On the Dimension of K^Δ

Katta G. Murty *

Department of Industrial and Operations Engineering

1205 Beal Avenue
The University of Michigan
Ann Arbor, MI 48109-2117, USA

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ABSTRACT

Let K be a convex polyhedron specified by a system of linear constraints. Let K^{Δ} be the convex hull of its extreme points. We derive a formula for the dimension of K^{Δ}, but show that computing it may be a hard problem.

INTRODUCTION

Let K denote a convex polyhedron specified by a system of linear constraints, and let K^Δ denote the set of its extreme points. Here we study the problem of determining the dimension of K^Δ, using the data specifying K. If the constraint system specifying K consists of equality constraints only, K is an affine space and has no extreme points, in this case $K^\Delta = \emptyset$, and the problem is trivial. So we assume that the constraint system specifying K consists of at least one inequality constraint. If there are any equality constraints in the system, using them some variables can be eliminated, thus reducing the system. So, for the sake of this study, we can, without any loss of generality, assume that the system specifying K consists of inequality constraints only. Suppose that K is the set of feasible solutions of

$$Dx \geq d \tag{1}$$

where D, d are given integer matrices of orders m×n and m×1 respectively. We assume that $K \neq \emptyset$.

We use the symbols $D_{i.}$, $D_{.j}$, to denote the ith row vector, the jth column vector of the matrix D respectively.

It is well known that the dimension of K is < n iff there exists at least one constraint in (1) which holds as a strict equality constraint at all points in K, that is, there exists an i such that $D_{i.} x = d_i$ for all $x \in K$. In that case, that constraint can be treated as an equality constraint, and a variable eliminated using it, and the process can be repeated. We assume that such reduction steps have been carried out as far as possible.

So, we assume that for each i = 1 to m, there exists an $x^i \in K$

satisfying $D_{i.} x > d_i$. Hence, the dimension of K is n.

It is well known (see, for example, [3]) that K has an extreme point

iff the set of column vectors of D is linearly independent. If K has no

extreme points, $K^\Delta = \emptyset$, and our problem becomes trivial. So, we assume that

this condition holds, that is that K has at least one extreme point.

Also, it is well known that K is bounded iff the system

$$D\xi \geqq 0 \tag{2}$$

has $\xi = 0$ as its unique solution. See [3]. If K is bounded, $K^\Delta = K$, in

this case the dimension of K^Δ = dimension of K. Our problem becomes inter-

esting if K is unbounded, that is, when (2) has at least one nonzero solution.

In this case K^Δ is a proper subset of K, it is the set of feasible solutions

of a system of constraints consisting of (1) and some additional constraints.

The number of these additional constraints needed to represent K^Δ could be

very large, but so far there is no systematic method known for generating them

in a reasonable manner. We derive a formula for the dimension of K^Δ, which

only needs the data in (1), and the solution to some problems on the convex

polyhedron K, however, these problems are NP-complete.

THE RESULTS

LEMMA 1: Let K, the set of feasible solutions of (1), be unbounded, and

suppose i is such that $D_{i.} x$ is unbounded above over K. The problem of

checking whether there exists of extreme point of K satisfying $D_{i.} x > d_i$,

is NP-complete.

PROOF: Clearly, this problem is in NP. Let F be the set of feasible

solutions of

$$D_{t.}x \geqq d_t, \quad t \neq i$$

Then by hypothesis, F is an unbounded convex polyhedron and $D_{i.}x$ is

unbounded above on it. Since the problem of finding the extreme point of

F that maximizes $D_{i.}x$ is NP-hard, see [1], the problem of checking whether

there exists an extreme point of F satisfying $D_{i.}x > d_i$ is NP-complete. By

the results of [2, 3], each extreme point of K has to belong to one of the

following types.

a) extreme points of F satisyfing $D_{i.}x > d_i$

b) extreme points of F satisyfing $D_{i.}x = d_i$

c) points of intersection of edges of F (bounded or unbounded)

   which do not totally lie in the hyperplane $\{x: \ D_{i.}x = d_i\}$,

   with that hyperplane.

So the only extreme points of K which satisfy $D_{i.}x > d_i$, are those of

type (a) above, that is, those extreme points of F satisfying $D_{i.}x > d_i$.

But from the argument made above, the problem of checking whether there

exists an extreme point of F satisfying $D_{i.}x > d_i$ is NP-complete. So the

problem of checking whether there exists an extreme point of K satisfying

$D_{i.}x > d_i$, is NP-complete. ▮

Since we assumed that the dimension of K, defined by (1), is n, there

exists an $\bar{x} \in K$ satisfying $D\bar{x} > d$, or equivalently, for each i = 1 to m

there exists an $x^i \in K$ satisfying $D_{i.}x > d_i$. We have the following result.

THEOREM 1: Let K, the set of feasible solutions of (1), be of dimension n.

The dimension of K^Δ is also n iff for each i = 1 to m, there exists an

extreme point $\hat{x}^i$ of K satisfying $D_{i.}x > d_i$.

PROOF: Since $K^\Delta \subset K$, every point in K^Δ satisfies (1), this implies that if

the dimension of K^Δ is n, there must exist a point $\bar{x} \in K^\Delta$ satisfying $D\bar{x} > d$.

3

Since $\bar{x} \, \varepsilon \, K^\Delta$, it is a convex combination of extreme points of K, so

$D\bar{x} > d$ holds iff for each i = 1 to m, there exists an extreme point $\hat{x}^i$ of K satisfying $D_{i.} x > d_i$.

Conversely, suppose for each i = 1 to m there exists an extreme point $\hat{x}^i$ of K satisfying $D_{i.} x > d_i$. It is well known that between every pair of extreme points of K, there exists an edge path of K joining them, consisting of only bounded edges of K. See [3]. Using this and the hypothesis, we can prove that the dimension of K^Δ is n.

Introducing the vector of slack variables $s = (s_1,\ldots,s_m)^T$, the system (1) can be expressed as

$$Dx - I_m s = d$$
$$s \geqq 0$$

where I_m is the unit matrix of order m. In this, the equality constraints can be used to eliminate the unrestricted variables $x_1,\ldots,x_n$. Suppose this leads to a system

$$Es = p \tag{3}$$
$$s \geqq 0$$

where E, p are matrices of orders r×m and r×1 respectively, where r = m - n, and E has rank r. Every extreme point of K corresponds to a basic feasible solution (BFS) of (3). Let $s^o = (s_1^o,\ldots,s_m^o)^T$ be a BFS of (3) corresponding to a basic vector $(s_1,\ldots,s_r)$ for (3). Suppose the system (3) is nondegenerate. In s^o, the nonbasic variables $s_{r+1},\ldots,s_m$ are all zero. By the hypothesis, for each t = r + 1 to m, there exists a BFS of (3) in which the variable $s_t > 0$. And each BFS of (3) is connected to s^o by an edge path as mentioned above. So, among the nonbasic variables $s_{r+1},\ldots,s_m$, at least some of them must enter the basic vector $(s_1,\ldots,s_r)$ of (3) leading to adjacent BFSs of s^o, and not to unbounded edges. Suppose these are the nonbasic variables s_{r+j}, j = 1 to q. Let $s^j = (s_1^j,\ldots,s_m^j)^T$ be the BFS

obtained when the nonbasic variable s_{r+j} is entered into the basic vector $(s_1, \ldots, s_m)$, for $j = 1$ to q. So

$$s_i^j = 0, \text{ for } i = r + 1 \text{ to } m, \ i \neq r + j$$
$$> 0, \text{ for } i = r + j \tag{4}$$

In each of the BFSs s^0, s^j, $j = 1$ to q, all the variables $s_{r+q+1}, \ldots, s_m$ are zero, and by the hypothesis there are BFSs of (3) in which these variables are > 0. By the edge path connectedness property, there must exist adjacent extreme points of s^j, $j = 1$ to q, in which exactly one of the variables among $s_{r+q+1}, \ldots, s_m$ is > 0 and the others are zero. Suppose these are BFSs

$$s^{q+t} = (s_1^{q+t}, \ldots, s_m^{q+t})^T, \ t = 1 \text{ to } u, \text{ such that}$$

$$s_i^{q+t} = 0 \text{ for } i = r+q+1 \text{ to } m, \ i \neq r+q+t$$
$$> 0 \text{ for } i = r+q+t \tag{5}$$

And this process can be repeated. Eventually we get BFSs $s^j = (s_1^j, \ldots, s_m^j)^T$ of (3), $j = 1$ to $m - r$, satisfying the property that

$$s_i^j = 0, \text{ for } i = r+j+1 \text{ to } m$$
$$> 0, \text{ for } i = r+j \tag{6}$$

By (6) we conclude that the rank of the set $\{s^j - s^0 : \ j = 1 \text{ to } m - r\}$ is $m - r$ which implies that the dimension of the convex hull of BFSs of (3) is $m - r = n$, and hence the dimension of the convex hull of extreme points of K is n.

A similar proof applies even when (3) is degenerate.

THEOREM 2: Let K be the set of feasible solutions of (1), and assume that K has at least one extreme point. Let $J = \{i: \text{ there exists no extreme point of } K \text{ satisfying } D_{i.}x > d_i\}$. Then the dimension of $K^\Delta = n - \text{rank of}$ $\{D_{i.} : \ i \ \epsilon \ J\}$.

PROOF: By the definition of the set J, all the extreme points of K satisfy

$$D_{i.} x \geqq d_i, \quad i \notin J$$

$$D_{i.} x = d_i, \quad i \in J \tag{7}$$

The result follows by applying Theorem 1 to the reduced system obtained by eliminating variables using the equality constraints in (7). ■

By Theorem 1, to check whether the dimension of K^Δ is n, we must check whether there exists an extreme point of K satisfying $D_{i.} x > d_i$, for each i = 1 to m. However, by Lemma 1, for any i, the problem of checking whether there exists an extreme point of K satisfying $D_{i.} x > d_i$, is NP-complete. This suggests that the problem of checking whether the dimension of K^Δ is n, or computing the dimension of K^Δ, may be hard problems.

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