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SIMILARITY ANALYSES OF PARTIAL DIFFERENTIAL EQUATIONS

Tsung-Yen Na
Douglas E. Abbott
Arthur G. Hansen

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FOREWORD

Tsung-Yen Na is Associate Professor of Mechanical Engineering, The University of Michigan, Dearborn Campus, Dearborn, Michigan.

Douglas E. Abbott is Associate Professor of Mechanical Engineering, Purdue University, Lafayette, Indiana.

Arthur G. Hansen is Dean of Engineering, Georgia Institute of Technology, Atlanta, Georgia.

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ABSTRACT

Methods for transforming partial differential equations into forms more suitable for analysis and solution are investigated. The idea of a Generalized Similarity Analysis is introduced and results applied to the equations of boundary-layer flow. A thorough presentation of the application of continuous transformation groups to the problem of similarity analyses (reduction of the number of independent variables of an equation) is given with new and very general methods evolved for determining the nature of transformations.

CHAPTER 1

GENERAL CONCEPTS AND SCOPE OF THE INVESTIGATION

1.0 INTRODUCTION

The investigations to be presented in this report are an outgrowth of a continuing study of ways in which partial differential equations associated with problems of physical interest may be simplified through transformation of independent and dependent variables. The original motivation for this study grew out of an interest on the part of investigators in the National Aeronautics and Space Agency as to how rather general methods might be formulated for reducing the number of independent variables of a partial differential equation (similarity analyses).

As the investigation proceeded, certain facts became clear. First it was noted that the usual types of transformation for simplifying partial differential equations of physical problems were usually of a rather special class. This led to the question of possibly generalizing the class of transformations noted. As will be seen significant success was achieved in this inquiry.

A second main area of investigation centered on finding methods of analyzing very general types of partial differential equations other than specific types such as viscous flow equations, etc. It was soon discovered that little of general nature could be uncovered in connection with the more commonly used methods found in current literature. Most of these methods treated problems for which not only the differential equations but the boundary values were also specified. Transformations evolved were based on both the equation and the nature of the boundary values. There was one exception to this, however, that gave promise of providing a very generalized method of analysis. This was the theory of continuous transformation groups. The method was by no means new. In fact, the basic ideas date back to the last century and are found in the work of the mathematician Sophus Lie. Moreover, the theory of groups has been employed quite extensively in recent times by investigators in the field of similarity analysis. Nevertheless, there was not in the literature a truly in depth study of the applications of Lie's theory to the similarity analysis of partial differential equations. Previous applications were quite limited in scope and it therefore seemed reasonable to find how far Lie's ideas might be pursued in formulating a very general approach to similarity analyses. The results that have been obtained and that are presented here are most encouraging. Very general methods of analysis have been formulated and the way made clear for further work.

The approach to be used in this report is the following. For the remainder of this chapter, a presentation of general concepts will be made, followed in Chapter 2 by a discussion of the form of general transformations for partial differential equations of problems having a physical basis. The use of transformation groups for similarity analyses will be introduced in Chapter 3, and basic ideas and applications presented. In Chapter 4, the methods employed in Chapter 3 will be generalized and a set of rather powerful techniques for analyzing partial differential equations will be developed.

1.1 GENERAL CONCEPTS

The concept of similarity solutions is very old and dates back to such well-known workers as Buckingham[1] and Bridgeman.[2] Originally, the term similarity analysis implied a procedure for finding some information about the solution of particular physical problem usually short of a complete mathematical answer. More recently, in the work of such analyists as Birkhoff[3] and Sedov,[4] similarity solutions have incorporated rigorous mathematical techniques which result in solutions in the engineering sense, that is, numerical answers. Of particular importance, the more recent work has yielded solutions to nonlinear partial differential equations which have been intractable to more standard solution techniques. In fact, exact solutions of the boundary-layer equations of fluid mechanics are almost universally based on similarity methods.

The various similarity solution methods can be broadly characterized as techniques which employ transformations of variables or parameters, or both. Typically transformations may linearize a problem (for example the Kirchhoff and hodograph transformations), reduce partial differential equations to ordinary differential equations (for example the Blasius similarity transformations), transform the system to one already solved (such as the Mangler transformation), or perform some other reduction of mathematical complexity. Sedov[4] formulates a mathematical theory of similarity on the basis that "two phenomena are similar, if the assigned characteristics of the other by a simple conversion, which is analogous to the transformation from one system of units of measurement to another." Thus, a certain analogy exists between the theories of dimensional analysis as formulated by Buckingham and Bridgeman and the geometric theory of invariants relative to a transformation of variables, a fundamental theory in modern mathematics and physics. Some authors have even found it convenient to define classes or types of similarity. Kline[5] defines "external similitude" as the similarity between problems of given class (a transformation in terms of parameters alone) and "internal similitude" as a similarity characterized by a mathematical relationship between points inside a single system of a given class (for example, a transformation which reduces the number of independent variables).

Inherent in any general theory of similarity, however, should be the recognition of specifying, in some sense or other, the complete physical problem to be analyzed. In fact, the term "similarity" implies a comparison between two or more complete and recognizable systems.

Thus, it is of prime importance to formulate a "complete" mathematical description of the physical problem under consideration before proceeding with a similarity solution. Gukhman[6] characterizes a complete problem as one which is expressed in terms of the governing equations together with a body of information termed the "conditions for uniqueness of solution" which may contain not only boundary and initial conditions in the usual sense, but also possible auxillary physical considerations such as conservation requirements. To be sure, it is not always possible to prescribe the uniqueness conditions a priori for all problems that engineers are called on to consider. In fact, it appears that much of the work that has been done in the past to develop similarity analyses has been motivated by an initial statement of completeness, or lack of it. For example, the early work of Buckingham and Bridgeman on dimensional analysis require only a recognition of the pertinent variables which apply, without any statement being made concerning the governing equations. Later workers such as Morgan,[7] Hansen,[8] Krzywoblocki,[9] and Wecker and Hayes[10] investigated similarity methods by considering the governing equations first, and only examining the boundary and initial conditions as a later step, if at all. Another group of workers developed similarity methods by starting with a complete mathematical formulation and, thus motivated, to examine less complete (and more general) problems; see for example Coles,[11] Abbott and Kline,[12] and Gukhman.[6]

An examination of these earlier works show that the initial problem statement, as far as assumed completeness, determined to a large extent the kind of mathematical approach employed. The more information that was known, the more direct was the method developed for finding a similarity solution and, at the same time, the less general were both the methods and the conclusions (as regards "general solutions"). It is not suggested that this dichotomy is necessarily bad. The more general techniques, such as group theory methods, have produced powerful theorems and yield results with a minimum of mathematical busy-work. On the other hand, the group theory methods are difficult for the average engineer to follow because their motivation is mathematical, not physical, and this has inhibited their wide use. Also, somewhat amazingly, the more powerful mathematical techniques have been to a degree more restrictive in some of their aspects (such as the "class of assumed transformations") than the less elegant methods.

By recognizing the differences in past motivation, and the resulting advantages and weaknesses, the suggestion naturally arises that perhaps a marriage of the two approaches might be fruitful; this on one goal of the present investigation. In the following chapters, the two viewpoints will be examined in more detail than has been done in the past. In the following chapter a technique based strongly on physical reasoning will be used to evolve a very broad definition of similarity. It will be shown that by introducing the concept of generalized similarity,* a unified method can be developed for deriving the majority of the well-known transformations employed in fluid dynamics.

*The term "generalized similarity" was found desirable because of the popular use of the term similarity to imply simply a reduction in the number of variables of a given problem.

CHAPTER 2

GENERAL TRANSFORMATION METHODS IN FLOW THEORY

2.0 GENERAL CONCEPTS

An appropriate change of variables is probably one of the most useful methods available for solving the partial differential equations of mathematical physics. The most general criterion for a transformation is simply to change a given problem into a simpler problem is some sense or other; that is, either to a form which will yield to more standard solution techniques, or possible to a form which has been previously solved in connection with a related or similar problem. Hodograph transformations and conformal transformations are well-known methods for transforming given problems into forms which yield to classical techniques (such as separation of variables). Ames[13] classifies the various transformation techniques into three groups: transformations only of the dependent variables, transformations only of the independent variables, and mixed transformations of both independent and dependent variables. However, all three groups have a common goal: to find a relation, or more specifically, a basis of comparison, between different physical (or mathematical) problems. This broad concept of comparison is the definition of the term "generalized similarity."

Generalized similarity might be applied in fluid mechanics to attempt to answer such questions as "Is there any similarity (i.e., basis of comparison) between compressible and incompressible flow problems, axisymmetric and planar flows, or in general, any more complicated and a less complicated flow?" By contrast, the usual term "similarity" as used by Birkhoff,[3] Morgan,[7] Hansen,[8] Abbott and Kline,[12] and others is defined in terms of independent variables of a problem. Thus it may ultimately refer to a physical similarity within a given problem, such as similarity of velocity or temperature profiles.

Much of the previous work on similarity was motivated by the desire to develop simple methods for reducing the number of independent variables. Out of this previous work came the realization that each of the proposed methods was based on an assumed class of transformations and the recognition that more general classes of transformations might lead to the solution of a wider class of problems. The first part of this chapter is concerned with seeking the most general class of transformations for particular types of problems. The two viewpoints ± discussed in Chapter 1 will be examined. First, the mathematical theory of transformations will be reviewed and it will be shown possible to postulate the so-called primitive transformation as the most general form of a class of transformations (under a certain assumption). Second, a separate approach, based on postulating the "complete physical problem," is examined for the special case of laminar boundary-layer flows and it is shown that this approach also yields the primitive transformation.

The second part of the chapter deals with the development of a technique for employing the primitive transformation to find the form of the variables for a wide variety of generalized similarity problems.

2.1 TYPICAL TRANSFORMATIONS USED IN FLUID MECHANICS

As a means of developing some insight into the question of "the most general class of similarity transformations" and the concept of "generalized similarity," Table 1 was compiles of examples of as many different types of known transformations as could be found to represent the field of fluid mechanics.

TABLE 1

Transformation[15]	Transformed Independent Variables
Similarity	$\xi(x) = x, \ \eta(x,y) = y\,\gamma(x)$
Meksyn-Görtler	$\xi(x) = \int_0^x u_1(x)dx, \ \eta(x,y) = u_1(x)y/\sqrt{x}$
von Mises	$\xi(x) = x, \ \eta(x,y) = \psi(x,y)$
Crocco	$\xi(x) = x, \ \eta(x,y) = u(x,y)$
Mangler	$\xi(x) = \int_0^x r_0^2(x)dx, \ \eta(x,y) = r_0(x)y$
Stewartson	$\xi(x) = \int_0^x a_1(x)p_1(x)dx,$ $\eta(x,y) = \int_0^x a_1(x)\rho(x,y)dy$
Dorodnitsyn	$\xi(x) = \int_0^x p_1(x)dx, \ \eta(x,y) = \int_0^x \rho(x,y)dy$

Similarity Rules of High Speed Flow (see, for example Ref. 16):

(i) Prandtl-Glauett (ii) Göthert (iii) von Kármán Transonic	$\xi(x) = x, \ \eta(y) = By$

By comparing the various transformations in the table, an interesting conclusion can be made: all of the transformed independent variables are of the general form $\xi = \xi(x)$ and $\eta = \eta(x,y)$ (that is, one of the new variables is a function of only one of the original variables). This realization raises the question "Is the general transformation $\xi(x)$, $\eta(x,y)$ the most general class of transformations that need be considered for physical problems?" In an attempt to answer this question, it will first prove useful to review the mathematical theory of transformations.

2.2 THE PRIMITIVE TRANSFORMATION

The following theorem can be found in the mathematical literature of general transformation theory (see, for example, Courant[14]):

THEOREM: An arbitrary one-to-one continuously differentiable transformation

$$\xi = \xi(x,y), \quad \eta = \eta(x,y)$$

of a region R in the x,y-plane onto a region R' in the ξ,η-plane can be resolved in the neighborhood of any point interior to R into one or more continuously differentiable "primitive" transformations, provided that throughout the region R, the Jacobian

$$J(\xi,\eta) = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \xi_x\eta_y - \xi_y\eta_x$$

differs from zero. A "primitive" transformation is of the form

$$\xi = \xi(x), \quad \eta = \eta(x,y) .$$

Now one-to-one transformations have an important interpretation and application in the representation of deformation or motions of continuously distributed systems, such as fluids. For example, if a fluid is spread out at a given time over a region R and then is deformed by motion, the motion of the fluid is described by the coordinates in the physical R-plane. If the fluid motion in R is characterized by the coordinates x,y, then the corresponding motion in the transformed region R' is characterized by coordinates ξ,η. The one-to-one character of the transformation obtained by bringing every point x,y into correspondence with a single point ξ,η is simply the mathematical expression of the physically obvious fact that the fluid motion in the physical R-plane must remain recognizable after transformation to the transformed R'-plane, i.e., that the corresponding motions remain distinguishable.

Physically, since most "practical" transformations of interest for solving physical problems should be one-to-one,* that is, have a unique inverse (except possibly at a finite number of singular points), it appears the primitive transformation or the resolution into primitive transformations, should be considered in the search for "the most general class of transformations."

For the sake of completeness, the following two properties of the primitive transformation are noted:

(i) If the primitive transformation

$$\xi = \xi(x), \quad \eta = \eta(x,y)$$

is continuously differentiable, and its Jacobian

$$J(\xi,\eta) = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \xi_x\eta_y - \xi_y\eta_x$$

differs from zero at a point $P(x_o,y_o)$, then in the neighborhood of P the transformation has a unique inverse, and this inverse is also a primitive transformation of the same type.

(ii) For primitive transformations, the sense of rotation in the x,y-plane is preserved or reversed in the ξ,η-plane according as the sign of the Jacobian is positive or negative, respectively.

In summary, the primitive transformation appears to be the most general class of transformations that need be considered, purely from a mathematical viewpoint, as long as it is required that a unique inverse of the transformation must exist. In the next section, the question of the most general transformation will be approached from a fundamentally physical viewpoint and it will be shown that again the primitive transformation appears to be a requirement from conclusions based on uniqueness arguments.

2.3 REQUIREMENTS IMPOSED BY THE PHYSICAL DESCRIPTION

The question of the most general class of transformations will now be

*An exception to this rule; for example, is the von Mises transformation of Table 1, which is singular along the x-axis. However, this transformation is used for computational (not physical) reasons, and hence its use is motivated by a completely different line of reasoning than that under consideration here.

examined from a more physical veiwpoint. Some mathematics will be involved, of course, but the physical problem will be kept near at hand and frequent reference to it will be made throughout the analysis.

Because in what follows, by definition, involves a particular physical problem, it will be convenient to introduce such a problem. As an example, consider the following equations describing the two-dimensional flow of a laminar incompressible boundary layer:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial y^2} = -\frac{dp}{dx} \tag{2.1}$$

or an equivalent single equation in terms of the stream function;

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^3 \psi}{\partial y^3} = -\frac{dp}{dx} . \tag{2.2}$$

For convience in the development, it is assumed that all of the variables are nondimensional. The objective is now to consider very general transformation of variables and see what conditions must be met by this transformation so as not to violate any known physical properties of the problem.

Let us begin by specifying a transformation of variables. For the independent variables, no restrictions at present will be placed on the assumed form; thus the form is specified simply as

$$\xi = \xi(x,y), \quad \eta = \eta(x,y) . \tag{2.3}$$

The dependent variable ψ (the stream function) is also transformed to a new function, say Ψ. Previous work on fluid flow tranformations has often assumed that the two stream functions, ψ and Ψ, should be the same at corresponding points and hence that streamlines in one plane are tranformed into streamlines in the other. However, this restriction is found to be unwarranted in many problems and will be avoided here. Instead, the relationship between ψ and Ψ will be specified in some weak sense by the form

$$\Psi(\xi,\eta) = g(x,y)\psi(x,y) . \tag{2.4}$$

It should be recognized that this assumption is not trivial and other forms could be considered. Nevertheless, Eq. (2.4) represents a more general case than usually employed and attention will be restricted to this form in the present report.

The next step is to carry out the transformation of Eqs. (2.3) and (2.4) on the left-hand side of Eq. (2.2). Application of the transformations (2.3) and (2.4) to the boundary-layer equation is not new and was considered previously by Coles[15] and others. However, the generalized similarity technique was not under consideration by the previous authors.

The results of the transformation of Eq. (2.2) is given in Table 2. For convenience in interpreting the physical terms after transformation, transformed velocities U,V have been defined by

$$U = \frac{\partial \Psi}{\partial \eta}, \quad V = -\frac{\partial \Psi}{\partial \xi} .$$

The transformed side of Eq. (2.2) given in Table 2 is essentially unmanageable in its present form. While it is unclear at the present time what conditions are either necessary or sufficient to ensure any particular mathematical behavior, it is interesting to consider an argument proposed by Coles.[15] Coles' argument is based on the a priori requirement that the transformed flow outside of the shear layer (i.e., for large values of y) should conform both physically and formally to the original flow. Thus, he argues, since the physical flow is bounded for large y and is, in fact, at most a function of x, then the transformed side of Eq. (2.2) must behave in a similar fashion. His argument is then that the substantial derivative terms and the U^2 terms become at most functions of ξ for large y, whereas the remaining terms behave either like y or y^2. His conclusion is to require the η and η^2 terms to vanish identically, which can be accomplished by requiring that $g = g(x)$, $\xi = \xi(x)$, and $\eta_y = \gamma(x)$. At the present time, about all that can be said is that this assumed form is one possible form of a transformation which will preserve a certain sense of physical correspondence between the given and transformed flow; this form of the transformation will be employed throughout the remainder of the present chapter.

In summary, the postulated general transformation, based on the physical mode, is:

$$\Psi(\xi,\eta) = g(x)\Psi(x,y)$$

$$\xi = \xi(x), \quad \eta = y(\gamma(x)) \tag{2.5}$$

TABLE 2

Transformation: $\Psi(\xi,\eta) = g(x,y)\psi(x,y)$; $\xi = \xi(x,y), \eta = \eta(x,y)$, $U = \frac{\partial\Psi}{\partial\eta}$, $V = -\frac{\partial\Psi}{\partial\xi}$

$$
\begin{aligned}
\frac{dp}{dx} &= \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^3\psi}{\partial y^3} = \left(U\frac{\partial U}{\partial\xi} + V\frac{\partial U}{\partial\eta}\right)(\eta_y - \xi_y)\frac{J(\xi,\eta)}{g^2} \\
&- U\left[\left(\frac{\eta_y}{g}\right)_{yy} - 2\left(\frac{g_y}{g^2}\right)_y \eta_y - \frac{g_y}{g^2}\eta_{yy}\right] \\
&+ V\left[\left(\frac{\xi_y}{g}\right)_{yy} - 2\left(\frac{g_y}{g^2}\right)_y \xi_y - \frac{g_y}{g^2}\xi_{yy}\right] \\
&+ U^2\frac{J(\eta_y/g,\eta)}{g} - UV\frac{J(\xi_y/g,\eta)+J(\eta_y/g,\xi)}{g} + V^2\frac{J(\xi_y/g,\xi)}{g} \\
&- \frac{\partial U}{\partial\xi}\left[\left(\frac{\xi_y}{g}\right)_y \eta_y - 2\frac{g_y\xi_y\eta_y}{g^2} + 2\left(\frac{\xi_y\eta_y}{g}\right)_y + \left(\frac{\eta_y}{g}\right)_y \xi_y\right] \\
&+ \frac{\partial V}{\partial\xi}\left[\left(\frac{\xi_y}{g}\right)_y \xi_y - \frac{g_y\xi_y^2}{g^2} + \left(\frac{\xi_y^2}{g}\right)_y\right] \\
&- \frac{\partial U}{\partial\eta}\left[\left(\frac{\eta_y}{g}\right)_y \eta_y - \frac{g_y}{g^2}\eta_y^2 + \left(\frac{\eta_y^2}{g}\right)_y\right] \\
&- \frac{\partial^2 U}{\partial\xi^2}\left[\frac{\xi_y^2}{g}\eta_y + 2\frac{\xi_y^2\eta_y}{g}\right] + \frac{\partial^2 V}{\partial\xi^2}\frac{\xi_y^3}{g} \\
&- \frac{\partial^2 U}{\partial\eta^2}\frac{\eta_y^3}{g} + \frac{\partial^2 V}{\partial\eta^2}\left[\frac{\eta_y^2}{g}\xi_y + 2\frac{\xi_y\eta_y^2}{g}\right] \\
&+ \Psi\left[\left(\frac{\partial U}{\partial\eta}\eta_y - \frac{\partial V}{\partial\eta}\xi_y\right)\frac{J(g,\eta)}{g^3} + \left(\frac{\partial U}{\partial\xi}\eta_y - \frac{\partial V}{\partial\xi}\xi_y\right)\frac{J(g,\xi)}{g^3} + \left(\frac{g_y}{g}\right)_{yy}\right] \\
&- \Psi U\left[\frac{J(\eta_y,g)+gJ(g_y/g,\eta)}{g^3}\right] \\
&+ \Psi V\left[\frac{J(\xi_y,g)+gJ(g_y/g,\xi)}{g^3}\right] \\
&+ \Psi^2\frac{J(g_y/g^2,g)}{g^2}
\end{aligned}
$$

It is of interest to note that the resulting form of the independent variables is a primitive transformation as discussed in the last section, but it has been shown by considering a particular problem and relying on physical arguments that it may be possible to restrict the form of η to be a linear function of y for this particular class of problems (boundary-layer flows). Thus, the two approaches, mathematical and physical, have lead to the same conclusion regarding a postulated "most general class of transformation." Of course it is necessary, as previously pointed out, to emphasize that there may be some cases of practical interest which will lie outside the realm of the mathematical conclusions presented here. However, it has been found that all of the cases listed in Table 1 are satisfied by the transformation of Eq. (2.5) (with the exception of the von Mises transformation previously discussed).

It is now worthwhile to return to Table 1 for a moment. Recall that from Table 1, it was noted that a wide class of different transformations were all primitive transformations. Further, note that one of these transformations, the similarity transformation, has been shown to yield to simple general analysis for its derivation for particular problems. In fact, at the present time there are two types of similarity analyses that are founded primarily on simple transformation theory: the free parameter method of Hansen[8] and the separation of variables method of Abbott and Kline.[12] It is thus of interest to speculate on the following question: Would it be possible to derive all of these "generalized similarity" transformations by the same technique that is used to derive the similarity transformation? The Generalized Similarity Analysis was developed as an attempt to answer this question.

This new method is based on a single assumption concerning the admissible class of transformations of the independent variables, namely that the assumed transformation should be one-to-one (that is, have a unique inverse). On the previous pages it was shown that this requirement would lead to the primitive transformation, and further that a particular primitive transformation may be obtained from physical arguments for a particular class of problems, namely $\xi = \xi(x)$, $\eta = y\gamma(x)$ for the boundary-layer flows. Of course the generality of the present ideas goes beyond a particular case, such as boundary-layer flow analysis and should be appliable to a wide class of problems.

2.4 THE GENERALIZED SIMILARITY ANALYSIS

The development of the generalized similarity analysis was motivated as an extension of the method of finding similarity variables (i.e., the reduction of the number of independent variables) to the problem of finding a transformation of variables which will convert a given physical problem into an alternate problem under certain prescribed conditions. For example, the prescribed condition for a similarity solution is that the number of independent variables must be reduced. By contrast, the prescribed condition for the Mangler transformation is that the axisymmetric boundary-layer equations are to be transformed into the planar form of the equations, and so forth, with similar statements for the rest of the examples in Table 1.

There are three distinct steps to the generalized similarity analysis. These steps are:

(a) The general mathemathical theory of transformations states that any continuous one-to-one transformation can be resolved into one or more primitive transformations of the form

$$\xi = \xi(x), \quad \eta = \eta(x,y).$$

Thus, a primitive transformation form is assumed a priori, where it is recognized that the general analysis has the possibility of being repeated more than once, depending on the particular problem at hand.

(b) The given equation, transformed under a primitive transformation to the new independent variables (ξ,η) is required to satisfy the state requirement; for example, that for a similarity analysis, the transformed equation should be a function only one of the new variables.

(c) Simultaneously the boundary conditions for the given equation are required to be satisfied when expressed in terms of the transformed variables.

These three steps will be shown to completely and uniquely determine the explicit form of the new variables $\xi(x)$ and $\eta(x,y)$ if, in fact, the original problem and associated boundary conditions do admit a generalized similarity solution.

As an example of the method, two problems will be examined in detail. First the classical Blasius flat plate boundary-layer problem will be solved to give a relatively simple motivation of the basic ideas. Second and more difficult Mangler transformation will be derived as an example of the broad applicability of the method.

Example 1

The boundary-layer equations for a steady, two-dimensional laminar flow with a zero pressure gradient (Blasius flat plate flow) can be written in the following form in terms of the stream function:

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \nu \frac{\partial^3\psi}{\partial y^3} \qquad (2.6)$$

(all variables are dimensional, where ν is the kinematic viscosity) with the boundary conditions

$$y = 0: \quad \frac{\partial\psi}{\partial y} = 0, \quad \frac{\partial\psi}{\partial x} = 0$$
$$y \to \infty: \quad \frac{\partial\psi}{\partial y} \to \text{constant} = u_0 \,. \tag{2.7}$$

To make the problem statement complete, it is necessary to state the given requirement for the transformation $x,y \to \xi,\eta$:

Requirement: under the transformation, the given equation and boundary conditions, (2.6) and (2.7), are to reduce to a function of a single independent variable (i.e , the similarity variable).

The three steps of the generalized similarity analysis are carried out in order as follows:

(a) Assume a primitive transformation

$$\xi = \xi(x) \text{ and } \eta = \eta(x,y).$$

Since a boundary-layer problem is under consideration, it was shown in the Section 2.3 that it is sufficient to choose the more specific transformation for η as $\eta = y\gamma(x)$, and further, for the present case of a similarity solution, ξ can be assumed in the simple form $\xi = x$ without loss of generality because the final result, by definition, is to be a function of the single variable $\eta(x,y)$, and not both ξ and η.* For the dependent variable $\psi(x,y)$, the transformed variable $\Psi(\xi,\eta)$ is assumed in the form

$$\Psi(\xi,\eta) = g(x)\psi(x,y)$$

as discussed in Section 2.3. Thus, the assumed transformation for the dependent and independent variables becomes

*The question naturally arises, why not assume the form $\xi = y$, $\eta = \eta(x,y)$? This alternative is easily eliminated for the present case by carrying out the corresponding analyses in an analogous manner as that presented here; it is then found that this assumed form will not satisfy the given boundary conditions.

$$\Psi(\xi,\eta) = g(x)\psi(x,y)$$

$$\xi = x, \; \eta = y\gamma(x) \; . \qquad (2.8)$$

Performing the transformation (2.8) on the given Eq. (2.6) yields the following results:

$$\frac{\partial\psi}{\partial y} = \frac{\partial\Psi}{\partial\eta}\frac{\eta_y}{g}$$

$$\frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2\Psi}{\partial\eta^2}\frac{\eta_y^2}{g}$$

$$\frac{\partial^3\psi}{\partial y^3} = \frac{\partial^3\Psi}{\partial\eta^3}\frac{\eta_y^3}{g}$$

$$\frac{\partial\psi}{\partial x} = -\Psi\frac{g'}{g^2} + \frac{\partial\Psi}{\partial\xi}\frac{1}{g} + \frac{\partial\Psi}{\partial\eta}\frac{\eta_x}{g}$$

$$\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial\Psi}{\partial\eta}\left(\frac{\eta_{xy}}{g} - \frac{g'}{g^2}\eta_y\right) + \frac{\partial^2\Psi}{\partial\xi\partial\eta}\frac{\eta_y}{g} + \frac{\partial^2\Psi}{\partial\eta^2}\frac{\eta_y\eta_x}{g}$$

where primes are used to indicate total derivatives. Substituting these results into Eq. (2.6) yields:

$$\nu\frac{\partial^3\Psi}{\partial\eta^3} + \left(\frac{g'}{g}\frac{1}{\eta_y} - \frac{\eta_{xy}}{\eta_y^2 g}\right)\left(\frac{\partial\Psi}{\partial\eta}\right)^2 + \left(\frac{\partial\Psi}{\partial\xi}\frac{\partial^2\Psi}{\partial\eta^2} - \frac{\partial\Psi}{\partial\eta}\frac{\partial^2\Psi}{\partial\xi\partial\eta}\right)\frac{1}{g\eta_y} - \Psi\frac{\partial^2\Psi}{\partial\eta^2}\frac{g'}{g}\frac{1}{\eta_y} = 0$$

with boundary conditions:

$$\eta = \eta(x,0) = 0: \quad \frac{\partial\Psi}{\partial\eta}\eta_y(x,0) = 0; \qquad -\Psi\frac{g'}{g} + \frac{\partial\Psi}{\partial\xi} + \frac{\partial\Psi}{\partial\eta}\eta_x(x,0) = 0$$

$$\eta = \eta(x,y) \to \infty: \quad \frac{\partial\Psi}{\partial\eta}\frac{\eta_y(x,y\to\infty)}{g} \to \text{constant} = u_o \; .$$

(b) Imposing the stated condition of similarity by requiring that

$$\Psi = \Psi(\eta)$$

that is, that the transformation must reduce the number of variables, and noting that $\eta_y = \gamma(x)$, the transformed equation becomes:

$$\nu\Psi''' + \left(\frac{g'}{g^2\gamma} - \frac{\gamma'}{\gamma^2 g}\right)\Psi'^2 - \frac{g'}{g^2}\Psi\Psi'' = 0 \tag{2.9}$$

with boundary conditions

$$\eta = 0: \quad \Psi'\gamma = 0, \qquad -\Psi\frac{g'}{g} + \Psi'\gamma = 0$$

$$\eta \to \infty: \quad \Psi'\frac{\gamma}{g} \to u_o \ . \tag{2.10}$$

(c) So far the functions $g(x)$ and $\gamma(x)$ are unspecified. However, only particular forms of these functions will satisfy both the boundary conditions and the stated requirement that Eq. (2.9) must be a function only of η. Examining the last boundary condition (2.9) yields

$$\Psi'(\infty)\,\frac{\gamma(x)}{g(x)} \to \text{constant} = u_o \ .$$

The only way that this can be true is if

$$\gamma(x) = u_o g(x) \ .$$

Also, the only way that Eq. (2.9) can be a function of the single variable η is if the coefficients of Ψ and its derivatives are constant. Hence, letting

$$\frac{g'}{g^2}\frac{1}{\gamma} = \frac{g'}{u_o g^3} = \text{constant} = c_1$$

then integration of this ordinary differential equation yields

$$g = [-2c_1 u_o(x+x_o)]^{-1/2}$$

where x_o is an undeterminable constant of integration,* and letting the arbitrary constant c_1 take the value $c_1 = -\nu/2$ yields

$$g = [\nu u_o(x + x_o)]^{-1/2}$$

$$\gamma = u_o[\nu u_o(x + x_o)]^{-1/2} \quad .$$

Evaluating the coefficient of Ψ'^2:

$$\frac{g'}{g^2\gamma} = c_1 \text{ and } \frac{\gamma'}{\gamma^2 g} = \frac{u_o g'}{u_o\gamma g^2} = c_1 \ .$$

Thus this coefficient is identically zero** and the final form of similarity solution is:

$$\Psi''' + \frac{1}{2}\Psi\Psi'' = 0$$

$$\Psi(0) = \Psi'(0) = 0, \ \Psi'(\infty) \to 1 \ . \qquad (2.11)$$

In summary, it has been shown that a unique choice for the transformation variables can be found for the similarity solution of the Blasius problem by carrying out the three steps of the generalized similarity analysis. The next example will show that the method can be applied to a much more generalized problem, that of finding the conditions for which an axisymmetric boundary layer is similar to a planar boundary layer.

*The meaning of x_o is clear from the physics of the problem; its magnitude is determined by conditions at the leading edge of the plate and cannot be derived from boundary-layer theory alone. Thus the appearance of x_o is a sufficient proof that the Blasius solution is a correct asymptotic downstream solution for laminar flow over a plate which holds independent of the initial conditions specifying the plate leading edge.

**The coefficient of $\Psi_2'^2$ in Eq. (2.9) is not independent of the relation $g'/g^2\gamma = c_1$. This can be seen as follows: for simplicity, let $\gamma = 1/\delta$ and $g = 1/h$. Then $g'/g^2\gamma = -h'\delta = c_1$ and letting $(g'/g^2\gamma - \gamma'/\gamma^2 g) = (-h'\delta + \delta'h) = c_2$, then $\delta'h$ is a constant, say c_3. The two equations $h'\delta = -c_1$ and $\delta'h = c_3$ may be combined to yield $\delta\delta'' = (c_1/c_3)\delta'^2 = 0$ which has the solution $\delta = |ax + b|^{1/(1-c_1/c_3)}$ for $c_1 \neq c_3$ and $\delta = ae^{bx}$ for $c_1 = c_3$. This provides a general solution, however we cannot evalute c_1 or c_3 without another condition. The conclusion, as found above, remains the same; the required condition must come from the boundary conditions.

Example 2

The well-known Mangler transformation[16] serves as a good example of the broad meaning of the idea of generalized similarity because the transformation answers the question "Under what conditions is an axisymmetric boundary-layer flow similar to a two-dimensional planar flow?" The answer lies in finding a transformation between the variables describing the two types of flow. The generalized similarity analysis is formulated in the same way as the simple similarity analysis of the preceding example; that is, by specifying the equations (and boundary conditions) and the stated requirement to be satisfied by the transformation.

The governing equations for a thin laminar axisymmetric boundary layer are:

$$\frac{\partial(r_o u)}{\partial x} + \frac{\partial(r_o v)}{\partial y} = 0 \tag{2.12}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1\frac{du_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{2.13}$$

where $r_o = r_o(x)$ is a given quantity (r_o specifies the body shape relative to the axis of symmetry) and ν, the kinematic viscosity, is a constant. The general boundary conditions are:

$$\begin{aligned} y = 0&: \quad u,v = 0 \\ y \to \infty&: \quad u \to u_1(x) \ . \end{aligned} \tag{2.14}$$

The problem statement is completed by writing down the required mathematical form of the desired equations, namely:

Requirement: under the transformation $u,v \to U,V$ and $x,y \to \xi,\eta$ the transformed equations are to be in the form of a planar boundary-layer flow; that is, of the form

$$\frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta} = 0 \tag{2.15}$$

$$U\frac{\partial U}{\partial \xi} + V\frac{\partial U}{\partial \eta} = U_1\frac{dU_1}{d\xi} + \nu\frac{\partial^2 U}{\partial \eta^2} \tag{2.16}$$

The three parts of the analysis are as follows:

(a) Assume a primitive transformation. Again, since the problem involves the boundary-layer equations, it is sufficient to assume the same general form of the transformation as given in Section 2.3, that is

$$\Psi(\xi,\eta) = g(x)\psi(x,y)$$
$$\xi = \xi(x), \qquad \eta = y\gamma(x) \tag{2.17}$$

where, for the present case, the stream function $\psi(x,y)$ is defined by the equations

$$u = \frac{1}{r_o}\frac{\partial\psi}{\partial y}, \qquad v = -\frac{1}{r_o}\frac{\partial\psi}{\partial x}$$

so as to satisfy Eq. (2.12). For the transformed flow to be in the form of planar equations, the transformed stream function $\Psi(\xi,\eta)$ is defined by

$$U = \frac{\partial\Psi}{\partial\eta}, \qquad V = -\frac{\partial\Psi}{\partial\xi} .$$

Using these definitions and the transformation (2.17), the velocities and their derivatives become

$$u(x,y) = \frac{\gamma}{r_o g} U(\xi,\eta)$$

$$v(x,y) = \frac{1}{r_o g}\frac{g'}{g}\Psi(\xi,\eta) + \frac{\xi_x}{r_o g}\mathbf{V}(\xi,\eta) - \frac{\eta_x}{r_o g}U(\xi,\eta)$$

$$\frac{\partial u}{\partial y} = \frac{\gamma}{r_o g}\gamma\frac{\partial U}{\partial\eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\gamma}{r_o g}\gamma^2\frac{\partial^2 U}{\partial\eta^2}$$

$$\frac{\partial U}{\partial x} = \left(\frac{\gamma}{r_o g}\right)' U + \frac{\gamma}{r_o g}\left(\frac{\partial U}{\partial\xi}\xi_x + \frac{\partial U}{\partial\eta}\eta_x\right) .$$

Substituting these expressions into Eqs. (2.12) and (2.13) yields, after rearrangement:

$$J(\xi,\eta)\left(\frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta}\right) = 0 \tag{2.18}$$

$$\xi_x\left(U\frac{\partial U}{\partial \xi} + V\frac{\partial U}{\partial \eta}\right) - U_1\frac{dU_1}{d\xi} = (\gamma r_o g)\,\nu\frac{\partial^2 U}{\partial \eta^2} + \left[\frac{\left(\frac{\gamma}{r_o g}\right)'}{\left(\frac{\gamma}{r_o g}\right)}(U_1^2 - U^2) - \frac{g'}{g}\Psi\frac{\partial U}{\partial \eta}\right]. \tag{2.19}$$

It is found that the continuity Eq. (2.18) is invariant under the transformation (2.17) as long as the Jacobian $J(\xi,\eta) = \gamma\xi_x$ differs from zero. Thus Eq. (2.18) does not force any requirements on the transformation. However, in comparing Eqs. (2.16) and (2.19), it is seen that for the latter equation to fulfill the given requirement of being identical to the planar form given by Eq. (2.16), it is necessary that

$$\xi_x = \gamma r_o g \tag{2.20}$$

and

$$\frac{\left(\frac{\gamma}{r_o g}\right)'}{\left(\frac{\gamma}{r_o g}\right)}(U_1^2 - U^2) - \frac{g'}{g}\Psi\frac{\partial U}{\partial \eta} = 0\,. \tag{2.21}$$

A sufficient condition for Eq. (2.21) to be satisfied is for

$$\frac{\gamma}{r_o g} = \text{constant} = c_1 \tag{2.22}$$

and

$$g = \text{constant} = c_2\,. \tag{2.23}$$

Although Eqs. (2.22) and (2.23) may not be necessary conditions for the satisfaction of (2.21), they at least provide <u>one</u> solution for the given requirement and this is usually satisfactory from an engineering veiwpoint. Since there are no further requirements to restrict a choice for the constants c_1 and c_2, they are normally chosen for dimensional reasons to assume the values $c_1 = 1$ and $g = 1/D$ where D is an arbitrary reference length. Hence, the final form for the transformation becomes:

$$\Psi(\xi,\eta) = \frac{1}{D} \psi(x,y) \tag{2.24}$$

$$\xi = \int_0^x \frac{r_0^2(x)}{D^2} dx \tag{2.25}$$

$$\eta = y \frac{r_0(x)}{D} . \tag{2.26}$$

(b) It has already been shown that for the present case, the unique form of the transformation, Eqs. (2.24), (2.25), and (2.26), is determined by requirements on the differential equation alone. Thus, the boundary conditions do not provide any additional information and, in fact, they are found to carry over directly as follows:

$$\begin{aligned} \eta = 0: &\quad U,V = 0 \\ \eta \to \infty: &\quad U \to U_1(\xi) \end{aligned} \tag{2.27}$$

The analysis is thus complete, Eqs. (2.24), (2.25), and (2.26) being known as the Mangler transformation.

2.5 CONCLUDING REMARKS

It has been shown that the primitive transformation appears ot be the most general class of transformations necessary to provide a transformation with a unique inverse. This result was obtained from the general mathematical theory of transformations, but was also supported by a physical argument for such cases as boundary-layer problems which showed that the transformation

$$\Psi(\xi,\eta) = g(x)\psi(x,y)$$

$$\xi = \xi(x), \ \eta = y\gamma(x)$$

is possibly sufficient to ensure proper behavior of the equations.

The generalized similarity analysis was then introduced to solve a wide class of problems which could be formulated as a comparison between two given subproblems. Two examples were given, the Blasius similarity problem and a derivation of the Mangler transformation, however all of the cases given in Table 1 can be derived in s aimilar fashion.

A few comments can be made concerning the role of the function g(x) appearing in the transformation of the dependent variable for the boundary-layer equations. In certain cases, the value of g(x) will be uniquely determined by the problem. For example, g(x) is proportional to $\sqrt{x}$ and $\sqrt{\xi(x)}$ for the similarity and Meksyn-Görtler transformations, respectively, and g is found to be constant for the Mangler transformation. However, in some cases, the choice for g is arbitrary; for example, g may take on any value for the von Mises transformation. Further, for the case of comparing compressible and incompressible forms of the boundary-layer equations, different choices for g lead to different, but nevertheless useful, results: Stewartson chooses g = constant and Dorodnitsyn chooses $g = \eta_y = \gamma(x)$. Coles discusses the significance of g for the difficult problem of compressible-incompressible transformations of the boundary-layer equations for turbulent flow in Ref. 11.

In summary, the ideas presented in this chapter are based on, or more appropriately, motivated by a physical description of a problem which is in some sense complete. Depending on the particular case under consideration, completeness may imply a knowledge of all necessary boundary conditions, or possibly only a statement of a particular requirement of the transformation. In any case, a fairly specific problem formulation is implied.

In the next chapter, a different technique will be examined which focuses attention on a more narrow application; the simple similarity problem (in the sense of reducing the number of independent variables). This technique, the group theory method, being less encumbered by statements of broad generality, will prove to yield very elegant and powerful mathematical results for finding similarity solutions for a wide range of applications.

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CHAPTER 3

SIMPLE GROUP THEORY APPLICATIONS

3.0 INTRODUCTION

In this chapter we will consider one of the more widely used methods of reducing the number of independent variables of a partial differential equation, the group-theoretical method. The foundation of the method is contained in the general theories of continuous transformation group that were introduced and treated extensively by Lie in the latter part of the last century. G. Birkhoff[2] was one of the first to apply this concept in searching for similarity solutions of partial differential equations. Subsequently, Morgan[5] investigated quite thoroughly the mathematical theories involved and essentially completed the development of the method now generally referred to as the group-theoretic method.[3]

In the classical group-theoretic similarity analysis, a one-parameter linear or spiral group of transformation is employed.* The variance of a given partial differential equation under the transformation group and the introduction of new invariant variables, enables the number of independent variables to be reduced by one. Once the theories involved are understood, the process of obtaining similarity solutions is considered to be the simplest as compared with other techniques. Extension of this method to multi-parameter linear and spiral groups of transformation has been developed in recent years, and will be discussed in this chapter along with the one-parameter approach.

3.1 ONE-PARAMETER GROUP-THEORETIC ANALYSIS

In this section the topic of group-theoretic analysis will be introduced by restricting attention to one-parameter methods. The material will have the twofold purpose of showing how the one-parameter method is used in similarity analyses and will provide a background for material in the remainder of this chapter and Chapter 4. By no means does the one-parameter similarity analysis plumb the depths of Lie's theory of continuous transformation groups. As will be seen, only two simple types of transformation groups will be considered—the linear and spiral groups. In the next chapter, a far more general method is presented that permits the construction of a greater variety of transformation groups. This section primarily reviews the work of Birkhoff[2] and Morgan.[5]

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*See Ref. 3 for a general introduction to group theory and this method.

3.1.1 Absolute Invariants

The first step in introducing the concept of one-parameter transformation groups is by establishing the group property of a transformation of coordinates. In general, if a transformation is defined by

$$X^i = f^i(x', \ldots, x^m; \ a', \ldots, a^n) \tag{3.1}$$

then there exists (m-1) functionally independent absolute invariants $\zeta^j(x', \ldots, x^m)$ where $j=1, \ldots, (m-1)$. By definition, an absolute invariant has the property

$$\zeta^j(x', \ldots, x^m) = \zeta (X', \ldots, X^m) \ . \tag{3.2}$$

Follwing Morgan,[5] we restrict ourselves to the one-parameter group*

$$\begin{aligned} T_a: \quad X^i &= f^i(x', \ldots, x^m; a) \\ Y^j &= h^j(y', \ldots, y^n; a) \end{aligned} \tag{3.3}$$

where $i=1, \ldots, m$ $(m \geq 2)$ and $j=1, \ldots, n$ $(n \geq 1)$.

For the subgroup $X^i = f^i(x', \ldots, x^m; a)$, there exists (m-1) functionally independent absolute invariants

$$\eta_k(x', \ldots, x^m), \quad k=1, \ldots, (m-1) \ . \tag{3.4}$$

For the group T_a as a whole, there are (m+n-1) functionally independent absolute invariants. We therefore add the following to the list:

$$g_\ell(x', \ldots, x^m; y', \ldots, y^n) \ , \qquad \ell=1, \ldots, n \ . \tag{3.5}$$

*The variables x^i and y^i will shortly be identified as independent and dependent variables. For now, they are just general variables.

3.1.2 Morgan's Theorem

Assuming that y^j are the dependent variables and x^i the independent variables in the one-parameter group of transformations

$$
\begin{aligned}
T_a: \quad X^i &= f^i(x', \ldots, x^m; a) \\
Y^j &= h^j(y', \ldots, y^n; a) \ ,
\end{aligned}
\qquad (3.6)
$$

the system of partial differential equations of order k

$$\phi_j\left(x', \ldots, x^m; y', \ldots, y^n; \frac{\partial^k y'}{\partial (x')^k}, \ldots, \frac{\partial^k y^n}{\partial (x^m)^k}\right) = 0 \qquad (3.7)$$

is invariant under this group of transformation, T_a, if each of the ϕ_j is "conformally invariant" under the transformation T_a^k. This means that

$$
\begin{aligned}
&\phi_j\left(X', \ldots, X^m; Y', \ldots, Y^n; \frac{\partial^k Y'}{\partial (X')^k}, \ldots, \frac{\partial^k Y^n}{\partial (X^m)^k}\right) \\
&= F\left(x', \ldots, x^m; y', \ldots y^n; \ldots; \frac{\partial^k y^n}{\partial (x^m)^k}; a\right) \\
&\times \ \phi_j\left(x', \ldots, x^m; y', \ldots y^n; \ldots; \frac{\partial^k y^n}{\partial (x^m)^k}\right) \ .
\end{aligned}
\qquad (3.8)
$$

In particular, if $F = F(a) = 1$, ϕ_j is said to be "absolute invariant" under this group of transformations.*

The above result leads to the very important theorem of Morgan:[5]

<u>Theorem</u>: Suppose that the forms ϕ_j are conformally invariant under the group T_a. Then the invariant solutions of $\phi_j = 0$ can be expressed in terms of solutions of a new system of partial differential equations

$$\phi_j\left(\eta_1, \ldots, \eta_{m-1}; F_1, \ldots, F_n; \ldots; \frac{\partial^k F_1}{\partial \eta_1^k}, \ldots, \frac{\partial^k F_n}{\partial \eta_{m-1}^k}\right) = 0 \qquad (3.9)$$

The η_i are the absolute invariants of the subgroup of transformations on the x^i alone and the variables F_j are such that

---

*See Ref. 3.

$$F_j(\eta_1,\ldots,\eta_{m-1}) = \text{function of } (x',\ldots,x^m) \,. \qquad (3.10)$$

An example is given in the next section to illustrate the method.

3.1.3 Application of One-Parameter Group Theory to Boundary-Layer Equations

Consider the steady, two-dimensional laminar boundary-layer equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3.11)$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = U\,\frac{dU}{dx} + \nu\,\frac{\partial^2 u}{\partial y^2} \qquad (3.12)$$

subject to the boundary conditions

$$y = 0: \quad u = v = 0$$

$$y = \infty: \quad u = U(x) \,.$$

Group theoretic methods will now be employed to reduce Eqs. (3.11) and (3.12) to ordinary differential equations. To this end, two groups of transformation (namely, the linear and the spiral groups) are considered. The method by Birkhoff and Morgan serves as the basis for the analysis that follows.

<u>Case 1</u>: The linear group.

The linear group of transformation is defined as

$$x = A^{\alpha_1}\bar{x}, \quad y = A^{\alpha_2}\bar{y}, \quad u = A^{\alpha_3}\bar{u}$$

$$v = A^{\alpha_4}\bar{v}, \quad U = A^{\alpha_5}\bar{U} \qquad (3.13)$$

where $\alpha_1,\ldots,\alpha_5$ are constants and A is the parameter of transformation. The function U will be considered as an independent variable with the defining relation $U = U(x)$.

Under this group of transformation, Eqs. (3.11) and (3.12) become

$$A^{\alpha_3-\alpha_1} \frac{\partial \bar{u}}{\partial \bar{x}} + A^{\alpha_4-\alpha_2} \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \qquad (3.14)$$

and

$$A^{2\alpha_3-\alpha_1} \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + A^{\alpha_3+\alpha_4-\alpha_2} \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = A^{2\alpha_5-\alpha_1} \bar{U} \frac{d\bar{U}}{d\bar{x}} + A^{\alpha_3-2\alpha_2} \nu \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} . \qquad (3.15)$$

It is seen that the differential equations will be invariant before and after the transformation if the powers of A in each term are the same. Thus, we get

$$\alpha_3 - \alpha_1 = \alpha_4 - \alpha_2 \qquad (3.16)$$

and

$$2\alpha_3 - \alpha_1 = \alpha_3 + \alpha_4 - \alpha_2 = 2\alpha_5 - \alpha_1 = \alpha_3 - 2\alpha_2 . \qquad (3.17)$$

Solution of Eqs. (3.16) and (3.17) then gives

$$\alpha_3 = \alpha_5 = \alpha_1 - 2\alpha_2 , \quad \alpha_4 = -\alpha_2 . \qquad (3.18)$$

Knowing the relation among the α's, the absolute invariants can be obtained by eliminating the parameter of transformation A. Thus, noticing that

$$\frac{y}{x^{\alpha}} = \frac{\bar{y}}{\bar{x}^{\alpha}} , \quad \alpha = \alpha_2/\alpha_1$$

$$\frac{u}{x^{1-2\alpha}} = \frac{\bar{u}}{\bar{x}^{1-2\alpha}}$$

$$\frac{v}{x^{-\alpha}} = \frac{\bar{v}}{\bar{x}^{-\alpha}}$$

$$\frac{U}{x^{1-2\alpha}} = \frac{\bar{U}}{\bar{x}^{1-2\alpha}} ,$$

these combination of variables are readily shown to be invariant under the linear group of transformations and so are absolute invariants. According to Morgan's theorem, Eqs. (3.11) and (3.12) can now be expressed in terms of these invariants; or in other words, these invariants are possible similarity variables. Therefore, we put

$$\eta = \frac{y}{x^{1-m/2}} ; \quad f(\eta) = \frac{u}{x^m} ; \quad g(\eta) = \frac{v}{x^{m-1/2}} \quad (m=1-2\alpha) \tag{3.19}$$

and

$$h(\eta) = c_1 = \frac{U}{x^m} \tag{3.20}$$

where f and g are functions of η and $h(\eta) = c_1$ is a constant since U, the mainstream velocity, is a function of x only and thus cannot be a nonconstant function of η which would introduce dependence on y.

From Eq. (3.20), we get the form of mainstream velocity for the establishment of similarity solutions, i.e.,

$$U(x) = c_1 x^m . \tag{3.21}$$

We may now make a check, the transformation of the boundary conditions. It is seen that the boundary conditions are transformed to:

$$\eta = 0: \quad f = g = 0$$

$$\eta = \infty: \quad f = \frac{U(x)}{x^m} = c_1 .$$

Thus, for constant values of η constant values of f and g result. It is now a simple matter to transform the differential equations. Thus, Eqs. (3.11) and (3.12) are transformed to

$$mf - \frac{1-m}{2}\eta f' + g' = 0 \tag{3.22}$$

and

$$mf^2 - \frac{1-m}{2}\eta ff' + gf' = mc_1^2 + \nu f'' \tag{3.23}$$

The boundary condition are those previously stated.

Case 2: The spiral group.

A spiral group of transformation is defined by

$$x = \bar{x} + \beta_1 b, \quad y = \bar{y}\, e^{\beta_2 b}, \quad u = \bar{u}\, e^{\beta_3 b}$$
$$v = \bar{v}\, e^{\beta_4 b}, \quad U = \bar{U}\, e^{\beta_5 b} \; . \tag{3.24}$$

Following the same steps as given in Case 1, we find that the parameters are related by

$$\beta_3 = \beta_5 = -\beta_2 \; , \quad \beta_4 = -2\beta_2 \tag{3.25}$$

and, after elimination of b, the absolute invariants are:

$$\eta = \frac{y}{e^{-\frac{1}{2}\beta x}} \; , \quad f = \frac{u}{e^{\beta x}} \; , \quad g = \frac{v}{e^{\frac{1}{2}\beta x}} \; , \quad (\beta = \beta_3/\beta_1) \tag{3.26}$$

and

$$h(\eta) = c_2 = \frac{U(x)}{e^{\beta x}} \; . \tag{3.27}$$

Again, c_2 must be a constant. Thus, the mainstream velocity should be of the form

$$U(x) = c_2 e^{\beta x} \tag{3.28}$$

for similarity solutions to exist. Checking the boundary conditions, we find that they are transformed as follows:

$$\eta = 0: \quad f = g = 0$$
$$\eta = \infty: \quad f = c_2 \; .$$

The last step is to transform the differential equations. Equations (3.11) and (3.12) become

$$\beta f + \frac{1}{2}\beta\eta f' + g' = 0 \qquad (3.29)$$

and

$$\beta f^2 + \frac{1}{2}\beta\eta f f' + g f' = \nu f'' . \qquad (3.30)$$

The above example illustrates the one-parameter group-theoretical method. Some further comments are now in order.

In both examples, the steps are as follows:

1. Define the group of transformation and substitute into the differential equations.

2. Require that the differential equation be invariant. Relations among the constants in the transformation are obtained.

3. Eliminate the parameter of transformation to give absolute invariants, which will become similarity variables.

4. Check the boundary conditions to see if they can be transformed into constant values.

5. Transform the differential equations.

It is seen that up to step 4, no differentiation is needed. This makes the group-theoretic method an advantage over other methods. As long as in step 2 zeros for all the constants (e.g., $\alpha_1, \ldots, \alpha_5$ in the linear group) are not obtained, the partial differential equations are transformable into ordinary differential equations.

Step 4 in this particular problem needs not be put before step 5; however, in general, if the unknown function appears in the boundary conditions alone (e.g., the diffusion equation with a boundary condition $u(0,t) = U(t)$ and the form of $U(t)$ is given), it ultimately may save effort to check the boundary conditions first as above.

Equations (3.11) and (3.12) can be reduced to one equation by solving Eq. (3.12) for g. However, we are illustrating the method and thus retain f and g in two separate equations.

Finally, comparison between Eqs. (3.18) and (3.25) shows that Eq. (3.25) can be obtained by putting $\alpha_1 = 0$ in Eq. (3.18). Thus, the spiral group is sometimes considered to be a special case of the linear group in which $\alpha_1=0$.

3.2 MULTIPARAMETER GROUPS

The method discussed in the previous article using a one-parameter group of transformation is usually applied to cases in which one independent variable is to be eliminated. If two variables are to be eliminated, the method should be applied twice, first reducing the variables by one and then applying the technique to reduce the variables of the transformed differential equations again. The simplicity of the method characterizing this technique is then lost. Thus it would be valuable to have a method in which the number of variables can be reduced by more than one in a single application. Such a method based on multiparameter groups was presented by Manohar[4] and later by Ames.[1] We will discuss this method but limit the discussion to the method for reduction of the number of independent variables by two. Generalization to more variables at a time can be made in a similar manner. Before proceeding with the nultiparameter group analysis, we shall first describe a special case where one-parameter groups may be employed to reduce the number of independent variables by more than one.

3.2.1 One-Parameter Group Method*

Let a transformation group Γ_{11} be defined by

$$\Gamma_{11}: \; x_1 = A^{\alpha_1}\bar{x}_1 , \quad x_2 = A^{\alpha_2}\bar{x}_2 , \quad x_3 = A^{\alpha_3}\bar{x}_3$$
$$y_j = A^{\alpha_j}\bar{y}_j , \qquad (j=4,\ldots,n) \tag{3.31}$$

where A is the parameter of transformation and the α's are constants to be determined from the condition that the given partial differential equations be constant conformally invariant under this group of transformation. We consider a partial differential equation in which the independent variables are x_1, x_2, and x_3.

In general, the group Γ_{11} can only reduce the variables by one. However, in the special case in which $\alpha_1 = \alpha_2 \neq 0$, reduction of the variables by two is possible. Absolute invariants can be shown to be

*Ames[1] considers the case in which $\alpha_1=\alpha_2=0$ as a separate case. However, since the form of absolute invariants in this case is the same as in Γ_{24} below, this case is omitted.

$$\eta = \frac{x_3}{(ax_1+bx_2)^{\alpha_3/\alpha_1}} \tag{3.32}$$

and

$$f_j(\eta) = \frac{y_j}{(ax_1+bx_2)^{\alpha_j/\alpha_1}} \tag{3.33}$$

where a and b are any arbitrary constants.

3.2.2 Two-Parameter Group Method

Let Γ_{21} be a two-parameter transformation group

$$\begin{aligned} \Gamma_{21}:\quad x_1 &= A^{\alpha_1}\overline{x}_1 , \quad x_2 = B^{\beta_1}\overline{x}_2, \quad x_3 = A^{\alpha_3}B^{\beta_3}\overline{x}_3 \\ y_j &= A^{\alpha_j}B^{\beta_j}\overline{y}_j \qquad (j=4,\ldots,n) . \end{aligned} \tag{3.34}$$

Again, A and B are parameters of transformation and α's and β's are constants to be determined from the condition that the given partial differential equations be conformally invariant under this group of transformation. Absolute invariants are found to be

$$\eta = \frac{x_3}{x_1^{\alpha_3/\alpha_1} x_2^{\beta_3/\beta_1}} \tag{3.35}$$

and

$$f_j(\eta) = \frac{y_j}{x_1^{\alpha_j/\alpha_1} x_2^{\beta_j/\beta_1}} . \tag{3.36}$$

Next, we consider the two-parameter group Γ_{22} where

$$\begin{aligned} \Gamma_{22}:\quad x_1 &= \overline{x}_1 + \alpha_1 A, \quad x_2 = B^{\beta_1}\overline{x}_2 , \quad x_3 = e^{\alpha_3 A}B^{\beta_3}\overline{x}_3 \\ y_j &= e^{\alpha_j A}B^{\beta_j}\overline{y}_j \qquad (j=4,\ldots,n) . \end{aligned} \tag{3.37}$$

Absolute invariants for this group are

$$\eta = \frac{x_3}{e^{\frac{\alpha_3}{\alpha_1}x_1} x_2^{\frac{\beta_3}{\beta_1}}} \tag{3.38}$$

and

$$f_j(\eta) = \frac{y_j}{e^{\frac{\alpha_j}{\alpha_1}x_1} x_2^{\frac{\beta_j}{\beta_1}}} \tag{3.39}$$

A third group Γ_{23} may be defined as

$$\Gamma_{23}: \quad x_1 = A^{\alpha_1}\bar{x}_1 , \quad x_2 = \bar{x}_2 + \beta_1 B, \quad x_3 = A^{\alpha_3} e^{\beta_3 B}\bar{x}_3$$
$$y_j = A^{\alpha_j} e^{\beta_j B}\bar{y}_j \qquad (j=4,\ldots,n) \; . \tag{3.40}$$

Absolute invariants for this group are:

$$\eta = \frac{x_3}{x_1^{\frac{\alpha_3}{\alpha_1}} e^{\frac{\beta_3}{\beta_1}x_2}} \tag{3.41}$$

and

$$f_j(\eta) = \frac{y_j}{x_1^{\frac{\alpha_j}{\alpha_1}} e^{\frac{\beta_j}{\beta_1}x_2}} \; . \tag{3.42}$$

Finally, let the group Γ_{24} be defined as

$$\Gamma_{24}: \quad x_1 = \bar{x}_1 + \alpha_1 A, \quad x_2 = \bar{x}_2 + \beta_1 B, \quad x_3 = e^{\alpha_3 A} e^{\beta_3 B}\bar{x}_3$$
$$y_j = e^{\alpha_j A} e^{\beta_j B}\bar{y}_j \qquad (j=4,\ldots,n) \; . \tag{3.43}$$

Absolute invariants for this group are:

$$\eta = \frac{x_3}{e^{\frac{\alpha_3}{\alpha_1}x_1} e^{\frac{\beta_3}{\beta_1}x_2}} \tag{3.44}$$

and

$$f_j(\eta) = \frac{y_j}{e^{\frac{\alpha_j}{\alpha_1} x_1} e^{\frac{\beta_j}{\beta_1} x_2}} \quad . \tag{3.45}$$

3.2.3 Example of a Similarity Analysis for a Three-Variable Problem

Consider the unsteady, two-dimensional boundary-layer equations expressed in terms of a stream function ψ:

$$\frac{\partial^2\psi}{\partial t\partial y} + \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^3\psi}{\partial y^3} \quad . \tag{3.46}$$

The boundary conditions are

$$y = 0: \quad \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial x} = 0$$

$$y = \infty: \quad \frac{\partial\psi}{\partial y} = U(x,t) \quad .$$

<u>One-Parameter</u> <u>Group</u>:

Let Γ_{11} be defined as

$$\Gamma_{11}: \quad t = A^{\alpha_1}\bar{t} , \quad x = A^{\alpha_2}\bar{x} , \quad y = A^{\alpha_3}\bar{y} ,$$
$$\psi = A^{\alpha_4}\bar{\psi} , \quad U = A^{\alpha_5}\bar{U} \quad . \tag{3.47}$$

From the invariance of Eq. (3.46) we get

$$\alpha_3 = \frac{1}{2}\alpha_1 , \quad \alpha_4 = \alpha_2 - \frac{1}{2}\alpha_1 , \quad \alpha_5 = \alpha_2 - \alpha_1 \; . \tag{3.48}$$

Reduction of the number of variables by two in a single step is possible only if $\alpha_1 = \alpha_2$, which gives

$$\alpha_3 = \frac{1}{2}\alpha_1 , \quad \alpha_4 = \frac{1}{2}\alpha_1 , \quad \alpha_5 = 0 \; . \tag{3.49}$$

Absolute invariants are:

$$\eta = \frac{y}{\sqrt{at+bx}} \tag{3.50}$$

and

$$f(\eta) = \frac{\psi}{\sqrt{(at+bx)}} \quad . \tag{3.51}$$

Since U is a function of t and x, the only possibility for similarity to exist is to assume

$$U = \text{constant} = U_0 \quad . \tag{3.52}$$

Using these variables, Eq. (3.46) can be easily transformed into an ordinary differential equation.

$$f''' + \frac{a}{2}\eta f'' + \frac{b}{2} ff'' = 0 \tag{3.53}$$

subject to the boundary conditions

$$\eta = 0: \quad f = f' = 0$$

$$\eta \to \infty: \quad f' = 1.$$

<u>Two-Parameter</u> <u>Group</u>:

Let Γ_{21} be defined as

$$\Gamma_{21}: \quad t = A^{\alpha_1}\bar{t}, \quad x = B^{\beta_1}\bar{x}, \quad y = A^{\alpha_3}B^{\beta_3}\bar{y}$$

$$\psi = A^{\alpha_4}B^{\beta_4}\bar{\psi} \; , \quad U = A^{\alpha_5}B^{\beta_5}\bar{U} \; . \tag{3.54}$$

From the invariance of Eq. (3.46), we get

$$\alpha_5 = -\alpha_1, \quad \alpha_3 = \frac{1}{2}\alpha_1, \quad \alpha_4 = -\frac{1}{2}\alpha_1$$

$$\beta_4 = \beta_5 = \beta_1, \quad \beta_3 = 0 \ . \tag{3.55}$$

Absolute invariants are:

$$\eta = \frac{y}{\sqrt{t}}, \quad f = \frac{\psi}{t^{-\frac{1}{2}}x}, \quad c = \frac{U}{t^{-1}x} \ . \tag{3.56}$$

Since U is a function of t and x, c cannot be a function of η. Then the only possibility is c to be a constant. Equation (3.46) is transformed into the following ordinary differential equation:

$$f''' + f' + \frac{1}{2}\eta f'' - f'^2 + ff'' + c(1-c) = 0 \tag{3.57}$$

subject to the boundary conditions

$$\eta = 0: \quad f = f' = 0$$

$$\eta = \infty: \quad f' = 1.$$

It can be shown that for the other three groups mentioned earlier, all α's and β's are zero. This means that similarity solutions do not exist for those groups.

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CHAPTER 4

GENERALIZED GROUP-THEORETIC ANALYSIS

4.0 INTRODUCTION

In the last chapter, the application of the simple group-theoretic method developed by Birkoff[2] and Morgan[14] was treated. Extension of this technique to multiparameter transformation by Manohar[12] and Ames[1] was also discussed.

As it turns out, these methods are probably the simplest to apply of the principal techniques. Whereas other methods may involve solutions of differential equations or other fairly complex mathematical manipulation, the group-theoretic method require straightforward algebraic procedures. One drawback of course, is that boundary conditions are not taken into account until the analysis is largely completed. However, checking for satisfaction of boundary conditions is a simple matter.

From the mathematical point of view, the method developed by Birkhoff[2] and Morgan[14] considered two particular groups of transformation, namely, the linear and the spiral groups of transformation. Application of this method of similarity analysis to a given partial differential equation therefore answers only the question as to whether similarity solutions exist for these two groups. Since there is no proof that these two groups are the only two possible for similarity solutions to exist for a given partial differential equation, it is still necessary to raise the question: Given a partial differential equation, what are all possible groups of transformation that make similarity solution possible? Are there groups other than the linear and spiral groups

To answer questions of this kind, we shall develop in this chapter a systematic procedure in searching for all possible groups of transformation to a given partial differential equation. The procedure is based on Lie's theory of "infinitesimal continuous group of transformation" and his concept of contact transformation. Although both concepts were introduced by Lie in the latter part of the nineteenth century, there has been little application to the solution of nonlinear partial differential equations. The present technique therefore can be considered as extension and application of Lie's theories of infinitesimal and contact transformations to the similarity solutions of boundary value problems.

The technique itself, instead of the mathematical theories, is emphasized in the development to follow. For a complete treatment of the theory, the reader is referred to the books by Lie[9-11] and other references.[3,15] In the first section, definitions and theorems are presented briefly. This is followed by three examples illustrating techniques. The diffusion equation is treated first. The goal of the analysis is to show how all the possible groups of transformation for a given partial differential equation (linear or nonlinear) may be found. By requiring that the given differential equation be invariant under a "infinitesimal transformation," the functional form of the so-called "characteristic function" can be determined. Once this is done, all possible groups of transformation can be obtained by the method developed in this chapter.

The second example shows the application of this method to problems containing an arbitrary function that is to be determined as part of the analysis. The equation discussed is the steady, two-dimensional laminar boundary-layer equations.

In the third example, the role of coordinate systems on similarity analyses is investigated by analyzing the Helmholtz equation in general orthogonal curvilinear coordinates. The problem encountered is the determination of the characteristic function mentioned earlier. By requiring that the Helmholtz equation be invariant under an infinitesimal transformation, a differential equation is obtained for the determination of the characteristic function. The equation contains the metric components (scaling factors) of a curvilinear coordinate system. The equation determines whether or not similarity solutions exist for a given coordinate system; or, given a group, what are the conditions under which the scale factors, h's, should satisfy so that similarity solutions exist.

4.1 DEFINITIONS AND THEOREMS

4.1.1 DEFINITIONS

A group is said to be continuous if, between any two operations of the group, a series of operations within the group can always be found of which the effect of any operation in the series differs from the effect of its previous operation only infinitesimally. Thus, the transformation

$$\begin{aligned} x' &= a_1 x + a_2 y \\ y' &= b_1 x + b_2 y \end{aligned} \tag{4.1}$$

is continuous if a_1, a_2, b_1, and b_2 are any real number. For example, if a_{10}, a_{20}, b_{10}, and b_{20} are values of a_1, a_2, b_1 and b_2, respectively, that carry (x_1, y_1) into (x_1', y_1'), a sequence of values of these parameters can be found to affect the same result. Each transformation can be made to differ from the previous one infinitesimally.

The concept of infinitesimal transformations comes as a natural consequence of the definition of a continuous transformation group. An infinitesimal transformation is one whose effects differ infinitesimally from the identical transformation. Any transformation of a finite continuous transformation group which contains the identical transformation can be obtained by infinite repetition of an infinitesimal transformation.

Let the identical transformation be

$$\begin{aligned} x_1 &= \phi(x,y,a_o) = x \\ y_1 &= \psi(x,y,a_o) = y \end{aligned} \qquad (4.2)$$

where a_o is a particular value of the general parameter a. Then the transformation

$$\begin{aligned} x_1 &= \phi(x,y,a_o+\delta\epsilon) \\ y_1 &= \psi(x,y,a_o+\delta\epsilon) , \end{aligned} \qquad (4.3)$$

where $\delta\epsilon$ is an infinitesimal quantity, defines an infinitesimal transformation in a broad sense. A slightly more restricted definition will shortly be given based on the concept of $\delta\epsilon$ being an "infinitesimal" i.e., a quantity such that higher orders of $\delta\epsilon$ than $\delta\epsilon$ itself may be neglected in a given operation.

Equation (4.3) can be expanded in Taylor series which gives

$$\begin{aligned} x_1 &= \phi(x,y,a_o+\delta\epsilon) \\ &= \phi(x,y,a_o) + \frac{\delta\epsilon}{1!}\left(\frac{\partial\phi}{\partial a}\right)_{a_o} + \frac{\delta\epsilon^2}{2!}\left(\frac{\partial^2\phi}{\partial a^2}\right)_{a_o} + \ldots \end{aligned} \qquad (4.4)$$

and

$$\begin{aligned} y_1 &= \psi(x,y,a) = \psi(x,y,a_o+\delta\epsilon) \\ &= \psi(x,y,a_o) + \frac{\delta\epsilon}{1!}\left(\frac{\partial\psi}{\partial a}\right)_{a_o} + \frac{\delta\epsilon^2}{2!}\left(\frac{\partial^2\psi}{\partial a^2}\right)_{a_o} + \ldots \end{aligned} \qquad (4.5)$$

Equations (4.4) and (4.5) can be written as

$$x_1 = x + \xi\delta\epsilon + \ldots \tag{4.6}$$

and

$$y_1 = y + \eta\delta\epsilon + \ldots \tag{4.7}$$

where

$$\xi = \xi(x,y,a_o) = \left(\frac{\partial\phi}{\partial a}\right)_{a_o} \tag{4.8}$$

and

$$\eta = \eta(x,y,a_o) = \left(\frac{\partial\psi}{\partial a}\right)_{a_o} . \tag{4.9}$$

The expression $\phi(x,y,a_o)$ and $\psi(x,y,a_o)$ in Eqs. (4.4) and (4.5) can be replaced, respectively, by x and y according to Eq. (4.2). Now since a_o is a fixed value of the parameter, a, both ξ and η will be function of x and y only. Also, since $\delta\epsilon$ is infinitesimal, we shall neglect higher order of $\delta\epsilon$ and write:

$$x_1 = x + \xi(x,y)\delta\epsilon \tag{4.10}$$

and

$$y_1 = y + \eta(x,y)\delta\epsilon . \tag{4.11}$$

This transformation shall be defined as infinitesimal transformation. It can be shown that any one-parameter group, G_1, contains only one unique infinitesimal transformation. The definition can be extended to a transformation containing any number of variables (but only one parameter).

Geometrically, an infinitesimal transformation transforms a point (x,y) to a neighboring point a distance

$$\sqrt{\{(x_1-x)^2+(y_1-y)^2\}} = \sqrt{(\xi^2+\eta^2)}\ \delta\epsilon \tag{4.12}$$

in the direction Θ where

$$\cos\Theta = \frac{\xi}{\sqrt{\xi^2+\eta^2}} \tag{4.13}$$

$$\sin\Theta = \frac{\eta}{\sqrt{\xi^2+\eta^2}} . \qquad (4.14)$$

4.1.2 Representations of Infinitesimal Transformations

Let f(x,y) be a generally analytic function of x and y. The effect of an infinitesimal transformation

$$x_1 = \phi(x,y,a_0+\delta\epsilon) \qquad (4.15)$$

and

$$y_1 = \psi(x,y,a_0+\delta\epsilon) \qquad (4.16)$$

on f(x,y) will be to produce the quantity $f(x_1,y_1)$ which, upon expanding in Taylor series,[5] becomes

$$\begin{aligned} f(x_1,y_1) &= f\{\phi(x,y,a_0+\delta\epsilon),\ \psi(x,y,a_0+\delta\epsilon)\} \\ &= f\{(x+\xi\delta\epsilon),(y+\eta\delta\epsilon)\} \\ &= f(x,y) + \frac{\delta\epsilon}{1!}\ \{\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}\} \\ &+ \frac{\delta\epsilon^2}{2!}\{\xi^2 \frac{\partial^2 f}{\partial x^2} + 2\xi\eta \frac{\partial^2 f}{\partial x \partial y} + \eta^2 \frac{\partial^2 f}{\partial y^2}\} \\ &+ \ldots \\ &+ \frac{\delta\epsilon^n}{n!}\ \{\xi^n \frac{\partial^n f}{\partial x^n} + \binom{n}{1} \xi^{n-1}\eta \frac{\partial^n f}{\partial x^{n-1}\partial y} + \ldots \\ &\quad \ldots + \eta^n \frac{\partial^n f}{\partial y^n}\} \end{aligned} \qquad (4.17)$$

Lie introduced the "symbol"*

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*It can readily be seen that

$$Uf = (\partial f(x_1,y_1)/\partial a)_{a_0}$$

and in particular, $Ux = \xi$ and $Uy = \eta$.

$$Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} \quad . \tag{4.18}$$

It should be noted that this symbol itself is not a transformation. It "represents" a transformation, however; and is determined by it.

It can be easily shown that the higher order term in the Taylor series expansion for $f(x_1,y_1)$ can be written in terms of U as:

$$U^2f = \{\xi^2 \frac{\partial^2 f}{\partial x^2} + 2\xi\eta \frac{\partial^2 f}{\partial x \partial y} + \eta^2 \frac{\partial^2 f}{\partial y^2}\} \tag{4.19}$$

$$U^nf = \{\xi^n \frac{\partial^n f}{\partial x^n} + \binom{n}{1}\xi^{n-1}\eta \frac{\partial^n f}{\partial x^{n-1}\partial y} + \ldots + \eta^n \frac{\partial^n f}{\partial y^n}\} \tag{4.20}$$

where U^nf represents operating on f by the operator

$$\xi(x,y) \frac{\partial}{\partial x} + \eta(x,y) \frac{\partial}{\partial y} \tag{4.21}$$

for n times.

Equation (4.17) then becomes

$$f(x_1,y_1) = f(x,y) + \frac{\delta\epsilon}{1!} Uf + \frac{\delta\epsilon^2}{2!} U^2f + \ldots. \tag{4.22}$$

By using this expansion, it will be possible to deduce the actual equations of the transformation generated by a given infinitesimal transformation. Some examples will make this point clear.

Example 4.1. Consider the infinitesimal transformation represented by

$$Uf = -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} \tag{4.23}$$

i.e.,

$$x_1 = x + \xi(x,y)\delta\epsilon$$

and

$$y_1 = y + \eta(x,y)\delta\epsilon$$

where

$$\xi(x,y) = -y$$

and

$$\eta(x,y) = x \text{ .}$$

We now seek expression for x_1 and y_1 as one-parameter transformation groups. This can be done quite readily as follows: First take f = x, then using Eq. (4.23),

$$\begin{aligned} Ux &= -y \\ U^2x &= -x \\ U^3x &= y \\ U^4x &= x \\ &\dots \end{aligned} \qquad (4.24)$$

Substituting into Eq. (4.22), we then get

$$\begin{aligned} x_1 &= x - \frac{\delta\epsilon}{1!} y - \frac{\delta\epsilon^2}{2!} x + \frac{\delta\epsilon^3}{3!} y + \frac{\delta\epsilon^4}{4!} x \\ &\dots \\ &= x\{1 - \frac{\delta\epsilon^2}{2!} + \frac{\delta\epsilon^4}{4!} \dots\} \\ &\quad - y\{\delta\epsilon - \frac{\delta\epsilon^3}{3!} + \frac{\delta\epsilon^5}{5!} \dots\} \\ &= x \cos \delta\epsilon - y \sin \delta\epsilon \text{ .} \end{aligned} \qquad (4.25)$$

Similarly, if f = y, Eq. (4.22) then gives

$$y_1 = y + \frac{\delta\epsilon}{1!} x - \frac{\delta\epsilon^2}{2!} y - \frac{\delta\epsilon^3}{3!} x + \frac{\delta\epsilon^4}{4!} y + \ldots$$

$$= x\{\delta\epsilon - \frac{\delta\epsilon^3}{3!} + \frac{\delta\epsilon^5}{5!} - \ldots\}$$

$$+ y\{1 - \frac{\delta\epsilon^2}{2!} + \frac{\delta\epsilon^4}{4!} - \ldots\}$$

$$= x \sin \delta\epsilon + y \cos \delta\epsilon \,. \qquad (4.26)$$

The results correspond to a form of the rotation group:

$$x_1 = x \cos a - y \sin a$$

$$y_1 = x \sin a + y \cos a$$

for which $a_0 = 0$ in our analysis.

Example 4.2. Consider the infinitesimal transformation represented by

$$Uf = c_1 x \frac{\partial f}{\partial x} + c_2 y \frac{\partial f}{\partial y}$$

i.e.,

$$x_1 = x + \xi(x,y)\delta\epsilon$$

and

$$y_1 = y + \eta(x,y)\delta\epsilon$$

where

$$\xi(x,y) = c_1 x$$

and

$$\eta(x,y) = c_2 y$$

we proceed as before to get expressions for x_1 and y_1: For $f = x$, we get

$$\begin{aligned} Uf &= c_1 x \\ U^2 f &= c_1^2 x \\ U^3 f &= c_1^3 x \\ &\ldots \end{aligned} \tag{4.27}$$

Therefore, Eq. (4.22) gives

$$\begin{aligned} x_1 &= x + \frac{c_1 \delta\epsilon}{1!} x + \frac{c_1^2 \delta\epsilon^2}{2!} x + \frac{c_1^3 \delta\epsilon^3}{3!} x + \ldots \\ &= x\left(1 + \frac{c_1 \delta\epsilon}{1!} + \frac{c_1^2 \delta\epsilon^2}{2!} + \ldots\right) \\ &= x e^{c_1 \delta\epsilon} \ . \end{aligned} \tag{4.28}$$

Similarly, for $f = y$, we get

$$y_1 = y e^{c_2 \delta\epsilon} \ . \tag{4.29}$$

If we let $a_o = 0$ in our analysis and define

$$A = e^a \qquad (\text{Note, } e^{a_o + \delta\epsilon} = e^{\delta\epsilon})$$

then Eqs. (4.28) and (4.29) become

$$x_1 = A^{c_1} x \ , \quad y_1 = A^{c_2} y \ . \tag{4.30}$$

This is seen to be the linear group.

Example 4.3. Consider the infinitesimal transformation represented by

$$Uf = c_1 \frac{\partial f}{\partial x} + c_2 y \frac{\partial f}{\partial y} \ . \tag{4.31}$$

We may easily obtain:

$$x_1 = x + c_1\delta\epsilon \tag{4.32}$$

and

$$y_1 = y + \frac{c_2\delta\epsilon}{1!} y + \frac{c_2^2\delta\epsilon^2}{2!} y + \ldots$$

$$= ye^{c_2\delta\epsilon} \tag{4.33}$$

This is seen to be a case of the spiral group:

$$x_1 = x + c_1 a$$

$$y_1 = ye^{c_2 a} \quad .$$

4.1.3 Functions Invariant Under a Given Group

A function $f(x,y)$ is said to be an invariant of a group if it is unaltered by the transformations of the group; therefore, for the function to be invariant, we have

$$f(x_1,y_1) = f(x,y) \tag{4.34}$$

It follows that the function is necessarily invariant under an infinitesimal transformation. From Eq. (4.22), we have

$$f(x_1,y_1) = f(x,y) + \frac{\delta\epsilon}{1!} Uf + \frac{\delta\epsilon^2}{2!} U^2 f + \ldots \tag{4.35}$$

But if $f(x,y)$ is to be invariant, the following condition must be met:

$$Uf = U^2 f = U^3 f = \ldots = 0 \ . \tag{4.36}$$

However, since the condition

$$Uf = 0 \tag{4.37}$$

implies

$$U^2f = U^3f = \dots = 0, \qquad (4.38)$$

Eq. (4.37) is the sufficient condition that Eq. (4.34) is true for all values of x, y and $\delta\epsilon$. Hence we may state the following theorem.

Theorem 4.1. The necessary and sufficient condition that f(x,y) be invariant under the group represented by Uf is Uf = 0.

In other words, the necessary and sufficient condition that f(x,y) be invariant under a one-parameter group is that it be left unaltered by the infinitesimal transformation of the group.

To determine the invariant function, f, it is sufficient to solve the equation

$$\boxed{Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} = 0} \qquad (4.39)$$

The equation can be solved using the method of Lagrange[4] in the theory of linear partial differential equations. The procedure in the case of three variables, x, y, and f, can be written as follows:

To find the general solution of

$$P_1 \frac{\partial f}{\partial x} + P_2 \frac{\partial f}{\partial y} = R \qquad (4.40)$$

where P_1, P_2 and R being functions of x, y, and f, solve

$$\frac{dx}{P_1} = \frac{dy}{P_2} = \frac{df}{R} \; . \qquad (4.41)$$

If the general solution of this system is

$$u_1 = c_1 \text{ and } u_2 = c_2 \; * \qquad (4.42)$$

*The system, Eq. (4.41), has only two independent solutions.

then

$$\phi_1(u_1,u_2) = 0 \tag{4.43}$$

or

$$u_1 = \phi_2(u_2) \ , \tag{4.44}$$

where ϕ_1 and ϕ_2 are arbitrary functions of u_1, u_2, and u_2 respectively, will be the general solution of Eq. (4.41).

As an illustration of this method, consider the equation

$$xf \frac{\partial f}{\partial x} + yf \frac{\partial f}{\partial y} = xy \ . \tag{4.45}$$

Based on the theorem, we consider the system

$$\frac{dx}{xf} = \frac{dy}{yf} = \frac{df}{xy} \ . \tag{4.46}$$

This system of equations has two <u>independent</u> solutions

$$xy - f^2 = c_1$$

and

$$\frac{x}{y} = c_2 \ .$$

Therefore, the general solution to Eq. (4.45) is of either of the following forms:

$$\phi_1(xy-f^2, \frac{y}{x}) = 0 \tag{4.47a}$$

or

$$\frac{x}{y} = \phi_2(xy-f^2) \tag{4.47b}$$

It is seen that application of this theorem to the solution of Eq. (4.39) for the invariant function, f, as a function of x and y forms a special case as, R = 0. Equation (4.40) becomes

$$p_1 \frac{\partial f}{\partial x} + p_2 \frac{\partial f}{\partial y} = 0 \tag{4.48}$$

clearly, one solution of this equation is

$$f = \text{constant}$$

and we may write

$$u_1(x,y,f) = f = \text{constant} \tag{4.49}$$

The general solution of Eq. (4.48) is therefore (cf, Eq. (4.44))

$$u_2 = \phi_3(f) = \phi_3 \, (\text{constant}) = \text{constant} \tag{4.50}$$

where u_2 is a solution to

$$\frac{dx}{p_1} = \frac{dy}{p_2} \tag{4.51}$$

We therefore get the following theorem:

Theorem 4.2. To find the general solution of

$$\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} = 0 \tag{4.52}$$

where ξ and η being functions of x and y, solve

$$\frac{dx}{\xi} = \frac{dy}{\eta} \ . \tag{4.53}$$

Let solution of this equation be expressed as

$$\Omega(x,y) = \text{constant.} \tag{4.54}$$

This function is the invariant function for the infinitesimal transformation*
represented by

$$Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} . \qquad (4.55)$$

Since Eq. (4.53) has only one solution depending upon a simple arbitrary constant, it follows that a one-parameter group in two variables, x, y, has one and only one independent invariant.

Example 4.4. Consider the rotation group represented by

$$Uf = -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} \qquad (4.56)$$

where

$$\xi(x,y) = -y \text{ and } \eta(x,y) = x .$$

Solution to the equation

$$\frac{dx}{-y} = \frac{dy}{x}$$

gives

$$x^2 + y^2 = \text{constant}. \qquad (4.57)$$

This is the invariant function of the rotation group.

Example 4.5. Consider the linear group

$$Uf = c_1 x \frac{\partial f}{\partial x} + c_2 y \frac{\partial f}{\partial y} \qquad (4.58)$$

where $\xi = c_1 x$ and $\eta(x,y) = c_2 y$. Solution to

*The function f is an invariant of the one-parameter group obtained from the infinitesimal transformation.

$$\frac{dx}{c_1 x} = \frac{dy}{c_2 y}$$

gives

$$\frac{y}{x^{c_2/c_1}} = \text{constant} \tag{4.59}$$

which is the invariant function of this group.

Example 4.6. Consider the spiral group representation

$$Uf = c_1 \frac{\partial f}{\partial x} + c_2 y \frac{\partial f}{\partial y} \tag{4.60}$$

where

$$\xi = c_1 \text{ and } \eta = c_2 y \ .$$

Solution to

$$\frac{dx}{c_1} = \frac{dy}{c_2 y}$$

gives

$$\frac{y}{e^{\frac{c_2}{c_1} x}} = \text{constant} \tag{4.61}$$

which is the invariant function of this group.

4.1.4 Extension of Two-Variable Analysis to n Variables

In the case of n variables, all the theories corresponding to two variables can be generalized following the same pattern. For example, if a function of n variables $f(x_1,\ldots,x_n)$ is invariant under the infinitesimal transformation

$$x_i' = x_i + \xi_i(x_1,\ldots,x_n)\delta\epsilon$$

$$(i=1,\ldots,n) \tag{4.62}$$

then a necessary and sufficient condition is again

$$Uf = 0 \tag{4.63}$$

which, in the case of n variables, takes the form

$$\xi_1(x_1,\ldots,x_n)\frac{\partial f}{\partial x_1} + \ldots + \xi_n(x_1,\ldots,x_n)\frac{\partial f}{\partial x_n} = 0 \ . \tag{4.64}$$

Following the same reasoning as in two-dimensional case, the invariant functions can be obtained by integrating

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \frac{dx_3}{\xi_3} = \ldots = \frac{dx_n}{\xi_n} \ . \tag{4.65}$$

Since there are (n-1) independent solutions to Eq. (4.65) a one-parameter group in n variables has (n-1) independent invariants. The invariant functions are therefore

$$\Omega_m(x_1,\ldots,x_n) = c_m, \quad (m=1,\ldots,n-1) \tag{4.66}$$

and are the solutions to the system of equations given by Eq. (4.65).

4.1.5 Relationships Satisfied by Differential Equations Admitting a Given Group of Infinitesimal Transformations

In this article a basic theorem on the determination of relationships satisfied by functions admitting a given group of infinitesimal transformation will be given. This theorem is comparable to the theorems of Morgan discussed in Chapter 3. It is this theorem which lays the groundwork for the reduction of the number of variables in a partial differential equations.

Consider now a function

$$F = F\left(x_1,\ldots,x_m; y_1,\ldots,y_n; \ldots, \frac{\partial^k y_1}{\partial(x_1)^k}, \ldots, \frac{\partial^k y_n}{\partial(x_m)^k}\right) \tag{4.67}$$

the arguments of which, assumed p in number, contain derivatives of y_j up to order k. Such a function is known as a differential form of the k-th order in m independent variables. Designate the arguments by $z_1, \ldots, z_p$, e.g.,

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \\ &\cdots \\ &\cdots \\ z_{p-1} &= \frac{\partial^k y_n}{\partial (x_{m-1})^k} \\ z_p &= \frac{\partial^k y_n}{\partial (x_m)^k} \quad . \end{aligned} \tag{4.67a}$$

Thus, Eq. (4.67) can be written in a simpler form as

$$F = F(z_1, \ldots, z_p) \tag{4.68}$$

The function $F(z_1, \ldots, z_p)$ is said to admit of a given group represented by

$$Uf = \xi_1(z_1, \ldots, z_p) \frac{\partial f}{\partial z_1} + \ldots + \xi_p(z_1, \ldots, z_p) \frac{\partial f}{\partial z_p} \tag{4.69}$$

if it is invariant under this group of transformation. Therefore, the function, F, admits of a group if

$$UF = 0 \tag{4.70}$$

or,

$$\xi_1 \frac{\partial F}{\partial z_1} + \xi_2 \frac{\partial F}{\partial z_2} + \ldots + \xi_p \frac{\partial F}{\partial z_p} = 0 \ . \tag{4.71}$$

It was shown in the preceding article that there are (p-1) functionally independent solutions, or invariants, to this equation. If they are denoted by

$$\eta_m = \Omega_m(z_1, \ldots, z_p) = \text{constant}, \quad m=1, \ldots, p-1 \ . \tag{4.72}$$

Equation (4.71) must be satisfied, i.e.,

$$\xi_1 \frac{\partial \Omega_m}{\partial z_1} + \dots + \xi_p \frac{\partial \Omega_m}{\partial z_p} = 0 \ . \tag{4.73}$$

Now, if a change of variable is made in the function F, given by Eq. (4.68). from $(z_1,\dots,z_p)$ to $(\eta_1,\dots,\eta_{p-1},z_p)$, we then get

$$F(z_1,\dots,z_p) = \psi(\eta_1,\dots,\eta_{p-1},z_p) \ . \tag{4.74}$$

The condition given in Eq. (4.71) will still have to be satisfied. Thus,

$$\begin{aligned} UF &= \xi_1 \frac{\partial F}{\partial z_1} + \xi_2 \frac{\partial F}{\partial z_2} + \dots + \xi_p \frac{\partial F}{\partial z_p} \\ &= \xi_1 \frac{\partial \psi}{\partial z_1} + \xi_2 \frac{\partial \psi}{\partial z_2} + \dots + \xi_p \frac{\partial \psi}{\partial z_p} = 0 \ . \end{aligned} \tag{4.75}$$

Since ψ is a function of $\eta_1,\dots,\eta_{p-1},z_p$, the chain rule of differentiation may be used to get

$$\begin{aligned} UF &= \xi_1 \left(\frac{\partial \psi}{\partial \eta_1} \frac{\partial \eta_1}{\partial z_1} + \dots + \frac{\partial \psi}{\partial \eta_{p-1}} \frac{\partial \eta_{p-1}}{\partial z_1} + \frac{\partial \psi}{\partial z_p} \frac{\partial z_p}{\partial z_1} \right) \\ &+ \xi_2 \left(\frac{\partial \psi}{\partial \eta_1} \frac{\partial \eta_1}{\partial z_2} + \dots + \frac{\partial \psi}{\partial \eta_{p-1}} \frac{\partial \eta_{p-1}}{\partial z_1} + \frac{\partial \psi}{\partial z_p} \frac{\partial z_p}{\partial z_2} \right) \\ &+ \dots \\ &+ \xi_p \left(\frac{\partial \psi}{\partial \eta_1} \frac{\partial \eta_1}{\partial z_p} + \dots + \frac{\partial \psi}{\partial \eta_{p-1}} \frac{\partial \eta_{p-1}}{\partial z_p} + \frac{\partial \psi}{\partial z_p} \frac{\partial z_p}{\partial z_p} \right) \\ &= \frac{\partial \psi}{\partial \eta_1} \left(\xi_1 \frac{\partial \eta_1}{\partial z_1} + \dots + \xi_p \frac{\partial \eta_1}{\partial z_p} \right) \\ &+ \dots \\ &+ \frac{\partial \psi}{\partial \eta_{p-1}} \left(\xi_1 \frac{\partial \eta_{p-1}}{\partial z_1} + \dots + \xi_p \frac{\partial \eta_{p-1}}{\partial z_p} \right) + \xi_n \frac{\partial \psi}{\partial z_p} \\ &= 0 \ . \end{aligned} \tag{4.76}$$

Since $\eta_1, \ldots, \eta_{p-1}$ are invariant functions, all the brackets equal to zero [cf. Eqs. (4.72) and (4.73)], the following important conclusion then results:

$$\frac{\partial\psi(\eta_1, \ldots, \eta_{p-1}, z_p)}{\partial z_p} = 0 . \qquad (4.77)$$

This means ψ is independent of z_p. Equation (4.74) can then be written as

$$F(z_1, \ldots, z_p) = \psi(\eta_1, \ldots, \eta_{p-1}) = 0 . \qquad (4.78)$$

Therefore, if a differential equation

$$F(z_1, \ldots, z_p) = 0$$

is invariant under the infinitesimal transformation, it must be expressible in terms of the (p-1) functionally independent solutions of the partial differential equations $Uf = 0$.

Like Morgan's theorem given in Chapter 3, this result is the foundation upon which the technique given later in this chapter is based.

4.1.6 The Extended Group Concept

Consider the one-parameter group of transformation

$$x_1 = \phi(x,y,a); \quad y_1 = \psi(x,y,a) . \qquad (4.79)$$

Suppose y is regarded as a function of x. Then if the differential coefficent $p(= dy/dx)$ be considered as a third variable, then under this group of transformations it will be transformed to p_1 where

$$p_1 = \frac{dy_1}{dx_1} = \frac{\dfrac{\partial\psi}{\partial x} + \dfrac{\partial\psi}{\partial y} p}{\dfrac{\partial\phi}{\partial x} + \dfrac{\partial\phi}{\partial y} p} = \chi(x,y,p,a) . \qquad (4.80)$$

It can be easily shown that the general transformation

$$x_1 = \phi(x,y,a); \quad y_1 = \psi(x,y,a); \quad p_1 = \chi(x,y,p,a) \tag{4.81}$$

form a group.[7] This group is known as the extended group of the group given in Eq. (4.79).

Now for a change $\delta\epsilon$ in the parameter a, we may express Eqs. (4.79) as

$$x_1 = x + \frac{\delta\epsilon}{1!}\xi(x,y) + \frac{\delta\epsilon^2}{2!}\left(\xi\frac{\partial\xi}{\partial x} + \eta\frac{\partial\xi}{\partial y}\right) + \dots \tag{4.82}$$

$$y_1 = x + \frac{\delta\epsilon}{1!}\eta(x,y) + \frac{\delta\epsilon^2}{2!}\left(\xi\frac{\partial\eta}{\partial x} + \eta\frac{\partial\eta}{\partial y}\right) + \dots \tag{4.83}$$

The transformed coefficient p_1 will be

$$\begin{aligned} p_1 &= \frac{dy + \frac{\delta\epsilon}{1!}\left(\frac{\partial\eta}{\partial x}dx + \frac{\partial\eta}{\partial y}dy\right) + \dots}{dx + \frac{\delta\epsilon}{1!}\left(\frac{\partial\xi}{\partial x}dx + \frac{\partial\xi}{\partial y}dy\right) + \dots} \\ &= p + \frac{\delta\epsilon}{1!}\left\{\frac{\partial\eta}{\partial x} + \left(\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x}\right)p - \frac{\partial\xi}{\partial y}p^2\right\} + \dots \\ &= p + \frac{\delta\epsilon}{1!}\zeta(x,y,p) + \dots \end{aligned} \tag{4.84}$$

Therefore, the extended infinitesimal transformation given by

$$x_1 = x + (\delta\epsilon)\xi, \quad y_1 = y + (\delta\epsilon)\eta \tag{4.85}$$

$$p_1 = p + (\delta\epsilon)\zeta$$

is represented by

$$UF = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial p} \tag{4.86}$$

where

$$\zeta = \frac{\partial\eta}{\partial x} + \left(\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x}\right)p - \frac{\partial\xi}{\partial y}p^2 . \tag{4.87}$$

Extension of this concept to higher order derivatives can be made by the same reasoning.

Example 1. For the rotation group represented by

$$Uf = -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} \qquad (4.88)$$

it can easily be shown that the extended group of infinitesimal transformation is

$$\begin{aligned} x_1 &= x + \xi(x,y)\delta\epsilon, \\ y_1 &= y + \eta(x,y)\delta\epsilon, \\ p_1 &= p + \zeta(x,y,p)\delta\epsilon \end{aligned} \qquad (4.89)$$

where

$$\begin{aligned} \xi &= -y \\ \eta &= x \end{aligned} \qquad (4.90)$$

and, from Eq. (4.87),

$$\zeta = \frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x}\right) p - \frac{\partial \xi}{\partial y} p^2 = 1 + p^2 . \qquad (4.91)$$

The symbol of this extended group of transformation is therefore

$$Uf = -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} + (1+p^2) \frac{\partial f}{\partial p} . \qquad (4.92)$$

Example 2. For the linear group, the extended group of transformation can be represented by

$$Uf = c_1 x \frac{\partial f}{\partial x} + c_2 y \frac{\partial f}{\partial y} + (c_2 - c_1) p \frac{\partial f}{\partial p} . \qquad (4.93)$$

Example 3. For the spiral group, the extended group of transformation can be represented by

$$Uf = c_1 \frac{\partial f}{\partial x} + c_2 y \frac{\partial f}{\partial y} + c_2 p \frac{\partial f}{\partial p} \quad . \qquad (4.94)$$

4.1.7 Contact Transformations

The theories developed up to this point are not complete. If a given partial differential equation

$$F = F(z_1, \ldots, z_p) = 0 \qquad (4.95)$$

is invariant under the infinitesimal transformation represented by Uf, then

$$UF = \xi_1 \frac{\partial F}{\partial z_1} + \ldots + \xi_p \frac{\partial F}{\partial z_p} = 0 \qquad (4.96)$$

and the differential equation can be expressed in terms of p-1 invariant functions. These invariant functions are solved by using the system of equations,

$$\frac{dz_1}{\xi_1} = \ldots = \frac{dz_p}{\xi_p} \qquad (4.97)$$

as discussed in article (4.14). In case a particular group of transformations is given, as in examples in the previous sections, the function $\xi_1, \ldots, \xi_p$ in Eq. (4.97) are known. The invariant functions can then be found by solving Eq. (4.97). These functions are the similarity variables. However, the results only give similarity solutions for this particular group. What is more important, is to derive the transformation groups and not specifying them. We shall now lay the groundwork for this type of analysis.

In the present section, Lie's theories of "contact transformations" are introduced. These theories make it possible to express the transformation functions, $\xi_1, \ldots, \xi_p$, in terms of a single function, known as the "characteristic function." Basically, Lie's theories of contact transformation deals with the transformation of a differential equation in a general, highly abstract, way. The abstractness and the complexity of the theories prevent any extensive discussion. Here, we merely state some of Lie's theorems without proof. This is sufficient for purposes of application.

Lie[9] defined a "lineal element" of the plane to be the ensemble of a point and a straight line passing through that point. In two-dimensional Cartesian coordinates, the coordinates of the lineal element consist of the coordinates, x and y, and the slope p of the line.

Thus any transformation in x, y, and p may be considered as a transformation of the lineal element.

Consider now a curve defined by $y = y(x)$ having a slope $dy/dx = p$ at the point (x,y). Let the variables x and y be transformed by

$$\begin{aligned} x_1 &= X(x,y) \\ y_1 &= Y(x,y) \ . \end{aligned} \tag{4.98}$$

The curve is transformed into a curve having a slope p_1 at (x_1,y_1) defined by

$$p_1 = \frac{dy_1}{dx_1} = \frac{\dfrac{\partial Y}{\partial x} + \dfrac{\partial Y}{\partial y} p}{\dfrac{\partial X}{\partial x} + \dfrac{\partial X}{\partial y} p} = \chi(x,y,p) \ . \tag{4.99}$$

Equation (4.99) can be considered to be a transformation that along with Eq. (4.98) transforms an element (x,y,p) into (x_1,y_1,p_1). Lie called this transformation an "extended transformation."

A property of an extended transformation is that it transforms the differential equation

$$dy - pdx = 0 \tag{4.100}$$

into the differential equation

$$dy_1 - p_1dx_1 = 0 \quad * \tag{4.101}$$

*The differential equation (4.100) is transformed into the differential equation (4.101) under the assumption that the Jacobian of the transformation (4.98) is nonvanishing.

We may say that every family of lineal elements (x,y,p) satisfying (4.100) is transformed by an extended transformation into a lineal element (x_1,y_1,p_1) satisfying (4.101).

For three-dimensional space, x, y, z, an "element" is defined by five quantities, namely, x, y, z, p, and q where p and q, like y' in two-dimensional space, are the slopes of a straight line passing through that point. The family of surfaces containing a given element would be defined by a single relation between the five quantities, that is, by a partial differential equation.

Lie therefore defines a "contact transformation" on a transformation of the lineal elements of the plane which leaves the Pfaffiian equation

$$dy - pdx = 0 \qquad (4.102)$$

invariant; that is, a transformation

$$\begin{aligned} x_1 &= X(x,y,p) \\ y_1 &= Y(x,y,p) \\ p_1 &= P(x,y,p) \end{aligned} \qquad (4.103)$$

is a contact transformation if the Pfaffiian equation, Eq. (4.102), is transformed to

$$dy_1 - p_1dx_1 = 0 \qquad (4.104)$$

after the transformation.

The above definition of contact transformation is extended by Lie[8] as follows.

When $Z,X_1,\ldots,X_n,P_1,\ldots,P_n$ are 2n+1 independent functions of the 2n+1 independent quantities $z,x_1,\ldots,x_n,p_1,\ldots,p_n$ such that the relation

$$dZ - P_i dX_i = \rho(dz - p_i dx_i) \; * \qquad (4.105)$$

*$P_i dX_i = \sum_{i=1}^{n} P_i dX_i$, i.e., the repetition of indices will indicate summation over all values of the index.

(where ρ does not vanish) is identically satisfied, then the transformation defined by

$$
\begin{aligned}
Z &= Z(z,x_1,\ldots,x_n,p_1,\ldots,p_n) \\
X_n &= X_n(z,x_1,\ldots,x_n,p_1,\ldots,p_n) \\
P_n &= P_n(z,x_1,\ldots,x_n,p_1,\ldots,p_n)
\end{aligned}
\tag{4.106}
$$

is called a contact transformation.

The contact transformation, Eq. (4.106) will transform a partial differential equation in $z,x_1,\ldots,x_n,p_1,\ldots,p_n$ into one in $Z,X_1,\ldots,X_n,P_1,\ldots,P_n$ and also the solution of the first partial differential equation into the solution of second.

In the next article, the property of a contact transformation, Eq. (4.105) will be used to derive expressions for the transformation functions in terms of characteristic functions.

4.1.8 Contact Transformation in Terms of the Characteristic Function

First, let us consider the infinitesimal transformation

$$Z = z + \delta\epsilon\zeta, \quad X_i = x_i + \delta\epsilon\xi_i, \quad P_i = p_i + \delta\epsilon\pi_i \tag{4.107}$$

where ζ, ξ_i and π_i are functions of $z,x_1,\ldots,x_n,p_1,\ldots,p_n$ and ϵ is a quantity so small that its square and higher powers may be neglected. We now wish to find the functions ζ, ξ_i, π_i in terms of a single function of z, x_i, p_i, known as the characteristic function.

From the definition of a contact transformation

$$dZ - P_i dX_i = \rho(dz - p_i dx_i) \tag{4.108}$$

we get

$$\frac{\partial Z}{\partial z} dz + \frac{\partial Z}{\partial x_i} dx_i + \frac{\partial Z}{\partial p_i} dp_i - P_i \left(\frac{\partial X_i}{\partial z} dz + \frac{\partial X_i}{\partial x_r} dx_r + \frac{\partial X_i}{\partial p_r} dp_r \right)$$

$$= \rho(dz - p_i dx_i) \quad . \tag{4.109}$$

Equating coefficients of dz, dx_i, and dp_i on both sides, we get

$$\begin{aligned} \frac{\partial Z}{\partial z} - P_i \frac{\partial X_i}{\partial z} &= \rho \\ \frac{\partial Z}{\partial p_r} - P_i \frac{\partial X_i}{\partial p_r} &= 0 \\ \frac{\partial Z}{\partial x_r} - P_i \frac{\partial X_i}{\partial x_r} &= -\rho p_r \end{aligned}$$

$$(r=1,\ldots,n) \quad . \tag{4.110}$$

For the infinitesimal transformation defined in Eq. (4.107), Eqs. (4.110) become

$$\begin{aligned} \frac{\partial \zeta}{\partial z} - p_i \frac{\partial \xi_i}{\partial z} &= \sigma \\ \frac{\partial \zeta}{\partial p_r} - p_i \frac{\partial \xi_i}{\partial p_r} &= 0 \\ \frac{\partial \zeta}{\partial x_r} - p_i \frac{\partial \xi_i}{\partial x_r} &= -\pi_r = -\sigma p_r \end{aligned} \tag{4.111}$$

where the relation $\rho = 1+\epsilon\sigma$ is used (σ, arbitrary). Now, p_i is independent of x_r, and z. Therefore, if we define a characteristic function W such that

$$W = p_i \xi_i - \zeta \quad . \tag{4.112}$$

Equations (4.111) will yield

$$-\frac{\partial W}{\partial z} = \sigma$$

$$\frac{\partial W}{\partial p_r} = \delta_{ir}\xi_i = \xi_r \quad *$$

$$-\frac{\partial W}{\partial x_r} = -\sigma p_r + \pi_r = p_r\frac{\partial W}{\partial z} + \pi_r \, . \qquad (4.113)$$

We then get

$$\xi_r = \frac{\partial W}{\partial p_r}$$

$$\zeta = p_i\frac{\partial W}{\partial p_i} - W$$

$$\pi_r = -\frac{\partial W}{\partial x_r} - p_r\frac{\partial W}{\partial z} \qquad (4.114)$$

and

$$\sigma = -\frac{\partial W}{\partial z} \, . \qquad (4.115)$$

Next, consider the infinitesimal transformation

$$Z_i = z_i + \delta\epsilon m_i(z_\nu, x_\mu, p_\mu^\nu)$$

$$X_k = x_k + \delta\epsilon\alpha_k(z_\nu, x_\mu, p_\mu^\nu)$$

$$P_k^i = p_k^i + \delta\epsilon\pi_k^i(z_\nu, x_\mu, p_\mu^\nu) \qquad (4.116)$$

where m_i, α_k and π_k^i are functions of $x_1,\ldots,x_m; z_1,\ldots,z_n$; and $p_1^1, p_2^1, \ldots, p_m^n$ $(= \partial z_n/\partial x_m)$ and ϵ is a quantity so small that its square and higher powers may be neglected. We now want to express the functions m_i, α_k, and π_k^i $(i=1,\ldots,n; k=1,\ldots,m)$ in terms of $W^i (i=1,\ldots,n)$, the characteristic functions.

From the definition of contact transformation we have

$$dZ_i - P_k^i dX_k = \rho_{i\nu}(dz_\nu - p_\mu^\nu dx_\mu) \qquad (4.117)$$

---

*By adding to both sides of the second equation in Eq. (4.111) the term

$$-\frac{\partial p_i}{\partial p_r}\xi_i .$$

we get

$$\left(\frac{\partial Z_i}{\partial z_\nu} dz_\nu + \frac{\partial Z_i}{\partial x_\mu} dx_\mu + \frac{\partial Z_i}{\partial p_\mu^\nu} dp_\mu^\nu\right)$$

$$- P_k^i \left(\frac{\partial X_k}{\partial z_\nu} dz_\nu + \frac{\partial X_k}{\partial x_\mu} dx_\mu + \frac{\partial X_k}{\partial p_\mu^\nu} dp_\mu^\nu\right)$$

$$= \rho_{i\nu} \, (dz_\nu - p_\mu^\nu dx_\mu) \ . \tag{4.118}$$

Equating coefficients of dz_ν, dx_μ, and dp_μ^ν we get

$$\frac{\partial Z_i}{\partial z_\nu} - P_k^i \frac{\partial X_k}{\partial z_\nu} = \rho_{i\nu}$$

$$\frac{\partial Z_i}{\partial x_\mu} - P_k^i \frac{\partial X_k}{\partial x_\mu} = -\rho_{i\nu} p_\mu^\nu$$

$$\frac{\partial Z_i}{\partial p_\mu^\nu} - p_k^i \frac{\partial X_k}{\partial p_\mu^\nu} = 0 \ . \tag{4.119}$$

For the infinitesimal transformation given in Eq. (4.116), Eqs. (4.119) become

$$\frac{\partial}{\partial z_\nu} (m_i - p_k^i \alpha_k) = R_{i\nu}$$

$$\frac{\partial}{\partial x_\mu} (m_i - p_k^i \alpha_k) = -p_\mu^\nu R_{i\nu} + \pi_\mu^i$$

$$\frac{\partial}{\partial p_\mu^\nu} (m_i - p_k^i \alpha_k) = \alpha_\mu \delta_{i\nu} \tag{4.120}$$

where the relation $\rho_{i\nu} = \delta_{i\nu} + \epsilon R_{i\nu}$ is used. Let us now define the characteristic function W_i as

$$W_i = p_k^i \alpha_k - m_i \ . \tag{4.121}$$

Equations (4.120) then become

$$\frac{\partial W_i}{\partial z_\nu} = R_{i\nu}$$

$$\frac{\partial W_i}{\partial x_\mu} = -p_\mu^\nu R_{i\nu} + \pi_\mu^i$$

$$\frac{\partial W_i}{\partial p_\mu^\nu} = \alpha_\mu \delta_{i\nu} \ . \qquad (4.122)$$

From these equations we then get

$$m_i = \frac{\partial W_i}{\partial p_\mu^\nu} p_\mu^\nu - W_i$$

$$\alpha_\mu = \frac{\partial W_i}{\partial p_\mu^i}$$

$$\pi_\mu^i = -\frac{\partial W_i}{\partial z_\nu} p_\mu^\nu - \frac{\partial W_i}{\partial x_\mu} \ . \qquad (4.123)$$

Having expressed m_i, α_μ, and π_μ^i in terms of the characteristic functions W_i, we now prove a very important property of m_i and α_μ.

For n=1, Eqs. (4.123) reduce to Eqs. (4.114) which are expressed in terms of a single characteristic function.

For $n > 1$, however, there is a very important property involving m_i and α_μ.[15] Now, differentiating the third equation in Eq. (4.120) with respect to p_σ^ρ, we get

$$\left\{\frac{\partial}{\partial p_\sigma^\rho} \frac{\partial}{\partial p_\mu^\nu} (m_i - p_k^i \alpha_k)\right\} = \frac{\partial}{\partial p_\sigma^\rho} (-\delta_{i\nu}\alpha_\mu)$$

or

$$\frac{\partial^2}{\partial p_\sigma^\rho \partial p_\mu^\nu} (m_i - p_k^i \alpha_k) = -\delta_{i\nu} \frac{\partial \alpha_\mu}{\partial p_\sigma^\rho} \ . \qquad (4.124)$$

The third equation in Eq. (4.120) can be written as

$$\frac{\partial}{\partial p_\sigma^\rho} (m_i - p_k^i \alpha_k) = -\delta_{i\rho}\alpha_\sigma \ . \qquad (4.125)$$

Differentiating Eq. (4.125) with respect to p^ν_μ, we get

$$\frac{\partial^2}{\partial p^\nu_\mu \partial p^\rho_\sigma}(m_i - p^i_k \alpha_k) = -\delta_{i\rho}\frac{\partial \alpha_\sigma}{\partial p^\nu_\mu} . \qquad (4.126)$$

Since the left side of Eqs. (4.124) and (4.126) are the same, we can equate their right sides and get

$$\delta_{i\nu}\frac{\partial \alpha_\mu}{\partial p^\rho_\sigma} = \delta_{i\rho}\frac{\partial \alpha_\sigma}{\partial p^\nu_\mu} . \qquad (4.127)$$

If we set $i=\nu\neq\rho$, Eq. (4.127) then gives

$$\frac{\partial \alpha_\mu}{\partial p^\rho_\sigma} = 0 . \qquad (4.128)$$

This means α_μ is independent of p^ρ_σ.

Next, the third equation in Eq. (4.120) gives

$$\begin{aligned}\frac{\partial m_i}{\partial p^\nu_\mu} &= \alpha_k \frac{\partial p^i_k}{\partial p^\nu_\mu} - \delta_{i\nu}\alpha_\mu \\ &= \alpha_k \delta_{k\mu} - \alpha_\mu = \alpha_\mu - \alpha_\mu = 0 . \qquad (4.129)\end{aligned}$$

This equation means m_i is independent of p^ν_μ.

As a result, the infinitesimal transformation defined in Eq. (4.116) should be modified to:

$$\begin{aligned} Z_i &= z_i + \delta\epsilon m_i(z_\nu, x_\mu) \\ X_k &= x_k + \delta\epsilon \alpha_k(z_\nu, x_\mu) \\ P^i_k &= p^i_k + \delta\epsilon \pi^i_k(z_\nu, x_\mu, p^\nu_\mu) \qquad (4.130)\end{aligned}$$

where m_i, α_k, and π^i_k are defined in Eqs. (4.123). Equations (4.123) also suggest that the dependence of W_i on p^ν_μ is linear—a very far-reaching and important conclusion which will be made use of later.

4.1.9 Contact Transformation of Higher Orders

Let us now consider the extended infinitesimal contact transformation defined by

$$
\begin{aligned}
Z_i &= z_i + \delta\epsilon m_i(z_\nu, x_\mu, p_\mu^\nu) \\
X_k &= x_k + \delta\epsilon\alpha_k(z_\nu, x_\mu, p_\mu^\nu) \\
P_k^i &= p_k^i + \delta\epsilon\pi_k^i(z_\nu, x_\mu, p_\mu^\nu) \\
P_{jk}^i &= p_{jk}^i + \delta\epsilon\pi_{jk}^i(z_\nu, x_\mu, p_\mu^\nu, p_{\mu s}^\nu) \\
P_{jk\ell}^i &= p_{jk\ell}^i + \delta\epsilon\pi_{jk\ell}^i(z_\nu, x_\mu, p_\mu^\nu, p_{\mu s}^\nu, p_{\mu st}^\nu)
\end{aligned}
\qquad (4.131)
$$

where α_k and m_i are independent of p_μ^ν for $i > 1$, and

$$
p_{jk}^i = \frac{\partial z_i}{\partial x_j \partial x_k}\text{, etc.} \qquad (4.132)
$$

We now want to express π_{jk}^i and $\pi_{jk\ell}^i$ as functions of the characteristic function W_i. The functions m_i, α_k^{jk}, and π_k^i are expressed in terms of W_i and given in Eqs. (4.123).

By definition,

$$
dP_k^i = P_{jk}^i dX_j \;. \qquad (4.133)
$$

Substituting the infinitesimal contact transformation from Eq. (4.131) into Eq. (4.133), we get

$$
d\pi_k^i = p_{jk}^i d\alpha_j + \pi_{jk}^i dx_j \quad . \qquad (4.134)
$$

Subtracting both sides of Eq. (4.134) by the quantity $\alpha_j p_{s\,jk}^i dx_s$, we then get

$$
d(\pi_k^i - p_{jk}^i\alpha_j) = \pi_{jk}^i dx_j - \alpha_j p_{s\,jk}^i dx_s \;. \qquad (4.135)
$$

The second term on the right side of Eq. (4.135) should be dropped since in π^i_{jk} derivatives of order higher than the second are omitted. We then get

$$\boxed{\pi^i_{jk} = \frac{d}{dx_j} (\pi^i_k - p^i_{jk}\alpha_j)} \tag{4.136}$$

or, in terms of α_j and π^i_k.

$$\boxed{\begin{aligned} \pi^i_{jk} &= \frac{\partial\pi^i_k}{\partial x_j} + \frac{\partial\pi^i_k}{\partial z_s} p^s_j + \frac{\partial\pi^i_k}{\partial p^\nu_\mu} p^\nu_{\mu j} \\ &- p^i_{jk}\left(\frac{\partial\alpha_j}{\partial x_j} + \frac{\partial\alpha_j}{\partial z_s} p^s_j + \frac{\partial\alpha_j}{\partial p^\nu_\mu} p^\nu_{\mu j}\right) \end{aligned}} \tag{4.137}$$

To express π^i_{jk} in terms of the characteristic functions W_i, we have to put expressions for π^i_k and α_j from Eqs. (4.123) into Eq. (4.137).

Similarly, for the third-order function $\pi^i_{jk\ell}$, we get

$$\boxed{\pi^i_{jk\ell} = \frac{d}{dx_\ell} (\pi^i_{jk} - p^i_{jkt}\alpha_t)} \tag{4.138}$$

or,

$$\boxed{\begin{aligned} \pi^i_{jk\ell} &= \frac{\partial\pi^i_{jk}}{\partial x_\ell} + \frac{\partial\pi^i_{jk}}{\partial z_s} p^s_\ell + \frac{\partial\pi^i_{jk}}{\partial p^\nu_\mu} p^\nu_{\mu\ell} + \frac{\partial\pi^i_{jk}}{\partial p^a_{bc}} p^a_{bc\ell} \\ &- p^i_{jkt}\left(\frac{\partial\alpha_t}{\partial x_\ell} + \frac{\partial\alpha_t}{\partial z_s} p^s_\ell + \frac{\partial\alpha_t}{\partial p^\nu_\mu} p^\nu_{\mu\ell} + \frac{\partial\alpha_t}{\partial p^a_{bc}} p^a_{bc\ell}\right) \end{aligned}} \tag{4.139}$$

In order to express $\pi^i_{jk\ell}$ in terms of the characteristic functions, we have to express π^i_{jk} and α_ℓ in terms of W_i, as in Eqs. (4.137) and (4.123), and substitute into Eq. (4.139).

The same principle can be extended to higher order, e.g.,

$$\boxed{\pi^i_{jk\ell m} = \frac{d}{dx_m} (\pi^i_{jk\ell} - p^i_{jk\ell f}\alpha_f)} \tag{4.140}$$

or

$$\boxed{\begin{aligned}\pi^i_{jk\ell m} = {} & \frac{\partial\pi^i_{jk\ell}}{\partial x_m} + \frac{\partial\pi^i_{jk\ell}}{\partial z_s}\,p^s_m + \frac{\partial\pi^i_{jk\ell}}{\partial p^\nu_\mu}\,p^\nu_{\mu m} + \frac{\partial\pi^i_{jk\ell}}{\partial p^a_{bc}}\,p^a_{bcm} + \frac{\partial\pi^i_{jk\ell}}{\partial p^a_{bcd}}\,p^a_{bcdm} \\ & - p^i_{jk\ell f}\left(\frac{\partial\alpha_f}{\partial x_m} + \frac{\partial\alpha_f}{\partial z_s}\,p^s_m + \frac{\partial\alpha_f}{\partial p^\nu_\mu}\,p^\nu_{\mu m} + \frac{\partial\alpha_f}{\partial p^a_{bc}}\,p^a_{bcm} + \frac{\partial\alpha_f}{\partial p^a_{bcd}}\,p^a_{bcdm}\right)\end{aligned}} \tag{4.141}$$

4.2 SIMILARITY ANALYSIS OF DIFFUSION EQUATION

We shall now look at the diffusion equation from another point of view, namely, the searching for all possible groups of transformation that will reduce the diffusion equation to an ordinary differential equation. In applying such a technique to a given differential equation, it may turn out that for some or all of the groups other than the linear and spiral groups, the boundary condition cannot be transformed although the partial differential equation can be transformed into an ordinary differential equation. For such cases, we are at least assured that the groups of transformations that remain are the groups possible for the given boundary value problems. A similarity analysis of the diffusion equation from this point of view is apparently not covered in the literature. The one-dimensional form of the diffusion equation in rectangular coordinate is chosen because of its simplicity. Extension of analyses to equations expressed in other coordinates can readily be made.

4.2.1 Analysis

Consider the diffusion equation

$$\frac{\partial u}{\partial t} - \nu\,\frac{\partial^2 u}{\partial y^2} = 0 \tag{4.142}$$

on which an infinitesimal transformation is to be made on the dependent and independent variables and derivatives of the dependent variable with respect to the independent variable. The infinitesimal transformation is

$$
\begin{aligned}
t' &= t + \delta\epsilon\xi(t,y,u,p,q) \\
y' &= y + \delta\epsilon\eta(t,y,u,p,q) \\
u' &= u + \delta\epsilon\zeta(t,y,u,p,q) \\
p' &= p + \delta\epsilon\pi_1(t,y,u,p,q) \\
q' &= q + \delta\epsilon\pi_2(t,y,u,p,q) \\
p_{22}' &= p_{22} + \delta\epsilon\pi_{22}(t,y,u,p,q,p_{11},p_{12},p_{22})
\end{aligned} \tag{4.143}
$$

where, in terms of the characteristic function, W,

$$
\begin{aligned}
\xi &= \frac{\partial W}{\partial p} \\
\eta &= \frac{\partial W}{\partial q} \\
\zeta &= p\,\frac{\partial W}{\partial p} + q\,\frac{\partial W}{\partial q} - W \\
-\pi_1 &= \frac{\partial W}{\partial t} + p\,\frac{\partial W}{\partial u} \\
-\pi_2 &= \frac{\partial W}{\partial y} + q\,\frac{\partial W}{\partial u} \\
-\pi_{22} &= \frac{\partial^2 W}{\partial y^2} + 2q\,\frac{\partial^2 W}{\partial y \partial u} + q^2\,\frac{\partial^2 W}{\partial u^2} + 2p_{12}\left(\frac{\partial^2 W}{\partial y \partial p} + q\,\frac{\partial^2 W}{\partial u \partial p}\right) \\
&\quad + 2p_{22}\left(\frac{\partial^2 W}{\partial y \partial q} + q\,\frac{\partial^2 W}{\partial u \partial q}\right) + p_{12}^2\,\frac{\partial^2 W}{\partial p^2} + 2p_{12}p_{22}\,\frac{\partial^2 W}{\partial p \partial q} \\
&\quad + p_{22}^2\,\frac{\partial^2 W}{\partial q^2} + p_{22}\,\frac{\partial W}{\partial u} \;.
\end{aligned} \tag{4.144}
$$

The characteristic function, W, is a function of t, y, u, p, and q. We note that

$$
p = \frac{\partial u}{\partial t}, \quad q = \frac{\partial u}{\partial y}, \quad p_{22} = \frac{\partial^2 u}{\partial y^2}, \quad p_{12} = \frac{\partial^2 u}{\partial t \partial y} \;. \tag{4.145}
$$

It is shown in article 4.1.3 that the necessary and sufficient condition that a partial differential equation $F(t,y,u,p,q,p_{12},p_{22}) = 0$ invariant under the group of transformation represented by Uf is UF = 0 which for the diffusion equation, is

$$U(p-\nu p_{22}) = 0 \tag{4.146}$$

or, expanding the expression by employing the operator U:

$$\xi \frac{\partial(\;)}{\partial t} + \eta \frac{\partial(\;)}{\partial y} + \zeta \frac{\partial(\;)}{\partial u} + \pi_1 \frac{\partial(\;)}{\partial p} + \pi_2 \frac{\partial(\;)}{\partial q}$$

$$+ \pi_{11} \frac{\partial(\;)}{\partial p_{11}} + \pi_{22} \frac{\partial(\;)}{\partial p_{22}} + \pi_{12} \frac{\partial(\;)}{\partial p_{12}} = 0 \tag{4.147}$$

where the parenthesis represents the differential equation

$$p - \nu p_{22} = 0 \tag{4.148}$$

Carrying out the operation in Eq. (4.147) yields:

$$\pi_1 - \nu\pi_{22} = 0. \tag{4.149}$$

Upon substituting expressions from Eq. (4.144) into Eq. (4.149) yields

$$- \frac{\partial W}{\partial t} - p \frac{\partial W}{\partial u} + \nu \frac{\partial^2 W}{\partial y^2} + 2\nu q \frac{\partial^2 W}{\partial y \partial u} + \nu q^2 \frac{\partial^2 W}{\partial u^2}$$

$$+ 2p_{12} \frac{\partial^2 W}{\partial y \partial p} \nu + 2p_{12}q \frac{\partial^2 W}{\partial u \partial p} \nu + 2p \frac{\partial^2 W}{\partial y \partial q} + pq \frac{\partial^2 W}{\partial u \partial q}$$

$$+ \nu p_{12}^2 \frac{\partial^2 W}{\partial p^2} + 2p_{12}p \frac{\partial^2 W}{\partial p \partial q} + \frac{p^2}{\nu} \frac{\partial^2 W}{\partial q^2} + p \frac{\partial W}{\partial u} = 0 . \tag{4.150}$$

Equation (4.150) is seen to be a linear partial differential equation in $W(t,y,u,p,q)$.

Since W is not a function of p_{12}, the coefficients of the terms involving p_{12} and p_{12}^2 should be zero. We then get

$$\frac{\partial^2 W}{\partial p^2} = 0 \qquad (4.151)$$

$$\frac{\partial^2 W}{\partial y \partial p} + q \frac{\partial^2 W}{\partial u \partial p} + \frac{p}{\nu} \frac{\partial^2 W}{\partial p \partial q} = 0 \,. \qquad (4.152)$$

Equation (4.151) indicates that W is a linear function of p. Thus we can write

$$W = W_1(t,y,u,q) + pW_2(t,y,u,q) \,. \qquad (4.153)$$

Substituting this form of W into Eq. (4.152), we get

$$\frac{\partial W_2}{\partial y} + q \frac{\partial W_2}{\partial u} + \frac{p}{\nu} \frac{\partial W_2}{\partial q} = 0 \,. \qquad (4.154)$$

Since W_2 is not a function of p, the coefficient of p in Eq. (4.154) must be zero. We therefore obtain two equations, namely,

$$\frac{\partial W_2}{\partial q} = 0 \qquad (4.155)$$

$$\frac{\partial W_2}{\partial y} + q \frac{\partial W_2}{\partial u} = 0 \,. \qquad (4.156)$$

Equation (4.155) indicates that W_2 is not a function of q, i.e., $W_2 = W_2(t,y,u)$, and so Eq. (4.156) can be broken into the two equations,

$$\frac{\partial W_2}{\partial y} = 0 \qquad (4.157a)$$

and

$$\frac{\partial W_2}{\partial u} = 0 \,. \qquad (4.157b)$$

This means W_2 is independent of both y and u and, as a result,

$$W_2 = W_2(t) \qquad (4.158)$$

and the characteristic function now takes the form

$$W = W_1(t,y,u,q) + pW_2(t) . \tag{4.159}$$

Putting this form of W into Eq. (4.150), we get

$$- \frac{\partial W_1}{\partial t} - p \frac{\partial W_2}{\partial t} + \nu \frac{\partial^2 W_1}{\partial y^2} + 2\nu q \frac{\partial^2 W_1}{\partial y \partial u} + \nu q^2 \frac{\partial^2 W_1}{\partial u^2}$$

$$+ 2p \frac{\partial^2 W_1}{\partial y \partial q} + pq \frac{\partial^2 W_1}{\partial u \partial q} + \frac{p^2}{\nu} \frac{\partial^2 W_1}{\partial q^2} = 0 . \tag{4.160}$$

Since both W_1 and W_2 are independent of p, Eq. (4.160) can be separated into three equations, corresponding to the coefficients of p^0, p^1, and p^2. We then get

$$p^0: \quad - \frac{\partial W_1}{\partial t} + \nu \frac{\partial^2 W_1}{\partial y^2} + 2\nu q \frac{\partial^2 W_1}{\partial y \partial u} + \nu q^2 \frac{\partial^2 W_1}{\partial u^2} = 0 \tag{4.161}$$

$$p^1: \quad - \frac{\partial W_2}{\partial t} + 2 \frac{\partial^2 W_1}{\partial y \partial q} + q \frac{\partial^2 W_1}{\partial u \partial q} = 0 \tag{4.162}$$

$$p^2: \quad \frac{\partial^2 W_1}{\partial q^2} = 0 . \tag{4.163}$$

From Eq. (4.163), W_1 is linearly dependent on q, therefore, it becomes

$$W_1 = W_{11}(t,y,u) + W_{12}(t,y,u)q . \tag{4.164}$$

Putting this form of W_1 into Eq. (4.162), we get

$$- \frac{dW_2}{dt} + 2 \frac{\partial W_{12}}{\partial y} + q \frac{\partial W_{12}}{\partial u} = 0 . \tag{4.165}$$

Both W_2 and W_{12} are independent of q, therefore, Eq. (4.165) becomes

$$- \frac{dW_2}{dt} + 2 \frac{\partial W_{12}}{\partial y} = 0 \tag{4.166}$$

$$\frac{\partial W_{12}}{\partial u} = 0 . \tag{4.167}$$

From Eq. (4.167), W_{12} is independent of u. Also, since W_2 is a function of t only, Eq. (4.166) shows that W_{12} depends linearly on y, i.e.,

$$W_{12} = W_{121}(t) + W_{122}(t)y \ . \tag{4.168}$$

Equation (4.162) then becomes

$$-\frac{dW_2}{dt} + 2W_{122} = 0 \ . \tag{4.169}$$

We will make use of this equation later.

The characteristic function, W, now becomes

$$W = W_{11}(t,y,u) + \{W_{121}(t)+W_{122}(t)y\}q + W_2(t)p \ . \tag{4.170}$$

Putting Eq. (4.164) into Eq. (4.161), we get

$$-\frac{\partial W_{11}}{\partial t} - \left(\frac{dW_{121}}{dt} + \frac{dW_{122}}{dt}y\right)q + \nu\frac{\partial^2 W_{11}}{\partial y^2} + 2\nu q\frac{\partial^2 W_{11}}{\partial y\partial u} + \nu q^2\frac{\partial^2 W_{11}}{\partial u^2} = 0 \tag{4.171}$$

Since W_{11}, W_{121} and W_{122} are independent of q, terms with different powers of q are grouped and their coefficients are put equal to zero. Three equations are obtained:

$$q^0: \quad -\frac{\partial W_{11}}{\partial t} + \nu\frac{\partial^2 W_{11}}{\partial y^2} = 0 \tag{4.172}$$

$$q^1: \quad -\frac{dW_{121}}{dt} - \frac{dW_{122}}{dt}y + 2\nu\frac{\partial^2 W_{11}}{\partial y\partial u} = 0 \tag{4.173}$$

$$q^2: \quad \frac{\partial^2 W_{11}}{\partial u^2} = 0 \ . \tag{4.174}$$

Equation (4.174) shows that W_{11} is linearly dependent on u. Therefore,

$$W_{11} = W_{111}(t,y) + W_{112}(t,y)u \ . \tag{4.175}$$

Equation (4.173) then gives

$$- \frac{dW_{121}}{dt} - \frac{dW_{122}}{dt} y + 2\nu \frac{\partial W_{112}}{\partial y} = 0 . \qquad (4.176)$$

Therefore, W_{112} can be written as

$$W_{112} = W_{1121}(t) + W_{1122}(t)y + W_{1123}(t)y^2 . \qquad (4.177)$$

Equation (4.176), then becomes

$$- \frac{dW_{121}}{dt} - \frac{dW_{122}}{dt} y + 2\nu W_{1122} + 4\nu y W_{1123} = 0 . \qquad (4.178)$$

Since all the W's in Eq. (4.178) are independent of y, we get

$$- \frac{dW_{121}}{dt} + 2\nu W_{1122} = 0 \qquad (4.179)$$

$$- \frac{dW_{122}}{dt} + 4\nu W_{1123} = 0 . \qquad (4.180)$$

Putting W_{11} into Eq. (4.172), we get

$$- \frac{\partial W_{111}}{\partial t} + \nu \frac{\partial^2 W_{111}}{\partial y^2}$$

$$- \left\{ \frac{dW_{1121}}{dt} + \frac{dW_{1122}}{dt} y + \frac{dW_{1123}}{dt} y^2 \right\} u$$

$$+ 2\nu W_{1123} u = 0 . \qquad (4.181)$$

Equation (4.181) can be separated into:

$$u^0: \quad - \frac{\partial W_{111}}{\partial t} + \nu \frac{\partial^2 W_{111}}{\partial y^2} = 0 \qquad (4.182)$$

$$u^1: \quad -\frac{dW_{1121}}{dt} + 2\nu W_{1123} = 0 \tag{4.183}$$

$$\frac{dW_{1122}}{dt} = 0 \tag{4.184}$$

$$\frac{dW_{1123}}{dt} = 0 \quad . \tag{4.185}$$

From Eqs. (4.184) and (4.185),

$$W_{1122} = c_1 \tag{4.186a}$$

and

$$W_{1123} = c_2 \quad . \tag{4.186b}$$

From Eq. (4.183),

$$W_{1121} = 2\nu c_2 t + c_3 \quad . \tag{4.187}$$

From Eqs. (4.179) and (4.180),

$$W_{121} = 2\nu c_1 t + c_4 \tag{4.188}$$

$$W_{122} = 4\nu c_2 t + c_5 \quad . \tag{4.189}$$

From Eq. (4.169),

$$W_2 = 4\nu c_2 t^2 + 2c_5 t + c_6 \quad . \tag{4.190}$$

The final form of the characteristic function is therefore

$$
\boxed{\begin{aligned} W(t,y,u,p,q) &= W_{111}(t,y) \\ &+ \{2\nu c_2 t + c_3 + c_1 y + c_2 y^2\}u \\ &+ \{2\nu c_1 t + c_4 + (4\nu c_2 t + c_5)y\}q \\ &+ \{4\nu c_2 t^2 + 2c_5 t + c_6\}p \end{aligned}} \tag{4.191}
$$

where $W_{111}(t,y)$ is any function satisfying Eq. (4.182), i.e.,

$$
\boxed{\frac{\partial W_{111}}{\partial t} - \nu \frac{\partial^2 W_{111}}{\partial y^2} = 0} \tag{4.192}
$$

The characteristic function, W, given in Eq. (4.191), will now be used to determine the absolute invariants.

4.2.2 Absolute Invariants and The Transformed Equations

With the characteristic function, W, obtained as in Eq. (4.191), we now make use of the general theory to find the absolute invariants. From Eq. (4.65), the following relations are obtained:

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = \frac{du}{\zeta} \tag{4.193}
$$

where the transformation functions ξ, η, and ζ can be obtained by putting into Eq. (4.144) the characteristic function, W, given by Eq. (4.191). Equation (4.193) then becomes

$$
\begin{aligned} \frac{dt}{4\nu c_2 t^2 + 2c_5 t + c_6} &= \frac{dy}{2\nu c_1 t + c_4 + (4\nu c_2 t + c_5)y} \\ &= \frac{du}{-W_{111}(t,y) - (2\nu c_2 t + c_3 + c_1 y + c_2 y^2)u} \ . \end{aligned} \tag{4.194}
$$

The number of possible groups are large, due to the fact that all c's are arbitrary and $W_{111}(t,y)$ is an arbitrary function satisfying Eq. (4.182). Therefore, we investigate a few special cases of the parameters. Other groups can be obtained in a similar manner.

Case 1. $W_{111}(t,y) = c_1 = c_2 = c_4 = c_6 = 0$

Equation (4.194) becomes

$$\frac{dt}{2c_5t} = \frac{dy}{c_5y} = \frac{du}{-c_3u} \quad . \tag{4.195}$$

The two independent solutions to Eq. (4.195) are

$$\frac{y}{\sqrt{t}} = \text{constant}$$

and

$$\frac{u}{t^\alpha} = \text{constant}; \qquad \alpha = -\frac{c_3}{2c_5} \quad .$$

According to the theories in the preceding articles, the diffusion equation can be expressed in terms of these two invariants, i.e.,

$$\eta = \frac{y}{\sqrt{t}} \tag{4.196a}$$

and

$$f(\eta) = \frac{u}{t^\alpha} \quad . \tag{4.196b}$$

The diffusion equation is then transformed into an ordinary differential equation

$$\nu f'' = \alpha f - \frac{1}{2}\eta f' \quad .$$

The transformation are seen to be the linear group of transformations.

Case 2. $W_{111}(t,y) = c_1 = c_2 = c_4 = c_5 = 0$

Equation (4.194) then becomes

$$\frac{dt}{c_6} = -\frac{du}{c_3u} \tag{4.197}$$

and

$$y \quad = \quad \text{constant.} \tag{4.198}$$

Following the same arguments as in Case 1, we get the absolute invariants as

$$\eta = y \text{ and } f(\eta) \quad = \quad \frac{u}{e^{\beta t}} \ ; \qquad (\beta = -\frac{c_3}{c_6})$$

and the diffusion equation is transformed to

$$\nu f'' - \beta f \quad = \quad 0 \ .$$

The transformations are seen to be the spiral group.

<u>Case 3</u>. $W_{111}(t,y) = c_2 = c_3 = c_4 = c_5 = c_6 = 0$

Equation (4.194) becomes

$$\frac{dy}{2\nu c_1 t} \quad = \quad \frac{du}{-c_1 yu} \tag{4.199}$$

and

$$t \quad = \quad \text{constant.} \tag{4.200}$$

The absolute invariants are

$$\eta = t \text{ and } f(\eta) \quad = \quad \frac{u}{e^{-\frac{y^2}{4\nu t}}}$$

and the transformed equation is

$$2\eta f' + f \quad = \quad 0 \ .$$

Case 4. $W_{111}(t,y) = c_1 = c_2 = c_5 = 0$

Equation (4.194) becomes

$$\frac{dt}{c_6} = \frac{dy}{c_4} = \frac{du}{-c_3 u} . \qquad (4.201)$$

The absolute invariants are

$$\eta = y - \frac{c_4}{c_6} t \text{ and } f = ue^{\frac{c_3}{c_6}t} .$$

The diffusion equation becomes

$$\nu f'' + \frac{c_4}{c_6} f' + \frac{c_3}{c_6} f = 0 .$$

The above four cases are examples of cases where the solution of Eq. (4.194) is straightforward, i.e., the two independent solutions can be solved by simple pairing of equations. The following cases are those in which the two solutions have to be solved in a sequence of steps.

Case 5. $W_{111}(t,y) = c_1 = c_3 = c_4 = c_5 = c_6 = 0$

Equation (4.194) now becomes

$$\frac{dt}{4\nu t^2} = \frac{dy}{4\nu ty} = \frac{du}{-(2\nu t+y^2)u} . \qquad (4.202)$$

The first of the two equations gives the solution

$$\frac{y}{t} = \frac{1}{k_1} \qquad (4.203)$$

Next, replacing t in the last two terms of Eq. (4.202) by $k_1 y$ [based on Eq. (4.203)], we get

$$-\frac{(2\nu k_1 y+y^2)}{4\nu k_1 y^2} dy = \frac{du}{u} . \qquad (4.204)$$

The solution to Eq. (4.204) is

$$uy^{1/2}\, e^{y/4\nu k_1} = \text{constant}$$

or, using Eq. (4.203) again,

$$uy^{1/2}\, e^{y^2/4\nu t} = k_2 \; . \qquad (4.204a)$$

According to the applicable theorems, the absolute invariants are, from Eqs. (4.203) and (4.204a)

$$\eta = \frac{y}{t} \text{ and } f(\eta) = uy^{1/2}\, e^{y^2/4\nu t} \; .$$

The diffusion equation is then transformed into

$$\eta^2 f'' - \eta f' + \frac{3}{4} f = 0 \; .$$

Case 6. $W_{111}(t,y) = c_2 = c_3 = c_4 = c_5 = 0$

Equation (4.194) becomes

$$\frac{dt}{c_6} = \frac{dy}{2\nu c_1 t} = \frac{du}{-c_1 yu} \; .$$

The first two terms give

$$y - \frac{\nu c_1}{c_6} t^2 = k_1 \qquad (4.205)$$

where k_1 is the constant of integration. Combining the first and the third term and making use of Eq. (4.205), we get

$$ue^{\frac{c_1}{c_6}\left(k_1 t + \frac{\nu c_1}{3c_6} t^3\right)} = k_2 \qquad (4.206)$$

where k_2 is the constant of integration. Using Eq. (4.205), we get

$$u e^{\frac{c_1}{c_6}\left(yt - \frac{2\nu c_1}{3c_6} t^3\right)} = k_2 \ . \tag{4.207}$$

Equations (4.205) and (4.207) give the invariants,

$$\eta = y - \frac{\nu c_1}{c_6} t^2$$

and

$$f(\eta) = u e^{\frac{c_1}{c_6}\left(yt - \frac{2\nu c_1}{3c_6} t^3\right)} \ .$$

The diffusion equation is transformed to

$$f'' + \frac{c_1}{c_6 \nu} \eta f = 0 \ .$$

Case 7. $W_{111}(t,y) = c_2 = c_3 = c_4 = c_6 = 0$

Equation (4.194) becomes

$$\frac{dt}{2c_5 t} = \frac{dy}{2\nu c_1 t + c_5 y} = \frac{du}{-c_1 y u} \ . \tag{4.208}$$

By following the same steps as in the two previous cases, the invariants are found to be

$$\eta = \frac{y}{\sqrt{t}} - \frac{2\nu c_1}{c_5} \sqrt{t}$$

and

$$f(\eta) = u e^{\frac{c_1}{c_5} y - \frac{\nu c_1^2}{c_5^2} t} \ .$$

The diffusion equation is then transformed to

$$\nu f'' + \frac{1}{2} \eta f' = 0 \ .$$

In all the above cases, $W_{111}(t,y)$ was taken to be zero. This is not necessary as we shall show in the following two cases.

Case 8. $W_{111}(t,y) = \nu a_1 t + \frac{1}{2} a_1 y^2$

$$c_1 = c_2 = c_3 = c_4 = c_6 = 0 .$$

It can be shown easily that this functional form of $W_{111}(t,y)$ satisfies Eq. (4.192). For this case, Eq. (4.194) becomes

$$\frac{dt}{2c_5 t} = \frac{dy}{c_5 y} = \frac{du}{-\nu a_1 t - \frac{1}{2} a_1 y^2} .$$

Using the same method as in previous cases, we find

$$\eta = \frac{y}{\sqrt{t}}$$

and

$$f(\eta) = u + \frac{a_1(2\nu t + y^2)}{4c_5} .$$

The diffusion equation is transformed to

$$\nu f'' + \frac{1}{2} \eta f' = 0 .$$

Case 9. $W_{111}(t,y) = a_o + \nu a_1 t + \frac{1}{2} a_1 y^2$

$$c_1 = c_2 = c_3 = c_4 = c_6 = 0 .$$

The only change made in this case is the addition of a constant term, a_o, to $W_{111}(t,y)$. The invariants in this case are

$$\eta = \frac{y}{\sqrt{t}}$$

and

$$f(\eta) = u + \frac{a_1(2\nu t+y^2)}{4c_5} - \frac{a_0}{2c_5} \ell nt .$$

The diffusion equation becomes

$$\nu f'' + \frac{1}{2} \eta f' - \frac{a_0}{2c_5} = 0 .$$

It is seen that as a result of the additional constant term, a_0, one more term is added to $f(\eta)$ and the transformed equation, as compared with Case 8.

4.3 SIMILARITY ANALYSIS OF STEADY, TWO-DIMENSIONAL, LAMINAR, BOUNDARY-LAYER EQUATIONS

This article is the second application of the theories given in article 4.1. The example treated in the previous article concerns the diffusion equation. The present article differs from that example in that an unknown function is involved in the differential equation which has to be determined. This factor introduces complexity in the process of searching for the possible groups of transformation. The method developed in this article can be used for similar problems.

We shall see that the boundary-layer equation may be transformed from a nonlinear partial differential equation to nonlinear ordinary differential equation by groups of transformation other than the linear and spiral groups. However, in transforming usual boundary conditions, the linear and the spiral groups are the only two groups possible.

4.3.1 Infinitesimal Transformation and the Characteristic Function

The laminar, incompressible boundary-layer equation, in terms of the stream function ψ, can be written as

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = u_e \frac{du_e}{dx} + \frac{\partial^3\psi}{\partial y^3} . \qquad (4.209)$$

With the boundary conditions

$$y = 0: \quad \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0$$

$$y = \infty: \quad \frac{\partial\psi}{\partial y} = u_e(x)$$

where all the quantities are in dimensionless form by the transformation

$$x = \frac{x'}{L}\ , \quad y = \frac{y'}{L}\sqrt{Re}\ , \quad \psi = \frac{\psi'}{LU_o}\sqrt{Re}\ , \quad u_e = \frac{u_e'}{U_o}\ , \quad Re = \frac{U_o L}{\nu}$$

where x', y', ψ', and u_e' refer to dimensional quantities and L and U_o are reference length and velocity.

Equation (4.209) can be put in a shorthand form as

$$p_2 p_{12} - p_1 p_{22} = \phi + p_{222} \qquad (4.210)$$

where

$$p_1 = \frac{\partial\psi}{\partial x}, \quad p_2 = \frac{\partial\psi}{\partial y}, \quad p_{12} = \frac{\partial^2\psi}{\partial x\partial y}\ , \text{ etc.},$$

and

$$\phi = u_e \frac{du_e}{dx}\ . \qquad (4.211)$$

Now, let us denote the following differential expression by F:

$$F = p_{222} + \phi - p_2 p_{12} + p_1 p_{22}\ . \qquad (4.212)$$

The equation F = 0 will be invariant under the infinitesimal transformation

$$
\begin{aligned}
x' &= x + \delta\epsilon\alpha_1(x,y,\psi,p_1,p_2) \\
y' &= y + \delta\epsilon\alpha_2(x,y,\psi,p_1,p_2) \\
\psi' &= \psi + \delta\epsilon m(x,y,\psi,p_1,p_2) \\
p_1' &= p_1 + \delta\epsilon\pi_1(x,y,\psi,p_1,p_2) \\
p_2' &= p_2 + \delta\epsilon\pi_2(x,y,\psi,p_1,p_2) \\
p_{12}' &= p_{12} + \delta\epsilon\pi_{12}(x,y,\psi,p_1,p_2,p_{11},p_{12},p_{22}) \\
p_{22}' &= p_{22} + \delta\epsilon\pi_{22}(x,y,\psi,p_1,p_2,p_{11},p_{12},p_{22}) \\
p_{222}' &= p_{222} + \delta\epsilon\pi_{222}(x,y,\psi,p_1,p_2,p_{11},p_{12},p_{22},p_{111},\ldots,p_{222})
\end{aligned}
\qquad (4.213)
$$

if

$$UF = 0 \qquad (4.214)$$

or, in expanded form,

$$\alpha_1 \frac{\partial F}{\partial x} + \alpha_2 \frac{\partial F}{\partial y} + m \frac{\partial F}{\partial \psi} + \pi_1 \frac{\partial F}{\partial p_1} + \pi_2 \frac{\partial F}{\partial p_2} + \pi_{ij} \frac{\partial F}{\partial p_{ij}} + \pi_{\ell mn} \frac{\partial F}{\partial p_{\ell mn}} = 0 \; . \qquad (4.215)$$

Putting F from Eq. (4.212) into Eq. (4.215) we get

$$-\pi_2 p_{12} - p_2\pi_{12} + \pi_1 p_{22} + p_1\pi_{22} + \phi'\alpha_1 + \pi_{222} = 0 \; . \qquad (4.216)$$

The next step is to express all the transformation functions, α_1, α_2, etc., in terms of the characteristic function, W. The functional form of W is then determined by Eq. (4.216). Now, from Eqs. (4.114), (4.137) and (4.139),

$$
\begin{aligned}
\alpha_1 &= \frac{\partial W}{\partial p_1} \\
\alpha_2 &= \frac{\partial W}{\partial p_2} \\
\pi_1 &= -\frac{\partial W}{\partial \psi} p_1 - \frac{\partial W}{\partial x} \\
\pi_2 &= -\frac{\partial W}{\partial \psi} p_2 - \frac{\partial W}{\partial y} \\
m &= p_i \frac{\partial W}{\partial p_i} - W
\end{aligned}
\qquad (4.217)
$$

and

$$-\pi_{12} = \frac{\partial^2 W}{\partial x \partial y} + p_2 \frac{\partial^2 W}{\partial x \partial \psi} + p_1\left(\frac{\partial^2 W}{\partial \psi \partial y} + p_2 \frac{\partial^2 W}{\partial \psi^2}\right) + p_{11}\left(\frac{\partial^2 W}{\partial p_1 \partial y} + p_2 \frac{\partial^2 W}{\partial p_1 \partial \psi}\right)$$

$$+ \; p_{21}\left(\frac{\partial^2 W}{\partial p_2 \partial y} + \frac{\partial W}{\partial \psi} + p_2 \frac{\partial^2 W}{\partial p_2 \partial \psi}\right)$$

$$+ \; p_{12}\left(\frac{\partial^2 W}{\partial x \partial p_1} + \frac{\partial^2 W}{\partial \psi \partial p_1} p_1 + \frac{\partial^2 W}{\partial p_1^2} p_{11} + \frac{\partial^2 W}{\partial p_2 \partial p_1} p_{21}\right)$$

$$+ \; p_{22}\left(\frac{\partial^2 W}{\partial x \partial p_2} + \frac{\partial^2 W}{\partial \psi \partial p_2} p_1 + \frac{\partial^2 W}{\partial p_1 \partial p_2} p_{11} + \frac{\partial^2 W}{\partial p_2^2} p_{21}\right) \qquad (4.218)$$

$$-\pi_{22} = \frac{\partial^2 W}{\partial y^2} + 2p_2 \frac{\partial^2 W}{\partial y \partial \psi} + p_2^2 \frac{\partial^2 W}{\partial \psi^2} + 2p_{12}\left(\frac{\partial^2 W}{\partial y \partial p_1} + p_2 \frac{\partial^2 W}{\partial \psi \partial p_1}\right)$$

$$+ \; 2p_{22}\left(\frac{\partial^2 W}{\partial y \partial p_2} + p_2 \frac{\partial^2 W}{\partial \psi \partial p_2}\right) + p_{12}^2 \frac{\partial^2 W}{\partial p_1^2} + 2p_{12}p_{22} \frac{\partial^2 W}{\partial p_1 \partial p_2}$$

$$+ \; p_{22}^2 \frac{\partial^2 W}{\partial p_2^2} + p_{22} \frac{\partial W}{\partial \psi} \qquad (4.219)$$

$$
\begin{aligned}
-\pi_{222} &= -\frac{\partial\pi_{22}}{\partial x_2} - \frac{\partial\pi_{22}}{\partial z}p_2 - \frac{\partial\pi_{22}}{\partial p_\mu}p_{\mu 2} - \frac{\partial\pi_{22}}{\partial p_{bc}}p_{bc2} \\
&+ p_{22t}\left(\frac{\partial\alpha_t}{\partial x_2} + \frac{\partial\alpha_t}{\partial z}p_2 + \frac{\partial\alpha_t}{\partial p_\mu}p_{\mu 2} + \frac{\partial\alpha_t}{\partial p_{bc}}p_{bc2}\right) \\
&= \left\{\frac{\partial^3 W}{\partial x_2^3} + 2p_2\frac{\partial^3 W}{\partial x_2^2\partial z} + p_2^2\frac{\partial^3 W}{\partial x_2\partial z^2} + 2p_{12}\left(\frac{\partial^3 W}{\partial x_2^2\partial p_1} + p_2\frac{\partial^3 W}{\partial x_2\partial z\partial p_1}\right)\right. \\
&+ 2p_{22}\left(\frac{\partial^3 W}{\partial x^2\partial p_2} + p_2\frac{\partial^3 W}{\partial x_2\partial z\partial p_2}\right) + p_{12}^2\frac{\partial^3 W}{\partial x_2\partial p_1^2} + 2p_{12}p_{22}\frac{\partial^3 W}{\partial x_2\partial p_1\partial p_2} \\
&\left.+ p_{22}^2\frac{\partial^3 W}{\partial x_2\partial p_2^2} + p_{22}\frac{\partial^2 W}{\partial x_2\partial z}\right\} \\
&+ p_2\left\{\frac{\partial^3 W}{\partial z\partial x_2^2} + 2p_2\frac{\partial^3 W}{\partial x_2\partial z^2} + p_2^2\frac{\partial^3 W}{\partial z^3} + 2p_{12}\left(\frac{\partial^3 W}{\partial z\partial x_2\partial p_1} + p_2\frac{\partial^3 W}{\partial z^2\partial p_1}\right)\right. \\
&+ 2p_{22}\left(\frac{\partial^3 W}{\partial z\partial x_2\partial p_2} + p_2\frac{\partial^3 W}{\partial z^2\partial p_2}\right) + p_{12}^2\frac{\partial^3 W}{\partial z\partial p_1^2} + 2p_{12}p_{22}\frac{\partial^3 W}{\partial z\partial p_1\partial p_2} \\
&\left.+ p_{22}^2\frac{\partial^3 W}{\partial z\partial p_2^2} + p_{22}\frac{\partial^2 W}{\partial z^2}\right\} \\
&+ p_{12}\left\{\frac{\partial^3 W}{\partial p_1\partial x_2^2} + 2p_2\frac{\partial^3 W}{\partial p_1\partial x_2\partial z} + p_2^2\frac{\partial^3 W}{\partial p_1\partial z^2} + 2p_{12}\left(\frac{\partial^3 W}{\partial x_2\partial p_1^2} + p_2\frac{\partial^3 W}{\partial z\partial p_1^2}\right)\right. \\
&+ 2p_{22}\left(\frac{\partial^3 W}{\partial p_1\partial x_2\partial p_2} + p_2\frac{\partial^2 W}{\partial p_1\partial z\partial p_2}\right) + p_{12}^2\frac{\partial^3 W}{\partial p_1^3} + 2p_{12}p_{22}\frac{\partial^3 W}{\partial p_1^2\partial p_2} \\
&\left.+ p_{22}^2\frac{\partial^3 W}{\partial p_1\partial p_2^2} + p_{22}\frac{\partial^2 W}{\partial p_1\partial z}\right\} \\
&+ p_{22}\left\{\frac{\partial^3 W}{\partial p_2\partial x_2^2} + 2\frac{\partial^2 W}{\partial x_2\partial z} + 2p_2\frac{\partial^3 W}{\partial p_2\partial x_2\partial z} + 2p_2\frac{\partial^2 W}{\partial z^2} + p_2^2\frac{\partial^3 W}{\partial p_2\partial z^2}\right. \\
&+ 2p_{12}\left(\frac{\partial^3 W}{\partial p_2\partial x_2\partial p_1} + \frac{\partial^2 W}{\partial z\partial p_1} + p_2\frac{\partial^3 W}{\partial p_2\partial z\partial p_1}\right) \\
&+ 2p_{22}\left(\frac{\partial^3 W}{\partial x_2\partial p_2^2} + \frac{\partial^2 W}{\partial z\partial p_2} + p_2\frac{\partial^3 W}{\partial z\partial p_2^2}\right) + p_{12}^2\frac{\partial^3 W}{\partial p_2\partial p_1^2} \\
&\left.+ 2p_{12}p_{22}\frac{\partial^3 W}{\partial p_1\partial p_2^2} + p_{22}^2\frac{\partial^3 W}{\partial p_2^3} + p_{22}\frac{\partial^2 W}{\partial p_2\partial z}\right\} \\
&+ 2p_{122}\left\{\frac{\partial^2 W}{\partial x_2\partial p_1} + p_2\frac{\partial^2 W}{\partial z\partial p_1} + p_{12}\frac{\partial^2 W}{\partial p_1^2} + p_{22}\frac{\partial^2 W}{\partial p_1\partial p_2}\right\}
\end{aligned}
$$

$$+ \quad p_{222} \left\{ 2 \left(\frac{\partial^2 W}{\partial x_2 \partial p_2} + p_2 \frac{\partial^2 W}{\partial z \partial p_2} \right) + 2p_{12} \frac{\partial^2 W}{\partial p_1 \partial p_2} + 2p_{22} \frac{\partial^2 W}{\partial p_2^2} + \frac{\partial W}{\partial z} \right\}$$

$$+ \quad p_{221} \left\{ \frac{\partial^2 W}{\partial x_2 \partial p_1} + \frac{\partial^2 W}{\partial z \partial p_1} p_2 + \frac{\partial^2 W}{\partial p_1^2} p_{12} + \frac{\partial^2 W}{\partial p_2 \partial p_1} p_{22} \right\}$$

$$+ \quad p_{222} \left\{ \frac{\partial^2 W}{\partial x_2 \partial p_2} + \frac{\partial^2 W}{\partial z \partial p_2} p_2 + \frac{\partial^2 W}{\partial p_1 \partial p_2} p_{12} + \frac{\partial^2 W}{\partial p_2^2} p_{22} \right\} \quad . \tag{4.220}$$

Now substituting Eqs. (4.217)-(4.220) into Eq. (4.216) and eliminating p_{222} by Eq. (4.210), we get

$$\begin{aligned} & f_0 + f_1 p_{12}^2 + f_2 p_{12} + f_3 p_{22} + f_4 p_{11} p_{12} + f_5 p_{22}^2 \\ & + f_6 p_{12} p_{22} + f_7 p_{12}^2 p_{22} + f_8 p_{12} p_{22}^2 + f_9 p_{22}^3 \\ & + f_{10} p_{122} + f_{11} p_{11} + f_{12} p_{11} p_{22} + f_{13} p_{12}^3 \\ & + f_{14} p_{221} = 0 \end{aligned} \tag{4.221}$$

where

$$\begin{aligned} f_0 = \; & \phi' \frac{\partial W}{\partial p_1} + p_2 \frac{\partial^2 W}{\partial x \partial y} + p_2^2 \frac{\partial^2 W}{\partial x \partial \psi} - p_1 \frac{\partial^2 W}{\partial y^2} - p_1 p_2 \frac{\partial^2 W}{\partial y \partial \psi} \\ & - \frac{\partial^3 W}{\partial y^3} - 3p_2 \frac{\partial^3 W}{\partial y^2 \partial \psi} - 3p_2^2 \frac{\partial^3 W}{\partial y \partial \psi^2} - p_2 \left(p_2^2 \frac{\partial^3 W}{\partial \psi^3} + p_2 \frac{\partial^3 W}{\partial \psi \partial y^2} \right. \\ & \left. + 2p_2^2 \frac{\partial^3 W}{\partial y \partial \psi^2} + p_2^3 \frac{\partial^3 W}{\partial \psi^3} \right) + \phi \left(3 \frac{\partial^2 W}{\partial y \partial p_2} + 3p_2 \frac{\partial^2 W}{\partial \psi \partial p_2} + \frac{\partial W}{\partial \psi} \right) \end{aligned} \tag{4.222}$$

$$f_1 = -2p_2 \frac{\partial^2 W}{\partial p_1 \partial p_2} - p_1 \frac{\partial^2 W}{\partial p_1^2} - 3 \left(\frac{\partial^3 W}{\partial y \partial p_1^2} + p_2 \frac{\partial^3 W}{\partial \psi \partial p_1^2} \right) \tag{4.223}$$

$$f_2 = p_2 \frac{\partial W}{\partial \psi} + \frac{\partial W}{\partial y} - 2p_2 \left(\frac{\partial^2 W}{\partial p_2 \partial y} + p_2 \frac{\partial^2 W}{\partial p_2 \partial \psi} \right) + p_2 \frac{\partial^2 W}{\partial x \partial p_1}$$

$$- p_1 \left(2 \frac{\partial^2 W}{\partial y \partial p_1} + p_2 \frac{\partial^2 W}{\partial \psi \partial p_1} \right) - 2 \left(\frac{\partial^3 W}{\partial y^2 \partial p_1} + p_2 \frac{\partial^3 W}{\partial y \partial \psi \partial p_1} \right) - \frac{\partial^3 W}{\partial p_1 \partial y^2}$$

$$- 2p_2 \left(\frac{\partial^3 W}{\partial \psi \partial y \partial p_1} + p_2 \frac{\partial^3 W}{\partial \psi^2 \partial p_1} \right) - 2p_2 \frac{\partial^3 W}{\partial p_1 \partial y \partial \psi} - p_2^2 \frac{\partial^3 W}{\partial p_1 \partial \psi^2} + 3\phi \frac{\partial^2 W}{\partial p_1 \partial p_2} \tag{4.224}$$

$$f_3 = \frac{\partial W}{\partial x} + p_2 \frac{\partial^2 W}{\partial x \partial p_2} + p_1 \left(\frac{\partial^2 W}{\partial y \partial p_2} + 2p_2 \frac{\partial^2 W}{\partial \psi \partial p_2} \right) - \left[2 \left(\frac{\partial^3 W}{\partial y^2 \partial p_2} + p_2 \frac{\partial^3 W}{\partial y \partial \psi \partial p_2} \right) \right.$$

$$+ \frac{\partial^2 W}{\partial y \partial \psi} + 2p_2 \left(\frac{\partial^3 W}{\partial \psi \partial y \partial p_2} + p_2 \frac{\partial^3 W}{\partial \psi^2 \partial p_2} \right) + p_2 \frac{\partial^2 W}{\partial \psi^2} + \frac{\partial^3 W}{\partial p_2 \partial y^2} + 2 \frac{\partial^2 W}{\partial y \partial \psi}$$

$$\left. + 2p_2 \frac{\partial^3 W}{\partial p_2 \partial y \partial \psi} + 2p_2 \frac{\partial^2 W}{\partial \psi^2} + p_2^2 \frac{\partial^3 W}{\partial p_2 \partial \psi^2} \right\} + p_1 \frac{\partial W}{\partial \psi} + 3\phi \frac{\partial^2 W}{\partial p_2^2} \tag{4.225}$$

$$f_4 = p_2 \frac{\partial^2 W}{\partial p_1^2} \tag{4.226}$$

$$f_5 = 2p_1 \frac{\partial^2 W}{\partial p_2^2} - 3 \left(\frac{\partial^3 W}{\partial y \partial p_2^2} + \frac{\partial^2 W}{\partial \psi \partial p_2} + p_2 \frac{\partial^3 W}{\partial \psi \partial p_2^2} \right) \tag{4.227}$$

$$f_6 = -2p_2 \frac{\partial^2 W}{\partial p_2^2} + p_1 \frac{\partial^2 W}{\partial p_1 \partial p_2} - 6 \left(\frac{\partial^3 W}{\partial p_1 \partial y \partial p_2} + p_2 \frac{\partial^2 W}{\partial p_1 \partial \psi \partial p_2} \right) - 3 \frac{\partial^2 W}{\partial p_1 \partial \psi} \tag{4.228}$$

$$f_7 = -3 \frac{\partial^3 W}{\partial p_1^2 \partial p_2} \tag{4.229}$$

$$f_8 = -3 \frac{\partial^3 W}{\partial p_1 \partial p_2^2} \tag{4.230}$$

$$f_9 = -\frac{\partial^3 W}{\partial p_2^3} \tag{4.231}$$

$$f_{10} = -\left\{2\frac{\partial^2 W}{\partial y \partial p_1} + 2p_2\frac{\partial^2 W}{\partial \psi \partial p_1} + 2p_{12}\frac{\partial^2 W}{\partial p_1^2} + 2p_{22}\frac{\partial^2 W}{\partial p_1 \partial p_2}\right\} \tag{4.232}$$

$$f_{11} = p_2\left(\frac{\partial^2 W}{\partial p_1 \partial y} + p_2\frac{\partial^2 W}{\partial p_1 \partial \psi}\right) \tag{4.233}$$

$$f_{12} = p_2\frac{\partial^2 W}{\partial p_1 \partial p_2} \tag{4.234}$$

$$f_{13} = -\frac{\partial^3 W}{\partial p_1^3} \tag{4.235}$$

$$f_{14} = -\left\{\frac{\partial^2 W}{\partial y \partial p_1} + \frac{\partial^2 W}{\partial \psi \partial p_1}p_2 + \frac{\partial^2 W}{\partial p_1^2}p_{12} + \frac{\partial^2 W}{\partial p_1 \partial p_2}p_{22}\right\} \tag{4.236}$$

Since W is a function of x, y, ψ, p_1, and p_2, Eq. (4.221) is satisfied identically only if all the coefficients are zero. Therefore $f_0, \ldots, f_{14}$ are all equal to zero, which gives fourteen equations

$$f_0 = f_1 = \ldots = f_{14} = 0 \tag{4.237}$$

from which the functional form of W is determined. From Eqs. (4.234) W can be separated into two terms as

$$W(x,y,\psi,p_1,p_2) = W_1(x,y,\psi,p_1) + W_2(x,y,\psi,p_2) . \tag{4.238}$$

Equation (4.226) implies W_1 is linearly dependent on p_1. Therefore,

$$W_1(x,y,\psi,p_1) = W_{11}(x,y,\psi)p_1 + W_{12}(x,y,\psi) . \tag{4.239}$$

Equations (4.232), (4.233) and (4.236) then give the same relation as follows:

$$\frac{\partial W_{11}}{\partial y} + p_2\frac{\partial W_{11}}{\partial \psi} = 0 . \tag{4.240}$$

Since W_{11} is independent of p_2, it means W_{11} is independent of both y and ψ and this is a function of x alone. Equation (4.223) is seen to be satisfied identically. W now becomes

$$W = W_{11}(x)p_1 + W_{12}(x,y,\psi) + W_2(x,y,\psi,p_2). \tag{4.241}$$

Substituting this form of W into Eq. (4.227), we get

$$2p_1 \frac{\partial^2 W_2}{\partial p_2^2} - 3\left(\frac{\partial^3 W_2}{\partial y_2 \partial p_2^2} + \frac{\partial^2 W_2}{\partial \psi \partial p_2} + p_2 \frac{\partial^3 W_2}{\partial \psi \partial p_2^2}\right) = 0 . \tag{4.242}$$

Since W_2 is independent of p_1, Eq. (4.242) gives

$$\frac{\partial^2 W_2}{\partial p_2^2} = 0 \tag{4.243}$$

and thus

$$\frac{\partial^2 W_2}{\partial \psi \partial p_2} = 0 . \tag{4.244}$$

Equation (4.243) implies W_2 is linearly dependent on p_2. Also, from Eq. (4.244), the coefficient of p_2 is independent of ψ. The characteristic function, W, can be written as

$$W(x,y,\psi,p_1,p_2) = W_{11}(x)p_1 + W_{21}(x,y)p_2 + W_3(x,y,\psi) . \tag{4.245}$$

Substituting this form of W into Eq. (4.244) we get

$$\frac{\partial W_3}{\partial y} + p_2\left(\frac{\partial W_3}{\partial \psi} - \frac{\partial W_{21}}{\partial y} + \frac{dW_{11}}{dx}\right) = 0 . \tag{4.246}$$

Since W_3, W_{11} and W_{21} are all independent of p_2, Eq. (4.246) is separated into two equations, namely,

$$\frac{\partial W_3}{\partial y} = 0 \tag{4.247}$$

$$\frac{\partial W_3}{\partial \psi} - \frac{\partial W_{21}}{\partial y} + \frac{dW_{11}}{dx} = 0 . \tag{4.248}$$

Equation (4.247) shows that W_3 is independent of y. Since both W_{21} and W_{11} are independent of ψ, Eq. (4.248) means W_3 is linearly dependent on ψ. Therefore,

$$W = W_{11}(x)p_1 + W_{21}(x,y)p_2 + W_{31}(x)\psi + W_{32}(x) . \qquad (4.249)$$

Employing this result in Eq. (4.248), we get

$$\frac{dW_{11}}{dx} + W_{31} = \frac{\partial W_{21}}{\partial y} . \qquad (4.250)$$

The left-hand side of Eq. (4.250) is a function of x alone, which in turn means W_{21} is linearly dependent on y, i.e.,

$$W_{21}(x,y) = W_{211}(x)y + W_{212}(x) . \qquad (4.251)$$

Equation (4.250) then becomes

$$\frac{dW_{11}}{dx} + W_{31} - W_{211} = 0 \qquad (4.252)$$

which will be used later.

The characteristic function now becomes

$$W(x,y,\psi,p_1,p_2) = W_{11}(x)p_1 + [W_{211}(x)y+W_{212}(x)]p_2 + W_{31}(x)\psi + W_{32}(x) . \qquad (4.253)$$

This form of W will satisfy Eqs. (4.228)-(4.231) and (4.235) identically. We have two equations left, namely, Eqs. (4.222) and (4.225). Substituting W from Eq. (4.253) into Eq. (4.222) gives

$$\phi' W_{11} + p_2^2 \frac{dW_{211}}{dx} + p_2^2 \frac{dW_{31}}{dx} + \phi(3W_{211}+W_{31}) = 0 \qquad (4.254)$$

which can, in turn, be separated into two equations:

$$\frac{dW_{211}}{dx} + \frac{dW_{31}}{dx} = 0 \tag{4.255}$$

$$\phi' W_{11} + \phi(3W_{211}+W_{31}) = 0 \ . \tag{4.256}$$

Similarly, putting W from Eq. (4.253) into Eq. (4.225) gives

$$-\left\{\frac{dW_{11}}{dx} p_1 + \left[\frac{dW_{211}}{dx} y + \frac{dW_{212}}{dx}\right] p_2 + \frac{dW_{31}}{dx} \psi + \frac{dW_{32}}{dx}\right\} + p_2 \left(\frac{dW_{211}}{dx} y + \frac{dW_{212}}{dx}\right) + p_1(W_{211}-W_{31}) = 0 \tag{4.257}$$

which then is separated into

$$\frac{dW_{31}}{dx} = 0 \tag{4.258}$$

$$\frac{dW_{32}}{dx} = 0 \tag{4.259}$$

$$\frac{dW_{11}}{dx} - W_{211} + W_{31} = 0 \ . \tag{4.260}$$

Equations (4.258) and (4.259) show that W_{31} and W_{32} are constants and so from Eq. (4.255), W_{211} is also a constant. Equations (4.252) and (4.260) give the same result, namely, W_{11} being linearly dependent on x.

To conclude, the characteristic function W can be written as

$$\boxed{\begin{aligned} &W(x,y,\psi,p_1,p_2) \\ &\quad = (a_0+a_1x)p_1 + [g(x)+b_1y]p_2 - [c_0+c_1\psi] \end{aligned}} \tag{4.261}$$

where a_0, a_1, b_1, c_0, and c_1 are constants and g(x) is an arbitrary function of x. One relation relating ϕ [defined in Eq. (4.221)] and the constants can be obtained from Eq. (4.256) as

$$(a_0+a_1x)\phi' - (c_1-3b_1)\phi = 0 \tag{4.262}$$

and also, Eq. (4.260) furnishes another relation among the constants, which is

$$a_1 - b_1 - c_1 = 0 \,. \qquad (4.263)$$

The absolute invariants and the restriction on ϕ will be discussed in the next section.

4.3.2 Limitation on the Mainstream Velocity

In the introduction the point was made that the type of problem involves the determination of an arbitrary function, ϕ. For such cases, not only the characteristic function W must be of the form given by Eq. (4.261) and Eq. (4.263) satisfied, but the arbitrary function ϕ has to satisfy Eq. (4.262) if the given differential equation is to be invariant.

From Eq. (4.262),

$$(a_0+a_1x)\phi' - (c_1-3b_1)\phi = 0 \,.$$

Upon integration, for $a_1 \neq 0$, $a_0 = 0$

$$\phi = (\text{constant})x^{(c_1-3b_1)/(c_1+b_1)} \qquad (4.264)$$

From Eq. (4.211),

$$\frac{1}{2}\frac{du_e^2}{dx} = (\text{constant})x^{(c_1-3b_1)/(c_1+b_1)} \qquad (4.265)$$

we then get

$$u_e = (\text{constant})x^{(c_1-b_1)/(c_1+b_1)} \,. \qquad (4.266)$$

This is seen to be the mainstream velocity associated earlier with the linear group.

For $a_1 = 0$, $a_0 \neq 0$,

$$\phi = (\text{constant})e^{\frac{c_1-3b_1}{a_0}x} \qquad (4.267)$$

and thus, using Eq. (4.211),

$$u_e(x) = (\text{constant})e^{\frac{c_1-3b_1}{2a_o}z} \qquad (4.268)$$

which is seen to be the mainstream for the spiral group.

For the general case in which both a_o and a_1 are nonzero, the mainstream velocity is found to be

$$u_e(x) = (\text{constant})(a_1x+a_o)^{(c_1+b_1)/(c_1+b_1)} \qquad (4.269)$$

which again belongs to the linear group.

We therefore conclude that the mainstream velocity, $u_e(x)$, belongs either to the linear or the spiral group, i.e., powers or exponentials of x. No other forms are possible—a point long assumed but not proved.

4.3.3 Absolute Invariants and the Transformed Differential Equation

With the characteristic function, W, obtained as in Eq. (4.261) we now make use of the general theory to find the absolute invariants and the transformed differential equation. We shall expect the invariants to correspond to linear and spiral groups. From Eq. (4.265), we obtain

$$\frac{dx}{\alpha_1} = \frac{dy}{\alpha_2} = \frac{d\psi}{m} \qquad (4.270)$$

where the functions α_1, α_2, and m can be obtained by putting Eq. (4.261) into Eq. (2.13) which then gives

$$\frac{dx}{a_o+a_1x} = \frac{dy}{g(x)+b_1y} = \frac{d\psi}{c_o+c_1\psi} . \qquad (4.271)$$

Since this equation has two independent solutions, we have two absolute invariants. Let us consider two special cases.

<u>Case 1</u>. $a_o = 0$, $g(x) = 0$, $c_o = 0$

Equation (4.271) becomes

$$\frac{dx}{a_1 x} = \frac{dy}{b_1 y} = \frac{d\psi}{c_1 \psi} . \tag{4.272}$$

The two independent solutions to Eq. (4.272) are

$$\frac{y}{x^{b_1/a_1}} = \text{constant}$$

and

$$\frac{\psi}{x^{c_1/a_1}} = \text{constant.}$$

The boundary-layer equation can be expressed in terms of these two invariants. Thus, we get the similarity transformation

$$\eta = \frac{y}{x^{b_1/a_1}} \quad \text{and} \quad f(\eta) = \frac{\psi}{x^{c_1/a_1}} . \tag{4.273}$$

The mainstream velocity for this case is given in Eq. (4.266) as

$$u_e(x) = k_1 x^{(c_1-b_1)/(c_1+b_1)} \tag{4.274}$$

where k_1 is a constant. Recalling that Eq. (4.263) has to be satisfied, Eqs. (4.273) and (4.274) can be written as

$$\eta = \frac{y}{x^{(1-m)/(2)}} \quad \text{and} \quad f(\eta) = \frac{\psi}{x^{(1+m)/(2)}} \tag{4.275}$$

and

$$u_e(x) = k_1 x^m \tag{4.276}$$

which is seen to be in the same form as the so-called Falkner and Skan similarity transformation (see Ref. 6). The transformed differential equation is well-known and can be written as

$$f''' + \frac{1}{2}(m+1)ff'' + m(1-f'^2) = 0 . \tag{4.277}$$

Case 2. $a_1 = 0$, $g(x) = 0$, $c_o = 0$

Equation (4.271) becomes

$$\frac{dx}{a_o} = \frac{dy}{b_1 y} = \frac{d\psi}{c_1\psi} \tag{4.278}$$

The two absolute invariants can be found to be

$$\eta = \frac{y}{e^{-\frac{\beta}{2}x}} \quad \text{and} \quad f(\eta) = \frac{\psi}{e^{\frac{\beta}{2}x}} \tag{4.279}$$

The mainstream velocity is given by Eq. (4.268) as

$$u_e(x) = k_2 e^{\beta x} . \tag{4.280}$$

The transformed differential equation is

$$f''' = \beta(f'^2 - \frac{1}{2} ff'') - \beta k_2^2 . \tag{4.281}$$

The transformation is seen to be the spiral group.

The more general cases in which both a_o and c_o are nonzero in Case 1 and c_o is not zero in Case 2 poses no problem. These assumptions merely introduce an extra constant in x and ψ in the transformation [cf. Eqs. (4.266) and (4.269)]. However, the case in which g(x) is not zero needs further investigation. For this case, Eq. (4.271) becomes:

$$\frac{dx}{a_o+a_1x} = \frac{dy}{b_1y+g(x)} = \frac{d\psi}{c_1\psi+c_o} . \tag{4.282}$$

Following the same steps as in Case 1, the absolute invariants are found to be

$$\eta = \frac{y}{(a_o+a_1x)^{\frac{b_1}{a_1}}} - \int \frac{g(x)dx}{(a_o+a_1x)^{\frac{b_1}{a_1}+1}} \tag{4.283}$$

and

$$f(\eta) = \frac{c_1\psi+c_0}{(a_0+a_1x)^{c_1/a_1}} \quad . \tag{4.284}$$

The mainstream velocity is given by

$$u_e(x) = k_3(a_0+a_1x)^{(c_1-b_1)/(c_1+b_1)} \tag{4.285}$$

based on Eq. (4.269).

The boundary-layer equation is transformed to an ordinary differential equation as

$$\frac{c_1-b_1}{a_1} f'^2 - \frac{c_1}{a_1} ff'' = f''' + k_3^2 \frac{c_1-b_1}{c_1+b_1} \quad . \tag{4.286}$$

By setting

$$u_e(x) = k_3(a_0+a_1x)^m \tag{4.287}$$

then

$$m = \frac{c_1-b_1}{c_1+b_1} \quad . \tag{4.288}$$

Also, Eq. (4.263) has to be satisfied, i.e.,

$$a_1 = b_1 + c_1 \quad . \tag{4.289}$$

Equation (4.283) to (4.286) become

$$\eta = \frac{y}{(a_0+a_1x)^{(1-m)/(2)}} - \int \frac{g(x)dx}{(a_0+a_1x)^{(3-m)/(2)}} \tag{4.290}$$

$$f(\eta) = \frac{c_1\psi+c_0}{(a_0+a_1x)^{1+m/2}} \tag{4.291}$$

and

$$f''' + mk_3^2 - mf'^2 + \frac{1+m}{2} ff'' = 0 . \qquad (4.292)$$

For $g(x) = 0$, this is seen to reduce to the Falkner-Skan's flow. In order to transform the boundary-layer equation to an ordinary differential equation, $g(x)$ can be any function of x; however, for $g(x) \neq 0$, the boundary condition at $y = 0$ cannot be transformed. We therefore conclude that for incompressible, two-dimensional, laminar boundary-layer equations, the linear and spiral groups are the only two groups possible. The group represented by Eq. (4.282) will transform the boundary layer equation from partial differential equation to an ordinary differential equation, it fails, however, to satisfy the boundary condition at $y = 0$.

4.4 SIMILARITY ANALYSIS OF THE HELMHOLTZ EQUATION IN GENERAL CURVILINEAR COORDINATES

In the present article the role of a coordinate frame relation on a similarity analysis is considered by examining the two-dimensional Helmholtz equation in general curvilinear coordinates. While the present method of analysis applies to both linear and nonlinear partial differential equations, there are at least two reasons for choosing this particular equation for presentation of the present method of analysis. The first is its simplicity and its wide use in the potential theories of a large number of physical problems. The second, and the most important, reason is that the separability of this equation has been studied thoroughly by Moon and Spencer.[13] By separation of variables we mean the original partial differential equation is reduced to two ordinary differential equations each having one of the original independent variables as an independent variable. In the case of similarity, the original partial differential equation is reduced to one ordinary differential equation where the independent variable is a function of the original independent variables. The present analysis is a parallel study to that of Moon and Spéncer[13] on the conditions of separability. Here we want to derive the conditions under which similarity is possible by requiring that the Helmholtz equation be invariant under the infinitesimal transformation. The basic problem is the determination of the characteristic function, W. The difference between this example and earlier examples is that the resulting differential equations for the solution of W involve unknown metric components of the curvilinear coordinates. These equations form the conditions for the existence of similarity solutions. They can be used in two ways. In the first case where a coordinate system is given, similarity solution is said to exist if substitution of the known metric components in these conditions result in solutions for the characteristic function, W. Once W is known, the searching of all possible groups of transformation can be made by following the same steps as in previous examples. In the second case where a group of transformation is given for which the characteristic function, W, is known, these conditions give limitation on the metric components for the general curvilinear coordinates. If substitution of the metric components of a given coordinate system satisfies these conditions, then similarity solutions exist for the coordinate system for that group of transformation. In both cases, of course, the boundary conditions have to be transformable for true similarity to exist.

4.4.1 Review of Separability Conditions Discussed by Moon and Spencer

The following is a short review of the separability conditions discussed by Moon and Spencer.[13] This is done for comparasion with results to be obtained from the theory of continuous transformations. We consider two dimensional form of the Helmholtz equation given by

$$\frac{1}{g^{1/2}}\left\{\frac{\partial}{\partial x_1}\left(\frac{g^{1/2}}{g_{11}}\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{g^{1/2}}{g_{22}}\frac{\partial\phi}{\partial x_2}\right)\right\} + \alpha_1\phi = 0 \qquad (4.293)$$

where g_{11} and g_{12} are components of the metric tensor and g is the determinant with elements g_{pq}.

Now, let the unknown function be expressed as a product of two functions:

$$\phi = U_1(x_1)U_2(x_2) \; . \qquad (4.294)$$

Substitution of Eq. (4.294) in Eq. (4.293) gives

$$\frac{1}{g^{1/2}}\left\{\frac{1}{U_1}\frac{\partial}{\partial x_1}\left(\frac{g^{1/2}}{g_{11}}\frac{dU_1}{dx_1}\right) + \frac{1}{U_2}\frac{\partial}{\partial x_2}\left(\frac{g^{1/2}}{g_{22}}\frac{dU_2}{dx_2}\right)\right\} + \alpha_1 = 0 \; . \qquad (4.295)$$

The most general condition that will allow separability is that $g^{1/2}/g_{ii}$ is a produce of two functions:

$$\frac{g^{1/2}}{g_{11}} = f_1(x_1)F_1(x_2) \qquad (4.296)$$

$$\frac{g^{1/2}}{g_{22}} = f_2(x_2)F_2(x_1) \qquad (4.297)$$

Substitution of Eqs. (4.296) and (4.297) into Eq. (4.295) gives

$$\frac{1}{g^{1/2}}\left[\frac{F_1}{U_1}\frac{d}{dx_1}\left(f_1\frac{dU_1}{dx_1}\right) + \frac{F_2}{U_2}\frac{d}{dx_2}\left(f_2\frac{dU_2}{dx_2}\right)\right] + \alpha_1 = 0 . \tag{4.298}$$

In this equation, g, F_i and f_i are determined by the coordinate system and are entirely independent of the boundary condition that characterize a particular problem. But in the solution, $\phi = U_1U_2$, the U's are functions of both the coordinate system and the separation constants. Now, suppose we differentiate Eq. (4.298) with respect to α_1 and α_2, we get

$$F_1\frac{\partial}{\partial\alpha_1}\left[\frac{1}{U_1}\frac{d}{dx_1}\left(f_1\frac{dU_1}{dx_1}\right)\right] + F_2\frac{\partial}{\partial\alpha_1}\left[\frac{1}{U_2}\frac{d}{dx_2}\left(f_2\frac{dU_2}{dx_2}\right)\right] + g^{1/2} = 0 \tag{4.299}$$

$$F_1\frac{\partial}{\partial\alpha_2}\left[\frac{1}{U_1}\frac{d}{dx_1}\left(f_1\frac{dU_1}{dx_1}\right)\right] + F_2\frac{\partial}{\partial\alpha_2}\left[\frac{1}{U_2}\frac{d}{dx_2}\left(f_2\frac{dU_2}{dx_2}\right)\right] = 0 . \tag{4.300}$$

Introducing the notation

$$\phi_{ij}(x_i) = -\frac{1}{f_i(x_i)}\frac{\partial}{\partial x_j}\left[\frac{1}{U_i}\frac{d}{dx_i}\left(f_i\frac{dU_i}{dx_i}\right)\right] . \tag{4.301}$$

Equations (4.299) and (4.300) becomes

$$f_1F_1\phi_{11}(x_1) + f_2F_2\phi_{21}(x_2) = g^{1/2} \tag{4.302}$$

$$f_1F_1\phi_{12}(x_1) + f_2F_2\phi_{22}(x_2) = 0 . \tag{4.303}$$

Equations (4.302) and (4.303) are solved for f_1F_1 and f_2F_2 and the results are (if $s \neq 0$)

$$f_1F_1 = g^{1/2}\frac{\phi_{22}}{s} \tag{4.304}$$

$$f_2F_2 = -g^{1/2}\frac{\phi_{12}}{s} \tag{4.305}$$

where

$$s = \begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{vmatrix} \qquad (4.306)$$

which is known as the Stäckel determinant.

Comparison of Eqs. (4.304) and (4.305) with Eqs. (4.296) and (4.297) shows that

$$g_{11} = \frac{s}{\phi_{22}} \qquad (4.307)$$

$$g_{22} = -\frac{s}{\phi_{12}} \qquad (4.308)$$

Equations (4.307) and (4.308) are called the first condition for simply separability.

Also, from Eqs. (4.304) and (4.305),

$$\frac{g^{1/2}}{s} = f_1(x_1)\left[\frac{F_1(x_2)}{\phi_{22}(x_2)}\right] = f_2(x_2)\left[\frac{F_1(x_1)}{-\phi_{12}(x_1)}\right] . \qquad (4.309)$$

This is possible only if

$$\frac{g^{1/2}}{s} = f_1(x_1)\cdot f_2(x_2) . \qquad (4.310)$$

Equation (4.310) is called the second condition for simple separability.

It can be shown that the two conditions are both necessary and sufficient for the simple separability of Helmholz's equation. Detail of the proof is given in the original work of Moon and Spencer.[13]

We now consider an example in which the cylindrical coordinate is considered, for which

$$x = r \cos \Theta, \quad y = r \sin \Theta \qquad (4.311)$$

From Eqs. (4.307) and (4.308),

$$\frac{s}{\phi_{22}} = 1 \tag{4.312}$$

and

$$\frac{s}{\phi_{12}} = -r^2 \ . \tag{4.313}$$

If ϕ_{22} is taken to be 1, then

$$s = 1 \quad \text{and} \quad \phi_{12} = -\frac{1}{r^2} \ . \tag{4.314a,b}$$

However, s is defined in Eq. (4.306) as

$$s = \begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{vmatrix} = \begin{vmatrix} \phi_{11} & -\frac{1}{r^2} \\ \phi_{21} & 1 \end{vmatrix} \ . \tag{4.315}$$

For s = 1, one possibility is

$$\phi_{11} = 1 \quad \text{and} \quad \phi_{21} = 0 \ . \tag{4.316}$$

The second condition then becomes

$$\frac{g^{1/2}}{s} = r = f_1(r)f_2(\theta) \ . \tag{4.317}$$

This condition is satisfied if

$$f_1(r) = r \quad \text{and} \quad f_2(\theta) = 1 \ . \tag{4.318a,b}$$

We therefore conclude that the Helmholtz equation in cylindrical coordinates is separable.

4.4.2 Similarity Analysis of the Equation

Let us put Eq. (4.293) in a slightly different form as

$$\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial\phi}{\partial x_2}\right) + \alpha_1 h_1 h_2 \phi = 0 \qquad (4.319)$$

where the following relations have been used:

$$g^{1/2} = h_1 h_2, \quad g_{11} = h_1^2, \quad g_{22} = h_2^2 .$$

We now want to find the conditions on the two unknown functions h_1 and h_2 which make similarity possible. In other words, the conditions thus obtained will enable one to decide if similarity solutions exist for the coordinate system under consideration.

Carrying out the differentiation and rearranging the terms, Eq. (4.319) becomes

$$\left(h_1 h_2^2 \frac{\partial h_2}{\partial x_1} - h_2^3 \frac{\partial h_1}{\partial x_1}\right) p_1 + \left(h_1^2 h_2 \frac{\partial h_1}{\partial x_2} - h_1^3 \frac{\partial h_2}{\partial x_2}\right) p_2$$

$$+ h_1 h_2^3 p_{11} + h_1^3 h_2 p_{22} + \alpha h_1^3 h_2^3 \phi = 0 \qquad (4.320)$$

where

$$p_1 = \frac{\partial\phi}{\partial x_1}, \quad p_2 = \frac{\partial\phi}{\partial x_2}, \quad p_{11} = \frac{\partial^2\phi}{\partial x_1^2}, \quad p_{22} = \frac{\partial^2\phi}{\partial x_2^2} . \qquad (4.321)$$

We now make the infinitesimal transformation:

$$x_i' = x_i + \epsilon\xi_i(x_1, x_2, \phi, p_1, p_2)$$

$$\phi' = \phi + \epsilon\zeta(x_1, x_2, \phi, p_1, p_2)$$

$$p_i' = p_i + \epsilon\pi_i(x_1, x_2, \phi, p_1, p_2)$$

$$p_{ij}' = p_{ij} + \epsilon\pi_{ij}(x_1, x_2, \phi, p_1, p_2, p_{11}, p_{12}, p_{22})$$

$$(i,j = 1,2) \qquad (4.322)$$

where the transformation functions ξ_i, ζ, π_i and π_{ij} can be expressed in terms of a characteristic function, W, as

$$\xi_i = \frac{\partial W}{\partial p_i} \qquad (4.323a)$$

$$\zeta = p_i \frac{\partial W}{\partial p_i} - W \qquad (4.323b)$$

$$-\pi_i = \frac{\partial W}{\partial x_i} + p_i \frac{\partial W}{\partial \phi} \qquad (4.323c)$$

$$-\pi_{11} = \frac{\partial^2 W}{\partial x_1^2} + 2p_1 \frac{\partial^2 W}{\partial x_1 \partial \phi} + p_1^2 \frac{\partial^2 W}{\partial \phi^2} + 2p_{11}\left(\frac{\partial^2 W}{\partial p_1 \partial x_1} + p_1 \frac{\partial^2 W}{\partial \phi \partial p_1}\right)$$

$$+ \; 2p_{12}\left(\frac{\partial^2 W}{\partial x_1 \partial p_2} + p_1 \frac{\partial^2 W}{\partial p_2 \partial \phi}\right) + p_{11}^2 \frac{\partial^2 W}{\partial p_1^2} + 2p_{11}p_{12} \frac{\partial^2 W}{\partial p_1 \partial p_2}$$

$$+ \; p_{12}^2 \frac{\partial^2 W}{\partial p_2^2} + p_{11} \frac{\partial W}{\partial \phi} \qquad (4.323d)$$

$$-\pi_{22} = \frac{\partial^2 W}{\partial x_2^2} + 2p_2 \frac{\partial^2 W}{\partial x_2 \partial \phi} + p_2^2 \frac{\partial^2 W}{\partial \phi^2} + 2p_{12}\left(\frac{\partial^2 W}{\partial x_2 \partial p_1} + p_2 \frac{\partial^2 W}{\partial \phi \partial p_1}\right)$$

$$+ \; 2p_{22}\left(\frac{\partial^2 W}{\partial x_2 \partial p_2} + p_2 \frac{\partial^2 W}{\partial \phi \partial p_2}\right) + p_{12}^2 \frac{\partial^2 W}{\partial p_1^2} + 2p_{12}p_{22} \frac{\partial^2 W}{\partial p_1 \partial p_2}$$

$$+ \; p_{22}^2 \frac{\partial^2 W}{\partial p_2^2} + p_{22} \frac{\partial W}{\partial \phi} \quad . \qquad (4.323e)$$

The Helmholtz equation, Eq. (4.320), denoted by F = 0, is invariant under the infinitesimal transformation if

$$UF = 0 \; . \qquad (4.324)$$

Expanding the operator in full, we get

$$\xi_i \frac{\partial F}{\partial x_i} + \zeta \frac{\partial F}{\partial \phi} + \pi_i \frac{\partial F}{\partial p_i} + \pi_{ij} \frac{\partial F}{\partial p_{ij}} = 0 \; . \qquad (4.325)$$

Putting the Helmholtz equation, Eq. (4.320) into this expression, we then get

$$
\begin{aligned}
&\xi_1 \Bigg\{ \left[\left(\frac{\partial h_1}{\partial x_1}\right) h_2^2 \frac{\partial h_2}{\partial x_1} + 2h_1h_2\left(\frac{\partial h_2}{\partial x_1}\right)^2 + h_1h_2^2 \left(\frac{\partial^2 h_2}{\partial x_1^2}\right) - 3h_2^2 \frac{\partial h_2}{\partial x_1}\frac{\partial h_1}{\partial x_1} - h_2^3 \frac{\partial^2 h_1}{\partial x_1^2}\right] p_1 \\
&+ \left[2h_1h_2 \frac{\partial h_1}{\partial x_1}\frac{\partial h_1}{\partial x_2} + h_1^2 \frac{\partial h_2}{\partial x_1}\frac{\partial h_1}{\partial x_2} + h_1^2h_2 \frac{\partial^2 h_1}{\partial x_1 \partial x_2} - 3h_1^2 \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - h_1^3 \frac{\partial^2 h_2}{\partial x_1 \partial x_2}\right] p_2 \\
&+ \left[h_2^3 \frac{\partial h_1}{\partial x_1} + 3h_1h_2^2 \frac{\partial h_2}{\partial x_1}\right] p_{11} + \left[3h_1^2h_2 \frac{\partial h_1}{\partial x_1} + h_1^3 \frac{\partial h_2}{\partial x_1}\right] p_{22} \\
&+ \alpha\phi \left[3h_1^2h_2^3 \frac{\partial h_1}{\partial x_1} + 3h_1^3h_2^2 \frac{\partial h_2}{\partial x_1}\right] \Bigg\} \\
&+ \xi_2 \Bigg\{ \left[h_2^2 \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1} + 2h_1h_2 \frac{\partial h_2}{\partial x_2}\frac{\partial h_2}{\partial x_1} + h_1h_2^2 \frac{\partial^2 h_2}{\partial x_2 \partial x_1} \right. \\
&\left. - 3h_2^2 \frac{\partial h_2}{\partial x_2}\frac{\partial h_1}{\partial x_1} - h_2^3 \frac{\partial^2 h_1}{\partial x_2 \partial x_1}\right] p_1 \\
&+ \left[2h_1h_2\left(\frac{\partial h_1}{\partial x_2}\right)^2 + h_1^2 \frac{\partial h_2}{\partial x_2}\frac{\partial h_1}{\partial x_2} + h_1^2h_2 \frac{\partial^2 h_1}{\partial x_2^2} - 3h_1^2 \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_2} - h_1^3 \frac{\partial^2 h_2}{\partial x_2^2}\right] p_2 \\
&+ \left(h_2^3 \frac{\partial h_1}{\partial x_2} + 3h_1h_2^2 \frac{\partial h_2}{\partial x_2}\right) p_{11} + \left(3h_1^2h_2 \frac{\partial h_1}{\partial x_2} + h_1^3 \frac{\partial h_2}{\partial x_2}\right) p_{22} \\
&+ \alpha\phi \left(3h_1^2h_2^3 \frac{\partial h_1}{\partial x_2} + 3h_1^3h_2^2 \frac{\partial h_2}{\partial x_2}\right) \Bigg\} + \zeta\alpha h_1^3h_2^3 \\
&+ \pi_1 \left(h_1h_2^2 \frac{\partial h_2}{\partial x_1} - h_2^3 \frac{\partial h_1}{\partial x_1}\right) + \pi_2 \left(h_1^2h_2 \frac{\partial h_1}{\partial x_2} - h_1^3 \frac{\partial h_2}{\partial x_2}\right) \\
&+ \pi_{11}h_1h_2^3 + \pi_{22}h_1^3h_2 = 0 . \qquad (4.326)
\end{aligned}
$$

Putting Eqs. (4.323) into Eq. (4.322) and we get:

$$
\begin{aligned}
&f_0p_{11} + f_1p_{12} + f_2p_{11}^2 + f_3p_{11}p_{12} + f_4p_{12}^2 \\
&+ f_5p_{12}p_{22} + f_6p_{22}^2 + f_7p_{22} + f_8 = 0 \qquad (4.327)
\end{aligned}
$$

where

$$f_0 = \frac{\partial W}{\partial p_1}\left(h_2^3 \frac{\partial h_1}{\partial x_1} + 3h_1h_2^2 \frac{\partial h_2}{\partial x_1}\right) + \frac{\partial W}{\partial p_2}\left(h_2^3 \frac{\partial h_1}{\partial x_2} + 3h_1h_2^2 \frac{\partial h_2}{\partial x_2}\right)$$

$$- h_1h_2^3 \left[2\left(\frac{\partial^2 W}{\partial p_1 \partial x_1} + p_1 \frac{\partial^2 W}{\partial \phi \partial p_1}\right) + \frac{\partial W}{\partial \phi}\right] \quad (4.328)$$

$$f_1 = - h_1h_2^3 \left[2\left(\frac{\partial^2 W}{\partial x_1 \partial p_2} + p_1 \frac{\partial^2 W}{\partial p_2 \partial \phi}\right)\right] - h_1^3h_2 \left[2\left(\frac{\partial^2 W}{\partial x_1 \partial p_1} + p_2 \frac{\partial^2 W}{\partial \phi \partial p_1}\right)\right] \quad (4.329)$$

$$f_2 = - h_1h_2^3 \frac{\partial^2 W}{\partial p_1^2} \quad (4.330)$$

$$f_3 = -2h_1h_2^3 \frac{\partial^2 W}{\partial p_1 \partial p_2} \quad (4.331)$$

$$f_4 = -h_1h_2^3 \frac{\partial^2 W}{\partial p_2^2} - h_1^3h_2 \frac{\partial^2 W}{\partial p_1^2} \quad (4.332)$$

$$f_5 = -2h_1^3h_2 \frac{\partial^2 W}{\partial p_1 \partial p_2} \quad (4.333)$$

$$f_6 = -h_1^3h_2 \frac{\partial^2 W}{\partial p_2^2} \quad (4.334)$$

$$f_7 = \frac{\partial W}{\partial p_1}\left(3h_1^2h_2 \frac{\partial h_1}{\partial x_1} + h_1^3 \frac{\partial h_2}{\partial x_1}\right) + \frac{\partial W}{\partial p_2}\left(3h_1^2h_2 \frac{\partial h_1}{\partial x_2} + h_1^3 \frac{\partial h_2}{\partial x_2}\right)$$

$$- 2h_1^3h_2\left(\frac{\partial^2 W}{\partial x_2 \partial p_2} + p_2 \frac{\partial^2 W}{\partial \phi \partial p_2}\right) - h_1^3h_2 \frac{\partial W}{\partial \phi} \quad (4.335)$$

$$
\begin{aligned}
f_8 = \frac{\partial W}{\partial p_1}\Bigg\{ & \Bigg[h_2^2 \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} + 2h_1h_2\left(\frac{\partial h_2}{\partial x_1}\right)^2 + h_1h_2^2\left(\frac{\partial^2 h_2}{\partial x_1^2}\right) \\
& - 3h_2^2 \frac{\partial h_2}{\partial x_1}\frac{\partial h_1}{\partial x_1} - h_2^3 \frac{\partial^2 h_1}{\partial x_1^2}\Bigg] p_1 \\
& + \Bigg[2h_1h_2 \frac{\partial h_1}{\partial x_1}\frac{\partial h_1}{\partial x_2} + h_1^2 \frac{\partial h_2}{\partial x_1}\frac{\partial h_1}{\partial x_2} + h_1^2h_2 \frac{\partial^2 h_1}{\partial x_1 \partial x_2} \\
& - 3h_1^2 \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - h_1^3 \frac{\partial^2 h_2}{\partial x_1 \partial x_2}\Bigg] p_2 \\
& + \alpha\phi\left[3h_1^2h_2^3 \frac{\partial h_1}{\partial x_1} + 3h_1^3h_2^2 \frac{\partial h_2}{\partial x_1}\right]\Bigg\} \\
+ \frac{\partial W}{\partial p_2}\Bigg\{ & \Bigg[h_2^2 \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1} + 2h_1h_2 \frac{\partial h_2}{\partial x_2}\frac{\partial h_2}{\partial x_1} + h_1h_2^2 \frac{\partial^2 h_2}{\partial x_2 \partial x_1} \\
& - 3h_2^2 \frac{\partial h_2}{\partial x_2}\frac{\partial h_1}{\partial x_1} - h_2^3 \frac{\partial^2 h_1}{\partial x_2 \partial x_1}\Bigg] p_1 \\
& + \Bigg[2h_1h_2\left(\frac{\partial h_1}{\partial x_2}\right)^2 + h_1^2 \frac{\partial h_2}{\partial x_2}\frac{\partial h_1}{\partial x_2} + h_1^2h_2 \frac{\partial^2 h_1}{\partial x_2^2} \\
& - 3h_1^2 \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_2} - h_1^3 \frac{\partial^2 h_2}{\partial x_2^2}\Bigg] p_2 \\
& + \alpha\phi\left(3h_1^2h_2^3 \frac{\partial h_1}{\partial x_2} + 3h_1^3h_2^2 \frac{\partial h_2}{\partial x_2}\right)\Bigg\} \\
& + \left(p_1 \frac{\partial W}{\partial p_1} + p_2 \frac{\partial W}{\partial p_2} - W\right)\alpha h_1^3h_2^3 \\
& - h_1h_2^3\left(\frac{\partial^2 W}{\partial x_1^2} + 2p_1 \frac{\partial^2 W}{\partial x_1 \partial \phi} + p_1^2 \frac{\partial^2 W}{\partial \phi^2}\right) \\
& - h_1^3h_2\left(\frac{\partial^2 W}{\partial x_2^2} + 2p_2 \frac{\partial^2 W}{\partial x_2 \partial \phi} + p_2^2 \frac{\partial^2 W}{\partial \phi^2}\right) \\
& - \left(\frac{\partial W}{\partial x_1} + p_1\frac{\partial W}{\partial \phi}\right)\left(h_1h_2^2 \frac{\partial h_2}{\partial x_1} - h_2^3 \frac{\partial h_1}{\partial x_1}\right) \\
& - \left(\frac{\partial W}{\partial x_2} + p_2 \frac{\partial W}{\partial \phi}\right)\left(h_1^2h_2 \frac{\partial h_1}{\partial x_2} - h_1^3 \frac{\partial h_2}{\partial x_2}\right) \quad . \qquad (4.336)
\end{aligned}
$$

Now, p_{22} in Eq. (4.327) can be eliminated by using Eq. (4.320) which, in shorthand form, can be written as

$$p_{22} = -(g_o+g_1p_{11}) \qquad (4.337)$$

where

$$g_0 = \frac{1}{h_1^3 h_2}\left\{\left(h_1 h_2^2 \frac{\partial h_2}{\partial x_1} - h_2^3 \frac{\partial h_1}{\partial x_1}\right)p_1 + \left(h_1^2 h_2 \frac{\partial h_1}{\partial x_2} - h_1^3 \frac{\partial h_2}{\partial x_2}\right)p_2 + \alpha h_1^3 h_2^3 \phi\right\} \quad (4.338)$$

$$g_1 = \frac{h_2^2}{h_1^2} \quad . \quad (4.339)$$

By substituting Eq. (4.337) into Eq. (4.327), we get

$$H_0 p_{11} + H_1 p_{12} + H_2 p_{11}^2 + H_3 p_{12}^2 + H_4 = 0 \quad (4.340)$$

where

$$H_0 = f_0 + 2f_6 g_0 g_1 - f_7 g_1 \quad (4.341)$$

$$H_1 = f_1 - f_5 g_0 \quad (4.342)$$

$$H_2 = f_2 + f_6 g_1^2 \quad (4.343)$$

$$H_3 = f_4 \quad (4.344)$$

$$H_4 = f_8 - g_0 f_7 + f_6 g_0^2 \quad . \quad (4.345)$$

Since the characteristic function, W, is a function of x_1, x_2, ϕ, p_1, and p_2, Eq. (4.340) is true only if the coefficients of each term are zero respectively; that is,

$$H_0 = h_1 = H_2 = H_3 = H_4 = 0 \quad . \quad (4.346)$$

The set of equations, represented by Eq. (4.346), forms the conditions for the existence of similarity solutions. These conditions can be used in two ways. In the first case where a coordinate system is given and we want to know if similarity solutions exist or not. Since now h_1 and h_2 are known, they can be substituted into these conditions and if there exists a solution for the characteristic function, W, similarity solutions then exist. The searching for all possible groups can be made by following exactly the same technique as given in the previous examples. In other words, as long as substitution of h_1 and h_2 for a given coordinate system into these conditions results in solutions to the characteristic function, W, the coordinate system under consideration possess similarity solutions.

In the second case, we consider a given group of transformation and we want to know the conditions h_1 and h_2 must satisfy for similarity to exist. For a given group of transformation, the characteristic function, W, is known. Substitution of this form of W into the conditions, Eq. (4.346) will result in five equations connecting h_1 and h_2. They are the conditions that a given coordinate system must satisfy if similarity solutions exist for this given group of transformation.

Two examples are given below.

4.4.3 Conditions of Similarity for a Given Coordinate System

As a simple example, consider the rectangular coordinate system in which both h_1 and h_2 are unity. Then the conditions for the existence of similarity solutions, Eq. (4.346), give

$$-\left(\frac{\partial^2 W}{\partial p_1 \partial x_1} + p_1 \frac{\partial^2 W}{\partial \phi \partial p_1}\right) - \frac{\partial^2 W}{\partial p_2^2}\alpha\phi + \left(\frac{\partial^2 W}{\partial x_2 \partial p_2} + p_2 \frac{\partial^2 W}{\partial \phi \partial p_2}\right) = 0 \tag{4.347}$$

$$-2\left(\frac{\partial^2 W}{\partial x_1 \partial p_2} + p_1 \frac{\partial^2 W}{\partial p_2 \partial \phi}\right) - 2\left(\frac{\partial^2 W}{\partial x_2 \partial p_1} + p_2 \frac{\partial^2 W}{\partial \phi \partial p_1}\right) + \alpha\phi \frac{\partial^2 W}{\partial p_1 \partial p_2} = 0 \tag{4.348}$$

$$\frac{\partial^2 W}{\partial p_1^2} + \frac{\partial^2 W}{\partial p_2^2} = 0 \tag{4.349}$$

$$\begin{aligned}\alpha\left(p_1 \frac{\partial W}{\partial p_1} + p_2 \frac{\partial W}{\partial p_2} - W\right) &- \left(\frac{\partial^2 W}{\partial x_1^2} + 2p_1 \frac{\partial^2 W}{\partial x_1 \partial \phi} + p_1^2 \frac{\partial^2 W}{\partial \phi^2}\right) \\ &- \left(\frac{\partial^2 W}{\partial x_2^2} + 2p_2 \frac{\partial^2 W}{\partial x_2 \partial \phi} + p_2^2 \frac{\partial^2 W}{\partial \phi^2}\right) \\ &- 2\alpha\phi\left(\frac{\partial^2 W}{\partial x_2 \partial p_2} + \frac{\partial^2 W}{\partial \phi \partial p_2} p_2\right) + \alpha\phi \frac{\partial W}{\partial \phi} \\ &- \alpha^2\phi^2 \frac{\partial^2 W}{\partial p_2^2} = 0 .\end{aligned} \tag{4.350}$$

This means if solution to Eqs. (4.347)-(4.350) for W exists, similarity solutions will then exist for this given coordinate frame. The searching for all possible groups of transformation satisfying these conditions shall not be our concern. The only conclusion being sought is whether similarity solutions exist and if they do what are the conditions the characteristic function, W, must satisfy. Equations (4.347)-(4.350) form those conditions and by trying various functions for solutions, the conclusion is easily drawn that similarity transformation do exist. An example will be given below.

Consider now the following form of the characteristic function, W,

$$W(x_1,x_2,\phi,p_1,p_2) = W_1(x_1)p_1 + W_2(x_2)p_2 + W_3(x_1,x_2,\phi) \ . \qquad (4.351)$$

For this form of W, Eqs. (4.348) and (4.349) are satisfied identically and Eqs. (4.347) and (4.350) give

$$-\frac{dW_1}{dx_1} + \frac{dW_2}{dx_2} = 0 \qquad (4.352)$$

$$-\alpha W_3 - \left(\frac{\partial^2 W_3}{\partial x_1^2} + 2p_1 \frac{\partial^2 W_3}{\partial x_1 \partial\phi} + p_1^2 \frac{\partial^2 W_3}{\partial\phi^2}\right) - \left(\frac{\partial^2 W_3}{\partial x_2^2} + 2p_2 \frac{\partial^2 W_3}{\partial x_2 \partial\phi} + p_2^2 \frac{\partial^2 W_3}{\partial\phi^2}\right) + \alpha\phi\left(2\frac{\partial W_2}{\partial x_2} + \frac{\partial W_3}{\partial\phi}\right) = 0 \ . \qquad (4.353)$$

Equation (4.352) gives

$$W_1 = c_1 x_1 + c_2 \qquad (4.354)$$

$$W_2 = c_1 x_2 + c_4 \ . \qquad (4.355)$$

These are substituted into Eq. (4.353) and we get

$$J_0 p_1 + J_1 p_2 + J_2 p_1^2 + J_3 p_2^2 + J_4 = 0 \qquad (4.356)$$

where

$$J_0 = \frac{\partial^2 W_3}{\partial x_1 \partial \phi} \tag{4.357a}$$

$$J_1 = \frac{\partial^2 W_3}{\partial x_2 \partial \phi} \tag{4.357b}$$

$$J_2 = \frac{\partial^2 W_3}{\partial \phi^2} \tag{4.357c}$$

$$J_3 = \frac{\partial^2 W_3}{\partial \phi^2} \tag{4.357d}$$

$$J_4 = -\alpha W_3 - \frac{\partial^2 W_3}{\partial x_1^2} - \frac{\partial^2 W_3}{\partial x_2^2} + 2\alpha\phi c_1 + \alpha\phi \frac{\partial W_3}{\partial \phi} \tag{4.357e}$$

Since all the J's are independent of the p's, Eq. (4.356) is satisfied if all the J's are zero; that is,

$$J_0 = J_1 = J_2 = J_3 = J_4 = 0 . \tag{4.358}$$

The first four conditions in Eq. (4.358) indicates that the function W_3, should be of the form

$$W_3 = C_5\phi + W_{31}(x_1,x_2). \tag{4.359}$$

Substitution of Eq. (4.359) into the last condition in Eq. (4.358) then gives

$$\frac{\partial^2 W_{31}}{\partial x_1^2} + \frac{\partial^2 W_{31}}{\partial x_2^2} + \alpha W_{31} - 2\alpha\phi c_1 = 0 . \tag{4.360}$$

However, since W_{31} does not depend on ϕ, Eq. (4.360) is satisfied if simultaneously

$$\alpha c_1 = 0 \tag{4.361}$$

$$\frac{\partial^2 W_{31}}{\partial x_1^2} + \frac{\partial^2 W_{31}}{\partial x_2^2} + \alpha W_{31} = 0 . \tag{4.362}$$

Since α is not zero, c_1 must be zero.

Therefore, the final form of the characteristic function is

$$W = c_2 p_1 + c_4 p_2 + c_5 \phi + W_{31}(x_1, x_2) \qquad (4.363)$$

where W_{31} is any function which satisfy Eq. (4.362).

To solve for the absolute invariants, it is necessary to solve

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \frac{d\phi}{\zeta} \qquad (4.364)$$

or, using Eqs. (4.323a,b) and (4.363),

$$\frac{dx_1}{c_2} = \frac{dx_2}{c_4} = \frac{d\phi}{-(c_5\phi + W_{31})} \qquad (4.365)$$

Let us next consider the case where W_{31} is a function of x_1 alone, then Eq. (4.362) is reduced to

$$\frac{d^2 W_{31}}{dx_1^2} + \alpha W_{31} = 0 . \qquad (4.366)$$

The absolute invariants are the two independent solutions of Eq. (4.365) which are

$$\eta = x_2 - \frac{c_4}{c_2} x_1 \qquad (4.367a)$$

$$f(\eta) = \phi e^{\frac{c_5}{c_2} x_1} + \frac{1}{c_2} \int W_{31} e^{\frac{c_5}{c_2} x_1} dx_1 . \qquad (4.367b)$$

Putting these transformations into the Helmholtz equation and making use of the condition, Eq. (4.366), we get

$$\left(1 + \frac{c_4^2}{c_2^2}\right) f'' + 2 \frac{c_4 c_5}{c_2^2} f' + \left(\alpha + \frac{c_5^2}{c_2^2}\right) f = 0 \qquad (4.368)$$

which is an ordinary differential equation.

The above is only a special case of all the possible groups of transformation which will reduce the original partial differential equation to an ordinary differential equation. Other solutions to W and the similarity solution for those groups can be obtained in a similar manner. We therefore conclude that similarity solutions do exist for Helmholtz equation in rectangular coordinates, in addition to its separability.

One final remark concerning the boundary conditions is necessary. When it is stated that similarity solutions exist, we mean that not only the partial differential equation is reduced to an ordinary differential equation, but also that boundary conditions can be transformed satisfactorily. For separation of variables in the usual sense (see the brief summary earlier), the boundary conditions can always be transformed. This is not true for similarity transformation. In the example just treated, Eqs. (4.367) and (4.368), the only possible similarity solution is when the boundary conditions are given as

$$\phi = 0 \quad \text{at} \quad x_2 - \frac{c_4}{c_2} x_1 = k_1$$

$$\phi = 0 \quad \text{at} \quad x_2 - \frac{c_4}{c_2} x_1 = k_2$$

where k_1 and k_2 are constants. By putting W_{31} to zero, Eq. (4.367) gives the following boundary conditions for Eq. (4.368),

$$f(k_1) = f(k_2) = 0 \quad . \tag{4.369}$$

4.4.4 Conditions of Similarity for a Given Group of Transformation

As an illustration, consider the spiral group of transformation where the spiral group of transformation where the characteristic function is given by

$$W(x_1, x_2, \phi, p_1, p_2) = c_1 p_1 + c_2 x_2 p_2 - c_3 \phi \quad . \tag{4.370}$$

The conditions for the existence of similarity solution, Eq. (4.346) then become

$$2c_1\left\{h_1h_2^2\frac{\partial h_2}{\partial x_1} - h_2^3\frac{\partial h_1}{\partial x_1}\right\} + 2c_2x_2\left\{h_1h_2^2\frac{\partial h_2}{\partial x_2} - h_2^3\frac{\partial h_1}{\partial x_2}\right\} + 2h_1h_2^3c_2 = 0 \tag{4.371}$$

and

$$G_0 + G_1p_1 + G_2p_2 + G_3\phi = 0 \tag{4.372}$$

where

$$G_0 = -c_1\left(h_1h_2^2\frac{\partial h_2}{\partial x_1} - h_2^3\frac{\partial h_1}{\partial x_1}\right) \tag{4.373}$$

$$\begin{aligned} G_1 = {} & c_1\left\{h_2^2\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_1} + 2h_1h_2\left(\frac{\partial h_2}{\partial x_1}\right)^2 + h_1h_2\left(\frac{\partial^2 h_2}{\partial x_1^2}\right)\right. \\ & \left. - 3h_2^2\frac{\partial h_2}{\partial x_1}\frac{\partial h_1}{\partial x_1} - h_2^3\frac{\partial^2 h_1}{\partial x_1^2}\right\} \\ & + c_2x_2\left\{h_2^2\frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1} + 2h_1h_2\frac{\partial h_2}{\partial x_2}\frac{\partial h_2}{\partial x_1} + h_1h_2^2\frac{\partial^2 h_2}{\partial x_2\partial x_1}\right. \\ & \left. - 3h_2^2\frac{\partial h_2}{\partial x_2}\frac{\partial h_1}{\partial x_1} - h_2^3\frac{\partial^2 h_1}{\partial x_2\partial x_1}\right\} \\ & + c_3\left(h_1h_2^2\frac{\partial h_2}{\partial x_1} - h_2^3\frac{\partial h_1}{\partial x_1}\right) - \frac{1}{h_1^3}\left(h_1h_2\frac{\partial h_2}{\partial x_1} - h_2^2\frac{\partial h_1}{\partial x_1}\right) \\ & (x)\left\{c_1\left(3h_1^2h_2\frac{\partial h_1}{\partial x_1} + h_1^3\frac{\partial h_2}{\partial x_1}\right) + c_2x_2\left(3h_1^2h_2\frac{\partial h_1}{\partial x_2} + h_1^3\frac{\partial h_2}{\partial x_2}\right)\right. \\ & \left. - 2c_2h_1^3h_2 + c_3h_1^3h_2\right\} \end{aligned} \tag{4.374}$$

$$G_2 = c_1\left(2h_1h_2 \frac{\partial h_1}{\partial x_1}\frac{\partial h_1}{\partial x_2} + h_1^2 \frac{\partial h_2}{\partial x_1}\frac{\partial h_1}{\partial x_2} + h_1^2h_2 \frac{\partial^2 h_1}{\partial x_1 \partial x_2}\right.$$

$$\left. - 3h_1^2 \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - h_1^3 \frac{\partial^2 h_2}{\partial x_1 \partial x_2}\right)$$

$$+ c_2x_2\left(2h_1h_2 \frac{\partial h_1}{\partial x_2}\frac{\partial h_1}{\partial x_2} + h_1^2 \frac{\partial h_2}{\partial x_2}\frac{\partial h_1}{\partial x_2} + h_1^2h_2 \frac{\partial^2 h_1}{\partial x_2^2}\right.$$

$$\left. - 3h_1^2 \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_2} - h_1^3 \frac{\partial^2 h_2}{\partial x_2^2}\right)$$

$$+ (c_3 - c_2)\left(h_1^2h_2 \frac{\partial h_1}{\partial x_2} - h_1^3 \frac{\partial h_2}{\partial x_2}\right) - \frac{1}{h_1h_2}\left(h_1^2h_2 \frac{\partial h_1}{\partial x_2} - h_1^3 \frac{\partial h_2}{\partial x_2}\right)$$

$$(x) \left\{c_1\left(3h_1^2h_2 \frac{\partial h_1}{\partial x_1} + h_1^3 \frac{\partial h_2}{\partial x_1}\right) + c_2x_2\left(3h_1^2h_2 \frac{\partial h_1}{\partial x_2} + h_1^3 \frac{\partial h_2}{\partial x_2}\right)\right.$$

$$\left. - 2c_2h_1^3h_2 + c_3h_1^3h_2\right\} \qquad (4.375)$$

$$G_3 = \alpha c_1\left(3h_1^2h_2^3 \frac{\partial h_1}{\partial x_1} + 3h_1^3h_2^2 \frac{\partial h_2}{\partial x_1}\right) + \alpha c_2x_2\left(3h_1^2h_2^3 \frac{\partial h_1}{\partial x_2}\right.$$

$$\left. + 3h_1^3h_2^2 \frac{\partial h_2}{\partial x_2}\right) + \alpha c_3h_1^3h_2^3$$

$$- \alpha h_2^2\left\{c_1\left(3h_1^2h_2 \frac{\partial h_1}{\partial x_1} + h_1^3 \frac{\partial h_2}{\partial x_1}\right) + c_2x_2\left(3h_1^2h_2 \frac{\partial h_1}{\partial x_2} + h_1^3 \frac{\partial h_2}{\partial x_2}\right)\right.$$

$$\left. - 2c_2h_1^3h_2 + c_3h_1^3h_2\right\} \ . \qquad (4.376)$$

It should be noted that for W in the form given in Eq. (4.370), the conditions $H_1 = 0$, $H_2 = 0$ and $H_3 = 0$ are satisfied identically. From Eq. (4.372) we conclude that the G's should all be zero. Thus, we get

$$G_o = G_1 = G_2 = G_3 = 0 \ . \qquad (4.377)$$

Equations (4.371) and (4.377) are the conditions to be satisfied for the h_1 and h_2 for similarity solutions to exist for the spiral group. As long as the functions h_1 and h_2 for a coordinate system satisfy these conditions, then the spiral group of similarity transformation exists for that coordinate system.

As an application of these conditions, Eqs. (4.371) and (4.377), let us ask if the spiral group of transformation exists for the rectangular coordinate frame. For such a coordinate system, both h_1 and h_2 are unity. Equation (4.371) becomes

$$c_2 = 0 \ . \qquad (4.378)$$

Equation (4.377), with $c_2 = 0$, are satisfied identically.

Thus, we conclude that the spiral group of transformation exists for the rectangular coordinate frame, since its h_1 and h_2 satisfys the conditions given in Eqs. (4.371) and (4.377) if $c_2 = 0$.

The characteristic function is therefore

$$W = c_1 p_1 - c_3 \phi \qquad (4.379)$$

and the absolute invariants can be obtained by solving

$$\frac{dx_1}{c_1} = \frac{d\phi}{c_3 \phi} \qquad (4.380)$$

and

$$x_2 = \text{constant} \qquad (4.381)$$

which gives

$$\eta = x_2 \quad \text{and} \quad f(\eta) = \frac{\phi}{e^{\frac{c_3}{c_1} x_1}}$$

and the Helmholtz equation is reduced to

$$f'' + (\alpha + \frac{c_3^2}{c_1^2})f = 0 \ .$$

By substituting into Eqs. (4.371) and (4.377) the h_1 and h_2 functions for a cylindrical coordinate ($h_1 = 1$, $h_2 = x_1$), it is seen that similarity solutions do not exist since they do not satisfy these conditions. Other coordinate frames can be tested in a similar manner.

4.5 CONCLUDING REMARKS

The method given in this chapter can be summarized as follows: Consider a partial differential equation

$$F(z_1, \ldots, z_p) = 0 \tag{4.382}$$

where

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \\ &\cdots \\ &\cdots \\ z_{p-1} &= \frac{\partial^k y}{(\partial x_{m-1})^k} \\ z_p &= \frac{\partial^k y}{(\partial x_m)^k} \ . \end{aligned}$$

This equation is said to be invariant under the infinitesimal contact transformation

$$z_i' = z_i + \xi_i \delta\epsilon \ ; \quad i=1,\ldots,p$$

if the following condition is satisfied

$$UF = \xi_1 \frac{\partial F}{\partial z_1} + \ldots + \xi_p \frac{\partial F}{\partial z_p} = 0 \ . \tag{4.383}$$

Since the functions, ξ_i, in the transformation are expressed in terms of a characteristic function, W, Eq. (4.383) is used to predict the form of W. The invariants can then be obtained by solving the following system of equations:

$$\frac{dz_1}{\xi_1} = \ldots = \frac{dz_p}{\xi_p} \ . \tag{4.384}$$

Finally, using the theorems given in article 4.1.5, the number of variables can be reduced by one using the invariants as new dependent and independent variables.

For simultaneous differential equations, the functions in the infinitesimal contact transformation are expressed in terms of characteristic functions, W_i where $i = 1,\ldots,m$ and m is the number of dependent variables.

The present method is seen to be a systematic way of searching for all possible groups of transformation which will reduce the number of variables by one. For reducing more variables, the same steps have to be repeated.

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