^{n} (u_j + v_j)$$

$$\begin{aligned} \text{subject to } & Ax \geqq b && && (P'_1) \\ & x_j - u_j + v_j = 0, && j = 1 \text{ to } n \\ & u_j, v_j \geqq 0, && j = 1 \text{ to } n \end{aligned}$$

$$\text{minimize } \theta$$

$$\begin{aligned} \text{subject to } & Ax \geqq b \\ & x_j - u_j + v_j = 0, && j = 1 \text{ to } n && (P'_\infty) \\ & \theta - u_j - v_j \geqq 0, && j = 1 \text{ to } n \\ & u_j, v_j \geqq 0, && j = 1 \text{ to } n \end{aligned}$$

We will refer to the optimum solutions of (P_1), (P_2), (P_∞) as the L_1-, L_2-, L_∞- minima or nearest points respectively. It should be understood that this refers to the nearest points in K to the origin.

Being the optimum solution of a strict convex QP, the L_2- nearest point in K is unique. Since the LPs (P'_1), (P'_∞) may have alternate optimum solutions, the L_1- and L_∞- nearest points in K may not be unique. We will use the following notation:

$\bar{x} = (\bar{x}_1, ..., \bar{x}_n)^T = L_2$- nearest point in K

Γ_1, Γ_∞ = sets of L_1-, L_∞- nearest points in K respectively

$\delta_r = ||x||_r$ for each $x \in \Gamma_r$, $r = 1, \infty$; and $\delta_2 = ||\bar{x}||_2$.

Since we assumed that $K \neq \phi$, $\bar{x}$ exists, and Γ_1 and Γ_∞ are both nonempty. Also, $\{\bar{x}\} = K \cap S_2(\delta_2)$, and $\Gamma_1 = S_1(\delta_1) \cap K$, $\Gamma_\infty = S_\infty(\delta_\infty) \cap K$. Obviously $\bar{x}$ is a boundary point of both K and $S_2(\delta_2)$. Γ_1, Γ_2 are both subsets of the boundary of K. Also Γ_1, Γ_2 are subsets of the boundaries of $S_1(\delta_1)$, $S_\infty(\delta_\infty)$ respectively.

Nearest point models of the type (P_1), (P_2), and (P_∞) are used very often in research studies involving statistics, science and engineering; for curve fitting, approximation of functions, etc. In a few instances, the choice of distance to use is preordained by circumstances, for example, for driving in a city with a rectangular grid of streets like Manhattan, NY, the rectilinear (L_1) distance is the only logical choice. However, in general in most applications all three distance measures make sense, and in such cases a widely debated issue, (Appa and Smith [1973]) is: which distance measure to use, L_1, L_2, or L_∞. Unfortunately, no solid scientific principles have come up to prefer one measure over the other. The choice is mostly a matter of personal preference. Sometimes researchers seem to prefer obtaining nearest points by all three measures, and select the best among them based on extraneous practical or political considerations. This opens the need to develop efficient procedures to obtain nearest points by one measure given a nearest point by another. McCormick and Sposito [1976] and Sklar and Armstrong [1982] have suggested some rules that can be used to obtain an optimum L_1- point given the L_2- minimum, in applications of constrained regression. These rules help in reducing the number of iterations required to solve the LP corresponding to the L_1- regression problem.

In this paper, we investigate the relationships between the sets $\{\bar{x}\}$, Γ_1, and Γ_∞, and ways in which the computation of the others can be simplified given one of them. We also derive results comparing the distances of the L_1-, L_2-, L_∞- minima measured using the same measure (one of L_1, L_2, or L_∞). Surprisingly, we find that these could differ by as a factor of approximately $\sqrt{n}$, or n asymptotically, so in higher dimensions they could be wide apart by any measure. The mysterious $\sqrt{n}$ or n factors add fuel to the controversy but leave the preferential nature of the choice among these measures unaltered. Philosophically this is unfortunate. It lends support to the skeptical view often expressed by practitioners, "the solution to a practical problem obtained from a correct and highly rigorous mathematical analysis, may depend critically on the way mathematics is used to model the problem."

2. SOME ILLUSTRATIVE EXAMPLES

$\bar{x}$ is a boundary point of K, and Γ_1, Γ_∞ are subsets of the boundary of K. Each of the sets Γ_1, Γ_∞ is a subset of a face of K. They could be proper subsets of faces of K as illustrated in Figure 2. Each of $\bar{x}$, Γ_1, Γ_∞ may lie in different faces of K with no intersection, this is illustrated in Figure 3.

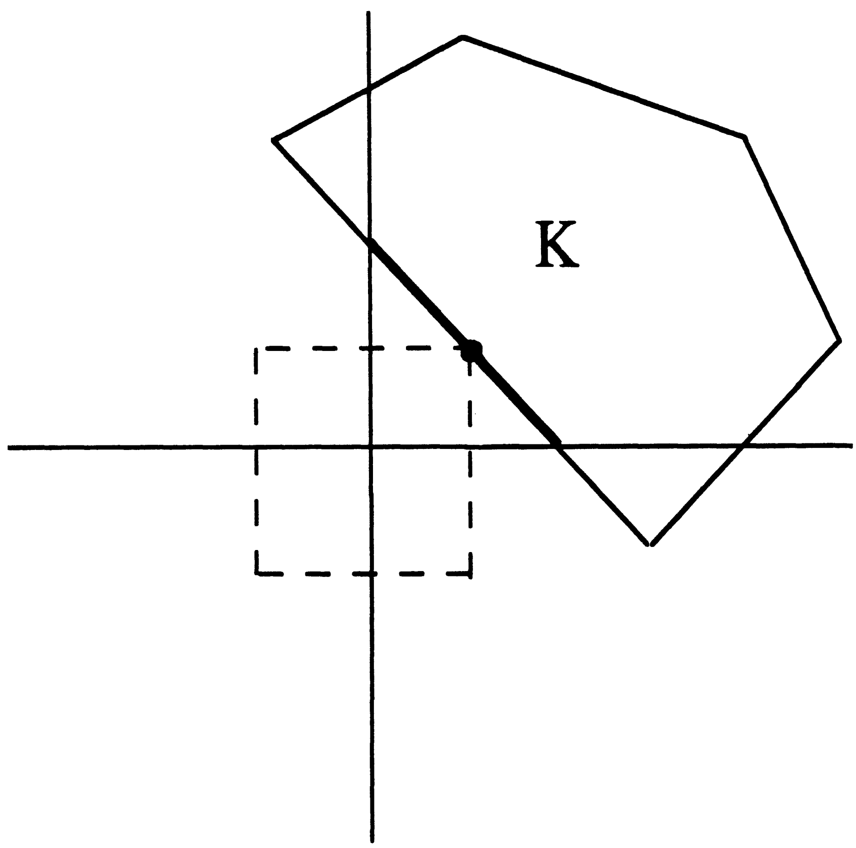

Figure 2: An example in R^2 where Γ_1 and Γ_∞ are both proper subsets of a face of K. Γ_1 is the thick line segment and Γ_∞ is the singleton set containing only the dotted point.

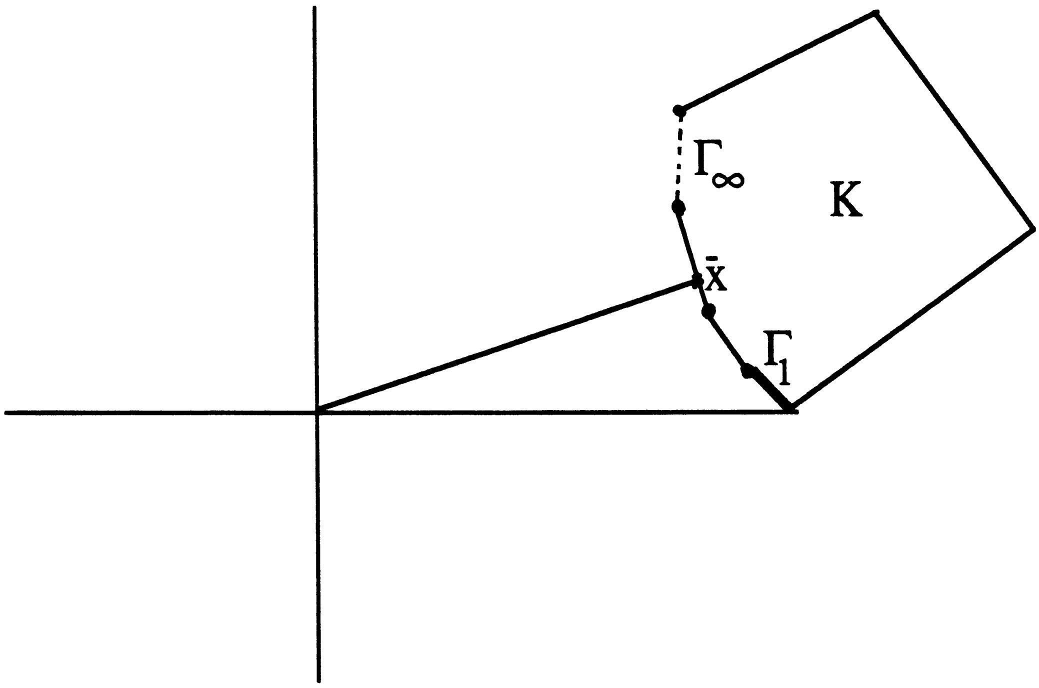

Figure 3: An example in R^2 with $\bar{x}$, Γ_1 (thick line segment), Γ_∞ (dashed line) lying in different nonintersecting faces of K.

3. RESULTS ON THE SETS Γ_1, $\{\bar{x}\}$, Γ_∞.

Remember that $K = \{ x : Ax \geqq b \}$. There may be no sign restrictions on x, and in general K may not be contained in a single orthant of R^n. However, we will show that Γ_1 is always contained in a single orthant of R^n.

THEOREM 1: Γ_1 lies in a single orthant of R^n.

PROOF: As discussed earlier $\Gamma_1 = S_1(\delta_1) \cap K$, and it is a subset of the boundaries of both K and $S_1(\delta_1)$. Let F be a facet of $S_1(\delta_1)$ containing a point of Γ_1, and H the hyperplane in R^n containing F. Hence H separates $S_1(\delta_1)$ from K, this implies that $S_1(\delta_1) \cap K = \Gamma_1 \subset F$, thus all of Γ_1 is in the orthant of R^n containing F (Figure 4).

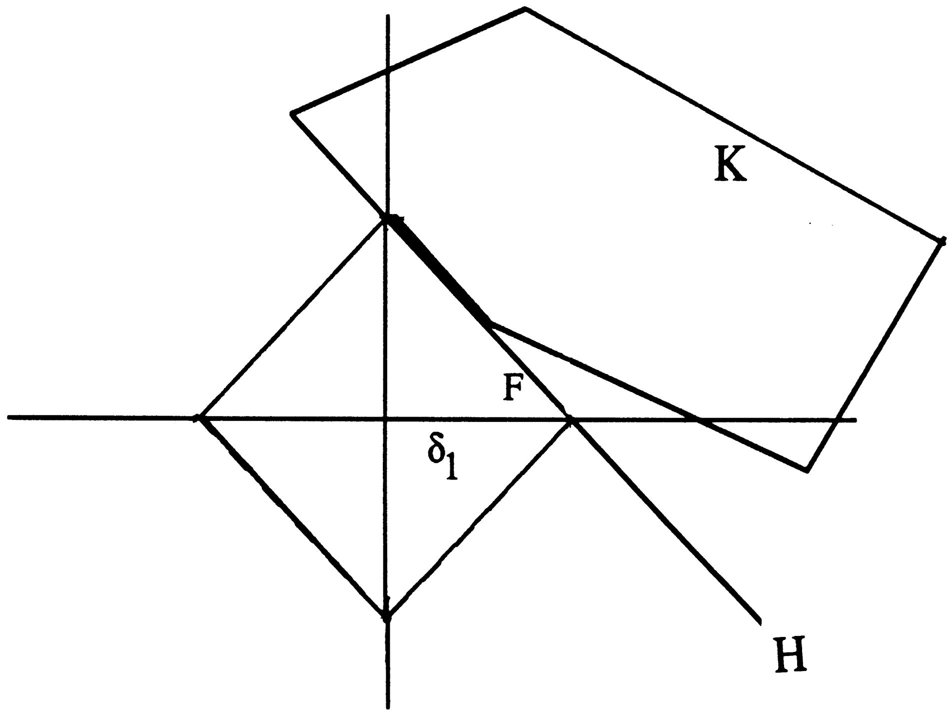

Figure 4: The thick line segment is $\Gamma_1 = F \cap K$.

RESULT: There may not be a single orthant containing the entire set Γ_∞. See Figure 5.

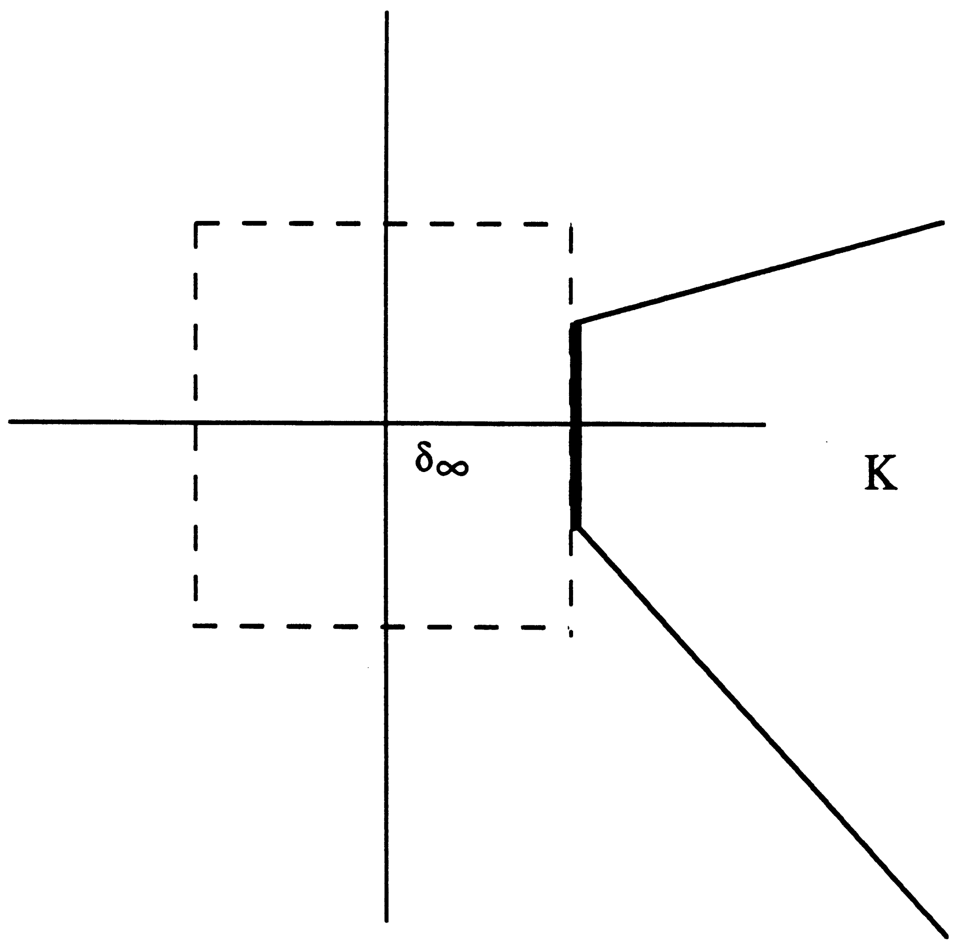

Figure 5: Example in R^2. Γ_∞ is the thick line segment.

4. RELATIONSHIPS AMONG δ_1, δ_2, δ_∞.

THEOREM 2:

(i) Either all three of δ_1, δ_2, δ_∞ are zero, or all three of them are strictly > 0.

(ii) (a) $\frac{\delta_1}{\sqrt{n}} \leqq \delta_2 \leqq \delta_1$

(b) $\delta_\infty \leqq \delta_2 \leqq \delta_\infty \sqrt{n}$

(c) $\delta_\infty \leqq \delta_1 \leqq n\, \delta_\infty$

(iii) $\delta_1 = \delta_2 = \delta_\infty$ iff $\bar{x}$ lies on a coordinate axis of R^n.

(iv) All the L_1-, L_2-, L_∞- nearest points coincide at a single point iff $\bar{x}$ lies on a coordinate axis.

PROOF:

(i) Follows very directly. All three of δ_1, δ_2, δ_∞ are zero if the origin is in K, otherwise all three are > 0.

(ii) (a): Let F be a facet of $S_1(\delta_1)$ containing Γ_1. The farthest points in F from the origin by the Euclidean distance are the points of intersection of the coordinate axes with F whose Euclidean distance is δ_1. Hence Γ_1 contains points whose Euclidean distance is $\leqq \delta_1$. Since $\bar{x}$ is the closest point in $K \supset \Gamma_1$ by the Euclidean distance, we have $\delta_2 \leqq \delta_1$.

H, the hyperplane containing F separates K from the origin, and the closest point in H to the origin by L_2- distance, has distance $\delta_1/\sqrt{n}$, so $\delta_2 \geqq \delta_1/\sqrt{n}$, completing the proof of (a).

(ii)(b,c) Let G be a facet of $S_\infty(\delta_\infty)$ containing Γ_∞. The Euclidean distance of points in G lies between δ_∞ and $\delta_\infty\sqrt{n}$, see Figure 6. This implies (b) by an argument similar to that in (a). The rectilinear distance of points in G lies between δ_∞ and n δ_∞, this implies (c) by an argument similar to that in (a).

(iii) Suppose $\bar{x}$ is on a coordinate axis of R^n. Then $||\bar{x}||_1$ and $||\bar{x}||_\infty$ are both equal to δ_2. So, from (ii)(a), $\delta_1 = \delta_2$ and $\bar{x}$ is also the L_1-minimum in K. Also, the tangent hyperplane to $S_2(\delta_2)$ at $\bar{x}$ is a facet of $S_\infty(\delta_2)$ (as $\bar{x}$ is a point on a coordinate axis) and it separates the origin and K, hence $\delta_\infty = \delta_2$ also, and $\bar{x}$ is also an L_∞- minimum in K.

Conversely, suppose $\delta_1 = \delta_2 = \delta_\infty$. The only points in $S_1(\delta_1)$ whose Euclidean distance is also δ_1 are the points on the coordinate axes. Hence $\delta_1 = \delta_2$ implies that the L_1- minimum in K must be a point on a coordinate axis, and by the above this implies that this same point is also the L_2- and L_∞- minimum in K.

(iv) This follows from (iii) and the fact that the only points which are common to the boundaries of L_2-, L_1-, and L_∞- spheres of the same radius are those on the coordinate axes.

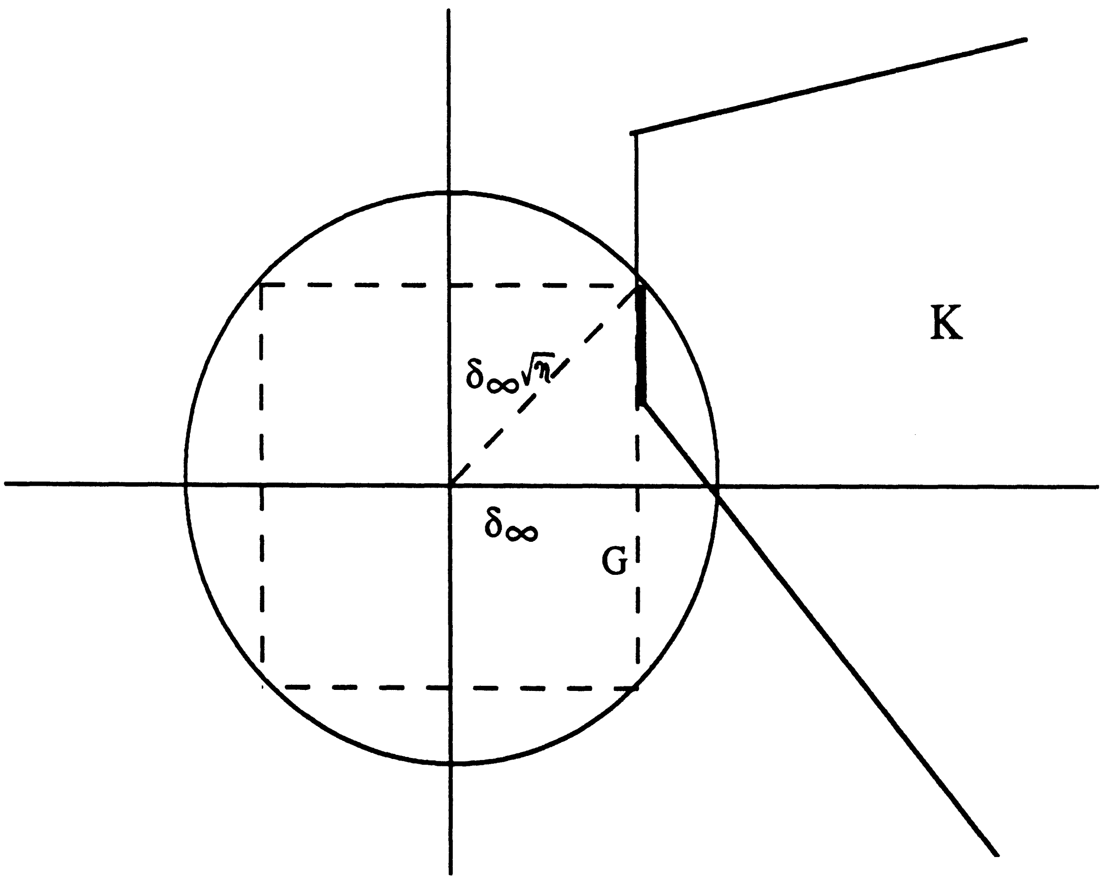

Figure 6.

5. COMPARISON OF THE DISTANCES OF THE THREE MINIMA, USING THE SAME MEASURE.

To help choose one of the three L_1-, L_2- or L_∞- distance measures, we need comparative figures on how far the three minima could be spread out. For example, if the distance of the L_2- minimum, $\bar{x}$, is measured in L_1- distance, how much greater than δ_1 can it be in the worst case? If this difference is small, we could consider it fairly unimportant which among L_1 or L_2 is selected as the distance measure. We obtain results on such comparative questions for every pair of distance measures in this section. We have the following definitions:

μ_1^2, μ_∞^2, the L_1- and L_∞ distances of $\bar{x}$.

μ_2^1, (μ_∞^1), the L_2- (L_∞-) distance of the nearest L_2- (L_∞-) point to the origin in Γ_1.

μ_1^∞, (μ_2^∞), the L_1- (L_2-) distance of the nearest L_1- (L_2-) point to the origin in Γ_∞.

So, in this notation, the superscript on μ gives the type of minimum (L_1, or L_2, or L_∞), and the subscript gives the distance measure used to measure its distance.

THEOREM 3: We have

$$\delta_1 \leqq \mu_1^2 \leqq \delta_1 \sqrt{n} \qquad (1)$$

PROOF: Clearly $\mu_1^2 \geqq \delta_1$. To prove the upper bound, we look for the maximum value of μ_1^2 with δ_1 fixed. The smallest L_2- sphere containing $S_1(\delta_1)$ is $S_2(\delta_1)$, and $\bar{x}$ must lie in it. The maximum value of $||x||_1$ over $S_2(\delta_1)$ is $\delta_1\sqrt{n}$, and hence $||\bar{x}||_1 \leqq \delta_1\sqrt{n}$, which implies $\mu_1^2 \leqq \delta_1\sqrt{n}$. ❐

EXAMPLE 1: A POLYHEDRON ATTAINING HALF OF THE UPPER BOUND IN (1) ASYMPTOTICALLY WITH n.

Let K_3 be the half-space defined by

$$\sum_{i=1}^{n-1}\left(\frac{x_i}{1+\sqrt{n}}\right) + x_n \geqq \delta_1 \qquad (2)$$

where $\delta_1 > 0$. The L_1- nearest point in K_3, is unique and it is $(0, ..., 0, \delta_1)$. The L_2-nearest point in K_3 is $\bar{x} = (\lambda/\alpha, ..., \lambda/\alpha, \lambda)^T$ where $\alpha = 1 + \sqrt{n}$ and $\lambda = \delta_1 / (1 + (n-1) / (1 + \sqrt{n})^2)$. So for K_3, we have

$$\mu_1^2 = \delta_1\left(1 + \frac{n-1}{1+\sqrt{n}}\right) \Big/ \left(1 + \frac{n-1}{(1+\sqrt{n})^2}\right) \qquad (3)$$

It can be verified that the right hand side of (3) approaches $\left(\frac{1}{2}\right)\delta_1\ (1 + \sqrt{n})$ asymptotically (i.e., as n gets larger).

NOTE 1: The proof of the upper bound $\delta_1 \sqrt{n}$ for μ_1^2 in Theorem 3 is extremely simple, but we believe that bound is not tight. We believe that the right hand side in (3), which approaches $\left(\frac{1}{2}\right)\delta_1 (1 + \sqrt{n})$ asymptotically is an upper bound for μ_1^2 (this is actually achieved in the polyhedron K_3 in Example 1) but its proof is messy.

THEOREM 4: We have

$$\delta_\infty \leqq \mu_\infty^2 \leqq \delta_\infty \sqrt{n} \tag{4}$$

PROOF: Clearly $\mu_\infty^2 \geqq \delta_\infty$. To prove the upper bound, we look for the maximum value of μ_∞^2 with δ_∞ fixed. K has a nonempty intersection with the boundary of $S_\infty(\delta_\infty)$ but not its interior, and the smallest L_2- sphere containing $S_\infty(\delta_\infty)$ is $S_2(\delta_\infty \sqrt{n})$, and $\bar{x}$ must lie in it. The maximum value of $||x||_\infty$ over $S_2(\delta_\infty \sqrt{n})$ is $\delta_\infty \sqrt{n}$, and hence $||\bar{x}||_\infty \leqq \delta_\infty \sqrt{n}$, which implies $\mu_\infty^2 \leqq \delta_\infty \sqrt{n}$. ❐

EXAMPLE 2: A POLYHEDRON ATTAINING HALF OF THE UPPER BOUND IN (4) ASYMPTOTICALLY WITH n.

Let K_4 be the half-space defined by

$$\sum_{i=1}^{n-1} \left(\frac{x_i - \delta_\infty}{\sqrt{n}}\right) + x_n \geqq \delta_\infty \tag{5}$$

Where $\delta_\infty > 0$. The L_∞- nearest point in K_4 is unique and it is $(\delta_\infty, \ldots, \delta_\infty)^T$. The L_2- nearest point in K_2 is $x = (\lambda/\sqrt{n}, \ldots, \lambda/\sqrt{n}, \lambda)^T$ where $\lambda = \delta_\infty\left(1 + \frac{n-1}{\sqrt{n}}\right)/\left(1 + \frac{n-1}{n}\right)$.

Its L_∞- distance is

$$\mu_\infty^2 = \lambda = \delta_\infty\left(\frac{\sqrt{n}(n - 1 + \sqrt{n})}{2n - 1}\right) \qquad (6)$$

It can be verified that the right hand side of (6) approaches $(\delta_\infty/2)\,(1 + \sqrt{n})$ asymptotically.

NOTE 2: Again, the proof of the upper bound $\delta_\infty \sqrt{n}$ for μ_∞^2 in Theorem 4 is very simple, but we believe that this bound is not tight. We conjecture that the right hand side of (6), which approaches $(\delta_\infty/2)\,(1 + \sqrt{n})$ asymptotically, is an upper bound for μ_∞^2, this is achieved in the polyhedron K_4 in Example 2.

THEOREM 5: We have

$$\delta_2 \leqq \mu_2^\infty \leqq \delta_2 \sqrt{n} \qquad (7)$$

PROOF: As in the proofs of Theorems 3, 4, this follows because the smallest L_∞- sphere containing $S_2(\delta_2)$ is $S_\infty(\delta_2)$ and the maximum value of $\|x\|_2$ over $S_\infty(\delta_2)$ is $\delta_2 \sqrt{n}$. ❐

THEOREM 6: We have

$$\delta_2 \leqq \mu_2^1 \leqq \delta_2 \sqrt{n} \tag{8}$$

PROOF: As in the proofs of Theorems 3, 4, this follows because the smallest L_1- sphere containing $S_2(\delta_2)$ is $S_1(\delta_2 \sqrt{n})$ and the maximum value of $||x||_2$ over $S_1(\delta_2 \sqrt{n})$ is $\delta_2 \sqrt{n}$. ❐

THEOREM 7: We have

$$\delta_\infty \leqq \mu_\infty^1 \leqq n\, \delta_\infty \tag{9}$$

PROOF: Again this follows because the smallest L_1- sphere containing $S_\infty(\delta_\infty)$ is $S_1(n\delta_\infty)$, and the maximum value of $||x||_\infty$ over $S_1(n\delta_\infty)$ is $n\delta_\infty$. ❐

THEOREM 8: We have

$$\delta_1 \leqq \mu_1^\infty \leqq n\delta_1 \tag{10}$$

PROOF: This follows because the smallest L_∞- sphere containing $S_1(\delta_1)$ is $S_\infty(\delta_1)$ and the maximum value of $||x||_1$ over $S_\infty(\delta_1)$ is $n\delta_1$. ❐

EXAMPLES TO SHOW THAT THE UPPER BOUNDS IN THEOREMS 5 TO 8 ARE TIGHT IN THE LIMIT.

For Theorem 5, take $S_2(\delta_2)$ and the tangent hyperplane, H_5, to it at a point on its boundary in the interior of R_+^n but very close to $(0, ..., 0, \delta_2)^T$. See Figure 7. Let K_5 be the half-space defined by H_5 not containing the origin. Then μ_2^∞ for K_5 can be verified to be very close to $\delta_2 \sqrt{n}$.

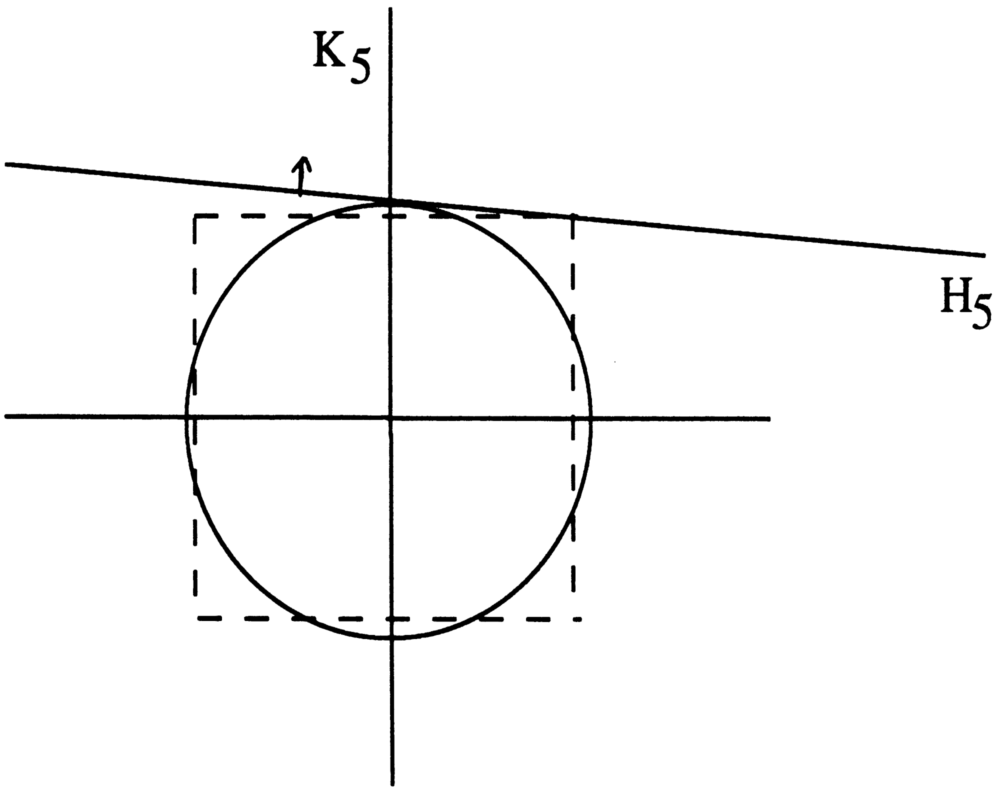

Figure 7: Construction to show that the upper bound in Theorem 5 is tight in the limit.

For Theorem 6, take $S_2(\delta_2)$, and the tangent hyperplane to it, H_6, at a boundary point close to $(\delta_2/\sqrt{n}, ..., \delta_2/\sqrt{n})^T$. See Figure 8. Let K_6 be the half-space defined by H_6 not containing the origin. Then for K_6, μ_2^1 can be verified to be very close to $\delta_2 \sqrt{n}$.

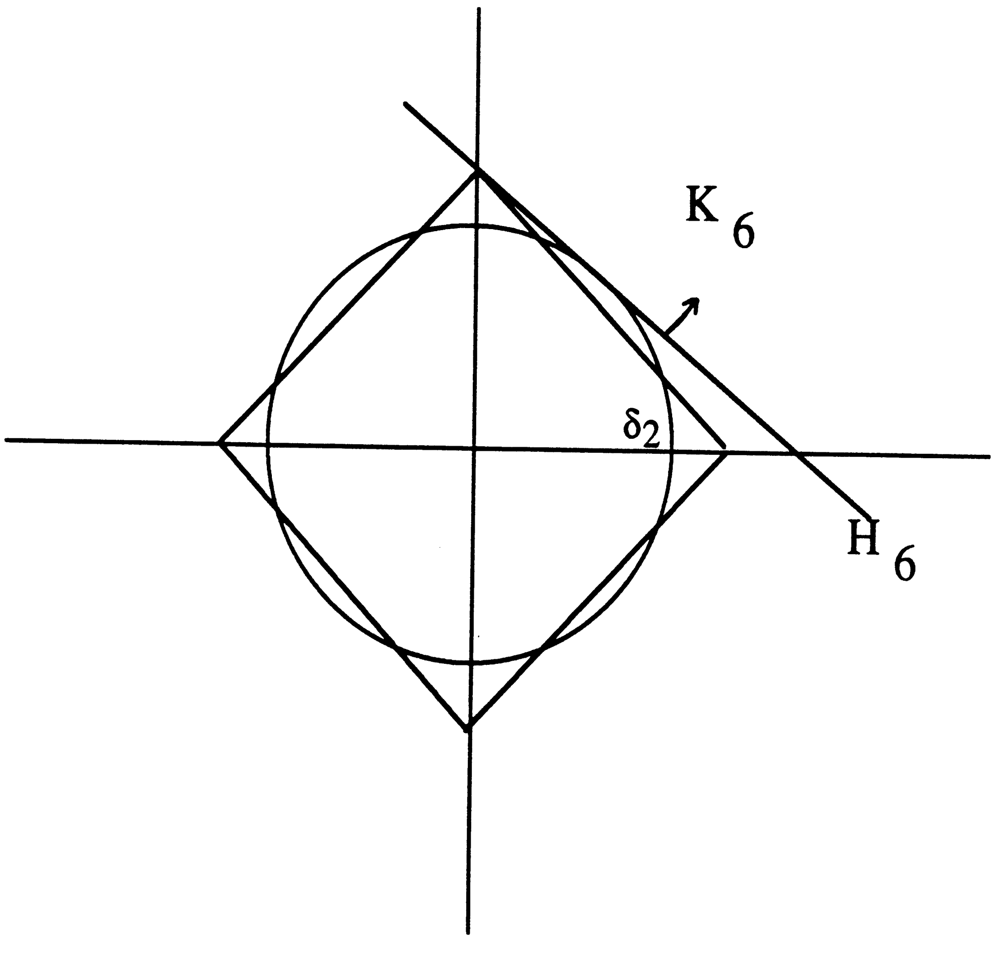

Figure 8: Construction to show the tightness of the upper bound in Theorem 6.

For Theorem 7, take H_7, the facetal hyperplane of $S_1(n\delta_\infty)$ in the nonnegative orthant, and tilt it slightly keeping the point $(\delta_\infty, ..., \delta_\infty)^T$ fixed. Let K_7 be the half-space defined by this hyperplane not containing the origin. See Figure 9. It can be verified that for K_7, μ^1_∞ is close to $n\delta_\infty$.

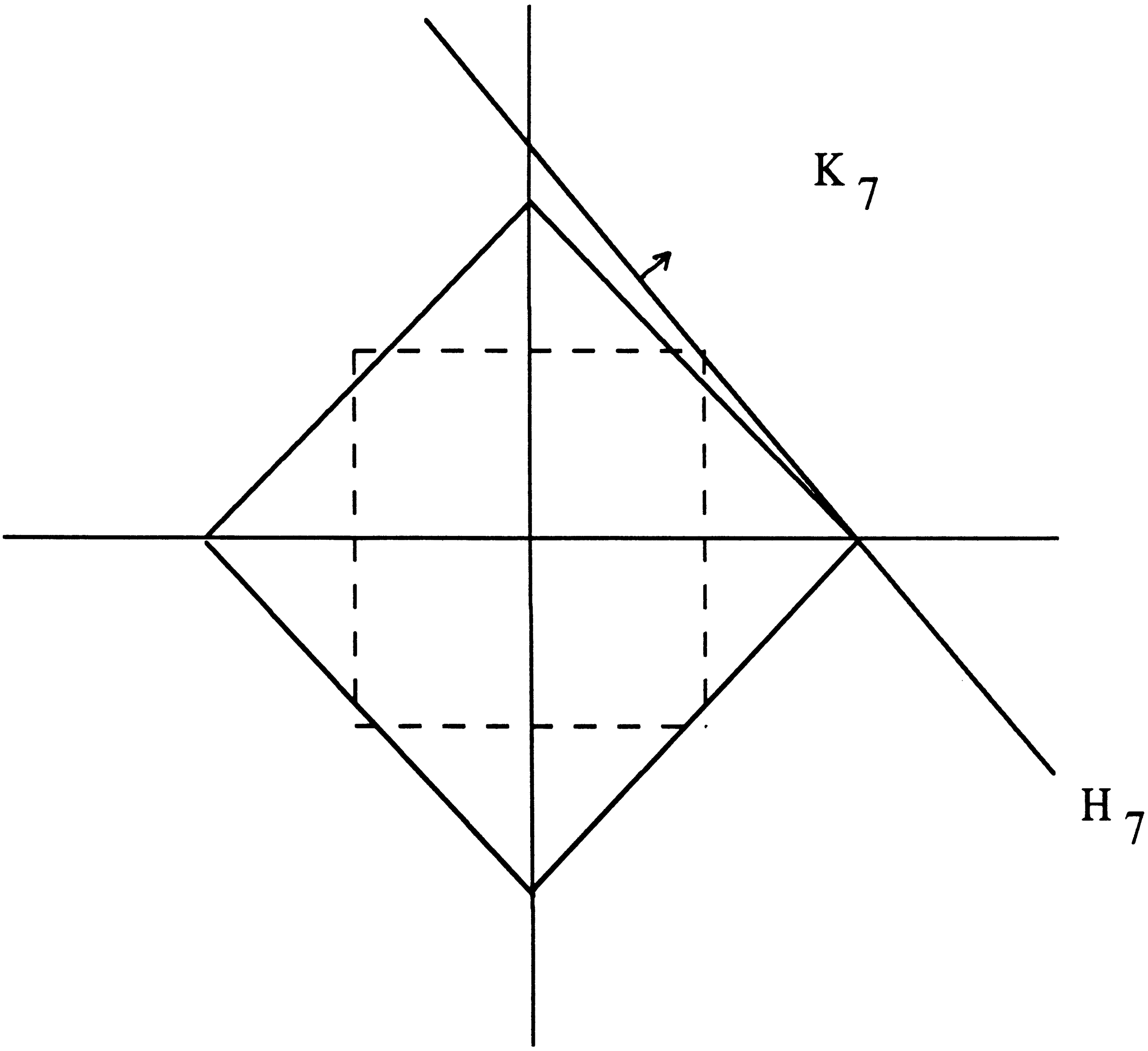

Figure 9: Construction to show the tightness of the upper bound in Theorem 7.

For Theorem 8, take $S_\infty(\delta_1)$, and keeping the point $(0, 0, ..., 0, \delta_1)^T$ fixed, tilt its facetal hyperplane containing this point slightly, and let K_8 be the half-space not containing the origin, defined by this tilted plane. See Figure 10. It can be verified that μ_1^∞ for K_8 is very close to $n\delta_1$.

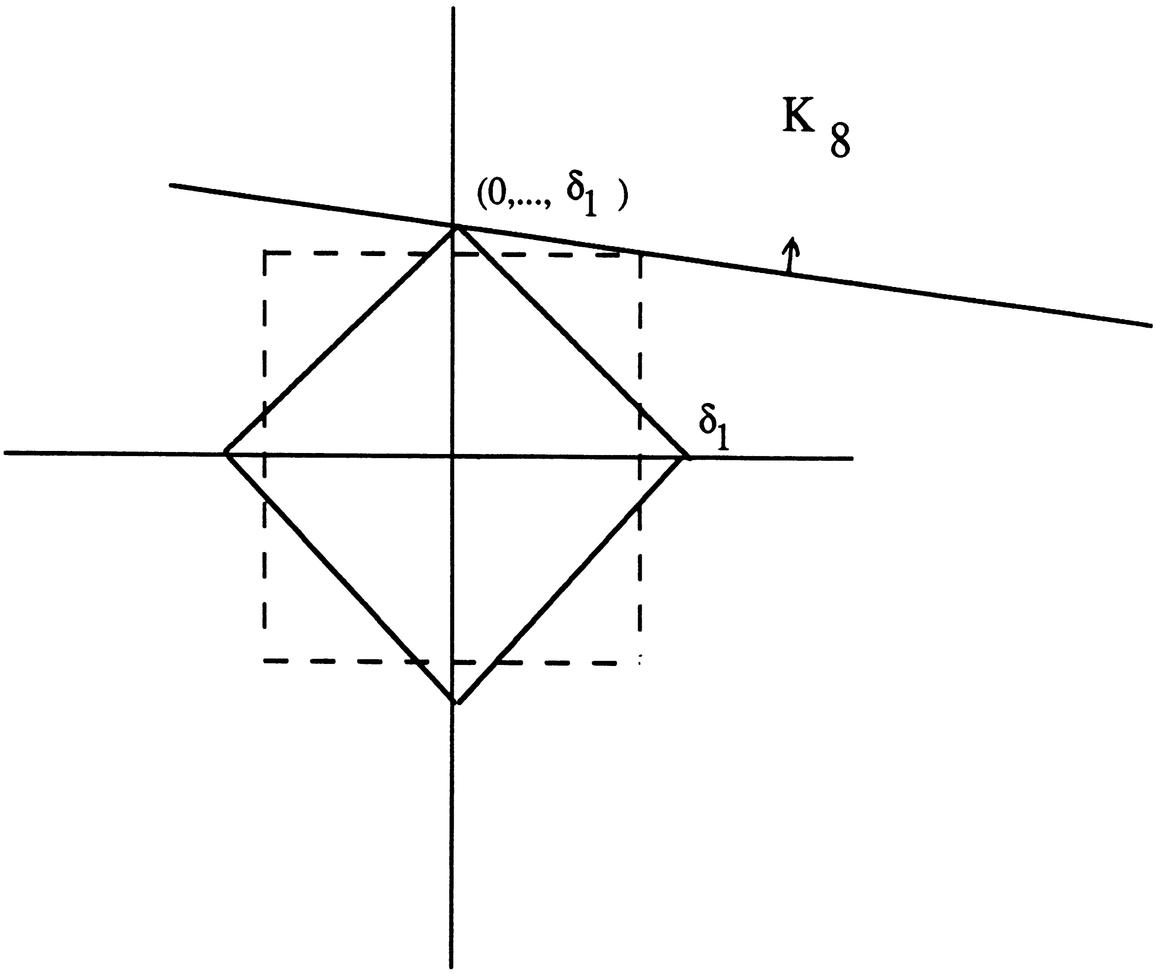

Figure 10: Construction to show the tightness of the upper bound in Theorem 8.

6. HOW TO USE A SOLUTION OF ONE OF THE PROBLEMS, AS A GOOD INITIAL POINT TO SOLVE THE OTHERS.

L_1-, L_∞- nearest point problems are LPs, while the L_2- nearest point problem is a QP. A variety of methods are available to solve each of these problems. Here we show how to use a given solution to one of these problems as a good initial point in algorithms to solve the others.

Suppose (P'_∞) is solved by a variant of the simplex algorithm, and let $\tilde{x} = (\tilde{x}_j)$ be the L_∞- nearest point obtained from an optimum basic feasible solution (BFS) of (P'_∞). Define $\tilde{v}_j = |\text{ minimum } \{0, \tilde{x}_j\}|$, $\tilde{u}_j = \text{maximum } \{0, \tilde{x}_j\}$, for j = 1 to n, $\tilde{u} = (\tilde{u}_j)$, $\tilde{v} = (\tilde{v}_j)$. Then $(\tilde{x}, \tilde{u}, \tilde{v})$ is a BFS of (P'_1) and can be used to initiate the solution of (P'_1) by the simplex algorithm. Similarly, if $(\hat{x}, \hat{u}, \hat{v})$ is an optimum BFS of (P'_1), define $\hat{\theta} = \text{Maximum } \{\hat{u}_j + \hat{v}_j : j = 1 \text{ to } n\}$, then $(\hat{x}, \hat{u}, \hat{v}, \hat{\theta})$ is a BFS of (P'_∞) and can be used to initiate the solution of (P'_∞) by the simplex algorithm.

Now, let w = Ax - b, the vector of slack variables, and z, the vector of Lagrange multipliers, associated with the constraints in (P_2). Then the linear complementarity problem (LCP) corresponding to (P_2) is

$$\begin{aligned} & w - AA^T z = -b \\ & w \geqq 0, z \geqq 0 \\ & w^T z = 0. \end{aligned} \tag{11}$$

and if $(\bar{w}, \bar{z})$ is a solution of the LCP (11), then $\bar{x} = A^T\bar{z}$ is an optimum solution of (P_2). Given x^1, an L_1- or L_∞- nearest point, let $w^1 = Ax^1 - b$, $g = -b - w^1$. Since $w^1 \geqq 0$, $g = 0$ iff $b \leqq 0$, in this case $0 \in K$ and hence 0 is the L_1-, L_2-, and L_∞- nearest point. Assuming that this is not the case, $g \neq 0$. Introduce an artificial variable z_0

associated with the column vector g in the system of linear equality constraints in (11) leading to

$$w - AA^Tz + z_0 g = -b \tag{12}$$

Then $(w, z, z_0) = (w^1, z^1 = 0, z_0^1 = 1)$ is a nonnegative solution of (12). From the results in Section 1, at least one component in w^1 is zero, hence (w^1, z^1, z_0^1) is an almost complementary basic feasible solution of (12) and the LCP (11) can be solved by applying the complementary pivot algorithm on (12) beginning with this almost complementary BFS [Lemke [1965], Cottle and Dantzig [1968], Murty [1988]].

Given the L_2- nearest point $\bar{x}$, a BFS of (P'_1) or (P'_∞) can be constructed from it by standard techniques, and this can be used to initiate the solution of those problems by the simplex algorithm.

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References

1. G. Appa, and C. Smith "On L_1 and Chebychev Estimation," *Mathematical Programming*, 5 (1973) 73-87.

2. R. W. Cottle and G. B. Dantzig, "Complementary Pivot Theory of Mathematical Programming," *Linear Algebra and Its Applications*, 1 (1968) 103-125.

3. C. Lemke, "Bimatrix Equilibrium Points and Mathematical Programming," *Management Science*, 11 (1965) 681-689.

4. G. F. McCormick, and V. A. Sposito "Using the L_2- Estimator in L_1-Estimation," *SIAM J. Numerical Analysis*, 13, No. 3 (June 1976) 377-342.

5. K. G. Murty, *Linear Complementarity, Linear and Nonlinear Programming* (Heldermann Verlag, West Berlin, 1988).

6. M. G. Sklar, and R. D. Armstrong "Least Absolute Value and Chebychev Estimation Utilizing Least Squares Results," *Mathematical Programming*, 24 (1982) 346-352.