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THE INFLUENCE OF LOCALIZED, NORMAL SURFACE OSCILLATIONS ON THE STEADY, LAMINAR FLOW OVER A FLAT PLATE

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NOMENCLATURE

a	thermal diffusivity of the fluid
$a_1 \ldots a_5$	coefficients of η^0 in the velocity and temperature profiles
$b_1 \ldots b_5$	coefficients of η^1 in the velocity and temperature profiles
$c_1 \ldots c_5$	coefficients of η^2 in the velocity and temperature profiles
c_{fx}	local drag coefficient
$d_1 \ldots d_5$	coefficients of η^3 in the velocity and temperature profiles
$e_1 \ldots e_5$	coefficients of η^4 in the velocity and temperature profiles
e	base of natural logarithm, 2.718281828
$F_1 \ldots F_5$	functions of X' or Pr
h_1, h_2	elements of metric tensor
h	coefficient of natural convection
i	$\sqrt{-1}$
$\vec{i}$	unit vector in the x'-direction
$\vec{j}$	unit vector in the y'-direction
$\vec{k}$	unit vector in the z'-direction
k	thermal conductivity
$\vec{\ell}$	displacement vector in the (x',y') coordinate system
L	distance of the disturbance from the leading edge of the plate
m	a constant
n	a constant
Nu_x	local Nusselt number

O( )	order of magnitude
p	pressure
Pr	Prandtl number
$\vec{r}$	distance vector
$\vec{R}$	distance vector
Re	Reynolds number
R{ }	real part of a complex quantity
t	time in (x,y) coordinate system
T	time in (X,Y) coordinate system
u	velocity in x-direction
U	velocity in X-direction
U_∞	velocity in the mainstream
v	velocity in the y-direction
V	velocity in the Y-direction
x	abscissa in (x,y) coordinate system
x'	abscissa in (x',y') coordinate system
X	abscissa in (X,Y) coordinate system
y	ordinate in (x,y) coordinate system
y'	ordinate in (x',y') coordinate system
Y	ordinate in (X,Y) coordinate system
Δ	inverse of the cubic root of Pr

Greek Letters

α	phase angle
β	an angle
δ	boundary layer thickness
ϵ	maximum amplitude of oscillation
η	dimensionless distance, Y^*/δ
η_t	dimensionless distance, Y^*/δ_t
Θ	temperature
λ_f	dimensionless parameter, $\epsilon'^2\omega'^4/Re$
λ_T	dimensionless parameter, $\epsilon'^2\omega'^2/Re$
μ	viscosity
ρ	density of fluid
σ	normal stress
τ	shearing stress
ϕ	disturbance function, $\exp[-m(X-L)^2+i\omega t]$
ϕ_s	steady disturbance function, $\exp[-m'(X'-1)^2]$
ψ	a function of X, defined by Eq. (152)
ω	frequency

Subscripts

0,1,2	zeroth, first and second approximation
20	steady component of the second approximation
∞	condition at infinity

c	complex function
I	imaginary part of a complex quantity
R	real part of a complex quantity
w	condition at wall
x',y'	local Cartesian coordinates
X,Y	accelerating coordinates

Superscripts

'	dimensionless quantities
*	dimensionless quantities multiplied by $Re^{1/2}$

ABSTRACT

The steady, laminar flow of an incompressible fluid over a semi-infinite flat plate with a localized vibration on the surface of the plate is analyzed. Formulation of the problem is made with reference to a set of coordinate frames fixed on the surface of the plate. The complete momentum, continuity, and energy equations are then simplified by the boundary layer assumptions. The simplified equations are then linearized by the perturbation procedures and the first three approximations are considered. The zeroth-order approximation is the well-known Blasius' solution. The first- and second-order approximations are solved by the integral method. Integration of the solutions with respect to time gives a steady component of both skin friction and heat transfer. This steady change depends on $(\epsilon'^2\omega'^4)/Re$ in the case of skin friction and $(\epsilon'^2\omega'^2)/Re$ in the case of heat transfer as well as the distance from the leading edge of the flat plate. The phase of skin friction is approximately in phase with the input disturbance, while that of heat transfer lags approximately $\pi/2$ radians. Integration of the steady change over the distance along the surface of the plate gives a net increase in skin friction and decrease in heat transfer.

CHAPTER I

INTRODUCTION

In the last two decades the unsteady boundary layer literature on the periodic and starting-ending type transients has grown rapidly. The existing literature may be divided into two classes, the problems dealing with the acoustic streaming and the problems having vibration. Actually, when an object vibrates, a sound wave is emitted. Therefore, these two areas are very closely related.

The phenomenon of acoustic streaming was first observed by Andrade[2] when he made a photographic study of isothermal streaming in a standing wave tube using tobacco smoke for visual identification. The existence of these steady streaming (secondary) flows was mathematically established by Schlichting[24] using a successive approximation technique. He predicted the existence of two regions of streaming in each quadrant around the cylinder; a thin layer next to the cylinder, called the d-c boundary layer, with streaming directed toward the surface along the propagation axis, and an outer streaming with direction away from the surface along this axis. The outer region forms a vortex when the fluid is bounded, but in unlimited space the outer core moves to infinity.

West[28] in his experiments on streaming patterns around small cylinders vibrating in air and in water, observed a very interesting phenomenon. Beginning with a feeble motion, streaming is increased gradually with increasing frequency and increasing amplitude to a certain point where suddenly a new type of circulation is started. Beyond this point, if the frequency is held constant and the amplitude is increased, the vortex system shrinks to a region near the cylinder surrounded by a vigorous circulation of the opposite sense. This phenomena has also been observed by Raney, et al.,[22] and Skavlem and Tjotta;[26] however, it has not been able to be predicted theoretically. Other experimental and analytical work on streaming in its general aspects include those of Westervelt,[29,30] Nyborg,[20] and Holtsmark, et al.[10]

The study of the effect of small oscillations in the main stream on the boundary layer is initiated by Lighthill.[18] He considered the problem of two-dimensional flow about a fixed cylindrical body when fluctuations in the external flow are produced by harmonic fluctuations in magnitude, but not in the direction of the oncoming stream. von Karman-Polhausen's technique was used to solve the resulting equations. He also discussed the temperature fluctuations using successive approximations. Rott and Rosenzweig[23] extended Lighthill's analysis in several ways. A practical method for obtaining the response to the laminar boundary layer to an impulsive change in velocity is presented. Hill and Stenning[9] consider the case of the boundary layer flow over a cylinder which undergoes a rotational oscillation. The amplitudes and frequencies of oscillation in all three papers are assumed to be small and the results are presented in terms of universal functions.

Lemlich[17] investigated the effect of transverse vibrations upon the heat transfer rates from horizontal heated wires to air. Anatanarayanan and Ramachandran[1] experimentally studied the effect of vibration on heat transfer from wires to air in forced parallel flow. An increase in heat transfer rates of 130% are observed.

Kubanski[14,15] has studied experimentally the influence of stationary sound fields on free convection from an electrically heated horizontal cylinder in air. The direction of sound wave in his experiments was longitudinal, i.e., parallel to the axis. For a test cylinder of 2.4 cm in diameter and 32.5 cm long subjected to an intensity of radiated vibrations in the center of the beam from 0.03 to 0.16 W/cm^2 and a frequency range of 8 to 30 kc, the free convective heat transfer coefficient is increased by approximately 75%. He also obtained heat transfer data for the case of a horizontal cylinder in a standing sound field with a superimposed horizontal cross flow.[16] The sound wave is perpendicular to the direction of forced flow and also to the axis of the cylinder. Fand and Kaye[6,7] performed photographic studies of the boundary layer flow near a horizontal cylinder in the presence of sound fields whose direction of propagation was horizontal and perpendicular to the axis of the cylinder. A new type of streaming was identified called "thermoacoustic streaming" which is characterized by the formation of two vortexes above the cylinder when the sound pressure level reaches a certain critical value.

Schoenhals and Clark[25] considered the response of velocity and temperature of a laminar incompressible fluid to a semi-infinite flat plate oscillating harmonically in a horizontal direction. The method of successive approximation and perturbation technique were employed to obtain analytical results. Experimental data on free convection over a finite plate were presented and the heat transfer coefficient is found to be increased as a result of the vibration. For the semi-infinite plate no increase could be observed experimentally. The analysis for this case was carried out to the second approximation only which is harmonic in character. Hence, it discloses no steady alternation to temperature and velocity profiles nor the heat transfer rate and shear stress. This work was continued by Blankenship and Clark[4,5] who considered an isothermal finite plate vibrating horizontally in a compressible fluid. Two cases are considered. In one case the buoyancy forces predominate over the inertia forces; in the other case the situation is reversed. They also extended the work of Schoenhals to a higher approximation for the semi-infinite plate. Eshghy[3] recently considered the same problem as Schoenhals and Clark except that the direction of vibration is vertical.

Hori[11-13] published recently a series of three papers in which the method of series expansion is used to solve various oscillation problems. Nanda and Sharma[19] solved the free convection problem with an oscillating wall temperature.

The problem of boundary layer flow with localized vibration apparently remained untreated. It is the purpose of this research to investigate the influence of localized vibration on the laminar flow of incompressible fluid over a semi-infinite flat plate.

CHAPTER II

THEORETICAL ANALYSIS

STATEMENT OF THE PROBLEM

The analytical model is shown in Figure 1. It consists of a semi-infinite flat plate extending from 0 to ∞ in the x-direction. A viscous fluid flows steadily over this plate producing a laminar boundary layer. At distance L from the leading edge a localized and periodic surface disturbance is introduced. The effect of this localized disturbance on the rate of heat transfer and the wall shearing stress are desired.

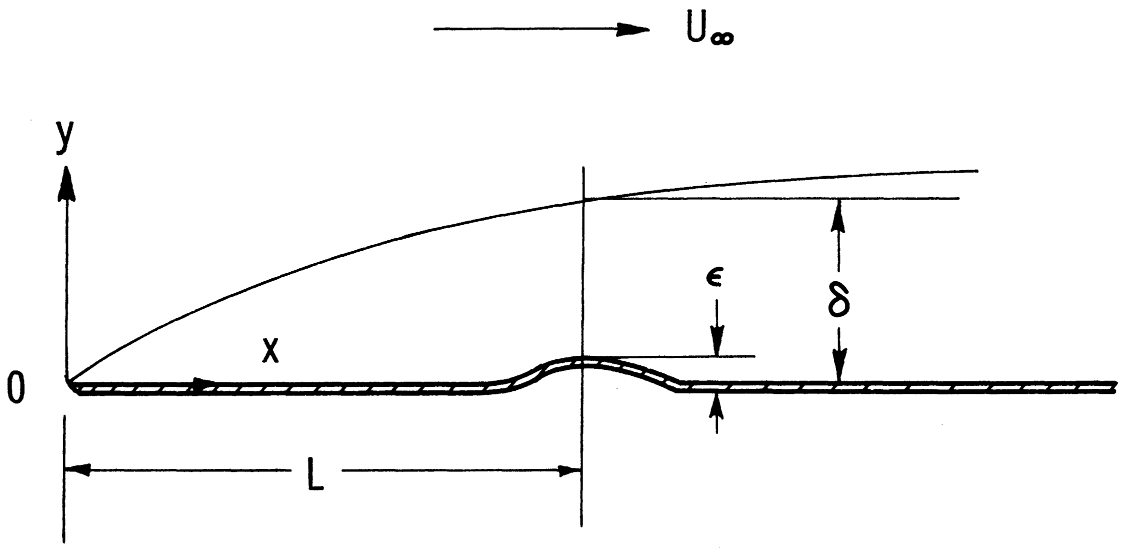

Figure 1. Model of the flat plate.

THE FORMULATION OF THE PROBLEM

The following assumptions are made in the formulation of the problem:

a. Distance L of the localized disturbance from the leading edge is taken to be much greater than the boundary layer thickness δ. Thus the effect of this disturbance on the boundary layer can be investigated within the restrictions of the boundary layer theory.

b. The fluid is incompressible so that the induced pressure waves travel with infinite velocity.

c. The surface boundary condition is simplified by replacing the plate and the disturbance by an analytical model consisting of a continuous curve, as shown in Figure 2. Using the probability curve, for example, for spacewise variations, the localized oscillations of the plate can be represented in the form of

$$y = \epsilon e^{-m(x-L)^2 + i\omega t} \tag{1}$$

where $\epsilon e^{-m(x-L)^2}$ is the amplitude of oscillation, and m is a parameter

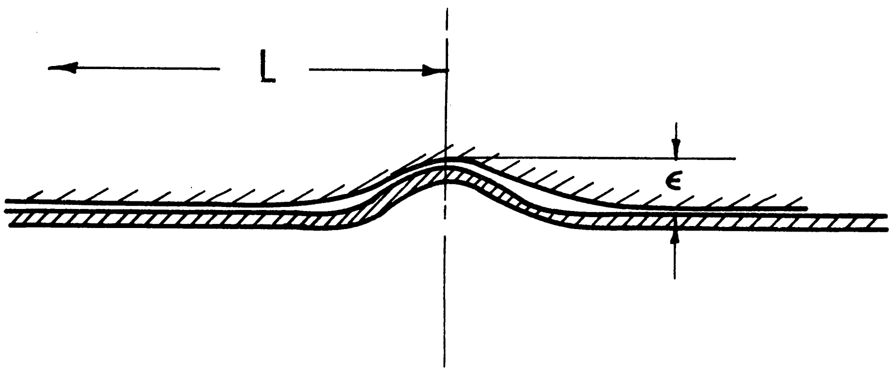

Figure 2. Analytical model of the disturbance.

to be selected according to the desired shape of the disturbance. The amplitude of oscillation is assumed to be small compared with the boundary layer thickness. The value of m is chosen such that the curvature of the wall remains small, allowing a curvilinear orthogonal system of coordinates to be used whose X-axis is in the direction of the wall, the Y-axis perpendicular to it. The desired steady periodic solutions suggest the use of the complex form of the surface oscillations. The selection of the probability curve is arbitrary, and can well be replaced by another curve if desired. The solution of the problem is a function of the selected curve. However, the effect of these classes of disturbances can very probably be demonstrated by any one of these curves.

d. The effect of localized oscillation on the potential flow is neglected. Hence, the potential flow is assumed to be the streaming flow of velocity U_∞.

e. The velocity component of the plate in the x-direction is zero.

With the above assumptions, the equations of momentum, continuity, and energy with respect to the Cartesian coordinates may be written as follows:

Momentum:

$$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (2)$$

$$\rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad (3)$$

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (4)$$

Energy:

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = a\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) \qquad (5)$$

The boundary conditions are

$$y = \epsilon e^{-m(x-L)^2 + i\omega t} ; \quad \theta = \theta_w , \; u = 0 , \; v = i\omega\epsilon e^{-m(x-L)^2 + i\omega t}$$

$$y \to \infty \quad ; \quad \theta = \theta_\infty , \; u = U_\infty \qquad (6)$$

KINEMATIC RELATIONS

A reference frame attached to the moving surface is used. Thus, the boundary conditions are greatly simplified at the expense of increased complexity of the governing differential equations. This compromise is usually more convenient for the mathematical solution. The formulation of the problem in the moving coordinate system (X,Y) (Figure 3) which accelerates with respect to fixed coordinate system (x,y) is as follows: First, the relation between the coordinates of the fixed and accelerating coordinate axes will be obtained.

At time t, the boundary is assumed to be at the position O'AX. The arc length of $\widehat{O'A}$, Figure 3,

$$X = \widehat{O'A} = \int_0^{x_1} \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx , \qquad (7)$$

using the particular value of $\left(\frac{dy}{dx}\right)$ from Eq. (1), can be written in the

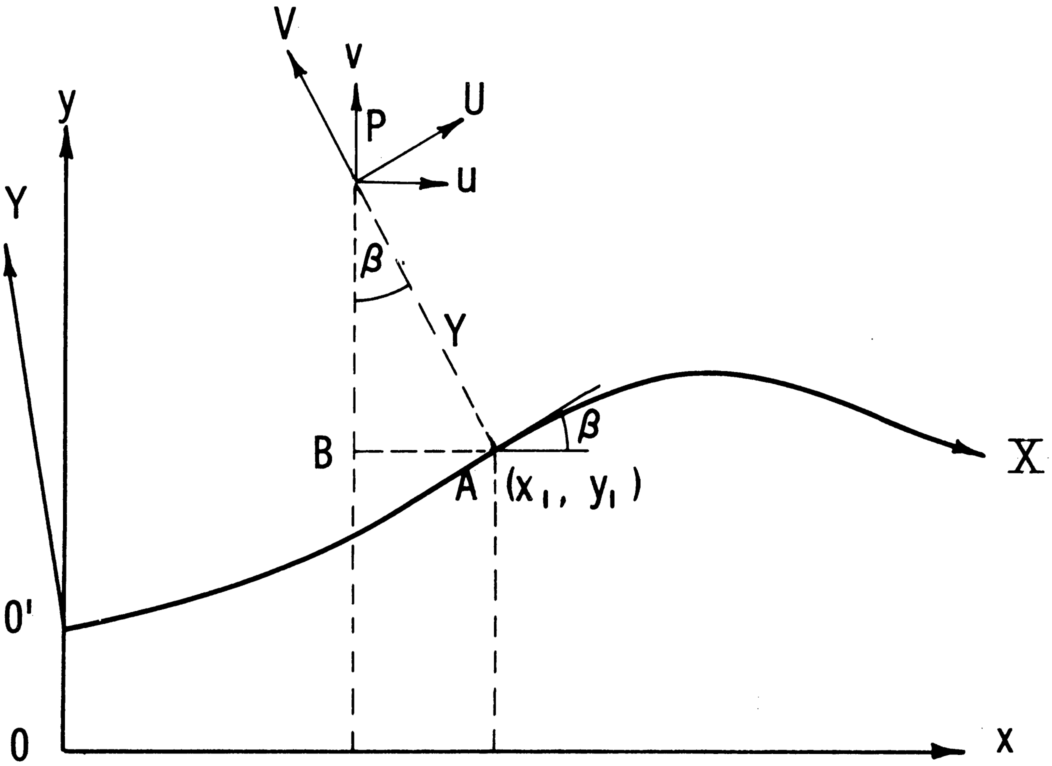

Figure 3. The fixed and accelerating coordinate system.

form:

$$\mathfrak{X} = \int_0^{x_1} \left[1 + 4m^2 (x-L)^2 \epsilon^2 e^{-2m(x-L)^2 + 2i\omega t} \right]^{\frac{1}{2}} dx \tag{8}$$

Expanding the integrand of Eq. (8) binominally,

$$\mathfrak{X} = x_1 + \epsilon^2 \left\{ 2m^2 e^{2i\omega t} \int_0^{x_1} (x-L)^2 e^{-2m(x-L)^2} dx \right\} + O(\epsilon^4) \tag{9}$$

On the other hand, we have

$$x = x_1 - Y \sin\beta = x_1 - Y \frac{\left(\frac{dy}{dx}\right)_1}{\left[1 + \left(\frac{dy}{dx}\right)_1^2\right]^{\frac{1}{2}}} \tag{10}$$

and

$$y = y_1 + Y \cos\beta = y_1 + Y \frac{1}{\left[1 + \left(\frac{dy}{dx}\right)_1^2\right]^{\frac{1}{2}}} \tag{11}$$

where subscript "1" indicates the values of x, y, and $\left(\frac{dy}{dx}\right)$ at location A.

Hence,

$$\left(\frac{dy}{dX}\right)_1 = -2m\epsilon(X_1-L)e^{-m(X_1-L)^2+i\omega t} = \epsilon\{-2m(X_1-L)\phi\} \tag{12}$$

Putting Eq. (12) into Eqs. (10) and (11), expanding them binominally and rearranging gives:

$$X = X_1 + \epsilon\{2m(X_1-L)Y\phi\} + O(\epsilon^3) \tag{13}$$

and,

$$Y = Y_1 + Y - \epsilon^2\{2m^2(X_1-L)^2 Y\phi^2\} + O(\epsilon^4) \tag{14}$$

Also, since (x_1,y_1) is on the surface of the plate, we can write

$$Y_1 = \epsilon e^{-m(X_1-L)^2+i\omega t} = \epsilon\phi \tag{15}$$

Therefore, we get four Eqs. (9), (13), (14), and (15), which may be used to get the relation between the two coordinate systems (x,y) and (X,Y) by simply eliminating the parameters x_1 and y_1. The integration of the second term of Eq. (9) is simplified by neglecting terms of order ϵ^2. Eliminating the parameters x_1 and y_1 from the four equations, we get

$$X = \bar{X} + \epsilon[2m(\bar{X}-L)Y\phi] \tag{16}$$

$$Y = Y + \epsilon\phi \tag{17}$$

where

$$\phi = e^{-m(X-L)^2 + i\omega T} \tag{18}$$

Equations (16) and (17) give the relations between these two coordinate systems. It should be remarked that the assumption of neglecting terms of order ϵ^2 and higher, or $X = x_1$, eliminates the change of length of the X-axis.

The velocity components of the boundary in the X- and Y-directions may now be evaluated. Since the velocity of the boundary is assumed to be in the y-direction, we readily have

$$V_W = \left(\frac{dy}{dt}\right)_W = \epsilon i\omega e^{-m(X-L)^2 + i\omega T} = \epsilon i\omega\phi \tag{19}$$

The velocity components in the X- and Y-directions are

$$\begin{aligned} V_{WY} = V_W \cos\beta &= (\epsilon i\omega\phi)\frac{1}{[1 + (\frac{dy}{dX})^2]^{\frac{1}{2}}} \\ &= (\epsilon i\omega\phi)\left\{1 - \frac{1}{2}\epsilon^2[2m(X-L)\phi]^2 + \cdots\cdots\right\} \\ &= \epsilon i\omega\phi + O(\epsilon^3) \end{aligned} \tag{20}$$

$$\begin{aligned} V_{WX} = V_W \sin\beta &= (\epsilon i\omega\phi)\frac{(\frac{dy}{dX})}{[1 + (\frac{dy}{dX})^2]^{\frac{1}{2}}} \\ &= (\epsilon i\omega\phi)\{-\epsilon 2m(X-L)\phi\}\left\{1 - \frac{\epsilon^2}{2}[2m(X-L)\phi]^2 + \cdots\cdots\right\} \\ &= O(\epsilon^2) \end{aligned} \tag{21}$$

Since the first two approximations are considered only,

$$V_{WY} = \epsilon i\omega\phi \tag{22}$$

$$V_{WX} = 0 \tag{23}$$

Next, let us find the relations between the velocity components relative to the fixed coordinate system (u,v) and that relative to the moving coordinate system (U,V). It is clear that the absolute velocities (u,v) equal to the sum of the relative velocities (U,V) and the velocities of the coordinate axes due to translation and rotation. To find these, consider Figure 4 in which a local Cartesian coordinate (x',y') is drawn at point Q with its axes tangential and normal to the

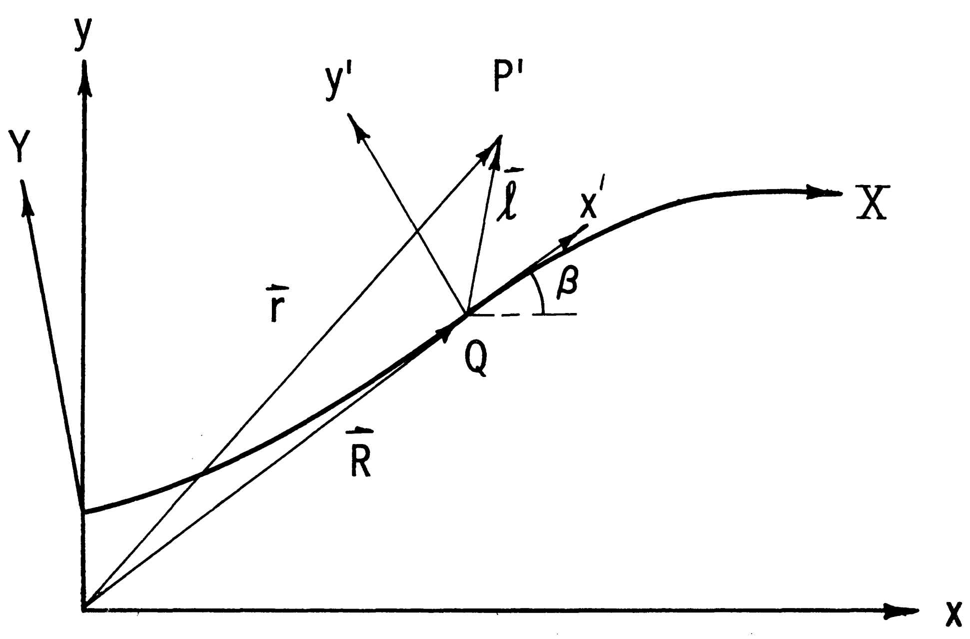

Figure 4. Illustration of the local Cartesian coordinate system.

X-axis. Assume that $\vec{i}$ and $\vec{j}$ be unit vectors in the x'- and y'-directions. Then

$$\vec{l} = x'\vec{i} + y'\vec{j} \tag{24}$$

The absolute velocity of the point P' can be obtained by differentiating the vector equation

$$\vec{r} = \vec{R} + \vec{\ell} \tag{25}$$

with respect to time t, i.e.,

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \frac{d\vec{R}}{dt} + \frac{d\vec{\ell}}{dt} \\
&= \frac{d\vec{R}}{dt} + \frac{d}{dt}(x'\vec{i} + y'\vec{j}) \\
&= \frac{d\vec{R}}{dt} + \left(\frac{dx'}{dt}\vec{i} + \frac{dy'}{dt}\vec{j}\right) + \left(x'\frac{d\vec{i}}{dt} + y'\frac{d\vec{j}}{dt}\right) \\
&= \frac{d\vec{R}}{dt} + u'\vec{i} + v'\vec{j} + (x'\vec{\dot{\beta}} \times \vec{i} + y'\vec{\dot{\beta}} \times \vec{j}) \\
&= \frac{d\vec{R}}{dt} + u'\vec{i} + v'\vec{j} + \vec{\dot{\beta}} \times \vec{\ell}
\end{aligned} \tag{26}$$

From Figure 4,

$$\vec{\dot{\beta}} = \dot{\beta}\,\vec{k}$$

$$\frac{d\vec{r}}{dt} = V_{P'x'}\vec{i} + V_{P'y'}\vec{j} \tag{27}$$

and

$$\frac{d\vec{R}}{dt} = V_{Wx'}\vec{i} + V_{Wy'}\vec{j}$$

Then, Eq. (26) becomes,

$$V_{P'x'} = u' + V_{WX} - y'\dot{\beta} \tag{28}$$

$$V_{P'y'} = v' + V_{WY} + x'\dot{\beta} \tag{29}$$

Therefore, at point P of Figure 3, the absolute velocities in the X- and Y-directions are

$$V_{PX} = U + V_{wX} - Y\dot{\beta} \tag{30}$$

$$V_{PY} = V + V_{wY} \tag{31}$$

v_{wX} and v_{wY} are given in Eqs. (22) and (23). Also, the angular velocity $\dot{\beta}$ may be found by differentiating β, Figure 3, with respect to time t. Since,

$$\beta = \frac{dY}{dX} = -2m(X-L)\epsilon\, e^{-m(X-L)^2 + i\omega T} = -2m(X-L)\epsilon\phi \tag{32}$$

we get,

$$\dot{\beta} = -2m(X-L)\, i\omega\phi\epsilon \tag{33}$$

Finally, the X and Y components of the absolute velocity at point P may be obtained by introducing Eqs. (22), (23), and (33) into Eqs. (30) and (31). The result is

$$V_{PX} = U + 2m(X-L)Y\, i\omega\phi\epsilon, \tag{34}$$

$$V_{PY} = V + \epsilon i\omega\phi. \tag{35}$$

It is now only a matter of geometry to write the absolute velocities in the x- and y-directions as a function of V_{PX} and V_{py}. Hence,

$$u = V_{PX}\cos\beta - V_{PY}\sin\beta \tag{36}$$

$$V = V_{PX} \sin\beta + V_{PY} \cos\beta \tag{37}$$

As before, we assume

$$\cos\beta = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+(\frac{dy}{dx})^2}} \doteq 1 \tag{38}$$

and

$$\sin\beta = \frac{\tan\beta}{\sqrt{1+\tan^2\beta}} = \frac{(\frac{dy}{dx})}{\sqrt{1+(\frac{dy}{dx})^2}} \doteq -2m(X-L)\epsilon\phi \tag{39}$$

Inserting Eqs. (34), (35), (38), and (39) into Eqs. (36) and (37), we get,

$$u = \bar{U} + 2m\epsilon(X-L)Yi\omega\phi + 2m(X-L)\epsilon\phi\bar{V} + O(\epsilon^2)$$

$$v = [\bar{U}+2m(X-L)Yi\omega\phi][-2m(X-L)\epsilon\phi] + (\bar{V}+\epsilon i\omega\phi) + O(\epsilon^2)$$

These, neglecting terms of order $O(\epsilon^2)$, may be reduced to

$$u = \bar{U} + \epsilon[2m(X-L)\phi](i\omega Y + \bar{V}) \tag{40}$$

$$v = \bar{V} + \epsilon[i\omega - 2m(X-L)\bar{U}]\phi \tag{41}$$

FORMULATION OF THE PROBLEM IN THE ACCELERATING COORDINATE SYSTEM

This formulation is first written with reference to a general orthogonal curvilinear coordinate system (X,Y) as follows:

Continuity:

$$\frac{\partial(\bar{U}h_Y)}{\partial X} + \frac{\partial(\bar{V}h_X)}{\partial Y} = 0 \tag{42}$$

Momentum:

$$\rho\left(\frac{\partial U}{\partial T} + \frac{U}{h_X}\frac{\partial U}{\partial X} + \frac{V}{h_Y}\frac{\partial U}{\partial Y} + \frac{UV}{h_X h_Y}\frac{\partial h_X}{\partial Y} - \frac{V^2}{h_X h_Y}\frac{\partial h_Y}{\partial X}\right) + a_X$$

$$= -\frac{1}{h_X}\frac{\partial p}{\partial X} + \frac{1}{h_X h_Y}\left[\frac{\partial}{\partial X}(h_Y \sigma_{XX}') + \frac{\partial}{\partial Y}(h_X \tau_{XY})\right]$$

$$+ \frac{\tau_{XY}}{h_X h_Y}\frac{\partial h_X}{\partial Y} - \frac{\sigma_{YY}'}{h_X h_Y}\frac{\partial h_Y}{\partial X} \tag{43}$$

$$\rho\left(\frac{\partial V}{\partial T} + \frac{U}{h_X}\frac{\partial V}{\partial X} + \frac{V}{h_Y}\frac{\partial V}{\partial Y} + \frac{UV}{h_X h_Y}\frac{\partial h_Y}{\partial X} - \frac{U^2}{h_X h_Y}\frac{\partial h_X}{\partial Y}\right) + a_Y$$

$$= -\frac{1}{h_Y}\frac{\partial p}{\partial Y} + \frac{1}{h_X h_Y}\left[\frac{\partial}{\partial X}(h_Y \tau_{XY}) + \frac{\partial}{\partial Y}(h_X \sigma_{YY}')\right]$$

$$+ \frac{\tau_{XY}}{h_X h_Y}\frac{\partial h_Y}{\partial X} - \frac{\sigma_{XX}'}{h_X h_Y}\frac{\partial h_X}{\partial Y} \tag{44}$$

where

U,V: velocity components relative to the moving coordinates (X,Y).

h_X, h_Y: elements of the metric tensor. Now,

$$x = X + \epsilon[2m(X-L)Y\phi] \tag{45}$$

$$y = Y + \epsilon\phi \tag{46}$$

and

$$h_X^2 = \left(\frac{\partial x}{\partial X}\right)^2 + \left(\frac{\partial y}{\partial X}\right)^2 \tag{47}$$

$$h_Y^2 = \left(\frac{\partial x}{\partial Y}\right)^2 + \left(\frac{\partial y}{\partial Y}\right)^2 \tag{48}$$

Putting Eqs. (45) and (46) into Eqs. (47) and (48), taking square roots and considering terms of ϵ^0 and ϵ', we get,

$$h_X = 1 + \epsilon(2mY)[1-2m(X-L)^2]\phi \tag{49}$$

$$h_Y = 1 \tag{50}$$

σ's:

$$\sigma'_{XX} = 2\mu\left(\frac{1}{h_X}\frac{\partial U'}{\partial X} + \frac{V'}{h_X h_Y}\frac{\partial h_X}{\partial Y}\right)$$

$$\sigma'_{YY} = 2\mu\left(\frac{1}{h_Y}\frac{\partial V'}{\partial Y} + \frac{U'}{h_X h_Y}\frac{\partial h_Y}{\partial X}\right)$$

With Eq. (50), these equations become

$$\sigma'_{XX} = 2\mu\left(\frac{1}{h_X}\frac{\partial U'}{\partial X} + \frac{V'}{h_X}\frac{\partial h_X}{\partial Y}\right) \tag{51}$$

$$\sigma'_{YY} = 2\mu\left(\frac{\partial V'}{\partial Y}\right) \tag{52}$$

τ_{xy}:

$$\tau_{XY} = \mu\left[\frac{h_Y}{h_X}\frac{\partial}{\partial X}\left(\frac{V'}{h_Y}\right) + \frac{h_X}{h_Y}\frac{\partial}{\partial Y}\left(\frac{U'}{h_X}\right)\right]$$

With Eq. (50), this equation becomes

$$\tau_{XY} = \mu\left[\frac{1}{h_X}\frac{\partial V'}{\partial X} + \frac{\partial U'}{\partial Y} - \frac{U'}{h_X}\frac{\partial h_X}{\partial Y}\right] \tag{53}$$

In Eqs. (51), (52), and (53),

$$U' = U + 2m(X-L)Y i\omega\phi\epsilon$$

$$V' = V + \epsilon i\omega\phi$$

This is derived in Appendix I.

a_X, a_Y: accelerations due to movement of coordinate axes. They are derived as follows: At any point P, there exists four accelerations,

$$\vec{\ddot{\beta}} \times \vec{\ell} \qquad \text{(angular acceleration)}$$

$$\vec{\dot{\beta}} \times (\vec{\dot{\beta}} \times \vec{\ell}) \qquad \text{(centrifugal acceleration)}$$

$$2\,\vec{\dot{\beta}} \times \vec{V} \qquad \text{(coriolis acceleration)}$$

$$\frac{d\dot{R}_Y}{dt}\,\vec{j} \qquad \text{(linear acceleration)}$$

Noting that

$$\vec{\dot{\beta}} = \dot{\beta}\,\vec{k}$$

$$\vec{\ell} = Y\,\vec{j}$$

$$\vec{V} = U\,\vec{i} + V\,\vec{j}$$

$$R_Y = \epsilon\, i\, \omega\, \phi$$

these accelerations become

$$\vec{\ddot{\beta}} \times \vec{\ell} = -Y\ddot{\beta}\,\vec{i} \tag{54}$$

$$\vec{\dot{\beta}} \times (\vec{\dot{\beta}} \times \vec{\ell}) = -Y\dot{\beta}^2\,\vec{j} \tag{55}$$

$$2\,\vec{\dot{\beta}} \times \vec{V} = -(2\dot{\beta}V)\,\vec{i} + (2\dot{\beta}U)\,\vec{j} \tag{56}$$

$$\frac{d\dot{R}_Y}{dt}\,\vec{j} = -\epsilon\,\omega^2\phi\,\vec{j} \tag{57}$$

Therefore, accelerations due to the movement of coordinate axes in the X-direction

$$a_X = -Y\ddot{\beta} - 2\dot{\beta}V \tag{58}$$

and in the Y-direction

$$a_Y = -Y\dot{\beta}^2 + 2\dot{\beta}U - \epsilon\omega^2\phi \qquad (59)$$

where $\dot{\beta}$ is given in Eq. (33).

Introducing Eqs. (51)-(53), (58), and (59), into Eqs. (42), (43), and (44), we get:

Continuity:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \epsilon(2m\phi)[1-2m(X-L)^2](V + Y\frac{\partial V}{\partial Y}) = 0 \qquad (60)$$

Momentum:

$$\begin{aligned}
&\rho\Big\{\Big(\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y}\Big) + \epsilon[2m(X-L)\phi](2i\omega V - Y\omega^2) \\
&\quad + \epsilon(2m\phi)[1-2m(X-L)^2](Y\frac{\partial U}{\partial T} + YV\frac{\partial U}{\partial Y} + UV)\Big\} \\
&= -\frac{\partial p}{\partial X} + \mu\Big(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\Big) + \mu 4m\phi\epsilon\frac{\partial V}{\partial X} \\
&\quad + \mu 2m(X-L)\phi\epsilon\Big\{-6mV + [2m(X-L)]^2 V - 4m(X-L)\frac{\partial V}{\partial X} + 4mY\frac{\partial U}{\partial X}\Big\} \\
&\quad + \mu 2m\phi[1-2m(X-L)^2]\epsilon\Big\{2mY(X-L)\frac{\partial U}{\partial X} - Y\frac{\partial^2 U}{\partial X^2} + \frac{\partial U}{\partial Y} + Y\frac{\partial^2 U}{\partial Y^2}\Big\} \\
&\quad - \mu 2mYi\omega\phi\epsilon[2m(X-L)][3-2m(X-L)^2]
\end{aligned} \qquad (61)$$

$$\begin{aligned}
&\rho\Big\{\Big(\frac{\partial V}{\partial T} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\Big) - 4m(X-L)i\omega\phi\epsilon U - \epsilon\omega^2\phi \\
&\quad + \epsilon(2m\phi)[1-2m(X-L)^2]\Big(Y\frac{\partial V}{\partial T} + YV\frac{\partial V}{\partial Y} - U^2\Big)\Big\} \\
&= -\{1+\epsilon 2mY[1-2m(X-L)^2]\phi\}\frac{\partial p}{\partial Y} + \mu\Big(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\Big) \\
&\quad + \mu 2m(X-L)\phi\epsilon\Big(4mY\frac{\partial V}{\partial X} + 4mU\Big) \\
&\quad - \mu 2m\phi[1-2m(X-L)^2]\epsilon\Big\{-Y\frac{\partial^2 V}{\partial X^2} - 3\frac{\partial U}{\partial X} + 2m(X-L)U \\
&\quad - i\omega + 2mY(X-L)\frac{\partial V}{\partial X} + Y\frac{\partial^2 V}{\partial Y^2}\Big\}
\end{aligned} \qquad (62)$$

Similarly, the energy equation written in terms of the accelerating coordinate system is

$$\frac{\partial\theta}{\partial T} + U\frac{\partial\theta}{\partial X} + V\frac{\partial\theta}{\partial Y} + \epsilon(2mY\phi)[1-2m(X-L)^2](\frac{\partial\theta}{\partial T} + V\frac{\partial\theta}{\partial Y})$$

$$= a\Big\{\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2} + \epsilon(2m\phi)[1-2m(X-L)^2][2m(X-L)Y\frac{\partial\theta}{\partial X}$$

$$-Y\frac{\partial^2\theta}{\partial X^2} + \frac{\partial\theta}{\partial Y} + Y\frac{\partial^2\theta}{\partial Y^2}] + \epsilon(2m\phi)Y[4m(X-L)]\frac{\partial\theta}{\partial X}\Big\} \qquad (63)$$

SIMPLIFICATION OF THE FORMULATION

At this stage it is expedient to introduce dimensionless variables by means of

$$U' = \frac{U}{U_\infty}, \qquad V' = \frac{V}{U_\infty}, \qquad X' = \frac{X}{L},$$

$$Y' = \frac{Y}{L}, \qquad \epsilon' = \frac{\epsilon}{L}, \qquad T' = \frac{T}{\left(\frac{L}{U_\infty}\right)},$$

$$p' = \frac{p}{\rho U_\infty^2}, \qquad \omega' = \frac{\omega}{\left(\frac{U_\infty}{L}\right)}, \qquad \theta' = \frac{\theta - \theta_\infty}{\theta_w - \theta_\infty}.$$

The formulation of the problem then appears to be:

Continuity:

$$\frac{\partial U'}{\partial X'} + \frac{\partial V'}{\partial Y'} + \epsilon'(2m'\phi')[1-2m'(X'-1)^2](V' + Y'\frac{\partial V'}{\partial Y'}) = 0 \qquad (64)$$

Momentum:

$$\left(\frac{\partial U'}{\partial T'}+U'\frac{\partial U'}{\partial X'}+V'\frac{\partial U'}{\partial Y'}\right)+\epsilon'(2m'\phi')[1-2m'(X'-1)^2]\left(Y'\frac{\partial U'}{\partial T'}+Y'V'\frac{\partial U'}{\partial Y'}+U'V'\right)$$

$$+\epsilon'[2m'(X'-1)\phi'](2i\omega'V'-Y'\omega'^2)$$

$$=-\frac{\partial p'}{\partial X'}+\frac{1}{Re}\left(\frac{\partial^2 U'}{\partial X'^2}+\frac{\partial^2 U'}{\partial Y'^2}\right)+\epsilon' 4m'\phi'\frac{\partial V'}{\partial X'}\frac{1}{Re}$$

$$+\epsilon' 2m'(X'-1)\phi'\left\{-6m'V'+[2m'(X'-1)]^2V'-4m'(X'-1)\frac{\partial V'}{\partial X'}+4m'Y'\frac{\partial U'}{\partial X'}\right\}\frac{1}{Re}$$

$$+\epsilon' 2m'\phi'[1-2m'(X'-1)^2]\left\{2m'Y'(X'-1)\frac{\partial U'}{\partial X'}-Y'\frac{\partial^2 U'}{\partial X'^2}+\frac{\partial U'}{\partial Y'}+Y'\frac{\partial^2 U'}{\partial Y'^2}\right\}\frac{1}{Re}$$

$$-\epsilon' 2m'Y'i\omega'\phi'[2m'(X'-1)][3-2m'(X'-1)^2]\frac{1}{Re} \qquad (65)$$

$$\left(\frac{\partial V'}{\partial T'}+U'\frac{\partial V'}{\partial X'}+V'\frac{\partial V'}{\partial Y'}\right)+\epsilon'(2m'\phi')[1-2m'(X'-1)^2]\left(Y'\frac{\partial V'}{\partial T'}+Y'V'\frac{\partial V'}{\partial Y'}-U'^2\right)$$

$$-\epsilon' 4m'(X'-1)i\omega'\phi'U'-\epsilon'\omega'^2\phi'$$

$$=-\left\{1+\epsilon'(2m'Y'\phi')[1-2m'(X'-1)^2]\right\}\frac{\partial p'}{\partial Y'}+\frac{1}{Re}\left(\frac{\partial^2 V'}{\partial X'^2}+\frac{\partial^2 V'}{\partial Y'^2}\right)$$

$$+\frac{\epsilon'}{Re}[2m'(X'-1)\phi']\left(4m'Y'\frac{\partial V'}{\partial X'}+4m'U'\right)$$

$$+\frac{\epsilon'}{Re}(2m'\phi')[1-2m'(X'-1)^2]\left\{-Y'\frac{\partial^2 V'}{\partial X'^2}-3\frac{\partial U'}{\partial X'}+2m'(X'-1)U'\right.$$

$$\left.-i\omega'+2m'Y'(X'-1)\frac{\partial V'}{\partial X'}+Y'\frac{\partial^2 V'}{\partial Y'^2}\right\} \qquad (66)$$

Energy:

$$\frac{\partial\theta'}{\partial T'}+U'\frac{\partial\theta'}{\partial X'}+V'\frac{\partial\theta'}{\partial Y'}+\epsilon'(2m'Y'\phi')[1-2m'(X'-1)^2]\left(\frac{\partial\theta'}{\partial T'}+V'\frac{\partial\theta'}{\partial Y'}\right)$$

$$=\frac{1}{Pr\,Re}\left\{\frac{\partial^2\theta'}{\partial X'^2}+\frac{\partial^2\theta'}{\partial Y'^2}+\epsilon'(2m'\phi')[1-2m'(X'-1)^2]\left[2m'(X'-1)Y'\frac{\partial\theta'}{\partial X'}\right.\right.$$

$$\left.\left.-Y'\frac{\partial^2\theta'}{\partial X'^2}+\frac{\partial\theta'}{\partial Y}+Y'\frac{\partial^2\theta'}{\partial Y'^2}\right]+\epsilon'(2m'\phi'Y')[4m'(X'-1)]\frac{\partial\theta'}{\partial X'}\right\} \qquad (67)$$

We are interested in the asymptotic form of the foregoing formulation corresponding to small viscosity or large Reynolds numbers. It is a well-known fact, however, that the terms having $\frac{1}{Re}$ as coefficient cannot be eliminated from the formulation because of the reduction of momentum equations to Eulerian equations. This difficulty may be circumvented by a new variable Y* defined such that

$$Y^* = Y' / Re^n \tag{68}$$

Let us now consider the terms, $V' \frac{\partial U'}{\partial Y'}$ and $\frac{1}{Re} \frac{\partial^2 U'}{\partial Y'^2}$. Employing Eq. (68), these terms may be rearranged as

$$V' \frac{\partial U'}{\partial Y'} = V' Re^{-n} \frac{\partial U'}{\partial Y^*} \tag{69}$$

$$\frac{1}{Re} \frac{\partial^2 U'}{\partial Y'^2} = \frac{1}{Re} \left(Re^{-2n} \frac{\partial^2 U'}{\partial Y^{*2}} \right) = Re^{-(1+2n)} \frac{\partial^2 U'}{\partial Y^{*2}} \tag{70}$$

From Eq. (70), it is seen that the term $\frac{\partial^2 U'}{\partial Y^{*2}}$ is retained as $Re \to \infty$ if $n = -\frac{1}{2}$. In order to retain the term $V' \frac{\partial U'}{\partial Y'}$, the following transformation is introduced

$$V^* = Re^{\frac{1}{2}} V' \tag{71}$$

which transforms Eq. (69) to $V^* \frac{\partial U'}{\partial Y^*}$. Based on this argument, the basic equations are transformed according to the relations:

$$Y^* = Re^{\frac{1}{2}} Y' \tag{72}$$

and

$$V^* = Re^{\frac{1}{2}} V' \tag{73}$$

The result is:

Continuity:

$$\frac{\partial U'}{\partial X'} + \frac{\partial V^*}{\partial Y^*} + \frac{\epsilon'}{\sqrt{Re}}(2m'\phi')[1-2m'(X'-1)^2](V^* + Y^*\frac{\partial V^*}{\partial Y^*}) = 0 \qquad (74)$$

Momentum:

$$\begin{aligned}
&(\frac{\partial U'}{\partial T'} + U'\frac{\partial U'}{\partial X'} + V^*\frac{\partial U'}{\partial Y^*}) + \epsilon'(2m'\phi')[1-2m'(X'-1)^2](Y^*\frac{\partial U'}{\partial T'} + Y^*V^*\frac{\partial U'}{\partial Y^*} + U'V^*)\frac{1}{\sqrt{Re}} \\
&\quad + \epsilon'[2m'(X'-1)\phi'](2i\omega'V^* - Y^*\omega'^2)\frac{1}{\sqrt{Re}} \\
&= -\frac{\partial \phi'}{\partial X'} + \frac{1}{Re}\frac{\partial^2 U'}{\partial X'^2} + \frac{\partial^2 U'}{\partial Y^{*2}} + \epsilon'(4m'\phi')\frac{\partial V^*}{\partial X'}\frac{1}{Re^{3/2}} \\
&\quad + \epsilon'[2m'(X'-1)\phi']\left\{-6m'V^* + [2m'(X'-1)]^2V^* - 4m'(X'-1)\frac{\partial V^*}{\partial X'} + 4m'Y^*\frac{\partial U'}{\partial X'}\right\}\frac{1}{Re^{3/2}} \\
&\quad + \epsilon'(2m'\phi')[1-2m'(X'-1)^2]\left\{\frac{1}{Re^{3/2}}[2m'Y^*(X'-1)\frac{\partial U'}{\partial X'} - Y^*\frac{\partial^2 U'}{\partial X'^2}] + \frac{1}{\sqrt{Re}}(\frac{\partial U'}{\partial Y^*} + Y^*\frac{\partial^2 U'}{\partial Y^{*2}})\right\} \\
&\quad - \epsilon'(2m'Y^*i\omega'\phi')[2m'(X'-1)][3-2m'(X'-1)^2]\frac{1}{\sqrt{Re}}
\end{aligned} \qquad (75)$$

$$\begin{aligned}
&\frac{1}{Re}(\frac{\partial V^*}{\partial T'} + U'\frac{\partial V^*}{\partial X'} + V^*\frac{\partial V^*}{\partial Y^*}) + \epsilon'(2m'\phi')[1-2m'(X'-1)^2][\frac{1}{Re^{3/2}}(Y^*\frac{\partial V^*}{\partial T'} + Y^*V^*\frac{\partial V^*}{\partial Y^*}) - \frac{U'^2}{\sqrt{Re}}] \\
&\quad - \frac{\epsilon'}{\sqrt{Re}}[4m'(X'-1)](i\omega'\phi'U') - \frac{\epsilon'}{\sqrt{Re}}\omega'^2\phi' \\
&= -\left\{1 + \frac{\epsilon'}{\sqrt{Re}}(2m'Y^*\phi')[1-2m'(X'-1)^2]\right\}\frac{\partial \phi'}{\partial Y^*} + \frac{1}{Re^2}\frac{\partial^2 V^*}{\partial X'^2} + \frac{1}{Re}\frac{\partial^2 V^*}{\partial Y^{*2}} \\
&\quad + \frac{\epsilon'}{\sqrt{Re}}[2m'(X'-1)\phi'](\frac{1}{Re^2}4m'Y^*\frac{\partial V^*}{\partial X'} + \frac{1}{Re}4m'U') \\
&\quad + \frac{\epsilon'}{\sqrt{Re}}(2m'\phi')[1-2m'(X'-1)^2]\left\{-\frac{1}{Re^2}Y^*\frac{\partial^2 V^*}{\partial X'^2} - \frac{3}{Re}\frac{\partial U'}{\partial X'} + \frac{1}{Re}[2m'(X'-1)U']\right. \\
&\quad \left. - \frac{1}{Re}i\omega' + \frac{1}{Re^2}[2m'Y^*(X'-1)]\frac{\partial V^*}{\partial X'} + \frac{1}{Re}Y^*\frac{\partial^2 V^*}{\partial Y^{*2}}\right\}
\end{aligned} \qquad (76)$$

Energy:

$$\frac{\partial \theta'}{\partial T'} + U'\frac{\partial \theta'}{\partial X'} + V^*\frac{\partial \theta'}{\partial Y^*} + \frac{\epsilon'}{\sqrt{Re}}(2m'Y^*\phi')[1-2m'(X'-1)^2](\frac{\partial \theta'}{\partial T'} + V^*\frac{\partial \theta'}{\partial Y^*})$$

$$= \frac{1}{Pr}\Big\{\frac{1}{Re}\frac{\partial^2\theta'}{\partial X'^2} + \frac{\partial^2\theta'}{\partial Y^{*2}} + \epsilon'(2m'\phi')[1-2m'(X'-1)^2][2m'(X'-1)Y^*\frac{\partial \theta'}{\partial X'}\frac{1}{Re^{3/2}}$$

$$- Y^*\frac{\partial^2\theta'}{\partial X'^2}\frac{1}{Re^{3/2}} + \frac{1}{\sqrt{Re}}\frac{\partial \theta'}{\partial Y^*} + \frac{1}{\sqrt{Re}}Y^*\frac{\partial^2\theta'}{\partial Y^{*2}}]$$

$$+ \frac{\epsilon'}{Re^{3/2}}(2m'\phi'Y^*)[4m'(X'-1)]\frac{\partial \theta'}{\partial X'}\Big\} \qquad (77)$$

Since we are looking for the asymptotic solutions of the above equations for large values of Re, this suggests the expansion of U', V*, p', and Θ' in terms of $\frac{1}{R^{1/2}}$ as follows:

$$U' = U_0' + \frac{1}{Re^{\frac{1}{2}}}U_1' + \frac{1}{Re}U_2' + \text{-------------} \qquad (78)$$

$$V^* = V_0^* + \frac{1}{Re^{\frac{1}{2}}}V_1^* + \frac{1}{Re}V_2^* + \text{-------------} \qquad (79)$$

$$p' = p_0' + \frac{1}{Re^{\frac{1}{2}}}p_1' + \frac{1}{Re}p_2' + \text{-------------} \qquad (80)$$

$$\theta' = \theta_0' + \frac{1}{Re^{\frac{1}{2}}}\theta_1' + \frac{1}{Re}\theta_2' + \text{-------------} \qquad (81)$$

Inserting Eqs. (78) through (81) into Eqs. (74) through (77), the successive orders of approximation are obtained. The zeroth-order equations are:

Continuity:

$$\frac{\partial U_0'}{\partial X'} + \frac{\partial V_0^*}{\partial Y^*} = 0 \qquad (82)$$

Momentum:

$$\frac{\partial U_0'}{\partial T'} + U_0'\frac{\partial U_0'}{\partial X'} + V_0^*\frac{\partial U_0'}{\partial Y^*} + \frac{\epsilon'}{\sqrt{Re}}[2m'(X'-1)\phi'][2i\omega' V_0^* - Y^*\omega'^2]$$

$$= -\frac{\partial p_0'}{\partial X'} + \frac{\partial^2 U_0'}{\partial Y^{*2}} - \frac{\epsilon'}{\sqrt{Re}}(2m'Y^* i\omega'\phi')[2m'(X'-1)][3-2m'(X'-1)^2] \qquad (83)$$

$$-\frac{\epsilon'}{\sqrt{Re}}[4m'(X'-1)i\omega'\phi]U_0' - \frac{\epsilon'}{\sqrt{Re}}\omega'^2\phi = -\frac{\partial p_0'}{\partial Y^*} \qquad (84)$$

Energy:

$$\frac{\partial \theta_0'}{\partial T'} + U_0'\frac{\partial \theta_0'}{\partial X'} + V_0^*\frac{\partial \theta_0'}{\partial Y^*} = \frac{1}{Pr}\frac{\partial^2 \theta_0'}{\partial Y^{*2}} \qquad (85)$$

It is seen that terms of order $\frac{1}{Re^{1/2}}$ and smaller are neglected, since these terms are small compared with the terms retained, which are of order 1. However, terms involving ω' are retained since the order of magnitude of them depends not only on Re but also on ω' and ϵ'. The order of magnitude of these terms is seen to be dependent on the following parameters

$$\frac{\epsilon'\omega'}{Re^{\frac{1}{2}}}, \quad \frac{\epsilon'\omega'^2}{Re^{\frac{1}{2}}} \quad \text{and} \quad \frac{\epsilon'\omega'}{Re^{\frac{3}{2}}} \qquad (86)$$

Since Re is large and ϵ' is small, these terms will have an order of magnitude of 1 only if ω' is large. For small ω', all terms involving ω' will be smaller than $\frac{1}{Re^{1/2}}$ and therefore can be dropped. This will reduce the basic equations to the Blasius' equations. The oscillation will have no effect on the flow.

It is easily shown that, for large ω',

$$\frac{\epsilon'\omega'^2}{Re^{\frac{1}{2}}} >> \frac{\epsilon'\omega'}{Re^{\frac{1}{2}}} \qquad (87)$$

and

$$\frac{\epsilon' \omega'^2}{Re^{\frac{1}{2}}} >> \frac{\epsilon' \omega'}{Re^{\frac{3}{2}}} \tag{88}$$

This means that only terms with $\frac{\epsilon' \omega'^2}{\sqrt{Re}}$ exist. The basic equations can be further simplified to:

Continuity:

$$\frac{\partial U'}{\partial X'} + \frac{\partial V^*}{\partial Y^*} = 0 \tag{89}$$

Momentum:

$$\frac{\partial U'}{\partial T'} + U' \frac{\partial U'}{\partial X'} + V^* \frac{\partial U'}{\partial Y^*} - \frac{\epsilon'}{\sqrt{Re}} [2m'(X'-1)\phi'] Y^* \omega'^2 = - \frac{\partial p'}{\partial X'} + \frac{\partial^2 U'}{\partial Y^{*2}} \tag{90}$$

$$\frac{\epsilon'}{\sqrt{Re}} \omega'^2 \phi' = \frac{\partial p'}{\partial Y^*} \tag{91}$$

Energy:

$$\frac{\partial \theta'}{\partial T'} + U' \frac{\partial \theta'}{\partial X'} + V^* \frac{\partial \theta'}{\partial Y^*} = \frac{1}{Pr} \frac{\partial^2 \theta'}{\partial Y^{*2}} \tag{92}$$

In these equations, the subscript "o" has been omitted for reason of simplicity. In Eq. (90), the inertia term represents the inertia force due to rotation of the normal coordinate axis. The inertia term in Eq. (91) is the inertia force due to the change of the velocity of vibration in the Y*-direction. Since this inertia force is a function of X', it will influence the pressure gradient in the X'-direction.

Complexity of the formulation of the problem suggests the Karman-Polhausen integral procedure be used most conveniently. Thus Eq. (91)

is integrated with respect to Y* and, based on assumption d,

$$p' = \frac{\epsilon' \omega'^2}{\sqrt{Re}} \phi' (Y^* - \delta') \tag{93}$$

where

$$\delta' = \frac{\delta}{L} \sqrt{Re} \tag{94}$$

Differentiating Eq. (93) with respect to X', we get

$$\frac{\partial p'}{\partial X'} = \frac{\epsilon' \omega'^2}{\sqrt{Re}} \phi' (\delta' - Y^*)[2m'(X'-1)] - \frac{\epsilon' \omega'^2}{\sqrt{Re}} \phi' \frac{d\delta'}{dX'} \tag{95}$$

Equation (95) is then substituted into Eq. (90) and the formulation becomes:

Continuity:

$$\frac{\partial U'}{\partial X'} + \frac{\partial V^*}{\partial Y^*} = 0 \tag{96}$$

Momentum:

$$\begin{aligned} \frac{\partial U'}{\partial T'} + U' \frac{\partial U'}{\partial X'} + V^* \frac{\partial U'}{\partial Y^*} + \frac{\epsilon' \omega'^2}{\sqrt{Re}} [2m'(X'-1)\phi] (\delta' - 2Y^*) \\ = \frac{\partial^2 U'}{\partial Y^{*2}} + \frac{\epsilon' \omega'^2}{\sqrt{Re}} \phi' \frac{d\delta'}{dX'} \end{aligned} \tag{97}$$

Energy:

$$\frac{\partial \theta'}{\partial T'} + U' \frac{\partial \theta'}{\partial X'} + V^* \frac{\partial \theta'}{\partial Y^*} = \frac{1}{Pr} \frac{\partial^2 \theta'}{\partial Y^{*2}} \tag{98}$$

with the boundary conditions

$$Y^* = 0 : \quad U' = V^* = 0 \quad , \quad \theta' = 1$$

$$Y^* = \delta' : \quad U' = 1 \quad , \quad \theta' = 0$$

The form of the formulation shows that this problem is equivalent to the unsteady flow of an incompressible fluid over a flat plate with a periodic body force, since all terms relating to the curvature effects of the coordinate system (X',Y*) are shown to be negligible.

PERTURBATION OF THE DIFFERENTIAL EQUATIONS

The basic equations are:

$$\frac{\partial U'}{\partial X'} + \frac{\partial V^*}{\partial Y^*} = 0 \tag{99}$$

$$\frac{\partial U'}{\partial T'} + U'\frac{\partial U'}{\partial X'} + V^*\frac{\partial U'}{\partial Y^*} + \frac{\epsilon'}{\sqrt{Re}}[2m'(X'-1)\phi']\omega'^2(\delta'-2Y^*)$$

$$= \frac{\partial^2 U'}{\partial Y^{*2}} + \frac{\epsilon'}{\sqrt{Re}}\omega'^2\phi'\frac{d\delta'}{dX'} \tag{100}$$

$$\frac{\partial \theta'}{\partial T'} + U'\frac{\partial \theta'}{\partial X'} + V^*\frac{\partial \theta'}{\partial Y^*} = \frac{1}{Pr}\frac{\partial^2\theta'}{\partial Y^{*2}} \tag{101}$$

with the boundary conditions:

$$Y^* = 0 \;:\quad U' = V^* = 0 \;,\quad \theta' = 1$$
$$Y^* = \delta' \;:\quad U' = 1 \;,\quad \theta' = 0$$

Let us assume that

$$U'(X',Y^*,T') = U_0'(X',Y^*) + \frac{\epsilon'}{\sqrt{Re}}U_1'(X',Y^*,T') + \frac{\epsilon'^2}{Re}U_2'(X',Y^*,T') + \cdots\cdots$$

$$V^*(X',Y^*,T') = V_0^*(X',Y^*) + \frac{\epsilon'}{\sqrt{Re}}V_1^*(X',Y^*,T') + \frac{\epsilon'^2}{Re}V_2^*(X',Y^*,T') + \cdots\cdots \tag{102}$$

$$\theta'(X',Y^*,T') = \theta_0'(X',Y^*) + \frac{\epsilon'}{\sqrt{Re}}\theta_1'(X',Y^*,T') + \frac{\epsilon'^2}{Re}\theta_2'(X',Y^*,T') + \cdots\cdots$$

By substituting Eq. (102) into Eqs. (99) through (101) and collecting various powers of $\frac{\epsilon'}{\sqrt{Re}}$, we get:

$(\frac{\epsilon'}{\sqrt{Re}})^0$:

$$\frac{\partial U_0'}{\partial X'} + \frac{\partial V_0^*}{\partial Y^*} = 0 \tag{103}$$

$$U_0'\frac{\partial U_0'}{\partial X'} + V_0^*\frac{\partial U_0'}{\partial Y^*} = \frac{\partial^2 U_0'}{\partial Y^{*2}} \tag{104}$$

$$U_0'\frac{\partial \theta_0'}{\partial X'} + V_0^*\frac{\partial \theta_0'}{\partial Y^*} = \frac{1}{Pr}\frac{\partial^2 \theta_0'}{\partial Y^{*2}} \tag{105}$$

$$Y^* = 0 : \quad U_0' = V_0^* = 0 \quad , \quad \theta_0' = 1$$
$$Y^* = \delta' : \quad U_0' = 1 \quad , \quad \theta_0' = 0$$

$(\frac{\epsilon'}{\sqrt{Re}})^1$:

$$\frac{\partial U_1'}{\partial X'} + \frac{\partial V_1^*}{\partial Y^*} = 0 \tag{106}$$

$$\frac{\partial U_1'}{\partial T'} + U_0'\frac{\partial U_1'}{\partial X'} + U_1'\frac{\partial U_0'}{\partial X'} + V_0^*\frac{\partial U_1'}{\partial Y^*} + V_1^*\frac{\partial U_0'}{\partial Y^*} + [2m'(X'-1)\phi']\omega'^2(\delta'-2Y^*)$$
$$= \frac{\partial^2 U_1'}{\partial Y^{*2}} + \omega'^2\phi'\frac{d\delta'}{dX'} \tag{107}$$

$$\frac{\partial \theta_1'}{\partial T'} + U_0'\frac{\partial \theta_1'}{\partial X'} + U_1'\frac{\partial \theta_0'}{\partial X'} + V_0^*\frac{\partial \theta_1'}{\partial Y^*} + V_1^*\frac{\partial \theta_0'}{\partial Y^*} = \frac{1}{Pr}\frac{\partial^2 \theta_1'}{\partial Y^{*2}} \tag{108}$$

$$Y^* = 0 : \quad U_1' = V_1^* = 0 \quad , \quad \theta_1' = 0$$
$$Y^* = \delta' : \quad U_1' = 0 \quad , \quad \theta_1' = 0$$

$(\frac{\epsilon'}{\sqrt{Re}})^2$:

$$\frac{\partial U_2'}{\partial X'} + \frac{\partial V_2^*}{\partial Y^*} = 0 \tag{109}$$

$$\frac{\partial U_2'}{\partial T'} + U_0'\frac{\partial U_2'}{\partial X'} + V_0^*\frac{\partial U_2'}{\partial Y^*} + U_2'\frac{\partial U_0'}{\partial X'} + V_2^*\frac{\partial U_0'}{\partial Y^*} - \frac{\partial^2 U_2'}{\partial Y^{*2}}$$
$$= -(U_1'\frac{\partial U_1'}{\partial X'} + V_1^*\frac{\partial U_1'}{\partial Y^*}) \tag{110}$$

$$\frac{\partial \theta_2'}{\partial T'} + U_0'\frac{\partial \theta_2'}{\partial X'} + U_2'\frac{\partial \theta_0'}{\partial X'} + V_0^*\frac{\partial \theta_2'}{\partial Y^*} + V_2^*\frac{\partial \theta_0'}{\partial Y^*} - \frac{1}{Pr}\frac{\partial^2 \theta_2'}{\partial Y^{*2}}$$

$$= -\left(U_1'\frac{\partial \theta_1'}{\partial X'} + V_1^*\frac{\partial \theta_1'}{\partial Y^*}\right) \tag{111}$$

$$Y^* = 0 : \quad U_2' = V_2^* = 0 \ , \quad \theta_2' = 0$$

$$Y^* = \delta' : \quad U_2' = 0 \ , \quad \theta_2' = 0$$

Since we are looking for the steady periodic solutions of the problem, we let:

$$U_1'(X', Y^*, T') = \mathcal{R}\left\{ U_{1c}'(X', Y^*)e^{i\omega' T'} \right\}$$

$$V_1^*(X', Y^*, T') = \mathcal{R}\left\{ V_{1c}'(X', Y^*)e^{i\omega' T'} \right\} \tag{112}$$

$$\theta_1'(X', Y^*, T') = \mathcal{R}\left\{ \theta_{1c}'(X', Y^*)e^{i\omega' T'} \right\}$$

where $\mathcal{R}$ denotes the real part of the complex quantity in the brackets. U_1', V_1^*, and Θ_1' given in Eq. (112) are then substituted into Eqs. (106) through (108). Since, in the resulting equations, the symbol $\mathcal{R}$ appears in front of each term, we can therefore drop it and get:

$$\frac{\partial U_{1c}'}{\partial X'} + \frac{\partial V_{1c}^*}{\partial Y^*} = 0 \tag{113}$$

$$i\omega' U_{1c}' + U_0'\frac{\partial U_{1c}'}{\partial X'} + U_{1c}'\frac{\partial U_0'}{\partial X'} + V_0^*\frac{\partial U_{1c}'}{\partial Y^*} + V_{1c}^*\frac{\partial U_0'}{\partial Y^*} + [2m'(X'-1)\phi']\omega'^2(\delta' - 2Y^*)$$

$$= \frac{\partial^2 U_{1c}'}{\partial Y^{*2}} + \omega'^2\frac{d\delta'}{dX'}\phi' \tag{114}$$

$$i\omega'\theta_{1c}' + U_0'\frac{\partial \theta_{1c}'}{\partial X'} + U_{1c}'\frac{\partial \theta_0'}{\partial X'} + V_0^*\frac{\partial \theta_{1c}'}{\partial Y^*} + V_{1c}^*\frac{\partial \theta_0'}{\partial Y^*} = \frac{1}{Pr}\frac{\partial^2 \theta_{1c}'}{\partial Y^{*2}} \tag{115}$$

$$Y^* = 0 : \quad U'_{1c} = V^*_{1c} = 0 \quad , \quad \theta'_{1c} = 0$$

$$Y^* = \delta' : \quad U'_{1c} = 0 \quad , \quad \theta'_{1c} = 0$$

Equation (112) is then substituted into the right-hand side of Eqs. (109), (110), and (111), and we get:

$$\frac{\partial U'_2}{\partial X'} + \frac{\partial V^*_2}{\partial Y^*} = 0 \tag{116}$$

$$\frac{\partial U'_2}{\partial T'} + U'_0 \frac{\partial U'_2}{\partial X'} + U'_2 \frac{\partial U'_0}{\partial X'} + V^*_0 \frac{\partial U'_2}{\partial Y^*} + V^*_2 \frac{\partial U'_0}{\partial Y^*} - \frac{\partial^2 U'_2}{\partial Y^{*2}}$$

$$= R\left\{U'_{1c} e^{i\omega' T'}\right\} \cdot R\left\{\frac{\partial U'_{1c}}{\partial X'} e^{i\omega' T'}\right\}$$

$$+ R\left\{V^*_{1c} e^{i\omega' T'}\right\} \cdot R\left\{\frac{\partial U'_{1c}}{\partial Y^*} e^{i\omega' T'}\right\} \tag{117}$$

$$\frac{\partial \theta'_2}{\partial T'} + U'_0 \frac{\partial \theta'_2}{\partial X'} + U'_2 \frac{\partial \theta'_0}{\partial X} + V^*_0 \frac{\partial \theta'_2}{\partial Y^*} + V^*_2 \frac{\partial \theta'_0}{\partial Y^*} - \frac{1}{Pr} \frac{\partial^2 \theta'_2}{\partial Y^{*2}}$$

$$= R\left\{U'_{1c} e^{i\omega' T'}\right\} \cdot R\left\{\frac{\partial \theta'_{1c}}{\partial X'} e^{i\omega' T'}\right\}$$

$$+ R\left\{V^*_{1c} e^{i\omega' T'}\right\} \cdot R\left\{\frac{\partial \theta'_{1c}}{\partial Y^*} e^{i\omega' T'}\right\} \tag{118}$$

with the boundary conditions:

$$Y^* = 0 : \quad U'_2 = V^*_2 = 0 \quad , \quad \theta'_2 = 0$$

$$Y^* = \delta' : \quad U'_2 = 0 \quad , \quad \theta'_2 = 0$$

The right-sides of Eqs. (117) and (118) need further simplification. Now, let us write

$$R\left\{U'_{1c} e^{i\omega' T'}\right\} = U'_{1R} \cos \omega' T' - U'_{1I} \sin \omega' T'$$

$$R\left\{V^*_{1c} e^{i\omega' T'}\right\} = V^*_{1R} \cos \omega' T' - V^*_{1I} \sin \omega' T'$$

where U'_{1R} and U'_{1I} are the real and imaginary parts of U'_{1c}. Similarly, V^*_{1R} and V^*_{1I} are the real and imaginary parts of V^*_{1c}. Then, we have

$$\mathcal{R}\left\{U'_{1c}\, e^{i\omega' T'}\right\} \cdot \mathcal{R}\left\{\frac{\partial U'_{1c}}{\partial X'}\, e^{i\omega' T'}\right\}$$

$$= U'_{1R}\,\frac{\partial U'_{1R}}{\partial X'}\cos^2\omega' T' + U'_{1I}\,\frac{\partial U'_{1I}}{\partial X'}\sin^2\omega' T'$$

$$-\frac{1}{2}\left(U'_{1R}\,\frac{\partial U'_{1I}}{\partial X'} + U'_{1I}\,\frac{\partial U'_{1R}}{\partial X'}\right)\sin 2\omega' T' \tag{119}$$

Using the trigonometric relation that

$$\sin^2\omega' T' = \frac{1}{2}\left(1 - \cos 2\omega' T'\right)$$

$$\cos^2\omega' T' = \frac{1}{2}\left(1 + \cos 2\omega' T'\right)$$

Eq. (119) becomes:

$$\mathcal{R}\left\{U'_{1c}\, e^{i\omega' T'}\right\} \cdot \mathcal{R}\left\{\frac{\partial U'_{1c}}{\partial X'}\, e^{i\omega' T'}\right\}$$

$$= \frac{1}{2}\left(U'_{1R}\,\frac{\partial U'_{1R}}{\partial X'} + U'_{1I}\,\frac{\partial U'_{1I}}{\partial X'}\right)$$

$$+\frac{1}{2}\left[\left(U'_{1R}\,\frac{\partial U'_{1R}}{\partial X'} - U'_{1I}\,\frac{\partial U'_{1I}}{\partial X'}\right)\cos 2\omega' T'\right.$$

$$\left.-\left(U'_{1R}\,\frac{\partial U'_{1I}}{\partial X'} + U'_{1I}\,\frac{\partial U'_{1R}}{\partial X'}\right)\sin 2\omega' T'\right] \tag{120}$$

Next, the following identities are used:

$$\overline{U'_{1c}}\,\frac{\partial U'_{1c}}{\partial X'} + U'_{1c}\,\frac{\partial \overline{U'_{1c}}}{\partial X'} = 2\left(U'_{1R}\,\frac{\partial U'_{1R}}{\partial X'} + U'_{1I}\,\frac{\partial U'_{1I}}{\partial X'}\right)$$

and

$$\mathcal{R}\left\{ U'_{1c} \frac{\partial U'_{1c}}{\partial X'} e^{2i\omega' T'} \right\} = \left(U'_{1R} \frac{\partial U'_{1R}}{\partial X'} - U'_{1I} \frac{\partial U'_{1I}}{\partial X'} \right) \cos 2\omega' T'$$
$$- \left(U'_{1R} \frac{\partial U'_{1I}}{\partial X'} + U'_{1I} \frac{\partial U'_{1R}}{\partial X'} \right) \sin 2\omega' T'$$

Equation (120) then reduces to:

$$\mathcal{R}\left\{ U'_{1c} e^{i\omega' T'} \right\} \cdot \mathcal{R}\left\{ \frac{\partial U'_{1c}}{\partial X'} e^{i\omega' T'} \right\}$$

$$= \frac{1}{4} \left(\overline{U'_{1c}} \frac{\partial U'_{1c}}{\partial X'} + U'_{1c} \frac{\partial \overline{U'_{1c}}}{\partial X'} \right) + \frac{1}{2} \mathcal{R}\left\{ U'_{1c} \frac{\partial U'_{1c}}{\partial X'} e^{2i\omega' T'} \right\} \qquad (121)$$

where the superscript bar ($^-$) denotes the complex conjugate. In a similar manner, the second term on the left of Eq. (117) can be reduced to:

$$\mathcal{R}\left\{ V^*_{1c} e^{i\omega' T'} \right\} \cdot \mathcal{R}\left\{ \frac{\partial U'_{1c}}{\partial Y^*} e^{i\omega' T'} \right\}$$

$$= \frac{1}{4} \left(\overline{V^*_{1c}} \frac{\partial U'_{1c}}{\partial Y^*} + V^*_{1c} \frac{\partial \overline{U'_{1c}}}{\partial Y^*} \right) + \frac{1}{2} \mathcal{R}\left\{ V^*_{1c} \frac{\partial U'_{1c}}{\partial Y^*} e^{2i\omega' T'} \right\} \qquad (122)$$

The right-hand terms in the energy equation, Eq. (118), can be reduced in a similar manner. Therefore, we can write:

$$\mathcal{R}\left\{ U'_{1c} e^{i\omega' T'} \right\} \cdot \mathcal{R}\left\{ \frac{\partial \theta'_{1c}}{\partial X'} e^{i\omega' T'} \right\}$$

$$= \frac{1}{4} \left(\overline{U'_{1c}} \frac{\partial \theta'_{1c}}{\partial X'} + U'_{1c} \frac{\partial \overline{\theta'_{1c}}}{\partial X'} \right) + \frac{1}{2} \mathcal{R}\left\{ U'_{1c} \frac{\partial \theta'_{1c}}{\partial X'} e^{2i\omega' T'} \right\} \qquad (123)$$

and

$$\mathcal{R}\left\{V_{1c}^{*} e^{i\omega' \tau'}\right\} \cdot \mathcal{R}\left\{\frac{\partial \theta_{1c}'}{\partial Y^*} e^{i\omega' \tau'}\right\}$$

$$= \frac{1}{4}\left(\overline{V_{1c}^{*}} \frac{\partial \theta_{1c}'}{\partial Y^*} + V_{1c}^{*} \overline{\frac{\partial \theta_{1c}'}{\partial Y^*}}\right) + \frac{1}{2}\mathcal{R}\left\{V_{1c}^{*} \frac{\partial \theta_{1c}'}{\partial Y^*} e^{2i\omega' \tau'}\right\} \qquad (124)$$

Substituting Eqs. (121) through (124) into Eqs. (117) and (118) results in:

$$\frac{\partial U_2'}{\partial \tau'} + U_0' \frac{\partial U_2'}{\partial X'} + U_2' \frac{\partial U_0'}{\partial X'} + V_0^{*} \frac{\partial U_2'}{\partial Y^*} + V_2^{*} \frac{\partial U_0'}{\partial Y^*} - \frac{\partial^2 U_2'}{\partial Y^{*2}}$$

$$= U_{10}' + \mathcal{R}\left\{U_{11}' e^{2i\omega' \tau'}\right\} \qquad (125)$$

and

$$\frac{\partial \theta_2'}{\partial \tau'} + U_0' \frac{\partial \theta_2'}{\partial X'} + U_2' \frac{\partial \theta_0'}{\partial X'} + V_0^{*} \frac{\partial \theta_2'}{\partial Y^*} + V_2^{*} \frac{\partial \theta_0'}{\partial Y^*} - \frac{1}{Pr} \frac{\partial^2 \theta_2'}{\partial Y^{*2}}$$

$$= \theta_{10}' + \mathcal{R}\left\{\theta_{11}' e^{2i\omega' \tau'}\right\} \qquad (126)$$

where

$$U_{10}' = -\frac{1}{4}\left(\overline{U_{1c}'} \frac{\partial U_{1c}'}{\partial X'} + U_{1c}' \frac{\partial \overline{U_{1c}'}}{\partial X'} + \overline{V_{1c}^{*}} \frac{\partial U_{1c}'}{\partial Y^*} + V_{1c}^{*} \frac{\partial \overline{U_{1c}'}}{\partial Y^*}\right) \qquad (127)$$

$$U_{11}' = -\frac{1}{2}\left(U_{1c}' \frac{\partial U_{1c}'}{\partial X'} + V_{1c}^{*} \frac{\partial U_{1c}'}{\partial Y^{*2}}\right) \qquad (128)$$

$$\theta_{10}' = -\frac{1}{4}\left(\overline{U_{1c}'} \frac{\partial \theta_{1c}'}{\partial X'} + U_{1c}' \frac{\partial \overline{\theta_{1c}'}}{\partial X'} + \overline{V_{1c}^{*}} \frac{\partial \theta_{1c}'}{\partial Y^*} + V_{1c}^{*} \frac{\partial \overline{\theta_{1c}'}}{\partial Y^*}\right) \qquad (129)$$

$$\theta_{11}' = -\frac{1}{2}\left(U_{1c}'\frac{\partial \theta_{1c}'}{\partial X'} + V_{1c}^*\frac{\partial \theta_{1c}'}{\partial Y^*}\right) \tag{130}$$

Inspection of Eqs. (125) and (126) reveals the steady periodic solutions of U_2', V_2^*, and Θ_2', may be written in the form

$$U_2' = U_{20}'(X', Y^*) + R\left\{U_{21}'(X', Y^*)e^{2i\omega'\tau'}\right\} \tag{131}$$

$$V_2^* = V_{20}^*(X', Y^*) + R\left\{V_{21}^*(X', Y^*)e^{2i\omega'\tau'}\right\} \tag{132}$$

$$\theta_2' = \theta_{20}'(X', Y^*) + R\left\{\theta_{21}'(X', Y^*)e^{2i\omega'\tau'}\right\} \tag{133}$$

in order to comply with the right-hand sides of Eqs. (125) and (126). In this way, the second approximations are separated into two parts, namely, the nonoscillatory and the oscillatory components. Substituting Eqs. (131) through (133) into Eqs. (116), (125), and (126), the following sets of equations are obtained.

Nonoscillatory components:

$$\frac{\partial U_{20}'}{\partial X'} + \frac{\partial V_{20}^*}{\partial Y^*} = 0 \tag{134}$$

$$U_0'\frac{\partial U_{20}'}{\partial X'} + U_{20}'\frac{\partial U_0'}{\partial X'} + V_0^*\frac{\partial U_{20}'}{\partial Y^*} + V_{20}^*\frac{\partial U_0'}{\partial Y^*} - \frac{\partial^2 U_{20}'}{\partial Y^{*2}}$$
$$= -\frac{1}{4}\left(U_{1c}'\frac{\partial \overline{U_{1c}'}}{\partial X'} + \overline{U_{1c}'}\frac{\partial U_{1c}'}{\partial X'} + V_{1c}^*\frac{\partial \overline{U_{1c}'}}{\partial Y^*} + \overline{V_{1c}^*}\frac{\partial U_{1c}'}{\partial Y^*}\right) \tag{135}$$

$$U_0'\frac{\partial \theta_{20}'}{\partial X'} + \theta_{20}'\frac{\partial U_0'}{\partial X'} + V_0^*\frac{\partial \theta_{20}'}{\partial Y^*} + \theta_{20}'\frac{\partial U_0'}{\partial Y^*} - \frac{1}{Pr}\frac{\partial^2 \theta_{20}'}{\partial Y^{*2}}$$
$$= -\frac{1}{4}\left(U_{1c}'\frac{\partial \overline{\theta_{1c}'}}{\partial X'} + \overline{U_{1c}'}\frac{\partial \theta_{1c}'}{\partial X'} + V_{1c}^*\frac{\partial \overline{\theta_{1c}'}}{\partial Y^*} + \overline{V_{1c}^*}\frac{\partial \theta_{1c}'}{\partial Y^*}\right) \tag{136}$$

with the boundary conditions

$$Y^* = 0 : \quad U'_{20} = V^*_{20} = 0 \ , \quad \theta'_{20} = 0$$

$$Y^* = \delta' : \quad U'_{20} = 0 \ , \quad \theta'_{20} = 0$$

Oscillatory components:

$$\frac{\partial U'_{21}}{\partial X'} + \frac{\partial V^*_{21}}{\partial Y^*} = 0 \tag{137}$$

$$2i\omega' U'_{21} + U'_0 \frac{\partial U'_{21}}{\partial X'} + U'_{21} \frac{\partial U'_0}{\partial X'} + V^*_0 \frac{\partial U'_{21}}{\partial Y^*} + V^*_{21} \frac{\partial U'_0}{\partial Y^*} - \frac{\partial^2 U'_{21}}{\partial Y^{*2}}$$

$$= -\frac{1}{2}\left(U'_{1c} \frac{\partial U'_{1c}}{\partial X'} + V^*_{1c} \frac{\partial U'_{1c}}{\partial Y^*}\right) \tag{138}$$

$$2i\omega' \theta'_{21} + U'_0 \frac{\partial \theta'_{21}}{\partial X'} + U'_{21} \frac{\partial \theta'_0}{\partial X'} + V^*_0 \frac{\partial \theta'_{20}}{\partial Y^*} + V^*_{21} \frac{\partial \theta'_0}{\partial Y^*} - \frac{1}{Pr} \frac{\partial^2 \theta'_{21}}{\partial Y^{*2}}$$

$$= -\frac{1}{2}\left(U'_{1c} \frac{\partial \theta'_{1c}}{\partial X'} + V^*_{1c} \frac{\partial \theta'_{1c}}{\partial Y^*}\right) \tag{139}$$

with boundary conditions

$$Y^* = 0 : \quad U'_{21} = V^*_{21} = 0 \ , \quad \theta'_{21} = 0$$

$$Y^* = \delta' : \quad U'_{21} = 0 \ , \quad \theta'_{21} = 0$$

SOLUTIONS OF THE DIFFERENTIAL EQUATIONS

Zeroeth-Order Approximation

The momentum equation (104) is integrated over the velocity boundary layer thickness δ':

$$\int_0^{\delta'} U'_0 \frac{\partial U'_0}{\partial X'} dY^* + \int_0^{\delta'} V^*_0 \frac{\partial U'_0}{\partial Y^*} dY^* = -\left(\frac{\partial U'_0}{\partial Y^*}\right)_{Y^*=0} \tag{140}$$

From the continuity equation (103), we have:

$$V_0^* = -\int_0^{Y^*} \frac{\partial U_0'}{\partial X'} dY^* \tag{141}$$

Equation (141) is substituted into Eq. (140) and it reduces to:

$$\frac{\partial}{\partial X'}\left\{U_\infty'^2 \int_0^{\delta'} \left(\frac{U_0'}{U_\infty'}\right)\left(1 - \frac{U_0'}{U_\infty'}\right) dY^*\right\} = \left(\frac{\partial U_0'}{\partial Y^*}\right)_{Y^*=0} \tag{142}$$

Similarly, the energy equation (105) is integrated over the thermal boundary layer thickness δ_t':

$$\int_0^{\delta_t'} U_0' \frac{\partial \theta_0'}{\partial X'} dY^* + \int_0^{\delta_t'} V_0^* \frac{\partial \theta_0'}{\partial Y^*} dY^* = -\frac{1}{Pr}\left(\frac{\partial \theta_0'}{\partial Y^*}\right)_{Y^*=0} \tag{143}$$

Again, using Eq. (141) and rearranging the terms, we get:

$$\frac{\partial}{\partial X'}\int_0^{\delta_t'} \theta_0' U_0' dY^* = -\frac{1}{Pr}\left(\frac{\partial \theta'}{\partial Y^*}\right)_{Y^*=0} \tag{144}$$

The solution of the momentum integral equation (142) is given first by Pohlhausen[21] and later by Holstein and Bohlen.[8] The velocity profile is found to be:

$$\frac{U_0'}{U_\infty'} = 2\eta - 2\eta^3 + \eta^4 \tag{145}$$

where $\eta = \frac{Y^*}{\delta'}$ and δ' is the velocity boundary layer thickness.

The solution of the thermal boundary is given by Squire.[27] The temperature profile is:

$$\theta_0' = 1 - 2\eta_T + 2\eta_T^3 - \eta_T^4 \tag{146}$$

where $\eta_T = \frac{Y^*}{\delta'_T}$ and δ'_T is the thermal boundary layer thickness. Now, let us define:

$$\Delta = \frac{\delta'_t}{\delta'} \tag{147}$$

Squire found that for laminar flow over a flat plate at zero incidence,

$$\Delta = P_r^{-\frac{1}{3}} \tag{148}$$

Therefore, the temperature profile can be written as:

$$\theta'_o = 1 - 2P_r^{\frac{1}{3}}\eta + 2P_r\eta^3 - P_r^{\frac{4}{3}}\eta^4 \tag{149}$$

These two approximate solutions have been checked with exact solutions and the errors are both less than 5%. Therefore, we take Eqs. (145) and (149) as the solutions of the zeroth-order approximation.

First-Order Approximation

The momentum equation for the first-order approximation is given in Eq. (114) as

$$i\omega' U'_{1c} + U'_o \frac{\partial U'_{1c}}{\partial X'} + U'_{1c}\frac{\partial U'_o}{\partial X'} + V^*_o \frac{\partial U'_{1c}}{\partial Y^*} + V^*_{1c}\frac{\partial U'_o}{\partial Y^*}$$

$$+ [2m'(X'-1)\phi'_s]\omega'^2(\delta' - 2Y^*)$$

$$= \frac{\partial^2 U'_{1c}}{\partial Y^{*2}} + \omega'^2\phi'_s\frac{d\delta'}{dX'}$$

It can be shown (see Appendix II) that the convective terms in Eq. (114) is small compared with the other terms. Therefore, Eq. (114)

becomes

$$i\omega' U_{1c}' + [2m'(X'-1)\phi_s']\,\omega'^2(\delta' - 2Y^*)$$

$$= \frac{\partial^2 U_{1c}'}{\partial Y^{*2}} + \omega'^2 \phi_s' \frac{d\delta'}{dX'} \tag{150}$$

The boundary conditions are

$$Y^* = 0 : \quad U_{1c}' = V_{1c}^* = 0$$

$$Y^* = \delta' : \quad U_{1c}' = 0$$

To solve Eq. (150) by means of the integral method, we assume a velocity profile in the form:

$$U_{1c}' = a_1 + b_1\eta + c_1\eta^2 + d_1\eta^3 + e_1\eta^4 \tag{151}$$

The constants are determined by the boundary conditions:

$$\eta = 0 : \quad U_{1c}' = V_{1c}^* = 0$$

$$\omega'^2[2m'(X'-1)\phi_s']\,\delta' = \left(\frac{\partial^2 U_{1c}'}{\partial Y^{*2}}\right)_{Y^*=0} + \omega'^2\phi_s' \frac{d\delta'}{dX'}$$

$$\eta = 1 : \quad U_{1c}' = 0 \quad , \quad \frac{\partial U_{1c}'}{\partial \eta} = 0$$

The second boundary condition is the so-called compatibility condition which is obtained by satisfying the momentum equation on the surface of the plate. With the boundary conditions, the constants are determined to be

$$a_1 = 0$$

$$c_1 = \psi(X)$$

$$d_1 = -3b_1 - 2\psi(X)$$

$$e_1 = 2b_1 + \psi(X)$$

where

$$\psi(X) = \omega'^2 \phi_s' \delta'^2 [m'(X'-1)\delta' - \frac{1}{2}\frac{d\delta'}{dX'}] \tag{152}$$

The velocity profile is therefore

$$U_{1c}' = b_1\eta + \psi\eta^2 - (3b_1 + 2\psi)\eta^3 + (2b_1 + \psi)\eta^4 \tag{153}$$

The profile for U_{1c}', Eq. (153), is put into the integrated form of Eq. (150),

$$i\omega' \int_0^{\delta'} U_{1c}' \, dY^* = -\left(\frac{\partial U_{1c}'}{\partial Y^*}\right)_{Y^*=0} + \omega'^2 \phi_s' \frac{d\delta'}{dX'} \tag{154}$$

which gives

$$i\omega'(0.15\, b_1 + 0.0333\, \psi) = -\frac{b_1}{\delta'} + \omega'^2 \phi_s' \delta' \frac{d\delta'}{dX'} \tag{155}$$

Solving for b_1, we get

$$b_1 = \frac{\omega'^2 \phi_s' \delta' \frac{d\delta'}{dX'} - 0.0333\, i\omega'\delta'\psi}{\frac{1}{\delta'} + 0.15\, i\omega'\delta'} \tag{156}$$

The boundary layer thickness δ' may be found be substituting Eq. (145) into Eq. (142), which gives

$$\delta' = 5.8493 \sqrt{\frac{X'}{U_\infty'}} \tag{157}$$

For the present case, U_∞ is assumed to be constant. Since, in nondimensionalizing the basic equations, the reference velocity is taken as U_∞, thus

$$U'_\infty = \frac{U_\infty}{U_\infty} = 1 \tag{158}$$

Using Eqs. (157) and (158), Eq. (156) becomes

$$b_1 = (44.453)\,\omega'^2 \sqrt{X'}\,\phi'_s\,[m'(X'-1)X' - 0.250]$$

$$- i\omega' \frac{\phi'_s}{\sqrt{X'}} \left\{(19.499) + (8.663)[m'(X'-1)X' - 0.250]\right\} \tag{159}$$

It has been discussed in simplifying the basic equations that, if the oscillation is to have any effect on the flow, ω' must be very large. Also, it is shown in Appendix III that the order-of-magnitude of ω' is approximately 400. Therefore, the imaginary part in Eq. (159) is small compared with its real part. Therefore, Eq. (159) becomes

$$U'_{1c} = (44.453)\, b'_1(-\eta + 4.502\eta^2 - 6.004\eta^3 + 2.502\eta^4) \tag{160}$$

where

$$b'_1 = \omega'^2 \phi'_s \sqrt{X'}\,[m'(X'-1)X' - 0.25] \tag{161}$$

The phase angle, α_1, is therefore zero, since Eq. (160) contains real part only.

To solve the energy equation, Eq. (115), the same procedure in obtaining solution for Eq. (114) is used. The temperature profile Θ'_{1c} is solved by

$$i\omega'\theta'_{1c} + U'_{1c}\frac{\partial \theta'_0}{\partial X'} + V^*_{1c}\frac{\partial \theta'_0}{\partial Y^*} = \frac{1}{Pr}\frac{\partial^2 \theta'_{1c}}{\partial Y^{*2}} \tag{162}$$

Integrating over thermal boundary layer thickness δ_t'

$$i\omega' \int_0^{\delta_t'} \theta_{1c}' \, dY^* + \int_0^{\delta_t'} U_{1c}' \frac{\partial \theta_0'}{\partial X'} dY^* + \int_0^{\delta_t'} V_{1c}^* \frac{\partial \theta_0'}{\partial Y^*} dY^* = -\frac{1}{P_r} \left(\frac{\partial \theta_{1c}'}{\partial Y^*}\right)_{Y^*=0} \qquad (163)$$

or,

$$i\omega' \int_0^{\delta_t'} \theta_{1c}' \, dY^* + \frac{d}{dX'} \int_0^{\delta_t'} \theta_0' U_{1c}' \, dY^* = -\frac{1}{P_r} \left(\frac{\partial \theta_{1c}'}{\partial Y^*}\right)_{Y^*=0} \qquad (164)$$

The profiles Θ_0' and U_{1c}' are given by Eqs. (146) and (160). Again, a fourth-order polynominal is assumed for Θ_{1c}':

$$\theta_{1c}' = a_3 + b_3 \eta_t + c_3 \eta_t^2 + d_3 \eta_t^3 + e_3 \eta_t^4 \qquad (165)$$

subjected to the boundary conditions that

$$\eta_t = 0: \quad \theta_{1c}' = 0 \;, \quad \frac{\partial^2 \theta_{1c}'}{\partial Y^{*2}} = 0$$

$$\eta_t = 1: \quad \theta_{1c}' = 0 \;, \quad \frac{\partial \theta_{1c}'}{\partial Y^*} = 0$$

The constants in Eq. (165) are determined by the above boundary conditions as

$$a_3 = 0$$

$$c_3 = 0$$

$$d_3 = -3 b_3$$

$$e_3 = 2 b_3$$

The profile is therefore

$$\theta_{1c}' = b_3 (\eta_t - 3\eta_t^3 + 2\eta_t^4) \qquad (166)$$

Equations (146), (160), and (166) are then substituted into Eq.

(164) and we get

$$i\omega'\delta_t'(0.15)b_3 + (260.02\,\omega'^2)F_1(Pr)F_2(X') = -\frac{b_3}{Pr\,\delta_t'} \tag{167}$$

where

$$F_1(Pr) = 0.834\,Pr^{-\frac{1}{3}} + 0.108\,Pr^{-\frac{2}{3}} - 0.064\,Pr^{-1} + 0.014\,Pr^{-\frac{4}{3}} \tag{168}$$

$$F_2(X') = \phi_s'\left\{[1-2m'(X'-1)X'][m'(X'-1)X' - 0.25] + X'm'(2X'-1)\right\} \tag{169}$$

Solving for b_3, we get

$$b_3 = -\frac{(260.02)F_1F_2\,\omega'^2}{\frac{1}{Pr^{\frac{2}{3}}\delta'} + i\omega'\frac{\delta'}{Pr^{\frac{1}{3}}}(0.15)}$$

or, using Eq. (157),

$$b_3 = -\frac{(296.36)\,Pr^{\frac{1}{3}}\omega' F_1F_2}{\sqrt{X'}}\,e^{-i\alpha_2} \tag{170}$$

where

$$\alpha_2 = \tan^{-1}(5.132\,\omega' Pr^{\frac{1}{3}}X') \tag{171}$$

Since ω' must be large in order to have effect on the flow (in the order of 400) it is seen that α_2 is very close to $\frac{\pi}{2}$. This means that the heat transfer lags approximately $\frac{\pi}{2}$ radians behind the input disturbances (cf., Appendix III).

Second-Order Approximation

It has been shown in Eqs. (131) through (133) that the second-order velocities and temperature consist of two terms, one steady component which is a function of X' and Y* only and a second term which oscillates harmonically with a frequency of $2\omega'$. The second term will be zero when integrated with respect to time, and is therefore of less importance. The steady component will contribute to the skin friction and heat transfer and are therefore solved in this paragraph.

The continuity, momentum, and energy equations of the steady components of the second approximation are given in Eqs. (134) through (136) as

$$\frac{\partial U'_{20}}{\partial X'} + \frac{\partial V^*_{20}}{\partial Y^*} = 0 \tag{134}$$

$$U'_0 \frac{\partial U'_{20}}{\partial X'} + U'_{20} \frac{\partial U'_0}{\partial X'} + V^*_0 \frac{\partial U'_{20}}{\partial Y^*} + V^*_{20} \frac{\partial U'_0}{\partial Y^*} - \frac{\partial^2 U'_{20}}{\partial Y^{*2}}$$
$$= -\frac{1}{4}\left(U'_{1c} \frac{\partial \overline{U'_{1c}}}{\partial X'} + \overline{U'_{1c}} \frac{\partial U'_{1c}}{\partial X'} + V^*_{1c} \frac{\partial \overline{U'_{1c}}}{\partial Y^*} + \overline{V^*_{1c}} \frac{\partial U'_{1c}}{\partial Y^*} \right) \tag{135}$$

$$U'_0 \frac{\partial \theta'_{20}}{\partial X'} + \theta'_{20} \frac{\partial U'_0}{\partial X'} + V^*_0 \frac{\partial \theta'_{20}}{\partial Y^*} + \theta'_{20} \frac{\partial U'_0}{\partial Y^*} - \frac{1}{Pr} \frac{\partial^2 \theta'_{20}}{\partial Y^{*2}}$$
$$= -\frac{1}{4}\left(U'_{1c} \frac{\partial \overline{\theta'_{1c}}}{\partial X'} + \overline{U'_{1c}} \frac{\partial \theta'_{1c}}{\partial X'} + V^*_{1c} \frac{\partial \overline{\theta'_{1c}}}{\partial Y^*} + \overline{V^*_{1c}} \frac{\partial \theta'_{1c}}{\partial Y^*} \right) \tag{136}$$

with the boundary conditions

$$Y^* = 0 : \quad U'_{20} = V^*_{20} = 0 \;, \quad \theta'_{20} = 0$$
$$Y^* = \delta' : \quad U'_{20} = 0 \;, \quad \theta'_{20} = 0$$

Since the oscillation is to have effect on the flow only for large ω', it can be shown by following the procedures in obtaining solution for Eq. (114) that the solutions of U'_{20}, V^*_{20}, and θ'_{20} can be obtained by the following simplified forms of Eqs. (134) through (136):

$$\frac{\partial U'_{20}}{\partial X'} + \frac{\partial V^*_{20}}{\partial Y^*} = 0 \tag{172}$$

$$\frac{\partial^2 U'_{20}}{\partial Y^{*2}} = \frac{1}{4}\left(U'_{1c} \frac{\partial \overline{U'_{1c}}}{\partial X'} + \overline{U'_{1c}} \frac{\partial U'_{1c}}{\partial X'} + V^*_{1c} \frac{\partial \overline{U'_{1c}}}{\partial Y^*} + \overline{V^*_{1c}} \frac{\partial U'_{1c}}{\partial Y^*}\right) \tag{173}$$

$$\frac{\partial^2 \theta'_{20}}{\partial Y^{*2}} = \frac{1}{4}\left(U'_{1c} \frac{\partial \overline{\theta'_{1c}}}{\partial X'} + \overline{U'_{1c}} \frac{\partial \theta'_{1c}}{\partial X'} + V^*_{1c} \frac{\partial \overline{\theta'_{1c}}}{\partial Y^*} + \overline{V^*_{1c}} \frac{\partial \theta'_{1c}}{\partial Y^*}\right) \tag{174}$$

with the boundary conditions

$$Y^* = 0 : \quad U'_{20} = V^*_{20} = 0 \ , \quad \theta'_{20} = 0$$

$$Y^* = \delta' : \quad U'_{20} = 0 \ , \quad \theta'_{20} = 0$$

As before, a fourth degree polynominal is assumed for U'_{20}. Thus, we have

$$U'_{20} = a_4 + b_4 \eta + c_4 \eta^2 + d_4 \eta^3 + e_4 \eta^4 \tag{175}$$

and the boundary conditions are

$$\eta = 0 : \quad U'_{20} = 0 \ , \quad \frac{\partial^2 U'_{20}}{\partial \eta^2} = 0$$

$$\eta = 1 : \quad U'_{20} = 0 \ , \quad \frac{\partial U'_{20}}{\partial \eta} = 0$$

The constants in Eq. (175) are then solved to be

$$a_4 = 0$$
$$c_4 = 0$$
$$d_4 = -3 b_4$$
$$e_4 = 2 b_4$$

Therefore, U'_{20} can be written as

$$U'_{20} = b_4 (\eta - 3\eta^3 + 2\eta^4) \tag{176}$$

Next, Eq. (173) is integrated over the boundary layer thickness δ' and, after using the integrated form of the equation of continuity

$$V_{20}^{*} = -\int_{0}^{Y^*} \frac{\partial \overline{U}_{20}'}{\partial X'} dY^* \tag{177}$$

it becomes

$$-\left(\frac{\partial \overline{U}_{20}'}{\partial Y^*}\right)_{Y^*=0} = \frac{1}{2}\frac{d}{dX'}\int_{0}^{\delta'} (\overline{U_{1c}'\,U_{1c}'})\,dY^* \tag{178}$$

Equation (176) is then substituted into Eq. (178), we then get

$$b_4 = -(53.749\,\omega'^4)\,F_3(X') \tag{179}$$

where

$$F_3(X') = X'\Phi_s'^2[m'(X'-1)X'-0.25]\{[1.50-4m'X'(X'-1)][m'(X'-1)X'-0.25]$$

$$+2X'm'(2X'-1)\} \tag{180}$$

$F_3(X')$ is shown in Figures 5(a) through 5(e) for different m's.

To solve the steady component of the second-order temperature profile, Eq. (174) is integrated over the thermal boundary layer thickness,

$$\left(\frac{\partial \theta_{20}'}{\partial Y^*}\right)_{Y^*=0} = -\frac{Pr}{4}\frac{d}{dX'}\int_0^{\delta_t'} (U_{1c}'\,\overline{\theta_{1c}'} + \overline{U_{1c}'}\,\theta_{1c}')\,dY^* \tag{181}$$

Again, a fourth-degree temperature is assumed to be in the form of

$$\theta_{20}' = a_5 + b_5\eta_t + c_5\eta_t^2 + d_5\eta_t^3 + e_5\eta_t^4 \tag{182}$$

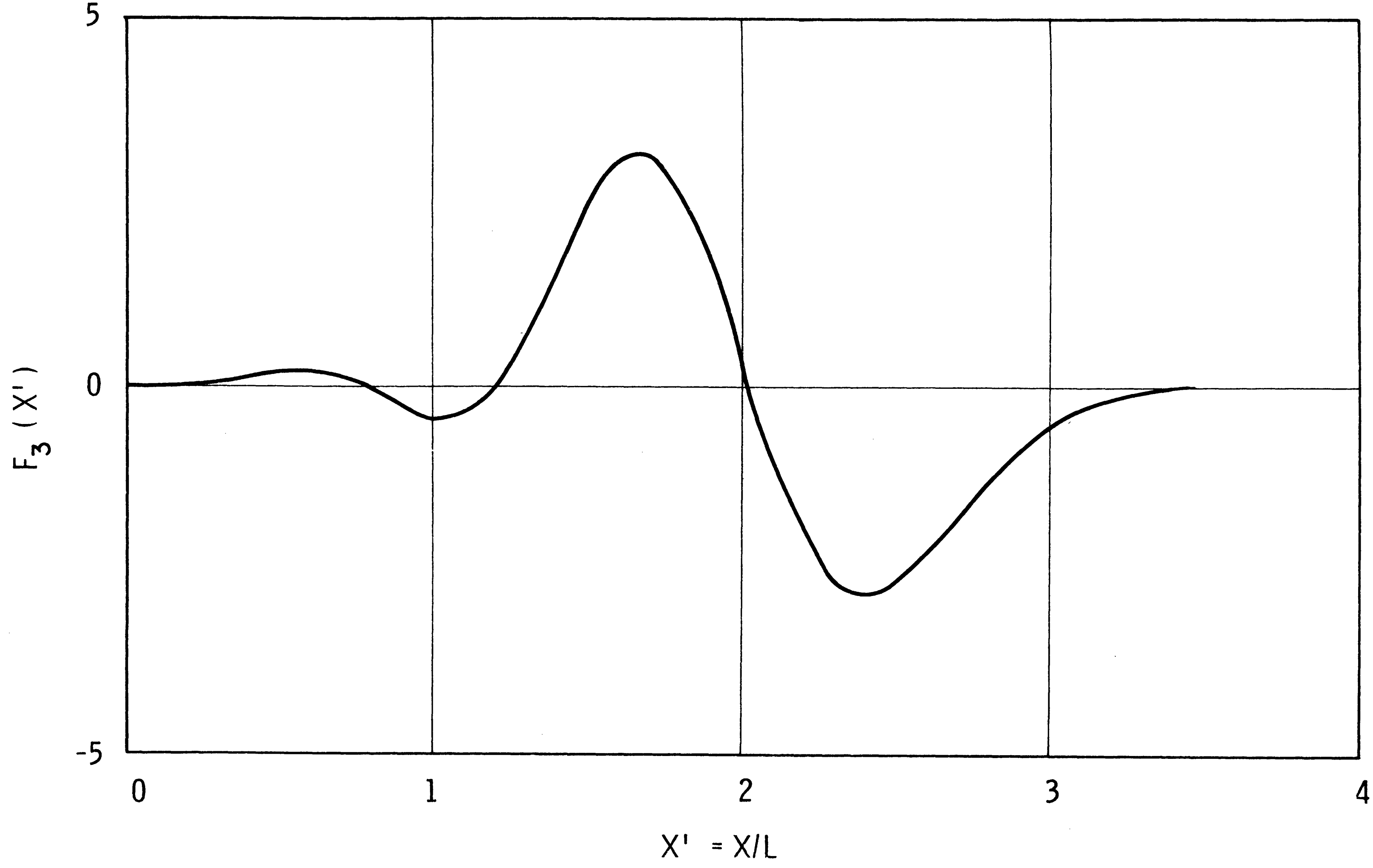

Figure 5(a). $F_3(X')$ as a function of $X'(m'=1)$.

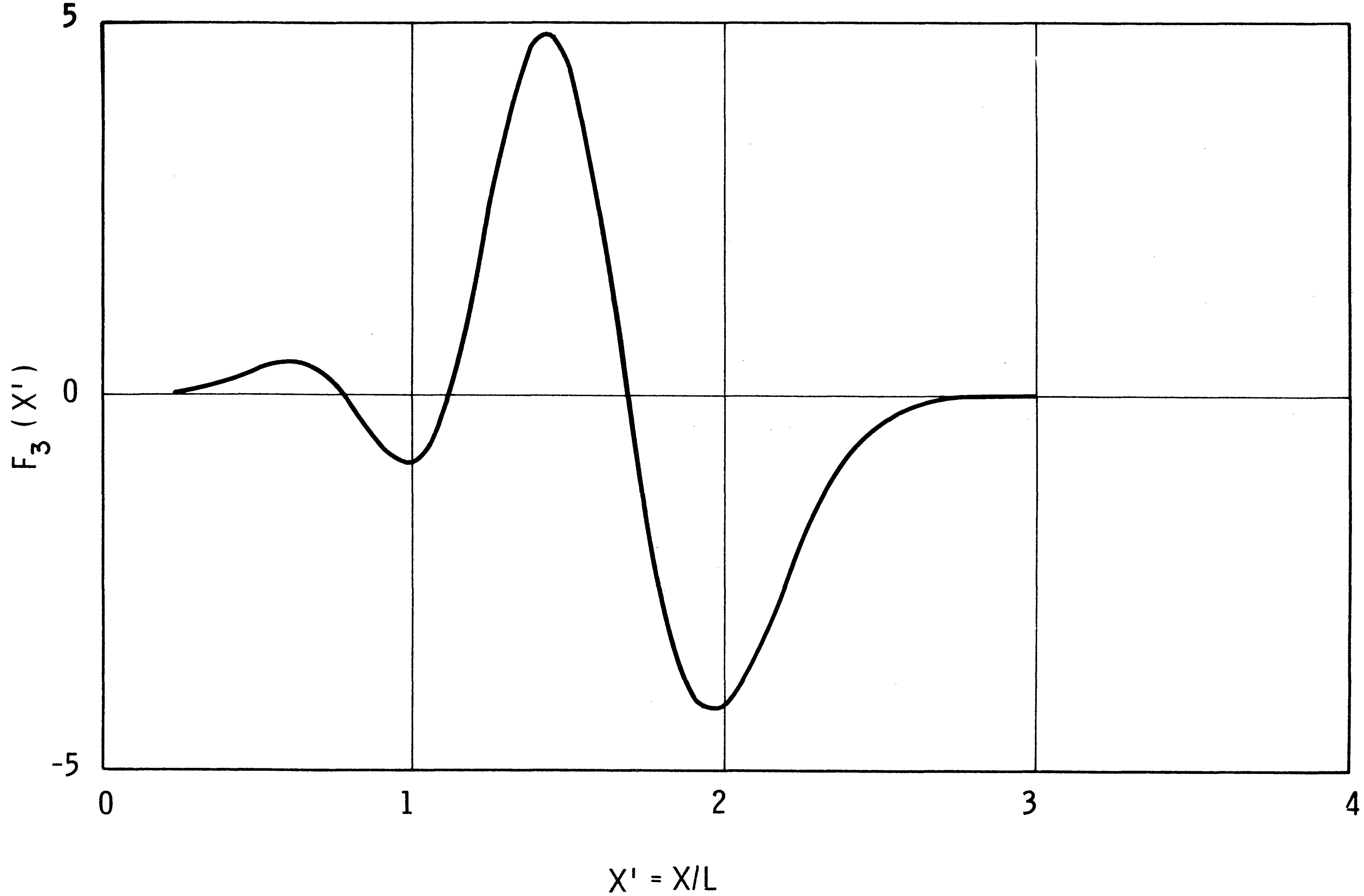

Figure 5(b). $F_3(X')$ as a function of $X'(m'=2)$.

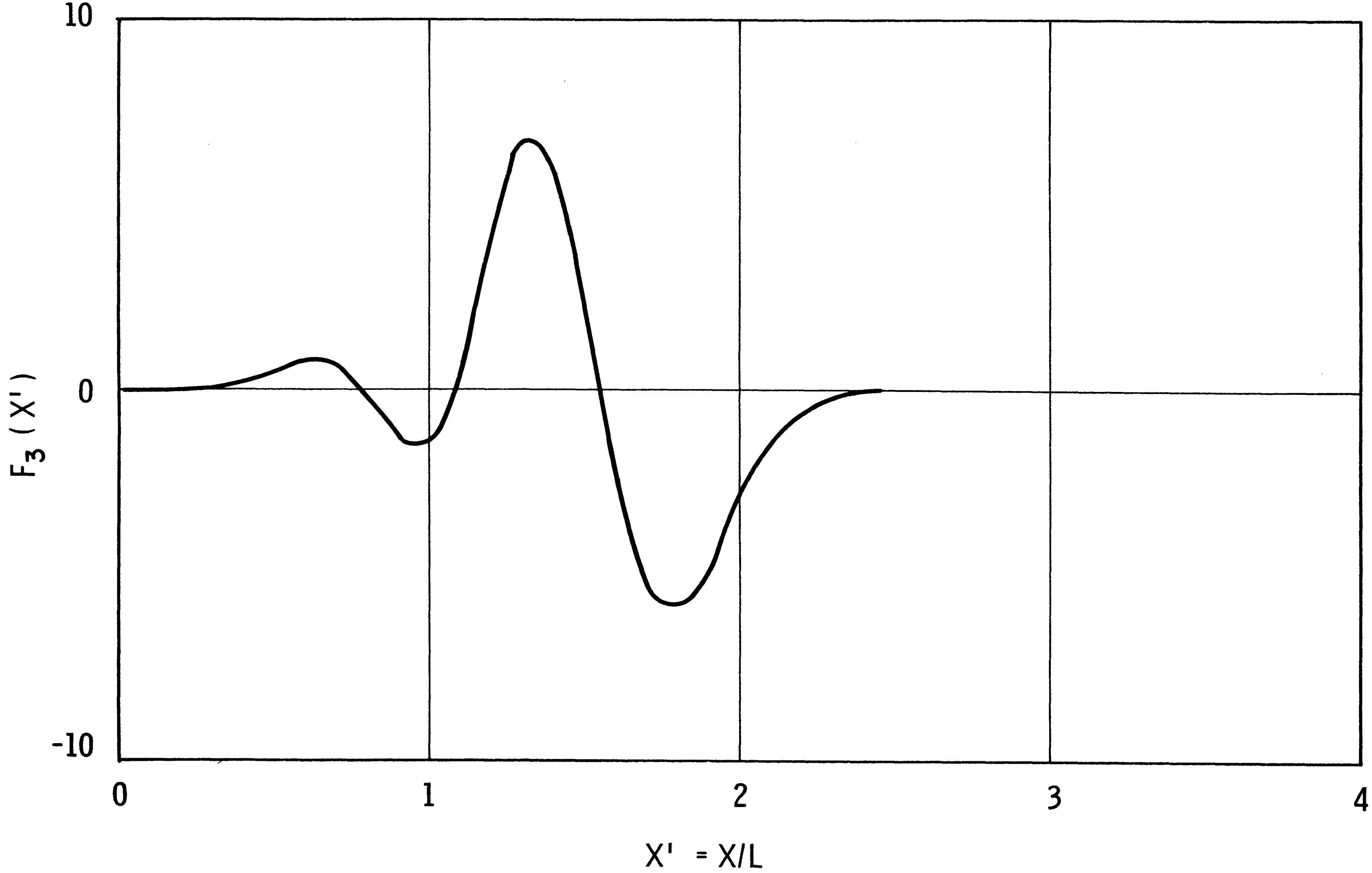

Figure 5(c). $F_3(X')$ as a function of $X'(m'=3)$.

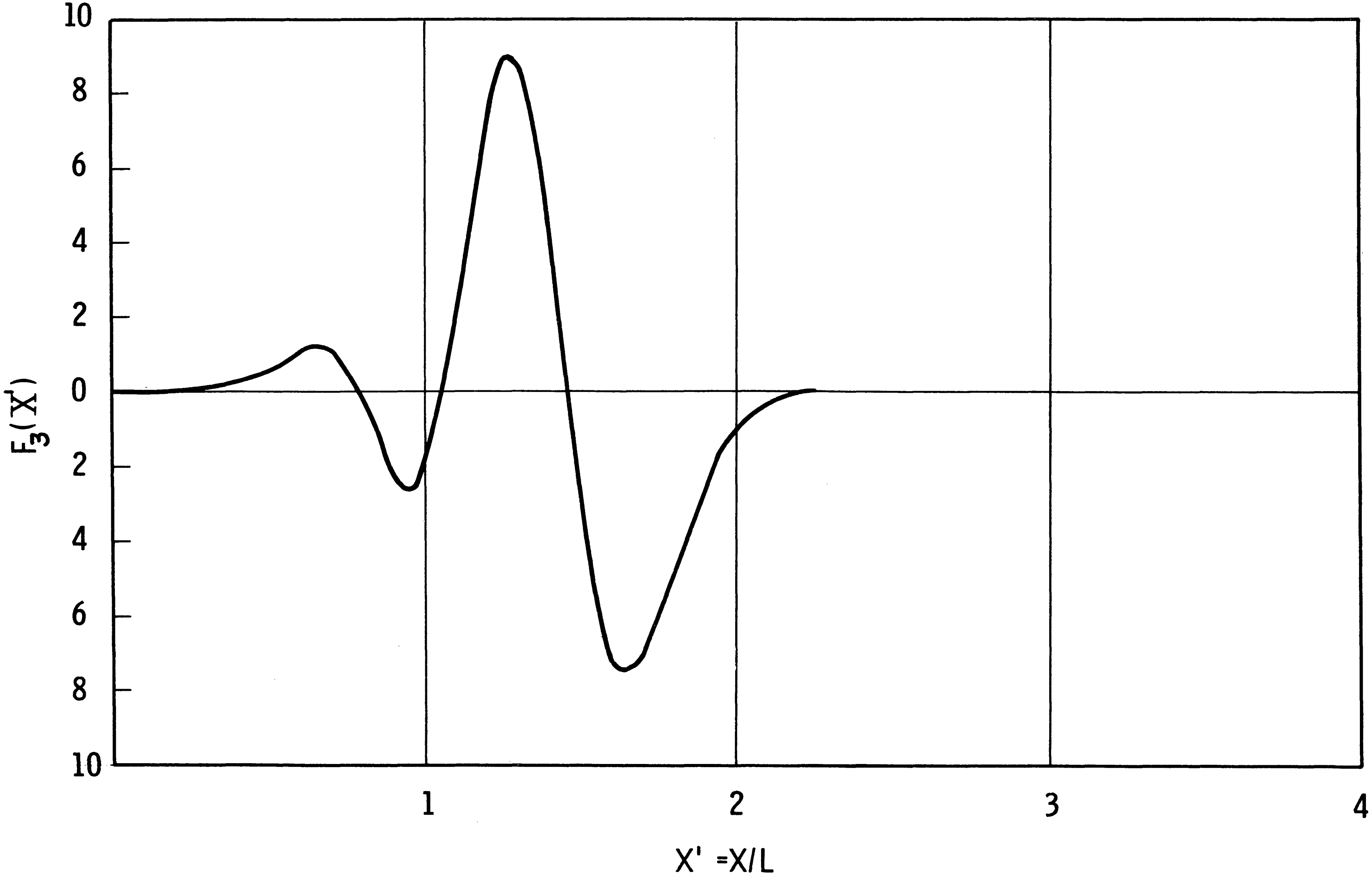

Figure 5(d). $F_3(X')$ as a function of $X'(m'=4)$.

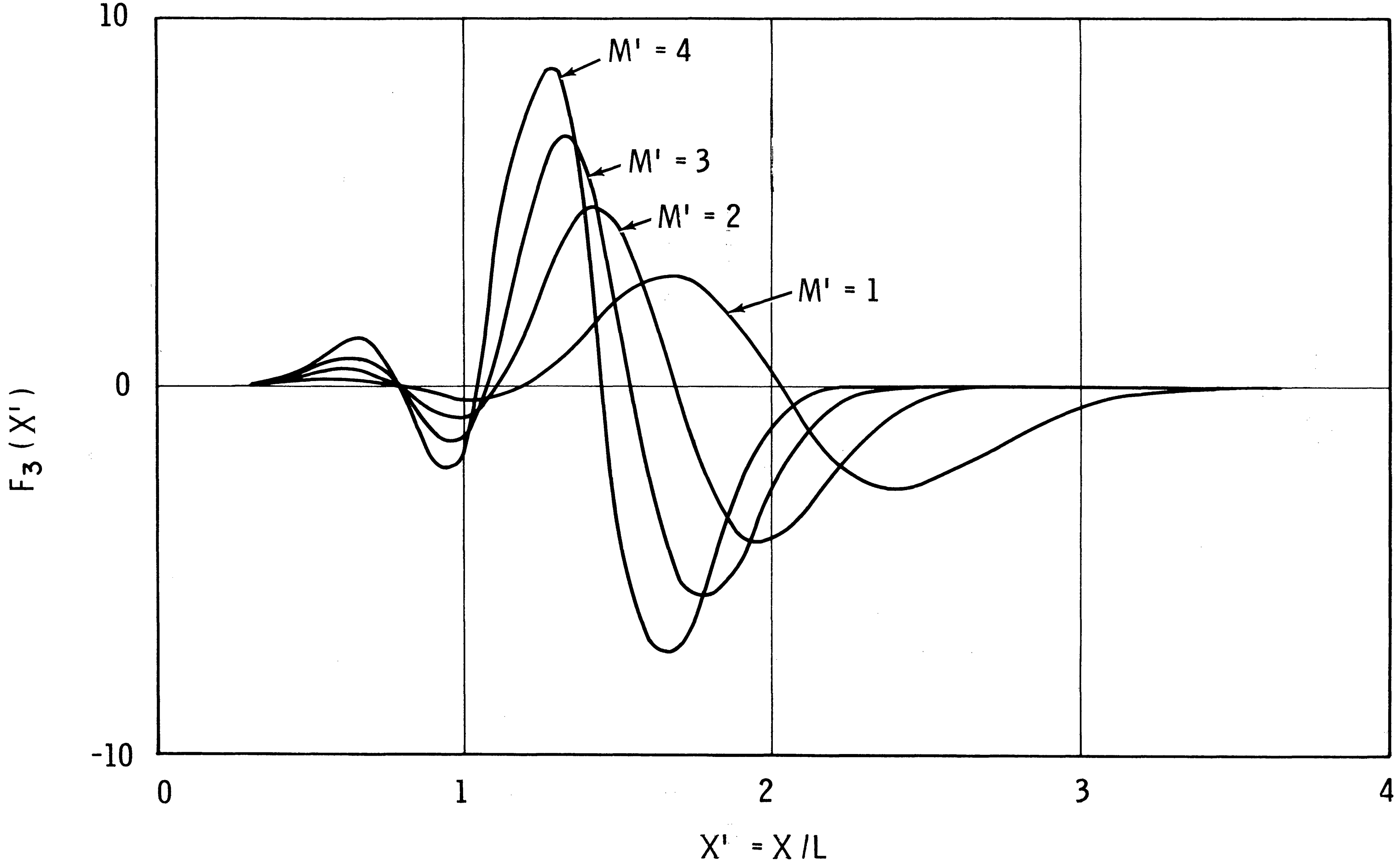

Figure 5(e). Comparison of $F_3(X')$ for different m's.

subjected to the following boundary conditions

$$\eta_t = 0 : \quad \theta'_{20} = 0 \quad , \quad \frac{\partial^2 \theta'_{20}}{\partial \eta_t^2} = 0$$

$$\eta_t = 1 : \quad \theta'_{20} = 0 \quad , \quad \frac{\partial \theta'_{20}}{\partial \eta_t} = 0$$

Then the constants are solved to be

$$a_5 = 0$$
$$c_5 = 0$$
$$d_5 = -3 b_5$$
$$e_5 = 2 b_5$$

Equation (182) then becomes

$$\theta'_{20} = b_5 (\eta_t - 3\eta_t^3 + 2\eta_t^4) \tag{183}$$

Equations (160), (166), and (183) are then substituted into Eq. (181) and the equation thus obtained is used to solve for b_5, we then get

$$b_5 = (43917)\,\omega'^2 F_1(P_r) F_4(P_r) F_5(X') \tag{184}$$

where

$$F_1(P_r) = 0.834\, P_r^{-\frac{1}{3}} + 0.108\, P_r^{-\frac{2}{3}} - 0.064\, P_r^{-1} + 0.014\, P_r^{-\frac{4}{3}} \tag{185}$$

$$F_4(P_r) = -0.067\, P_r^{-\frac{1}{3}} + 0.162\, P_r^{-\frac{2}{3}} - 0.129\, P_r^{-1} + 0.035\, P_r^{-\frac{4}{3}} \tag{186}$$

$$\begin{aligned} F_5(X') = \frac{\phi_s'^2}{X'} \Big\{ & -8m'^4 X'^8 + 32 m'^4 X'^7 - (48 m'^4 + 13 m'^3) X'^6 \\ & + (32 m'^4 + 17 m'^3) X'^5 + (8 m'^4 - 19 m'^3 + 15.5 m'^2) X'^4 \\ & + (3 m'^3 - 19.5 m'^2) X'^3 + (5 m'^2 - 2.0625 m') X'^2 \\ & + 0.8125\, m' X' + 0.0313 \Big\} \end{aligned} \tag{187}$$

$F_5(X')$ is plotted in Figures 6(a) through 6(e) for different m's.

RESULTS OF THE ANALYSIS

With the solutions of the basic equations obtained in the preceding sections, we can proceed to determine the time-averaged velocity and temperature profiles, skin friction, and heat transfer.

The time-averaged velocity profile in the boundary layer is

$$\bar{U}' = \bar{U}_0' + \frac{\epsilon'^2}{Re} \bar{U}_{20}' \tag{188}$$

in which the oscillating terms in the first- and second-order approximations become zero upon integration with respect to time. The terms U_0' and U_{20}' are given in Eqs. (145) and (176), respectively. Using them, Eq. (188) becomes

$$\bar{U}' = (2\eta - 2\eta^3 + \eta^4) - \lambda_f (53.749) F_3(X')(\eta - 3\eta^3 + 2\eta^4) \tag{189}$$

where

$$\lambda_f = \frac{\epsilon'^2 \omega'^4}{Re} \tag{190}$$

In Figure 5, it is seen that $F_3(X')$ is equal to zero at three points. For example, at m' = 1 these points are at X' = 0.75, 1.18, and 2.02. This means the velocity profile is the same as Blasius' velocity profile. The location of these points does not depend on λ_f, being dependent on m' only. The surface of the flat plate is divided into four regions by these three points. The effect of oscillation on

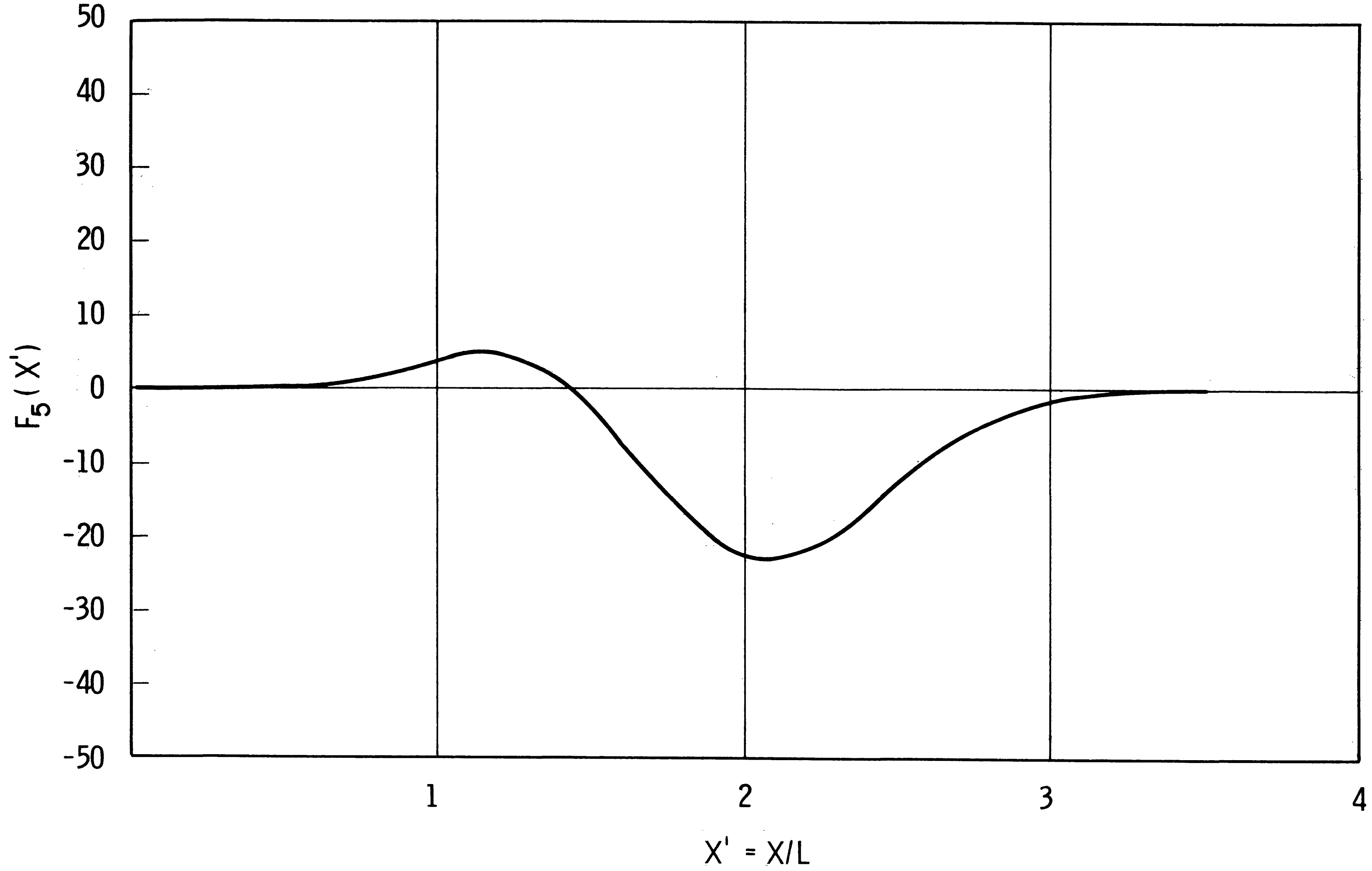

Figure 6(a). $F_5(X')$ as a function of $X'(m'=1)$.

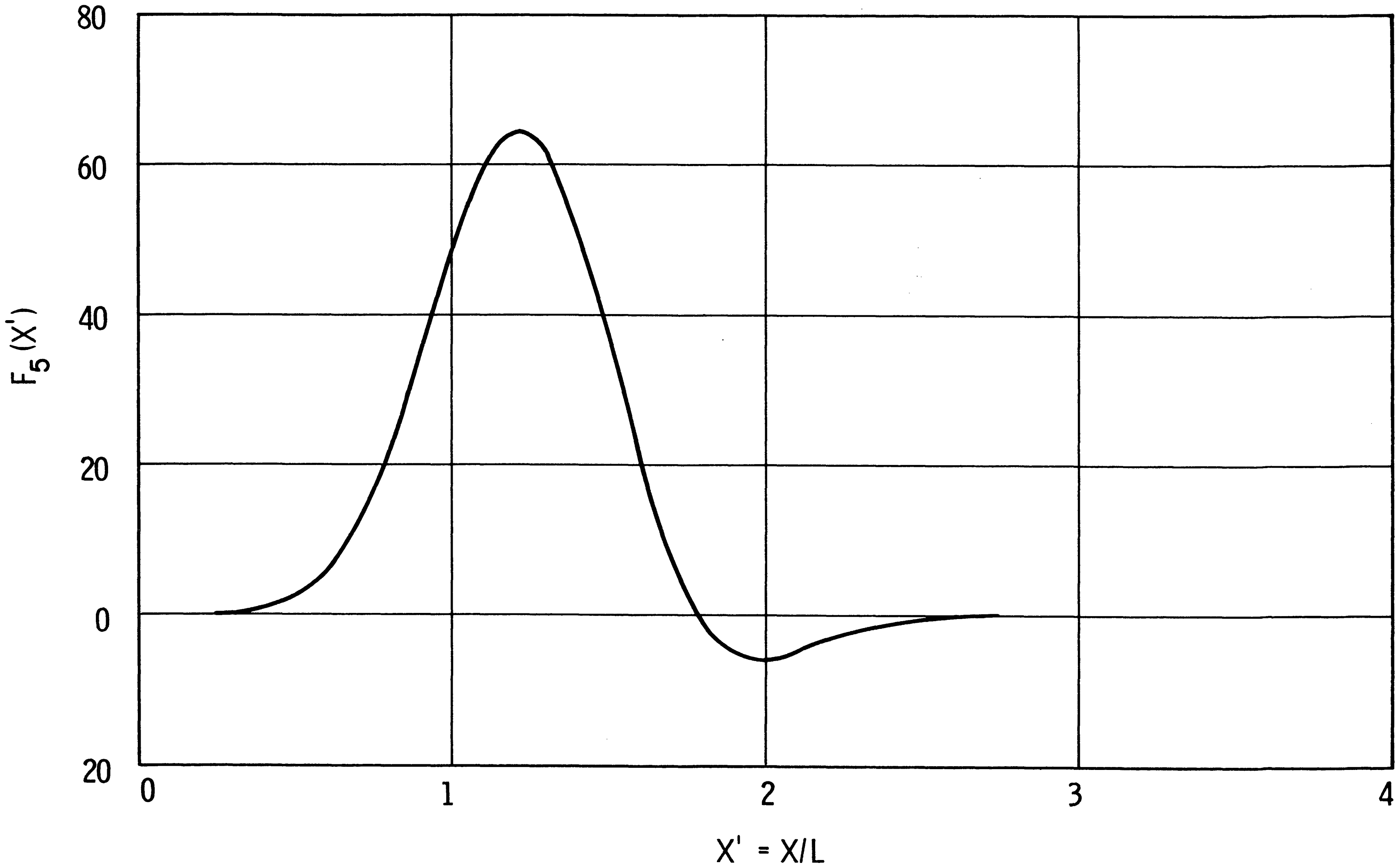

Figure 6(b). $F_5(X')$ as a function of $X'(m'=2)$.

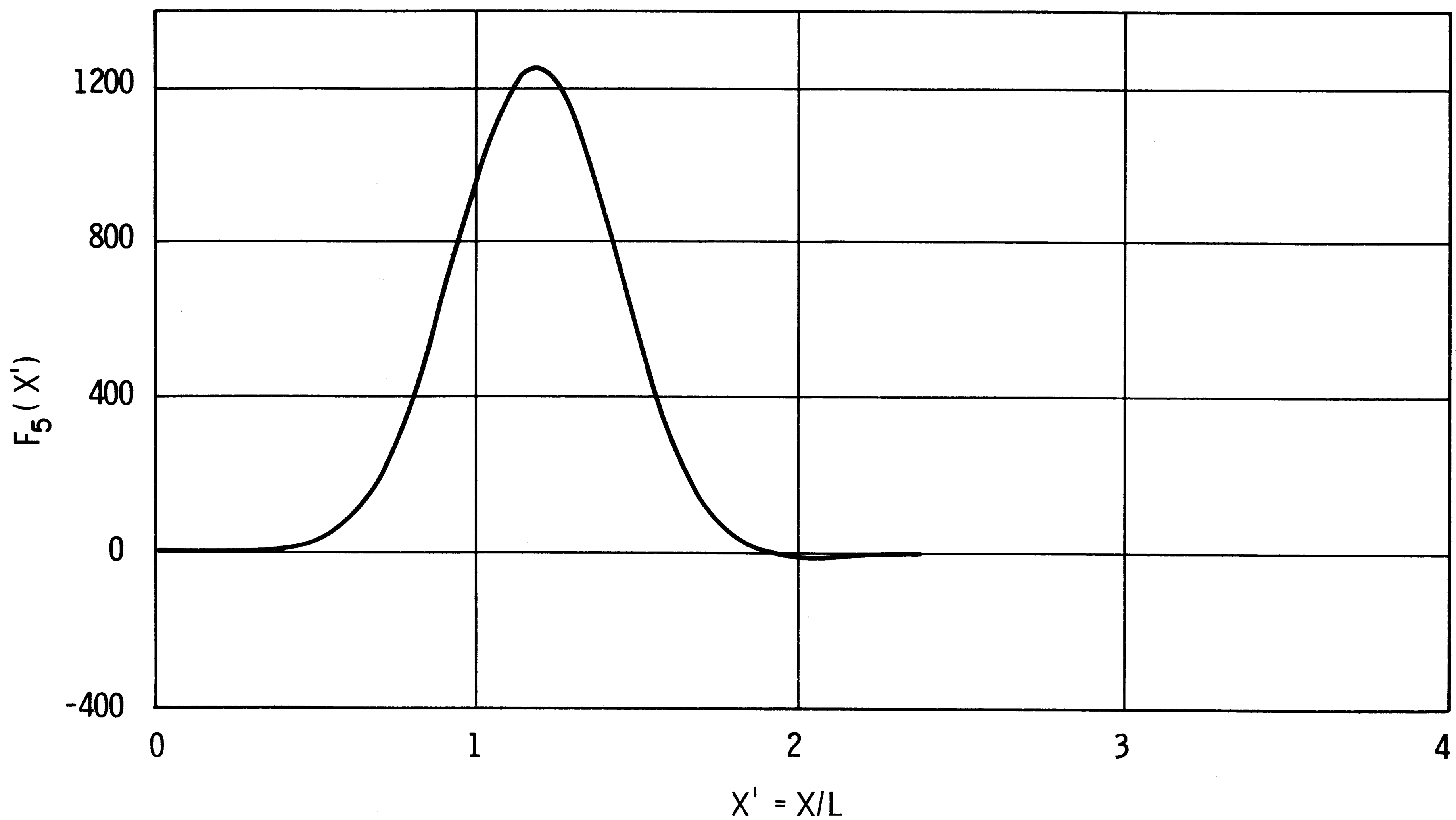

Figure 6(c). $F_5(X')$ as a function of $X'(m'=3)$.

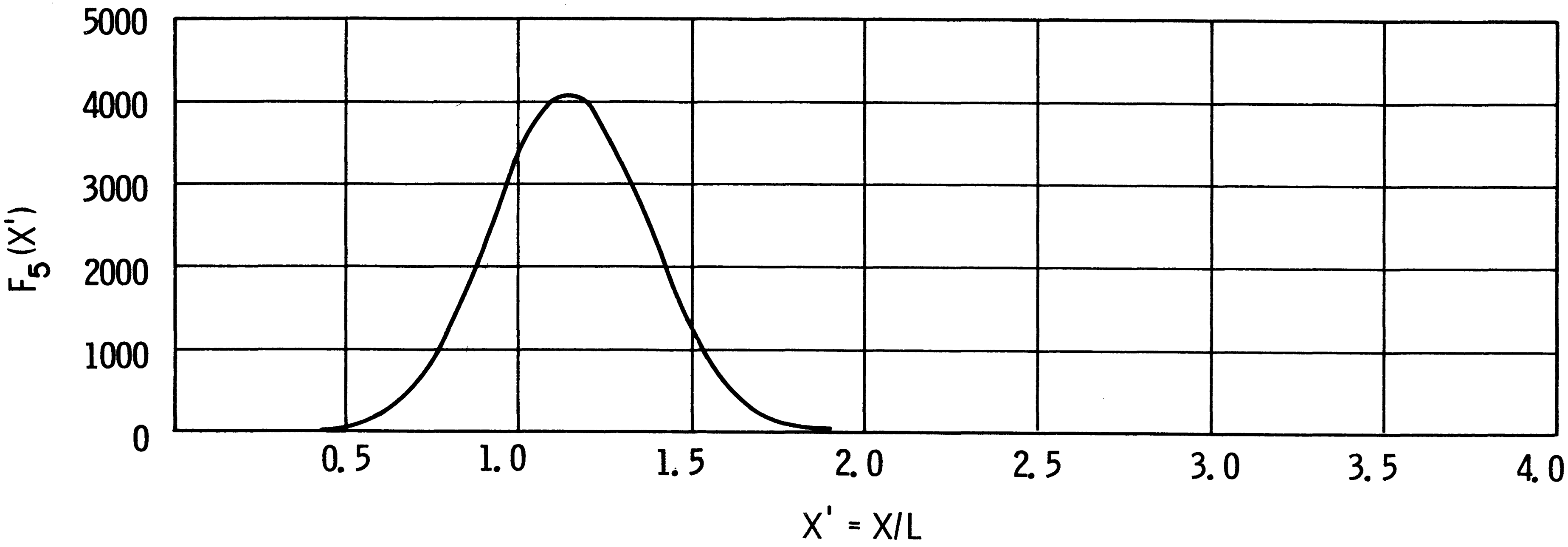

Figure 6(d). $F_5(X')$ as a function of $X'(m'=4)$.

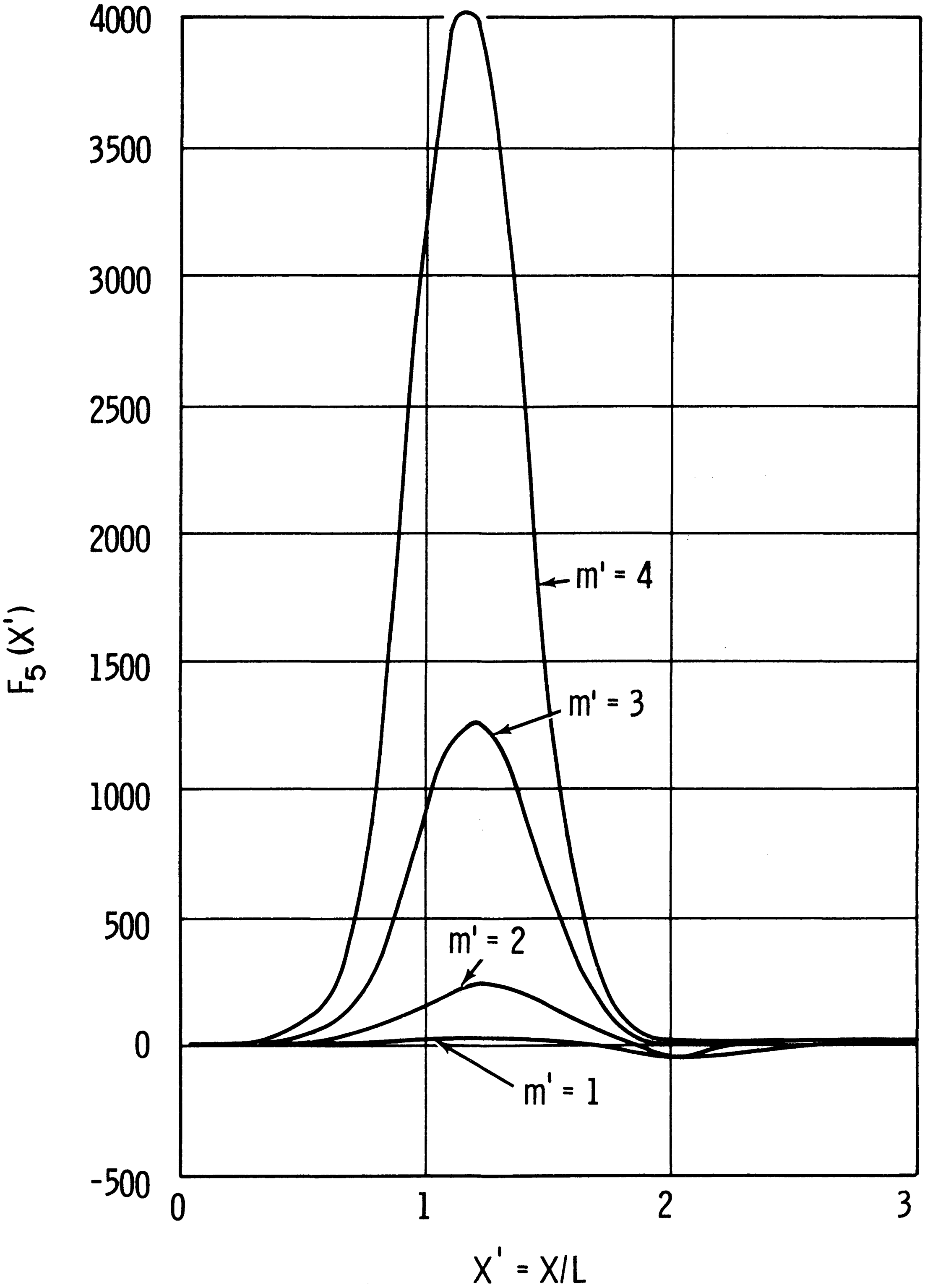

Figure 6(e). Comparison of $F_5(X')$ for different m's.

the flow being different in each region. Figures 7(a) through 7(d) show the velocity profiles in each of the four regions.

By definition, the local skin friction is

$$\tau_x = \mu \left(\frac{\partial U}{\partial Y}\right)_{Y=0}$$

so the drag coefficient is

$$C_{fx} = \frac{\tau_x}{\frac{\rho U_\infty^2}{2}}$$

In terms of dimensionless quantities, it becomes

$$C_{fx} = \frac{2}{\sqrt{Re}} \left(\frac{\partial U'}{\partial Y^*}\right)_{Y^*=0} \tag{191}$$

Equation (189) is put into Eq. (191) and we then get

$$C_{fx} = \frac{2}{\sqrt{Re}} \left\{ \frac{0.3419}{\sqrt{X'}} - \lambda_f (9.189) \frac{F_3(X')}{\sqrt{X'}} \right\} \tag{192}$$

C_{fx} is plotted in Figures 8(a) to 8(d).

The net effect of the oscillation on the surface of the plate is obtained by integrating the second term of Eq. (192) with respect to X' between the upstream and downstream penetration depths and dividing it by the distance between them. The result is

$$\Delta \bar{C}_f = - \frac{1}{L_{fd} - L_{fu}} \int_{L_{fu}}^{L_{fd}} \frac{2}{\sqrt{Re}} \lambda_f (9.189) \frac{F_3(X')}{\sqrt{X'}} dX' \tag{193}$$

in which the two penetration depths are determined numerically as follows: In Figure 5 it is seen that at m' = 1 the expression

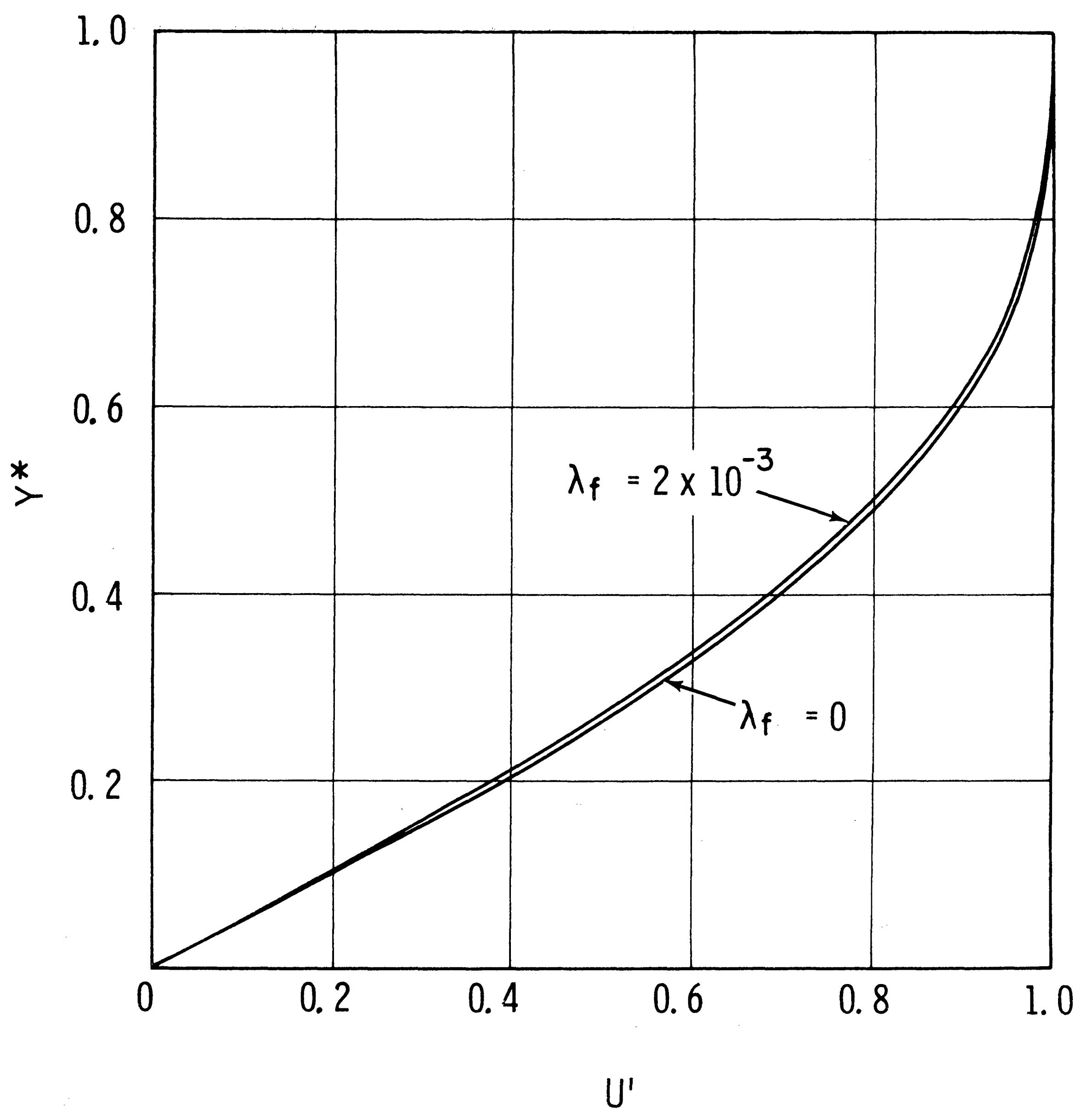

Figure 7(a). Velocity profile at X' = 0.5(m'=1).

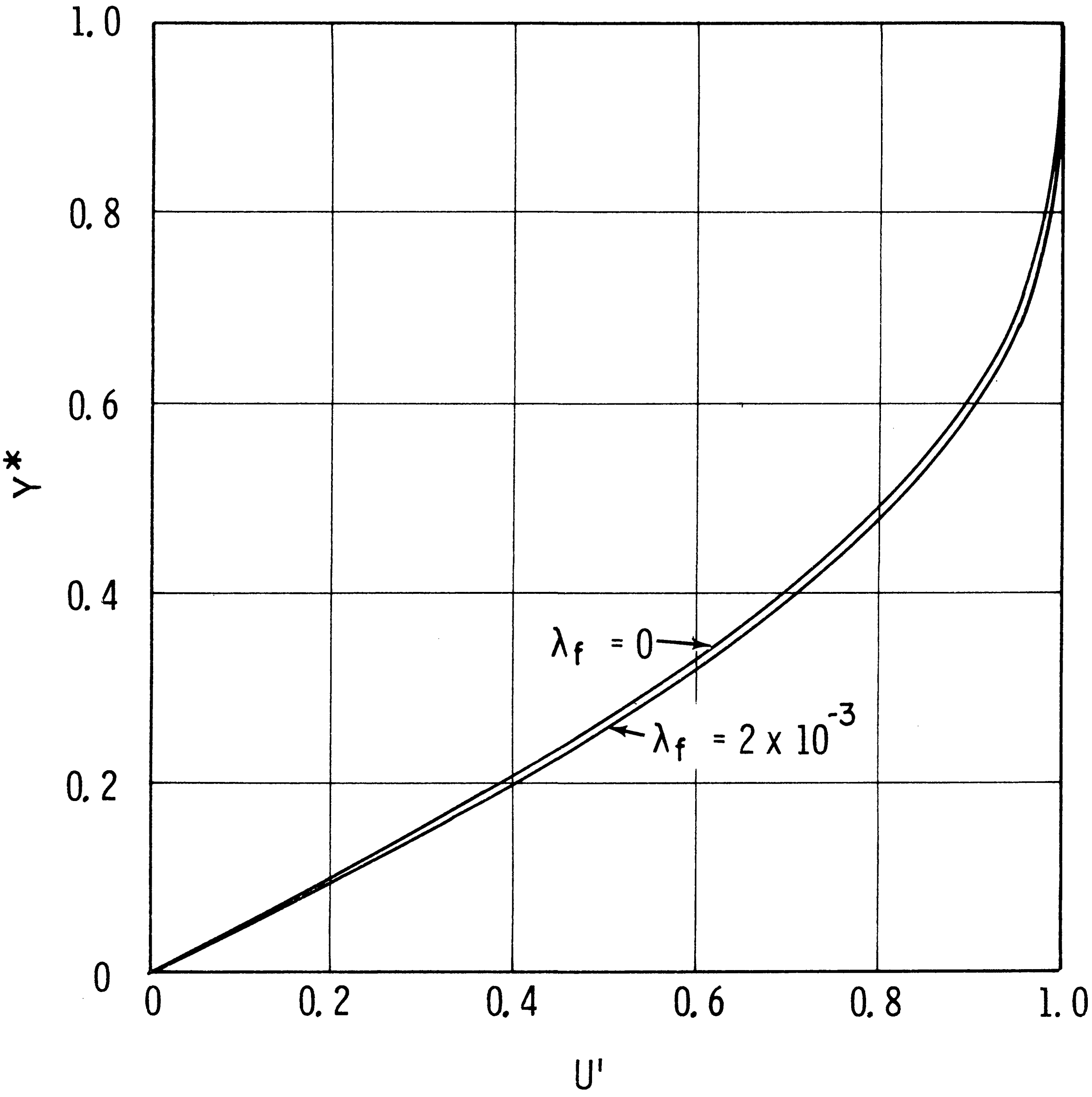

Figure 7(b). Velocity profile at X' = 1.1(m'=1).

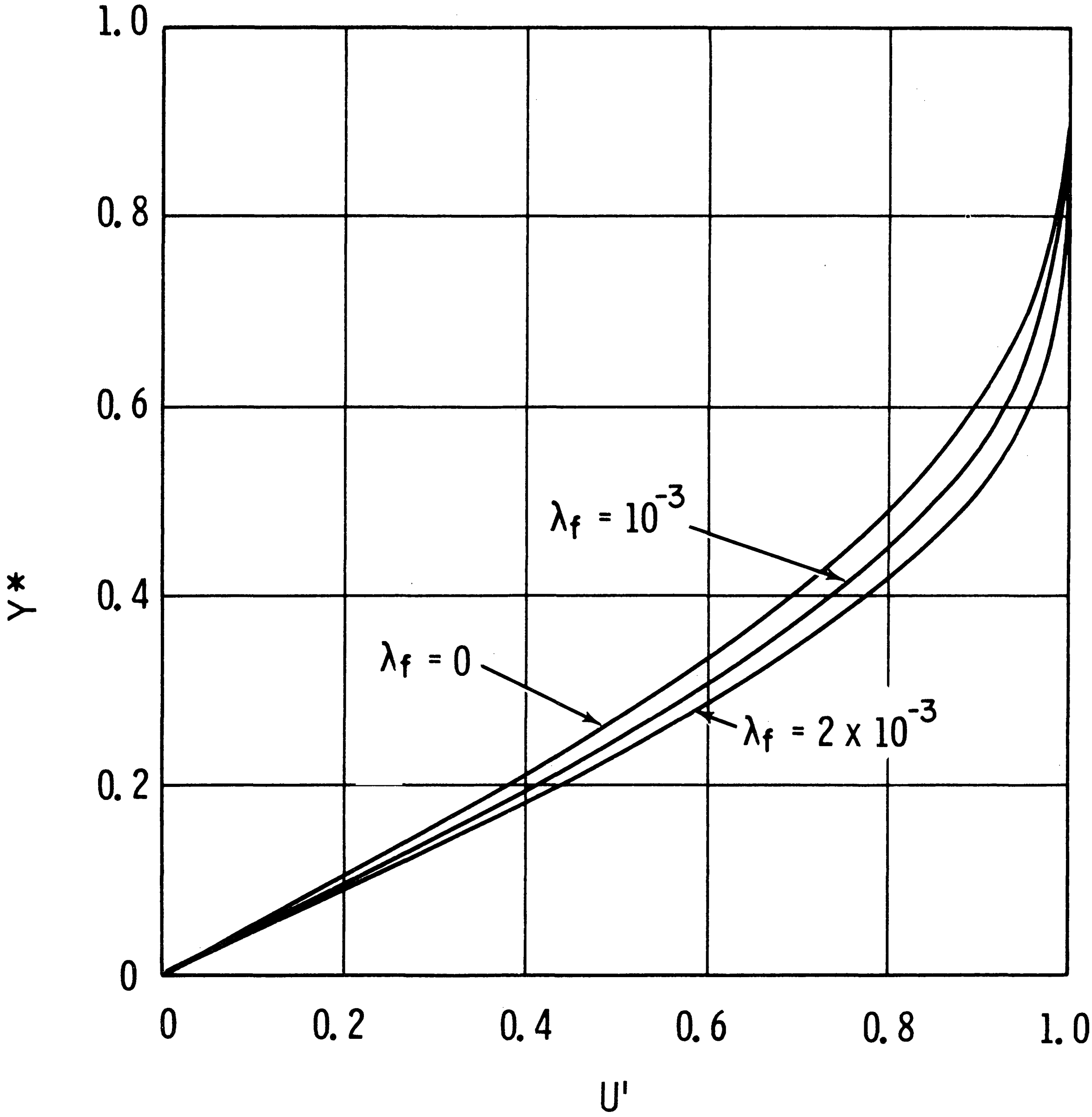

Figure 7(c). Velocity profile at X' = 1.6(m'=1).

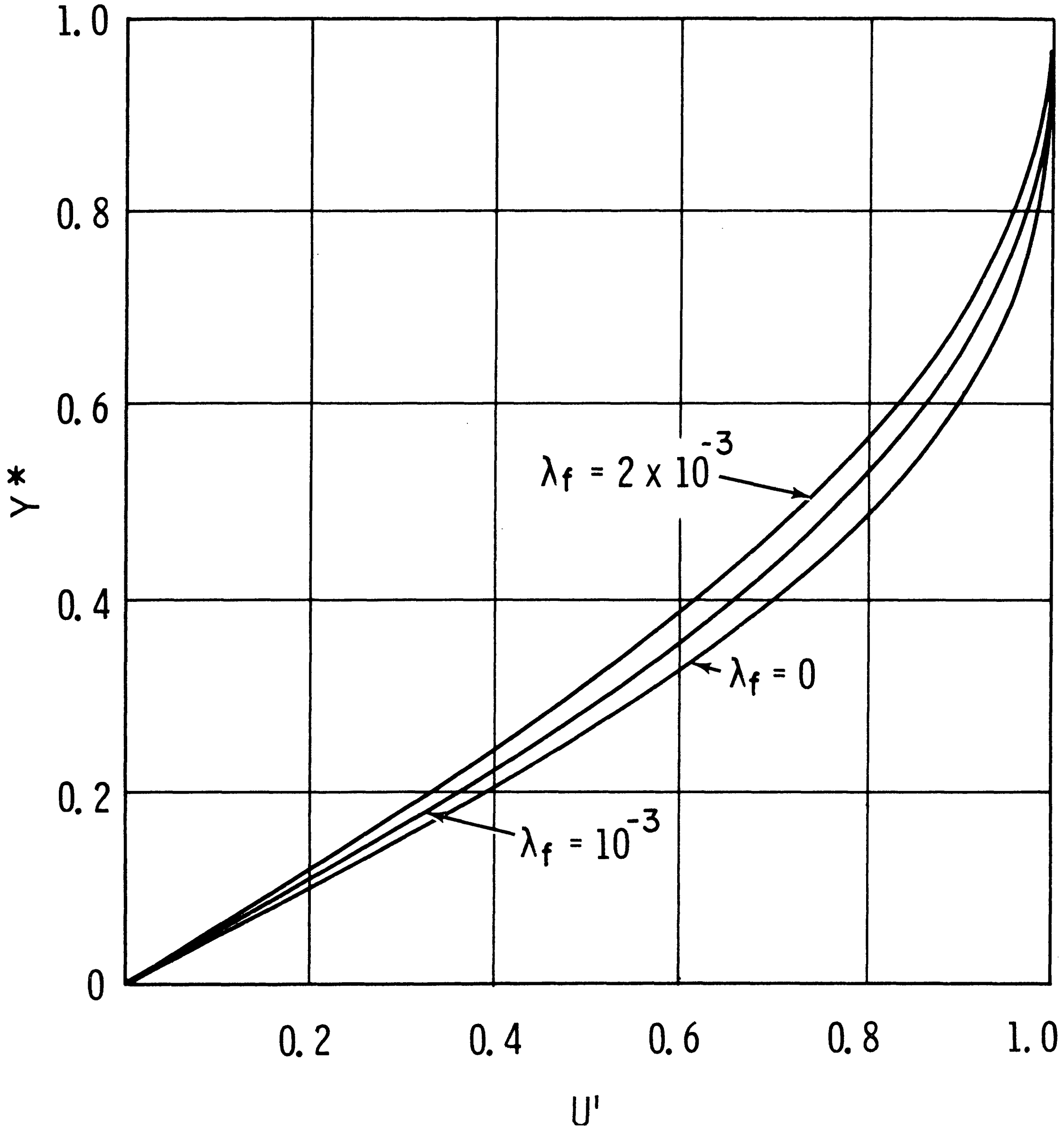

Figure 7(d). Velocity profile at X' = 2.4(m'=1).

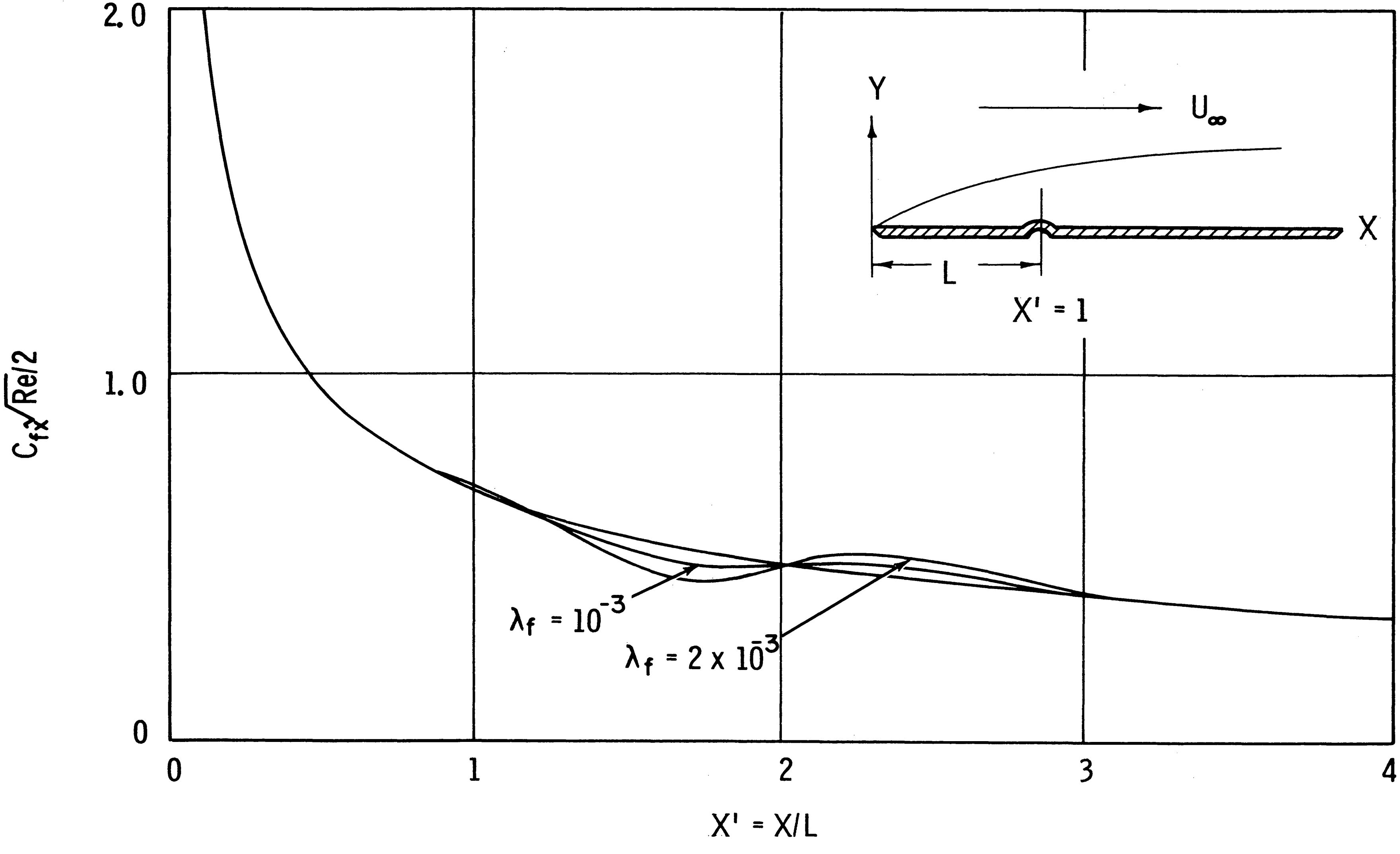

Figure 8(a). Local drag coefficient as a function of X'(m'=1).

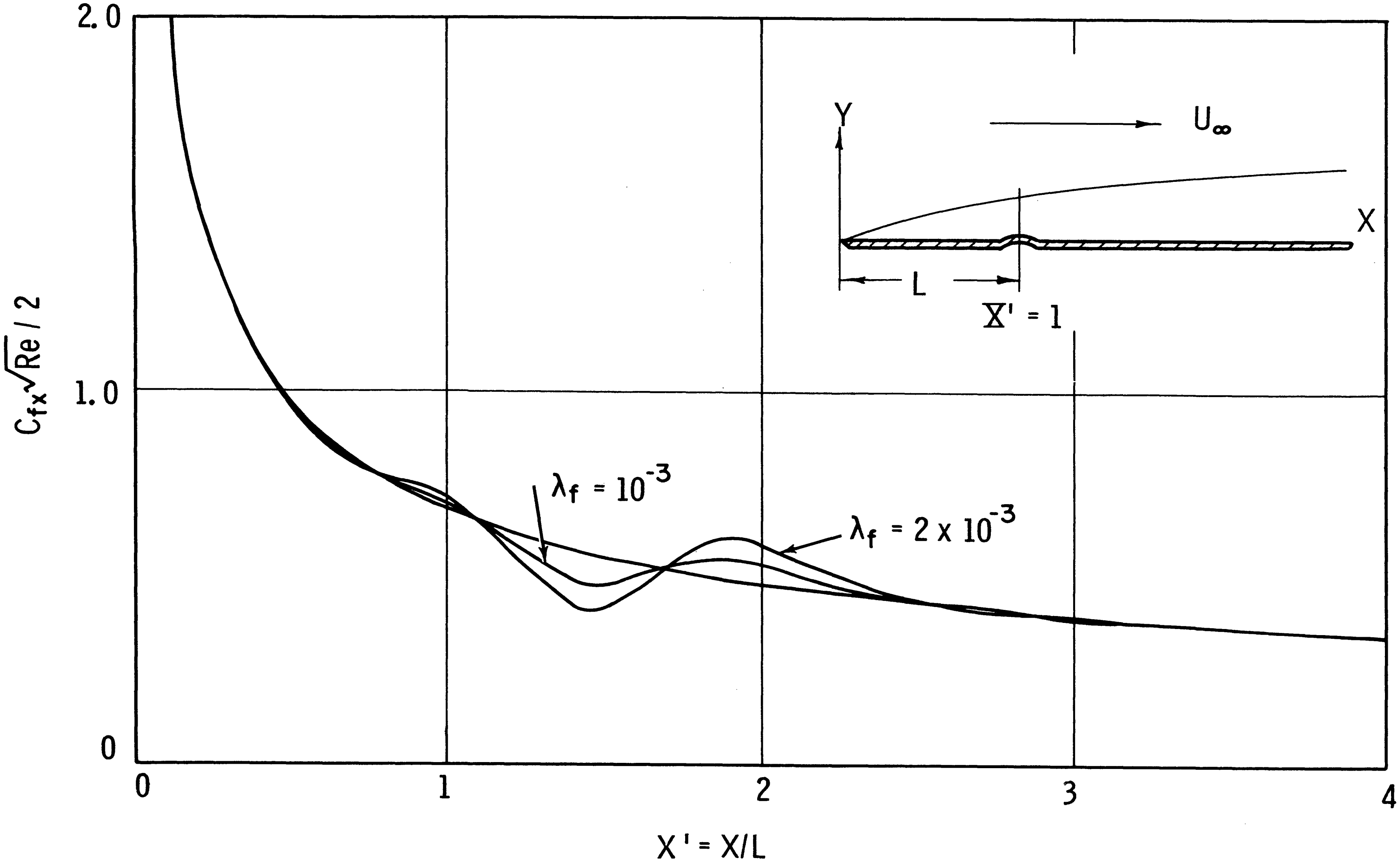

Figure 8(b). Local drag coefficient as a function of X'(m'=2).

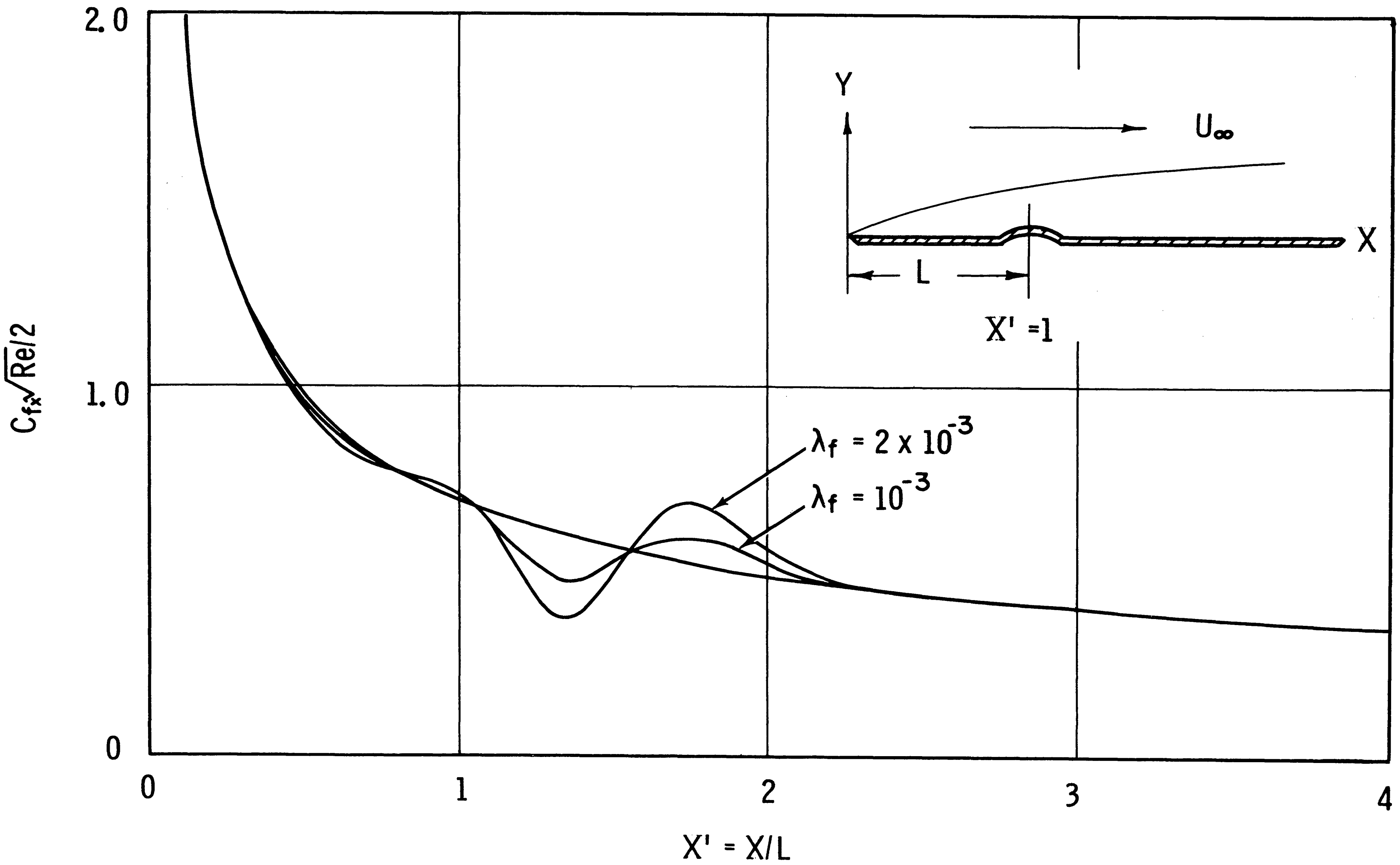

Figure 8(c). Local drag coefficient as a function of X'(m'=3).

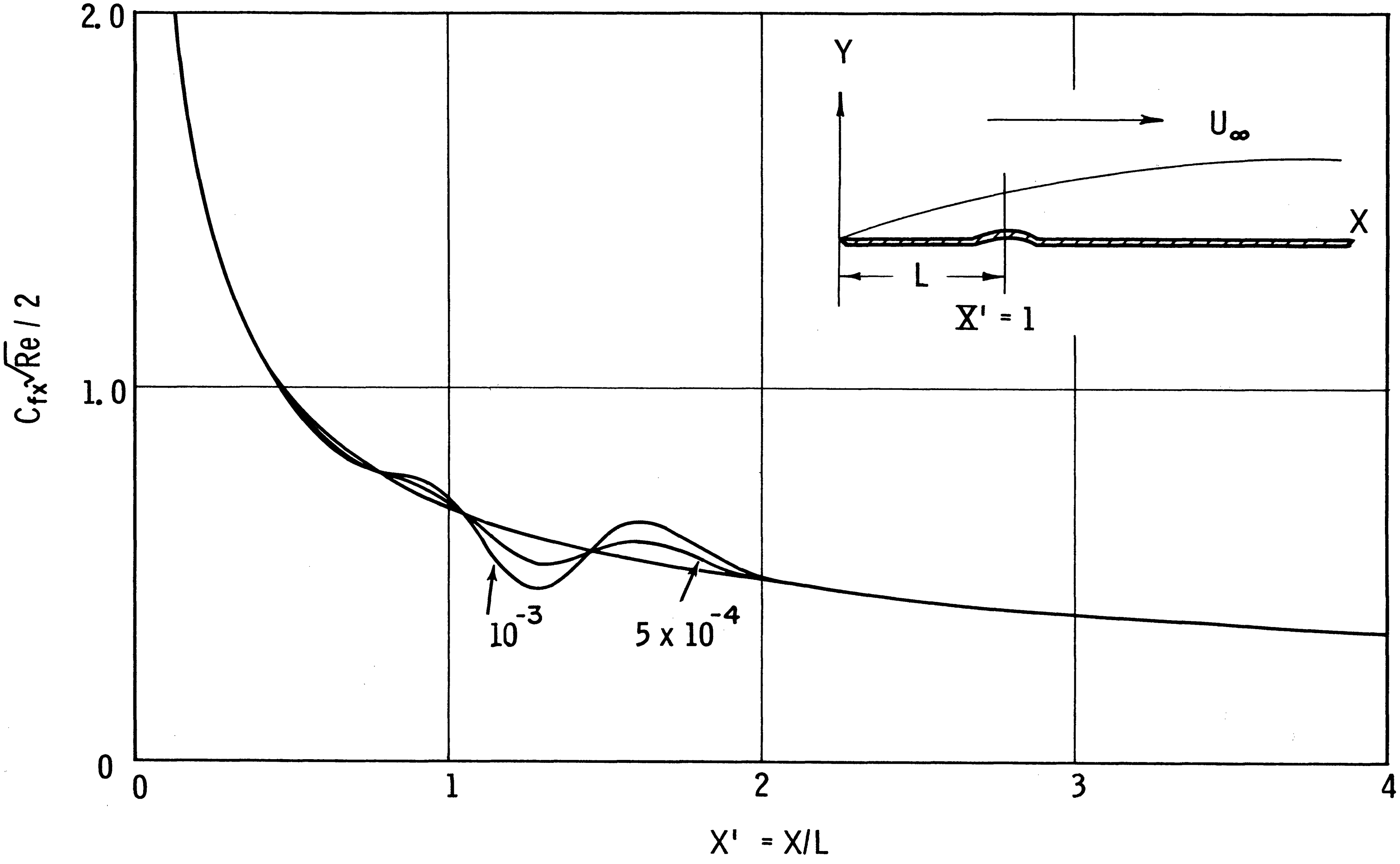

Figure 8(d). Local drag coefficient as a function of X'(m'=4).

$$\frac{F_3(X')}{\sqrt{X'}}$$

equals to its maximum value of 0.917 at X' = 1.7. It is also seen that this expression equals to 0.01 (approximately 1% of its maximum value) at X' = 0.07 and -0.01 at X' = 3.47. Therefore, we take L_{fu} = .07 and L_{fd} = 3.47.

Numerical integration of Eq. (193) then yields the following results:

$$\begin{aligned}
\Delta \overline{C_f} &= .02260 \frac{\lambda_f}{\sqrt{Re}} \qquad \text{for} \qquad m' = 1 \\
&= .01215 \frac{\lambda_f}{\sqrt{Re}} \qquad \text{for} \qquad m' = 2 \\
&= .00555 \frac{\lambda_f}{\sqrt{Re}} \qquad \text{for} \qquad m' = 3 \\
&= .00229 \frac{\lambda_f}{\sqrt{Re}} \qquad \text{for} \qquad m' = 4
\end{aligned} \tag{194}$$

The positive sign for $\Delta\overline{C_f}$ means that the oscillation results in an steady increase in skin friction. However, if the first term on the right of Eq. (192) is integrated over the same penetration depths, the result will be of the order of $2.5/\sqrt{Re}$. In viewing of the inherent limitation of the perturbation method, the maximum value of λ_f is of the order of 10^{-3} (see Appendix III). Therefore, the net change is always smaller than 1% of its value without this localized disturbance.

The time-averaged temperature profile in the thermal boundary layer is

$$\theta' = \theta'_0 + \frac{\epsilon'^2}{Re} \theta'_{20} \tag{195}$$

Again, the oscillating terms in the first- and second-order approximations become zero upon integration with respect to time. The terms Θ_0' and Θ_{20}' are given in Eqs. (149) and (183), respectively. Using them, Eq. (195) becomes

$$\theta' = (1 - 2\eta_T + 2\eta_T^3 - \eta_T^4)$$

$$+ (439/7\lambda_T) F_1 F_4 F_5 (\eta_T - 3\eta_T^3 + 2\eta_T^4) \tag{196}$$

where

$$\lambda_T = \frac{\epsilon'^2 \omega'^2}{Re} \tag{197}$$

From Figure 6, it is seen that $F_5(X')$ changes from positive to negative at a certain X'. For example at m' = 1 this point is at X' = 2.02. At this point, the temperature profile is not influenced by the oscillation. Figures 9(a) and 9(b) show the temperature profiles in the boundary layer.

To find an expression for the Nusselt number, it is defined that

$$Nu_x = \frac{hL}{k} = - \frac{\partial\left(\frac{T - T_\infty}{T_s - T_\infty}\right)}{\partial\left(\frac{Y}{L}\right)}$$

In terms of dimensionless quantities, it becomes

$$Nu_x = -\sqrt{Re}\left(\frac{\partial \theta'}{\partial Y^*}\right)_{Y^* = 0} \tag{198}$$

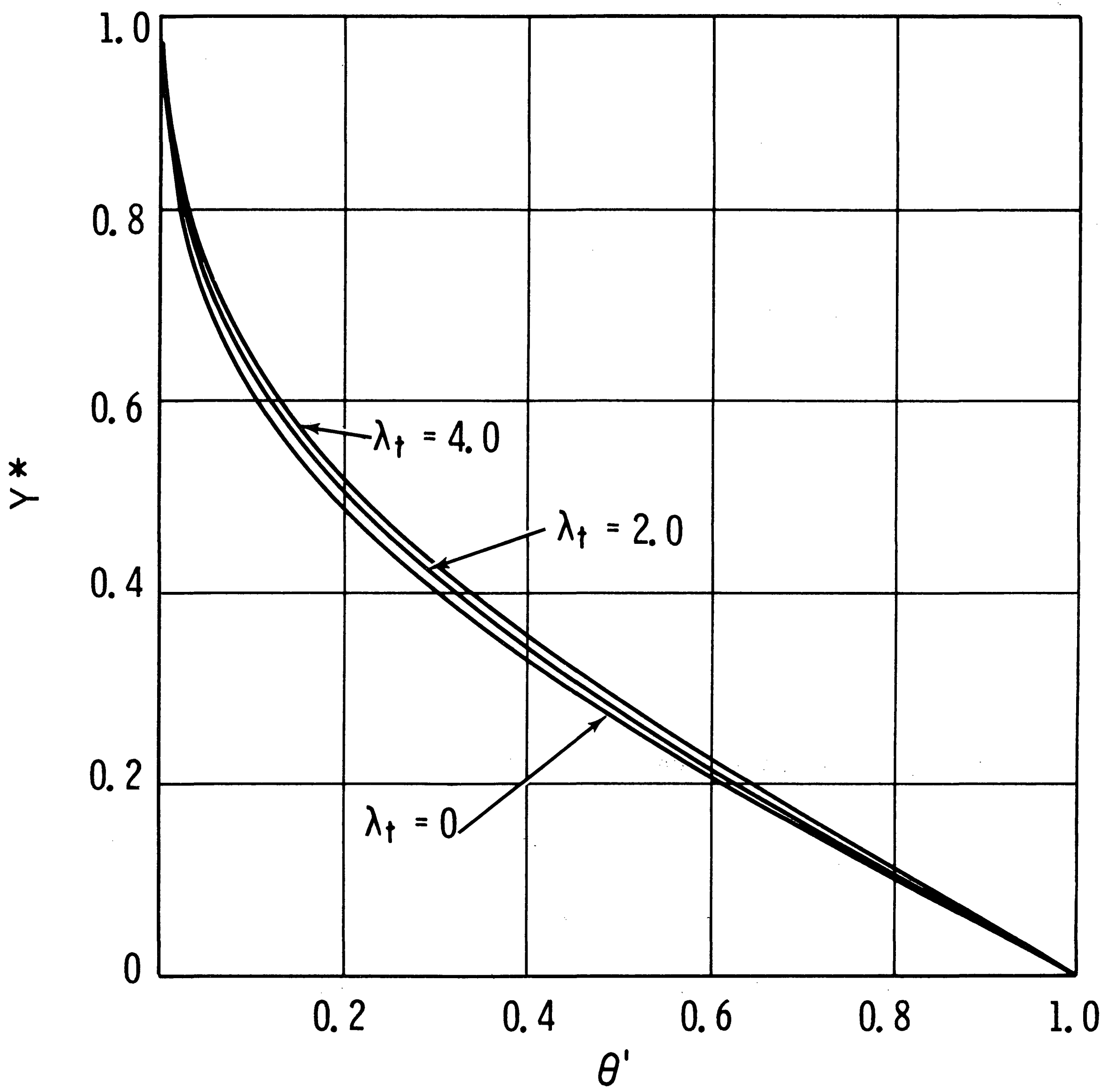

Figure 9(a). Temperature profile at X' = 1(m'=1).

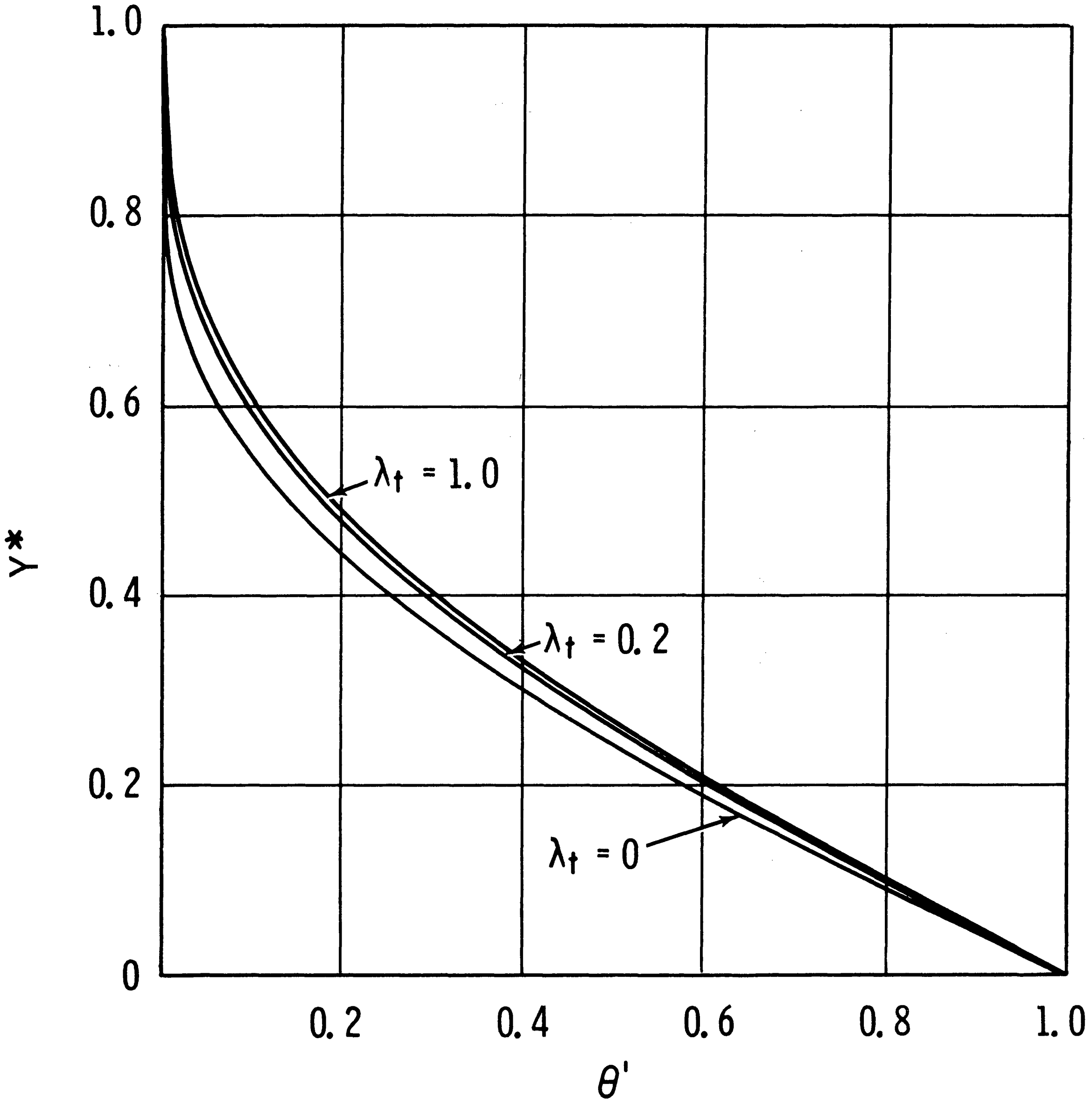

Figure 9(b). Temperature profile at X' = 2(m'=2).

Substituting Eq. (196) into Eq. (198), we then get

$$Nu_x = \sqrt{Re}\left\{\frac{0.3419 Pr^{\frac{1}{3}}}{\sqrt{X'}} - (7508\lambda_T Pr^{\frac{1}{3}}) F_1 F_4 \frac{F_5(X')}{\sqrt{X'}}\right\} \tag{199}$$

Nu_x is plotted in Figures 10(a) through 10(d).

As before, the average effect of the oscillation of the Nusselt number is obtained by integrating the second term in Eq. (199) between the two penetration depths. Then we have

$$\Delta Nu = -\frac{1}{L_{td} - L_{tu}}\int_{L_{tu}}^{L_{td}} \sqrt{Re}\,\lambda_T (7508 Pr^{\frac{1}{3}}) F_1 F_4 \frac{F_5(X')}{\sqrt{X'}} dX' \quad . \tag{200}$$

After numerically integrating it by the same method as in computing Eq. (194), we get

$$\begin{aligned}
\Delta Nu &= -(34,200)\sqrt{Re}\,\lambda_T Pr^{\frac{1}{3}} F_1 F_4 && \text{for} \quad m' = 1 \\
&= -(388,000)\sqrt{Re}\,\lambda_T Pr^{\frac{1}{3}} F_1 F_4 && \text{for} \quad m' = 2 \\
&= -(2,205,000)\sqrt{Re}\,\lambda_T Pr^{\frac{1}{3}} F_1 F_4 && \text{for} \quad m' = 3 \\
&= -(6,650,000)\sqrt{Re}\,\lambda_T Pr^{\frac{1}{3}} F_1 F_4 && \text{for} \quad m' = 4
\end{aligned} \tag{201}$$

The minus sign indicates a decrease in heat transfer.

Integration of the first term on the right of Eq. (199) between the same penetration depth shows that it is of the order of $P_r^{1/3}\sqrt{Re}$. The maximum value of λ_f above which the perturbation method being no longer valid was given as 10^{-3}. Therefore, the maximum value of λ_T is of the order of $\frac{\lambda_f}{\omega'^2}$, which is 10^{-8} (see Appendix III). Again, the net change in heat transfer is seen to be less than 1% of its undisturbed value.

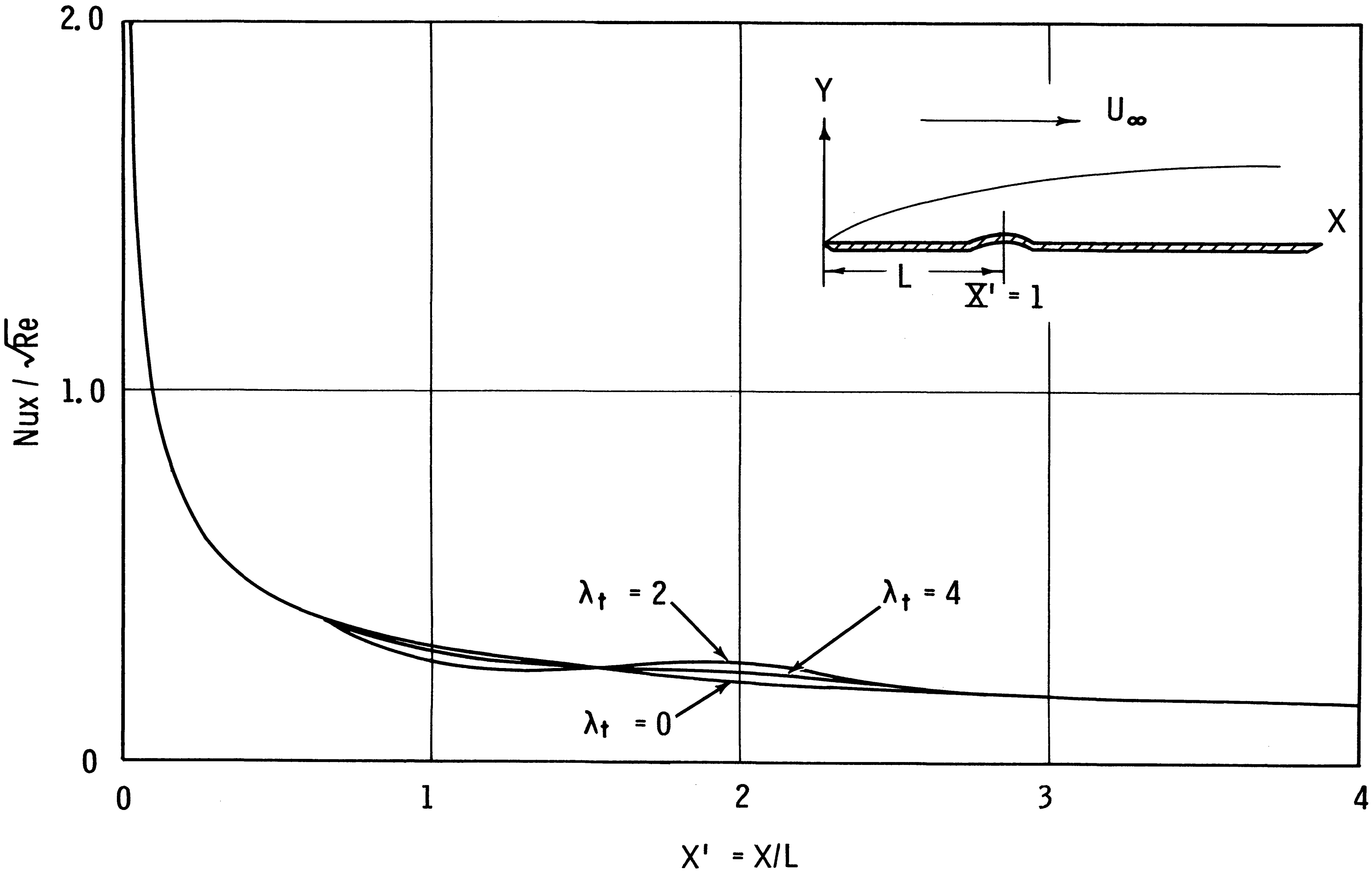

Figure 10(a). Local Nusselt number as a function of X'(m'=1).

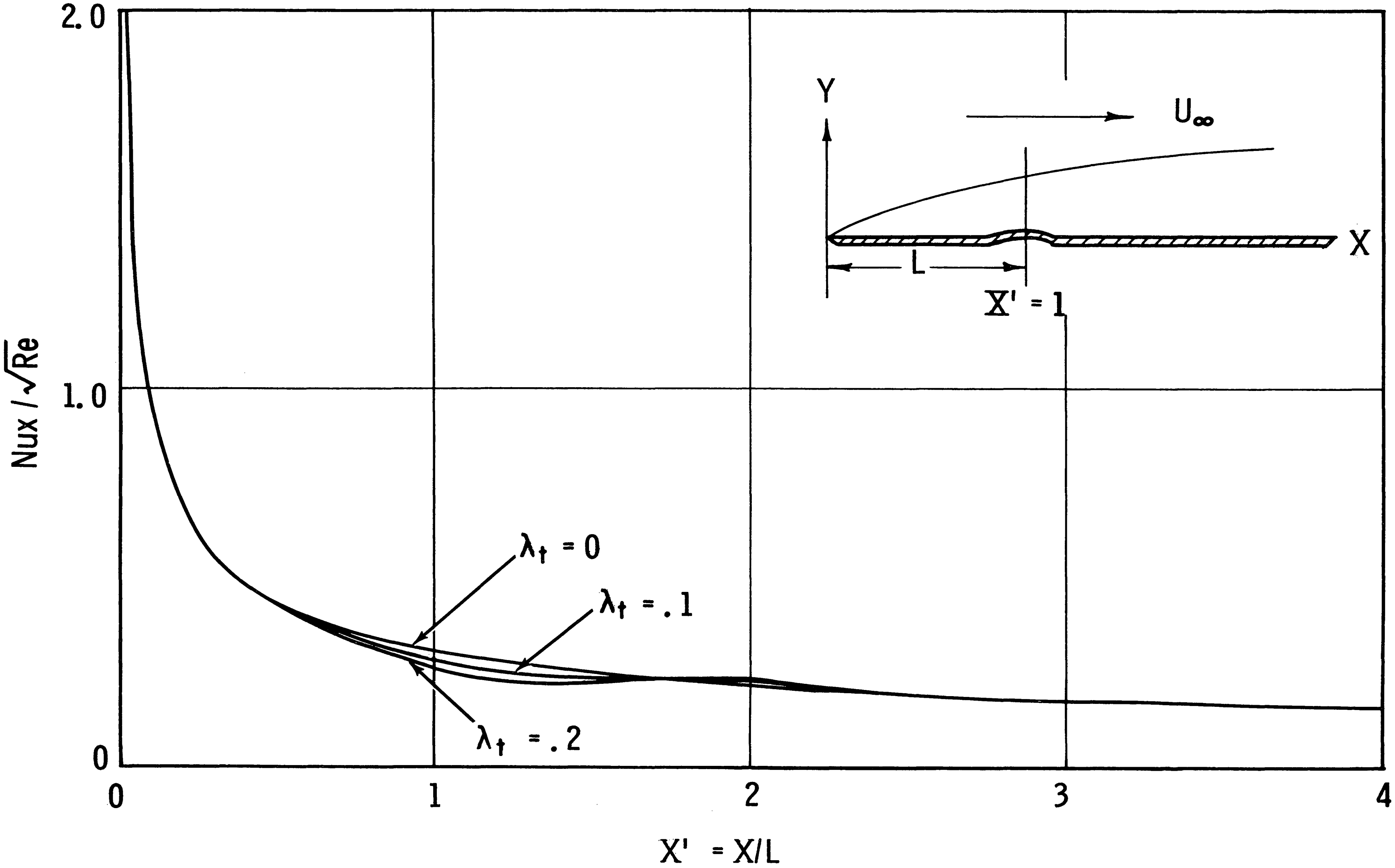

Figure 10(b). Local Nusselt number as a function of X'(m'=2).

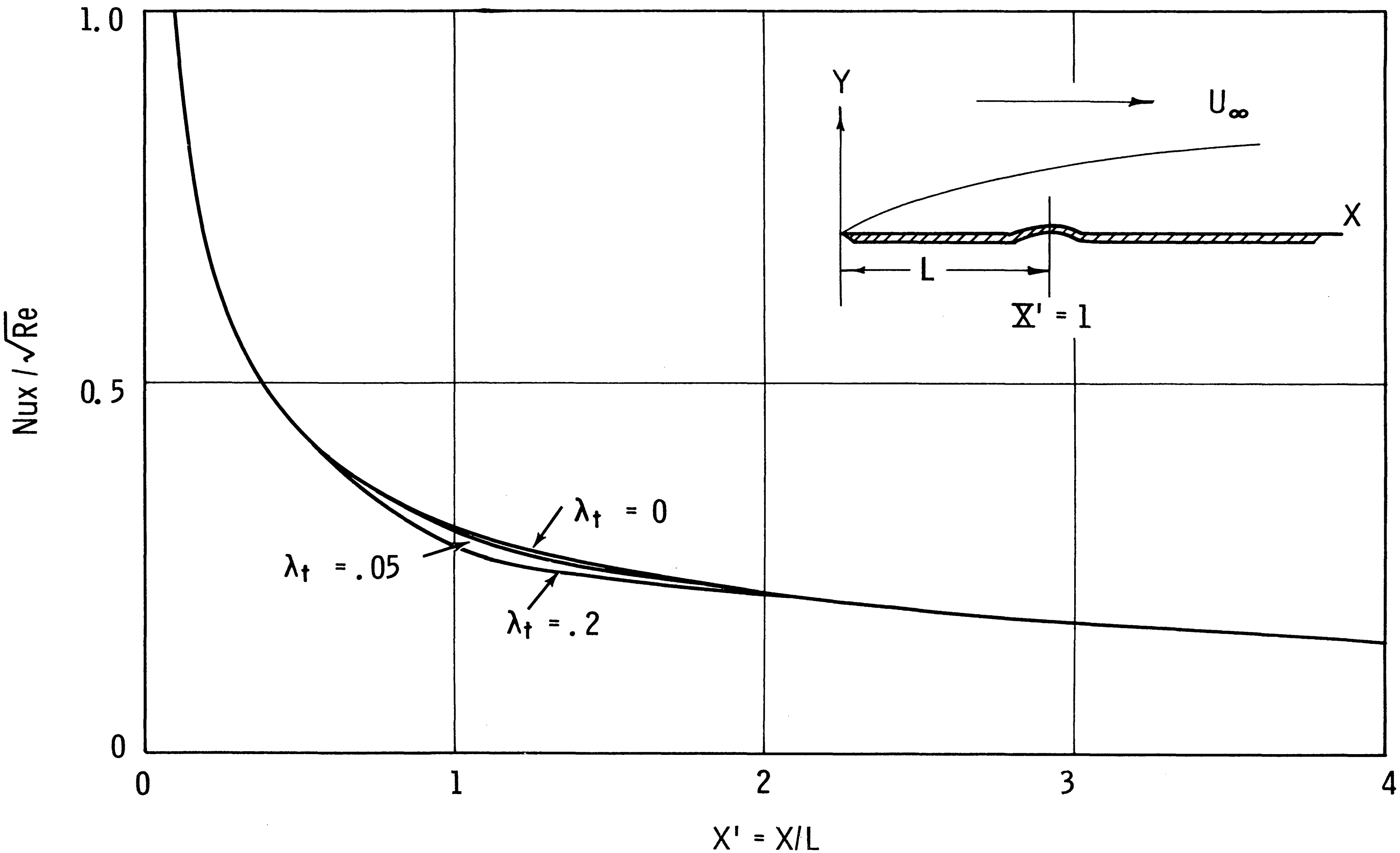

Figure 10(c). Local Nusselt number as a function of X'(m'=3).

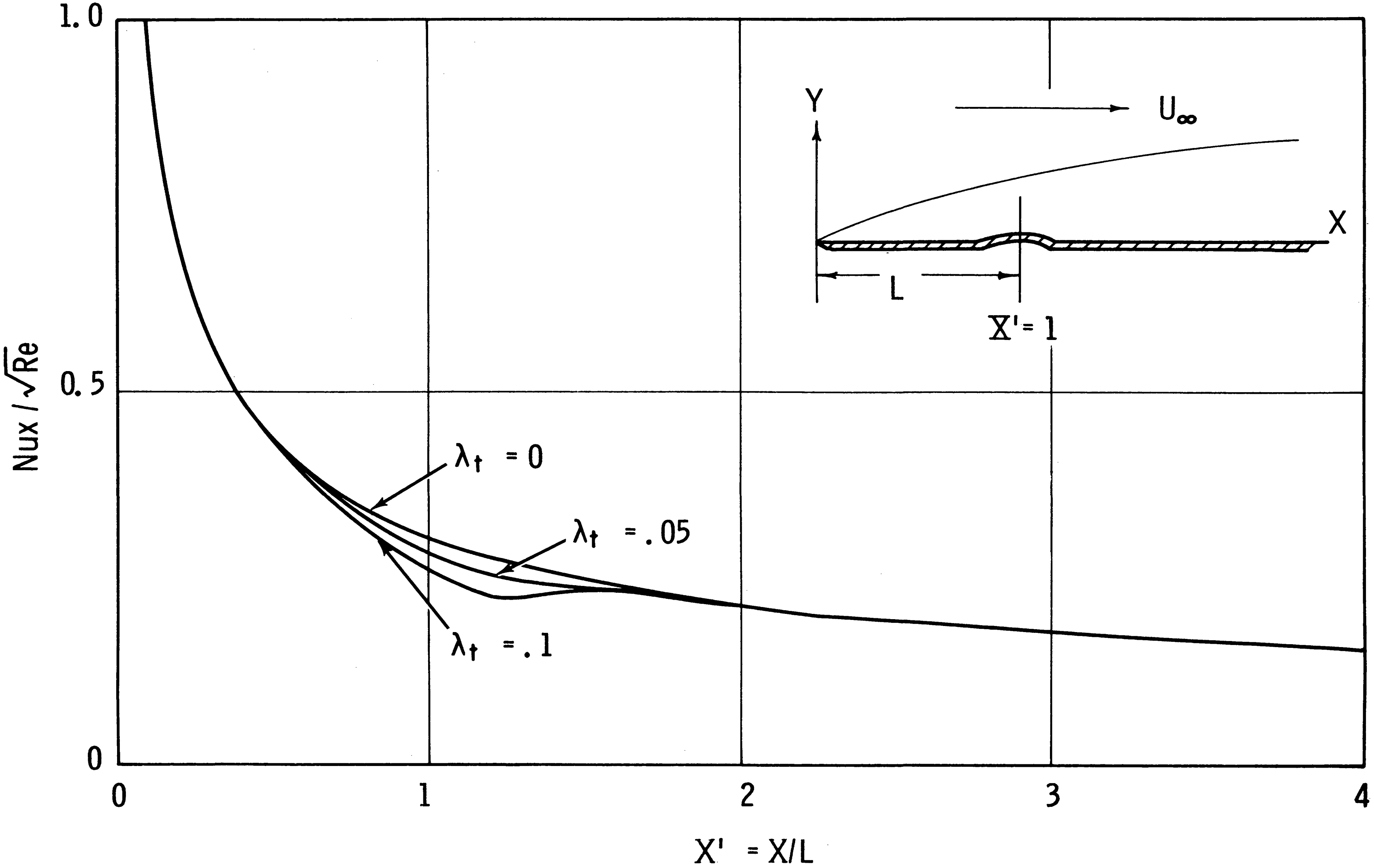

Figure 10(d). Local Nusselt number as a function of X'(m'=4).

CHAPTER III

CONCLUSIONS

In the preceding chapters, the problem is solved by the perturbation method. The various orders of approximations are then solved by the integral method which is known to be accurate in the case of uniform flow over a flat plate. In spite of the inherent limitations of the perturbation method, the following important conclusions are drawn as a result of this analysis.

a. The phase of skin friction is approximately in phase with the input disturbance, while that of heat transfer lags approximately $\frac{\pi}{2}$ radians.

b. The existence of a steady change of skin friction and heat transfer is shown in Eqs. (192) and (199) and also plotted in Figures 8 and 10. The four branches of the skin friction is expected since the equation of the boundary shows that it consists of two points of inflection and a point of zero slope at $X' = 1$. These points divide the plate into four parts, each producing a different effect on the skin friction. The governing parameters for the steady change in skin friction and heat transfer are λ_f and λ_T, respectively. Increasing the curvature of oscillation, indicated by an increase of m', tends to shrink the effect closer to the point of disturbance.

c. Integration of the steady local skin friction and heat transfer over X' shows that the net effect of the prescribed oscillation is an increase in total skin friction and a decrease in heat transfer from the plate to the fluid. The magnitude of these net effects are always less than 1% of the undisturbed flow within the limitations of the perturbation analysis.

A numerical example is given in Appendix III which helps to check the validity of the various assumptions made in the analysis.

APPENDIX I

DERIVATION OF EQUATIONS (43) AND (44)

At a certain instant, the boundary will be at a poistion shown in Figure 3. The normal velocity of the boundary is equal to $\epsilon i\omega\phi$ and the tangential component of it is of the order of $O(\epsilon^2)$. It is therefore neglected. To derive the momentum equations, we have to consider the inertia, pressure, body, and viscous forces separately.

The rate of change of any vector quantity $\vec{Q}$ may be written as

$$\frac{D\vec{Q}}{DT} = \frac{\partial\vec{Q}}{\partial T} + (\vec{V}\cdot\nabla)\vec{Q} + \vec{Q}\,(\nabla\cdot\vec{V}) \tag{I-1}$$

where, in two-dimensional form,

$$\vec{Q} = \vec{i}_X\, Q_X + \vec{i}_Y\, Q_Y \tag{I-2}$$

If we substitute $\vec{Q} = \rho\vec{V}$, i.e.,

$$\begin{aligned} \vec{Q} &= \vec{i}_X\, Q_X + \vec{i}_Y\, Q_Y \\ &= \vec{i}_X\, \rho U + \vec{i}_Y\, \rho V \end{aligned} \tag{I-3}$$

then Eq. (I-1) becomes:

$$\frac{D(\rho\vec{V})}{DT} = \frac{\partial(\rho\vec{V})}{\partial T} + (\vec{V}\cdot\nabla)\rho\vec{V} + \rho\vec{V}\,(\nabla\cdot\vec{V}) \tag{I-4}$$

where

$$\vec{V} = \vec{i}_X\, U + \vec{i}_Y\, V \tag{I-5}$$

and for incompressible fluid, we take

$$\rho = constant \tag{I-6}$$

Equations (I-5) and (I-6) are substituted into Eq. (I-4) and we get, for incompressible flow,

$$\begin{aligned} \frac{D(\rho\vec{V})}{DT} &= \rho\,\vec{i}_X\left\{\frac{\partial U}{\partial T} + (\vec{V}\cdot\nabla)U + U(\nabla\cdot\vec{V})\right\} \\ &+ \rho\,\vec{i}_Y\left\{\frac{\partial V}{\partial T} + (\vec{V}\cdot\nabla)V + V(\nabla\cdot\vec{V})\right\} \\ &+ \rho U(\vec{V}\cdot\nabla)\,\vec{i}_X + \rho V(\vec{V}\cdot\nabla)\,\vec{i}_Y \end{aligned} \tag{I-7}$$

The operators in this equation

$$\vec{V}\cdot\nabla = \frac{U}{h_X}\frac{\partial}{\partial X} + \frac{V}{h_Y}\frac{\partial}{\partial Y} \tag{I-8}$$

and

$$\nabla\cdot\vec{V} = \frac{1}{h_X h_Y}\left[\frac{\partial(h_Y U)}{\partial X} + \frac{\partial(h_X V)}{\partial Y}\right] \tag{I-9}$$

Putting into Eq. (I-7) we get

$$\begin{aligned} \frac{D(\rho\vec{V})}{DT} &= \rho\,\vec{i}_X\left\{\frac{\partial U}{\partial T} + \frac{U}{h_X}\frac{\partial U}{\partial X} + \frac{V}{h_Y}\frac{\partial U}{\partial Y}\right\} \\ &+ \rho\,\vec{i}_Y\left\{\frac{\partial V}{\partial T} + \frac{U}{h_X}\frac{\partial V}{\partial X} + \frac{V}{h_Y}\frac{\partial V}{\partial Y}\right\} \\ &+ \rho U(\vec{V}\cdot\nabla)\,\vec{i}_X + \rho V(\vec{V}\cdot\nabla)\,\vec{i}_Y \end{aligned} \tag{I-10}$$

Finally, we have to find out the last two terms. From vector calculus, we have

$$\frac{\partial \vec{i}_k}{\partial X_j} = \frac{\vec{i}_j}{h_k} \frac{\partial h_j}{\partial X_k} \qquad (k \neq j) \tag{I-11}$$

and

$$\frac{\partial \vec{i}_k}{\partial X_k} = \frac{\vec{i}_k}{h_k} \frac{\partial h_k}{\partial X_k} - \nabla \cdot h_k \tag{I-12}$$

Therefore,

$$\frac{\partial \vec{i}_X}{\partial X} = \frac{\vec{i}_X}{h_X} \frac{\partial h_X}{\partial X} - \frac{\vec{i}_X}{h_X} \frac{\partial h_X}{\partial X} - \frac{\vec{i}_Y}{h_Y} \frac{\partial h_X}{\partial Y} = - \frac{\vec{i}_Y}{h_Y} \frac{\partial h_X}{\partial Y} \tag{I-13}$$

$$\frac{\partial \vec{i}_X}{\partial Y} = \frac{\vec{i}_Y}{h_Y} \frac{\partial h_X}{\partial Y} \tag{I-14}$$

Similarly,

$$\frac{\partial \vec{i}_Y}{\partial X} = \frac{\vec{i}_X}{h_Y} \frac{\partial h_X}{\partial Y} \tag{I-15}$$

and

$$\frac{\partial \vec{i}_Y}{\partial Y} = - \frac{\vec{i}_X}{h_X} \frac{\partial h_Y}{\partial X} \tag{I-16}$$

Therefore,

$$(\vec{V} \cdot \nabla) \vec{i}_X = - \frac{U}{h_X} \frac{\vec{i}_Y}{h_Y} \frac{\partial h_X}{\partial Y} + \frac{V}{h_Y} \frac{\vec{i}_Y}{h_X} \frac{\partial h_Y}{\partial X} \tag{I-17}$$

and

$$(\vec{V} \cdot \nabla) \vec{i}_Y = \frac{U}{h_X} \frac{\vec{i}_X}{h_Y} \frac{\partial h_X}{\partial Y} - \frac{V}{h_Y} \frac{\vec{i}_X}{h_X} \frac{\partial h_Y}{\partial X} \tag{I-18}$$

Equations (I-17) and (I-18) are put into Eq. (I-10) and the result becomes,

$$\frac{D(\rho\vec{V})}{Dt} = \rho\vec{i}_X\left\{\frac{\partial U}{\partial T} + \frac{U}{h_X}\frac{\partial U}{\partial X} + \frac{V}{h_Y}\frac{\partial U}{\partial Y} + \frac{1}{h_X h_Y}\left[UV\frac{\partial h_X}{\partial Y} - V^2\frac{\partial h_Y}{\partial X}\right]\right\}$$

$$+\rho\vec{i}_Y\left\{\frac{\partial V}{\partial T} + \frac{U}{h_X}\frac{\partial V}{\partial X} + \frac{V}{h_Y}\frac{\partial V}{\partial Y} + \frac{1}{h_X h_Y}\left[UV\frac{\partial h_Y}{\partial X} - U^2\frac{\partial h_X}{\partial Y}\right]\right\} \qquad \text{(I-19)}$$

Therefore, the inertial force per unit volume in the X-direction is

$$\rho\left\{\frac{\partial U}{\partial T} + \frac{U}{h_X}\frac{\partial U}{\partial X} + \frac{V}{h_Y}\frac{\partial U}{\partial Y} + \frac{1}{h_X h_Y}\left(UV\frac{\partial h_X}{\partial Y} - V^2\frac{\partial h_Y}{\partial X}\right)\right\} \qquad \text{(I-20)}$$

and in the Y-direction is

$$\rho\left\{\frac{\partial V}{\partial T} + \frac{U}{h_X}\frac{\partial V}{\partial X} + \frac{V}{h_Y}\frac{\partial V}{\partial Y} + \frac{1}{h_X h_Y}\left(UV\frac{\partial h_Y}{\partial X} - U^2\frac{\partial h_X}{\partial Y}\right)\right\} \qquad \text{(I-21)}$$

The pressure forces per unit volume is independent of the coordinate axis chosen and is therefore equal to

$$-\frac{1}{h_X}\frac{\partial p}{\partial X} \qquad \text{and} \qquad -\frac{1}{h_Y}\frac{\partial p}{\partial Y} \qquad \text{(I-22)}$$

respectively.

Finally, we have to derive the viscous force terms. In two-dimensional form, we can write the viscous force per unit volume as,

$$\frac{\mu}{h_X h_Y}\left(\frac{\partial \vec{S}_X h_Y}{\partial X} - \frac{\partial \vec{S}_Y h_X}{\partial Y}\right) \qquad \text{(I-23)}$$

where

$$\vec{S}_X = \vec{i}_X \sigma'_{XX} + \vec{i}_Y \tau_{XY} \tag{I-24}$$

$$\vec{S}_Y = \vec{i}_X \tau_{YX} + \vec{i}_Y \sigma'_{YY} \tag{I-25}$$

Substituting Eqs. (I-24) and (I-25) into Eq. (I-23) we get

$$\begin{aligned} F_{viscous} = \frac{\mu}{h_X h_Y} \Big[& \vec{i}_X \frac{\partial(\sigma'_{XX} h_Y)}{\partial X} + \vec{i}_Y \frac{\partial(\tau_{XY} h_Y)}{\partial X} \\ & + \sigma'_{XX} h_Y \frac{\partial \vec{i}_X}{\partial X} + \tau_{XY} h_Y \frac{\partial \vec{i}_Y}{\partial X} \\ & + \vec{i}_X \frac{\partial(\tau_{YX} h_X)}{\partial Y} + \vec{i}_Y \frac{\partial(\sigma'_{YY} h_X)}{\partial Y} \\ & + \tau_{YX} h_X \frac{\partial \vec{i}_X}{\partial Y} + \sigma'_{YY} h_X \frac{\partial \vec{i}_Y}{\partial Y} \Big] \end{aligned} \tag{I-26}$$

Using Eqs. (I-11) through (I-14), we get

$$\begin{aligned} F_{viscous} = & \frac{\mu \vec{i}_X}{h_X h_Y} \Big[\frac{\partial(\sigma'_{XX} h_Y)}{\partial X} + \frac{\partial(\tau_{YX} h_X)}{\partial Y} + \tau_{XY} \frac{\partial h_X}{\partial Y} - \sigma'_{YY} \frac{\partial h_Y}{\partial X} \Big] \\ & + \frac{\mu \vec{i}_Y}{h_X h_Y} \Big[\frac{\partial(\tau_{XY} h_Y)}{\partial X} + \frac{\partial(\sigma'_{YY} h_X)}{\partial Y} + \tau_{YX} \frac{\partial h_Y}{\partial X} - \sigma'_{XX} \frac{\partial h_X}{\partial Y} \Big] \end{aligned} \tag{I-27}$$

Therefore, in the X- and Y-directions:

$$\begin{aligned} & F_{viscous,X} \\ & = \frac{\mu}{h_X h_Y} \Big[\frac{\partial(\sigma_{XX} h_Y)}{\partial X} + \frac{\partial(\tau_{YX} h_X)}{\partial Y} - \sigma'_{YY} \frac{\partial h_Y}{\partial X} + \tau_{XY} \frac{\partial h_X}{\partial Y} \Big] \end{aligned} \tag{I-28}$$

and

$F_{viscous,Y}$

$$= \frac{\mu}{h_X h_Y}\left[\frac{\partial(\tau_{XY} h_Y)}{\partial X} + \frac{\partial(\sigma_{YY}' h_X)}{\partial Y} + \tau_{YX}\frac{\partial h_Y}{\partial X} - \sigma_{XX}'\frac{\partial h_X}{\partial Y}\right] \qquad \text{(I-29)}$$

where, from the general Stokes' law,

$$\sigma_{XX}' = 2\mu\left[\frac{1}{h_X}\frac{\partial U'}{\partial X'} + \frac{V'}{h_X h_Y}\frac{\partial h_X}{\partial Y}\right] \qquad \text{(I-30)}$$

$$\sigma_{YY}' = 2\mu\left[\frac{1}{h_Y}\frac{\partial V'}{\partial Y} + \frac{U'}{h_X h_Y}\frac{\partial h_Y}{\partial X}\right] \qquad \text{(I-31)}$$

$$\tau_{XY} = \tau_{YX} = \mu\left[\frac{h_Y}{h_X}\frac{\partial}{\partial X}\left(\frac{V'}{h_Y}\right) + \frac{h_X}{h_Y}\frac{\partial}{\partial Y}\left(\frac{U'}{h_X}\right)\right] \qquad \text{(I-32)}$$

In these three equations, the absolute velocities U' and V' are used which are given in Eqs. (34) and (35) as:

$$U' = U + 2m(X-L)Y\, i\omega\phi\, \epsilon$$

and

$$V' = V + i\omega\phi\,\epsilon$$

The reason for this is that any physical law must be given referring to fixed, or absolute, coordinate system.

The acceleration terms that must be added on the left side of the moment equation is discussed in the text.

APPENDIX II

SOLUTION OF EQUATION (114)

In solving Eq. (114), a method combining the methods of successive approximation and momentum integral is used. The major simplification obtained is that the convective terms are small compared with the other terms. The idea was given by Lighthill,[18] but without a detailed mathematical proof. It is the purpose of this appendix to prove it mathematically.

It has been discussed before that ω' is a large parameter. According to the theory of differential equations containing a large parameter, we can retain only terms having this parameter as a factor and the highest order term to get a first approximation. This reduces Eq. (114) to

$$i\omega' U_{1c}'^{(1)} + [2m'(X'-1)\phi_s']\omega'^2(\delta' - 2Y^*)$$

$$= \frac{\partial^2 U_{1c}'^{(1)}}{\partial Y^{*2}} + \omega'^2 \phi_s' \frac{d\delta'}{dX'} \qquad \text{(II-1)}$$

where the superscript (1) means first approximation to Eq. (114). Comparing Eqs. (114) and (II-1) shows that the terms dropped are the convective terms. The question now is the accuracy of this approximation. In order to investigate it, let us write the solution to Eq. (114) as

$$U'_{1c} = U'^{(1)}_{1c} + U'^{(2)}_{1c} \tag{II-2}$$

The first approximation is solved from Eq. (II-1). To solve the second approximation, we substitute Eq. (II-2) into Eq. (114) and subtract Eq. (II-1) from it. Then we get

$$i\omega' U'^{(2)}_{1c} - \frac{\partial^2 U'^{(2)}_{1c}}{\partial Y^{*2}}$$

$$= -\left\{ U'_0 \frac{\partial U'_{1c}}{\partial X'} + U'_{1c} \frac{\partial U'_0}{\partial X'} + V^*_0 \frac{\partial U'_{1c}}{\partial Y^*} + V^*_{1c} \frac{\partial U'_0}{\partial Y^*} \right\} \tag{II-3}$$

The right-hand side of Eq. (II-3) is calculated approximately by replacing U'_{1c} and V^*_{1c} by their first approximation $U'^{(1)}_{1c}$ and $V^{*(1)}_{1c}$. Therefore, we have

$$i\omega' U'^{(2)}_{1c} - \frac{\partial^2 U'^{(2)}_{1c}}{\partial Y^{*2}}$$

$$= -\left\{ U'_0 \frac{\partial U'^{(1)}_{1c}}{\partial X'} + U'^{(1)}_{1c} \frac{\partial U'_0}{\partial X'} + V^*_0 \frac{\partial U'^{(1)}_{1c}}{\partial Y^*} + V^{*(1)}_{1c} \frac{\partial U'_0}{\partial Y^*} \right\} \tag{II-4}$$

The boundary conditions are

$$Y^* = 0 : \quad U'^{(2)}_{1c} = V^{*(2)}_{1c} = 0$$

$$Y^* = \delta' : \quad U'^{(2)}_{1c} = 0$$

In order to compare the two solutions, $U'^{(1)}_{1c}$ and $U'^{(2)}_{1c}$, let us first integrate Eq. (II-4) over the boundary layer thickness δ'. Thus we get

$$i\omega' \int_0^{\delta'} U_{1c}'^{(2)} dY^* + \left(\frac{\partial U_{1c}'^{(2)}}{\partial Y^*}\right)_{Y^*=0}$$

$$= \frac{d}{dX'} \int_0^{\delta'} (2U_0' - 1)\, U_{1c}'^{(1)} dY^* \qquad \text{(II-5)}$$

Again, the velocity profile is assumed to be

$$U_{1c}'^{(2)} = a_2 + b_2 \eta + c_2 \eta^2 + d_2 \eta^3 + e_2 \eta^4 \qquad \text{(II-6)}$$

with the boundary conditions

$$Y^* = 0 \quad : \quad U_{1c}'^{(2)} = 0 \quad , \quad \frac{\partial^2 U_{1c}'^{(2)}}{\partial Y^{*2}} = 0$$

$$Y^* = \delta' \quad : \quad U_{1c}'^{(2)} = 0 \quad , \quad \frac{\partial U_{1c}'^{(2)}}{\partial Y^*} = 0$$

The constant are therefore

$$a_2 = 0$$

$$c_2 = 0$$

$$d_2 = -3b_2$$

$$e_2 = 2b_2$$

The velocity profile then becomes

$$U_{1c}'^{(2)} = b_2 (\eta - 3\eta^3 + 2\eta^4) \qquad \text{(II-7)}$$

The velocity profiles of U_0', $U_{1c}'^{(1)}$, and $U_{1c}'^{(2)}$ given by Eqs. (145), (160), and (II-7), respectively are substituted into Eq. (II-5). We then get

$$i\omega'(0.15)\delta' b_2 + \frac{b_2}{\delta'} = (0.0331)\frac{d(\delta' b_1)}{dX'} + (0.0175)\frac{d(\delta' \psi)}{dX'} \qquad \text{(II-8)}$$

Solving for b_2, we get

$$b_2 = \frac{(156.486)\,\omega'^2 \Phi_s' \sqrt{X'}\left\{-2m'^2X'^4 + 4m'^2X'^3 + (3.5m' - 2m'^2)X'^2 - 2.5m'X' - 0.25\right\} e^{-\alpha_2}}{\sqrt{1 + (26.337)\,\omega'^2 X'^2}} \tag{II-9}$$

where

$$\alpha_2 = \tan^{-1}(5.132)\,\omega' X' \tag{II-10}$$

It can be shown numerically that $U'^{(2)}_{1c}$ is always less than 1% of $U'^{(1)}_{1c}$. Therefore, we can take $U'^{(1)}_{1c}$ as the solution to Eq. (114) and the error introduced is thus less than 1%. This is shown numerically in Appendix III.

APPENDIX III

A NUMERICAL EXAMPLE

In order to show the validity of the various assumptions made in the analysis and also get a more clear idea of the effect of a localized vibration on the flow, a numerical example is worked out in this appendix.

Let us consider a semi-infinite flat plate with air flows at 48 fps. The localized vibration is put at x = 2 ft from the leading edge. The viscosity of air is taken to be 0.144×10^{-3} fps.

From the given conditions, it is seen that

$$Re = \frac{U_\infty L}{\nu} = \frac{48 \times 2}{0.144 \times 10^{-3}} = 6.67 \times 10^5$$

$$\delta = 5X\sqrt{\frac{\nu}{U_\infty X}} = (5 \times 2)\sqrt{\frac{(0.144 \times 10^{-3})}{48 \times 2}} = 0.0122 \text{ ft} = 0.146 \text{ in.}$$

Following assumption c which limits ϵ to be small compared with δ, i.e., $\epsilon \ll \delta$, ϵ can be selected as 0.015 in. in this example. Then,

$$\epsilon' = \frac{\epsilon}{L} = \frac{0.0015}{2 \times 12} = 6.25 \times 10^{-5}$$

$$\omega' = \omega\left(\frac{L}{U_\infty}\right) = \omega\left(\frac{2}{48}\right) = \frac{\omega}{24}$$

LIMITATIONS OF λ_f

From Eq. (192) and Figure 8, the steady change of skin friction is seen to be directly proportional to λ_f. When λ_f is so small as to introduce only a maximum change of 1% or less, the oscillation may be considered to be of no effect to the unperturbed flow. This gives the lower limit of λ_f which is approximately at

$$\lambda_f = \frac{\epsilon'^2 \omega'^4}{Re} = 10^{-4}$$

So,

$$\omega' = \left[\frac{(6.67 \times 10^5)(10^{-4})}{(6.25 \times 10^{-5})^2}\right]^{-4} = 360$$

or,

$$\omega = 24\,\omega' = 8600 \ {}^{rad}/_{sec} = 82000 \ cpm$$

Due to the limitation of the perturbation analysis, the maximum deviation must be kept small compared with the unperturbed value in order to make sure of the convergence of the series expansion (102). Taking 20% maximum deviation, λ_f is seen to be approximately

$$\lambda_f = \frac{\epsilon'^2 \omega'^4}{Re} = 20 \times 10^{-4}$$

So,

$$\omega' = 760$$

or,

$$\omega = 18200 \ {}^{rad.}/_{sec.} = 174,000 \ cpm$$

This gives the upper limit of λ_f in the analysis, above which the perturbation analysis being invalid.

ORDER-OF-MAGNITUDE

The order-of-magnitude of the various terms in Eqs. (64) through (67) can be grouped in accordance with their coefficients as follows:

$$[1] = 1$$

$$\left[\frac{\epsilon'}{\sqrt{Re}}\right] = 10^{-7}$$

$$\left[\frac{\epsilon'\omega'}{\sqrt{Re}}\right] = 10^{-5}$$

$$\left[\frac{1}{Re}\right] = 10^{-6}$$

$$\left[\frac{\epsilon'}{Re^{\frac{3}{2}}}\right] = 10^{-13}$$

$$\left[\frac{\epsilon'\omega'^2}{\sqrt{Re}}\right] = 10^{-2}$$

$$\left[\frac{\epsilon'}{Re^{\frac{5}{2}}}\right] = 10^{-19}$$

In the analysis, only terms of order-of-magnitude of 1 and 10^{-2} are retained. The other terms are seen to be small compared with these terms. This confirms the simplification made in Eqs. (74) through (92).

THE FIRST-ORDER VELOCITY

From Eq. (II-2), U'_{1c} is given as

$$U'_{1c} = U'^{(1)}_{1c} + U'^{(2)}_{1c}$$

where $U'^{(1)}_{1c}$ is given by Eq. (153) as

$$U'^{(1)}_{1c} = b_1\eta + \psi\eta^2 - (3b_1 + 2\psi)\eta^3 + (2b_1 + \psi)\eta^4$$

and $U'^{(2)}_{1c}$ given in Eq. (II-7) as:

$$U'^{(2)}_{1c} = b_2(\eta - 3\eta^3 + 2\eta^4)$$

To compare the order-of-magnitude of $U'^{(1)}_{1c}$ and $U'^{(2)}_{1c}$, it is only necessary to compare b_1 and b_2. Now, b_1 is given in Eq. (159) as

$$b_1 = \omega'^2 (f_{1r} - i f_{1i})$$

where

$$f_{1r} = (44.453)\sqrt{\bar{X}'}\,\phi_s' \,[(X'-1)X' - 0.25]$$

$$f_{1i} = \frac{1}{\omega'} \frac{\phi_s'}{\sqrt{\bar{X}'}} \left\{ 19.499 + (8.663)[(X'-1)X' - 0.25] \right.$$

and also b_2 is given by Eq. (II-9) as

$$b_2 = \frac{(156.486)\omega'^2 \phi_s' \sqrt{\bar{X}'}\,[-2m'^2 X'^4 + 4m'^2 X'^3 + (3.5m' - 2m'^2)X'^2 - 2.5\, m'X' - 0.25]\, e^{-\alpha_2}}{\sqrt{1 + (26.337)\omega'^2 X'^2}}$$

where

$$\alpha_2 = \tan^{-1}(5.132)\,\omega' X'$$

Before comparing b_1 and b_2, b_1 and b_2 themselves can be simplified. To simplify b_1, we compare f_{1r} and f_{1i} in Table III-1. The value of ω' is taken to be 360, its minimum value. It is seen that the error in neglecting f_{1i} is only 1%, i.e.,

$$f_{1r} >> f_{1i}$$

as shown in Table III-1. Therefore, we can write

$$b_1 = \omega'^2 (44.453)\sqrt{\bar{X}'}\,\phi_s' \,[(X'-1)X' - 0.25]$$

TABLE III-1

TABULATION OF F_{1r}, F_{1i}, AND b_2

X	F_{1r}	F_{1i}	b_2
.200000E 00	-.429785E 01	.522297E-01	-.660317E-01
.400000E 00	-.961128E 01	.467422E-01	-.752335E-01
.600000E 00	-.143776E 02	.466146E-01	-.563923E-01
.800000E 00	-.156624E 02	.475844E-01	-.556825E-02
1.000000E 00	-.111133E 02	.481479E-01	.635250E-01
.120000E 01	-.467866E 00	.472948E-01	.124418E 00
.140000E 01	.138944E 02	.443809E-01	.151452E 00
.160000E 01	.278531E 02	.392983E-01	.133929E 00
.180000E 01	.374228E 02	.325421E-01	.813180E-01
.200000E 01	.404725E 02	.250442E-01	.165248E-01
.220000E 01	.373359E 02	.178388E-01	-.371958E-01
.240000E 01	.301682E 02	.117294E-01	-.667636E-01
.260000E 01	.216656E 02	.710763E-02	-.718796E-01
.280000E 01	.139541E 02	.396549E-02	-.606875E-01
.300000E 01	.810871E 01	.203593E-02	-.432158E-01
.320000E 01	.426934E 01	.961644E-03	-.267813E-01
.340000E 01	.204306E 01	.417850E-03	-.146862E-01
.360000E 01	.890719E 00	.167030E-03	-.719876E-02
.380000E 01	.354438E 00	.614302E-04	-.317519E-02
.400000E 01	.128920E 00	.207893E-04	-.126610E-02

To simplify b_2, it is seen that

$$(26.337)\,\omega'^2 X'^2 >> 1$$

and that

$$\tan^{-1}(5.132\,\omega' X') \geq \tan^{-1}(185)$$

since the minimum values of ω' and X' in this analysis is 360 and 0.1, respectively. Therefore, α_2 is approximate $\frac{\pi}{2}$ and

$$b_2 = (30.492)\,\omega' \frac{\Phi_3'}{\sqrt{X'}}\left[-2m'^2 X'^4 + 4m'^2 X'^3 + (3.5m' - 2m'^2)X'^2 - 2.5m'X' - 0.25\right]$$

The minimum value of ω' is 360. For this value of ω', the value of b_2 is seen to be about 1% or less than f_{1r}, as shown in Table III-1. Therefore, we known that

$$U_{1c}'^{(2)} \ll U_{1c}'^{(1)}$$

and we can write

$$U_{1c}' = U_{1c}'^{(1)} = b_1'(44.453)(-\eta + 4.502\eta^2 - 6.004\eta^3 + 2.502\eta^4) \qquad \text{(III-1)}$$

where

$$b_1' = \omega'^2 \phi_s' \sqrt{X'}\,[m'(X'-1)X' - 0.25] \qquad \text{(III-2)}$$

PHASE ANGLES

From Eq. (III.1), the phase angle in skin friction is seen to be zero. In the case of heat transfer, the phase angle is given by Eq. (171) as

$$\alpha_2 = \tan^{-1}(5.132\,\omega' P_r^{\frac{1}{3}} X')$$

For $P_r = 0.72$ and $\omega' = 360$, this equation becomes

$$\alpha_2 = \tan^{-1}(1655\,X') \qquad \text{(III-3)}$$

The smallest value of X' in this analysis may be taken as 0.2, then

$$\alpha_2 = \tan^{-1}(331)$$

Considering that $\tan 89^\circ = 57.3$, we see that α_2 is approximately equal to 90° for all other ω' and X'.

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