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GENERAL GROUP—THEORETIC TRANSFORMATIONS FROM NONLINEAR TO LINEAR DIFFERENTIAL EQUATIONS

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FOREWORD

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ABSTRACT

A systematic method using S. Lie's continuous contact transformation groups is developed in this report which enables one to derive the class of transformations from nonlinear to linear differential equations. Examples are worked out in detail as illustrations of the procedure.

1. INTRODUCTION

The present report treats an important class of transformations, namely, the transformation from nonlinear to linear differential equations. A general method based on Lie's concept of infinitesimal contact transformation is developed. This transformation can be derived by a simple procedure involving mostly algebraic manipulations.

This report is one of a series exploring possible applications of continuous transformation groups to solving differential equations. The first report [8] in the series treats the class of transformation which reduces the number of variables in a partial differential equation by one (i.e., the similarity transformation). In the second report [9], the concept is applied to transform boundary value into initial value problems. The classes of transformations discussed (seeminly unrelated) have been shown to be based essentially on the same concept.

There are many transformations in the literature which will transform a certain nonlinear equation to its equivalent linear form. Table 1 gives a few examples of such transformations. Certain transformations have the property of raising the order of an equation [1,10], or transforming the given equation to a new equation of the same order [1-3,12]. Transformation can also be classified according to whether the independent variable is transformed (the dependent variable is always transformed). Some transformations reduce one non-linear equation to another which has known solutions, or known to be reducible to linear form [7]. Special equations may be linearized by more than one form of transformation proposed. A good example is the Riccati equation [10-13].

In general, there is no systematic method of transforming equations to a linear form. It is therefore of great interest if such a method can be developed in which a step-by-step procedure can be followed and the proper transformation derived. Such a method is developed in this report.

TABLE 1

EXAMPLES OF TRANSFORMATIONS FROM NONLINEAR TO LINEAR EQUATIONS

Nonlinear ODE	Transformations	Linear Form	Reference
$y' + B(x)y = R(x)y^k$	$Z = y^{1-k}$	$Z' + B(1 - k)Z = R(1 - k)$	Leibnitz [4]
$y' + B(x)y = R(x)y^k$	$y = \left\{\frac{Rg(1 - k)}{g'}\right\}^{1/1-k}$	$g'' + [B(1 - k) + (R'/R)]g' = 0$	Sugai [10]
$y' + B(x)y + Q(x)y^2 = R(x)$	$y = Z'/QZ$	$Z'' + [B - (Q'/Q)]Z' - QRZ = 0$	Sugai [10]
$y'' + AF_2(y)y' + BF_2(y) \int F_2(y)dy + CF_2(y) = 0$	$\frac{df}{dy} = \frac{d\eta}{dx} = F_2(y)$	$f'' + Af' + Bf + C = 0$	Dasarathy and Srinivasan [2]
$yy'' - 2y'^2 + 3\kappa y' - \kappa^2 - \kappa y = 0$	$x = \eta,\ y = \frac{f'}{\kappa(\eta)f}$	$f''' - 2\frac{\kappa'}{\kappa}f'' - (\frac{\kappa''}{\kappa} - 2\frac{\kappa'^2}{\kappa^2})f' = 0$	Sugai [10]

2. REVIEW OF THE CONCEPT OF INFINITESIMAL CONTACT TRANSFORMATION GROUPS AND THE GENERAL METHOD

In this section, a brief review of the concept of the infinitesimal contact transformation group will be presented. Only those concepts closely related to the present problem will be given here. A detailed treatment may be found in Ref. 8. It will be shown at the end of this section how these concepts can be applied to the class of transformations from nonlinear to linear equations.

2.1. THE INFINITESIMAL CONTACT TRANSFORMATIONS

2.1.1. Infinitesimal Transformation

Let the transformation be

$$\begin{aligned} \phi(x,y,a_o) &= x \\ \psi(x,y,a_o) &= y \end{aligned} \tag{2.1}$$

Let the infinitesimal transformation be defined as

$$\begin{aligned} x_1 &= \phi(x,y,a_o + \delta\epsilon) \\ &= \phi(x,y,a_o) + \frac{\delta\epsilon}{1!}\left(\frac{\partial\phi}{\partial a}\right)_{a_o} + \frac{(\delta\epsilon)^2}{2!}\left(\frac{\partial^2\phi}{\partial a^2}\right)_{a_o} + \ldots \end{aligned} \tag{2.2}$$

$$\begin{aligned} y_1 &= \psi(x,y,a_o + \delta\epsilon) \\ &+ \psi(x,y,a_o) + \frac{\delta\epsilon}{1!}\left(\frac{\partial\psi}{\partial a}\right)_{a_o} + \frac{(\delta\epsilon)^2}{2!}\left(\frac{\partial^2\psi}{\partial a^2}\right)_{a_o} + \ldots \end{aligned} \tag{2.3}$$

Since $\delta\epsilon$ is infinitesimal, higher-order terms of $\delta\epsilon$ can be neglected in Eqs. (2.2) and (2.3). Also, using the definition of the identity transformation, Eqs. (2.2) and (2.3) become:

$$x_1 = x + \xi(x,y)\delta\epsilon$$
$$y_1 = y + \eta(x,y)\delta\epsilon \tag{2.4}$$

2.1.2. Notation for the Infinitesimal Transformation

The employment of the infinitesimal transformation

$$x_1 = x + \xi\delta\epsilon \quad \text{and} \quad y_1 = y + \eta\delta\epsilon$$

in conjunction with the function $f(x,y)$ will be to transform $f(x,y)$ into $f(x_1,y_1)$ which, upon expanding in Taylor series, becomes

$$\begin{aligned} f(x_1,y_1) &= f(x + \xi\delta\epsilon, y + \eta\delta\epsilon) \\ &= f(x,y) + \frac{\delta\epsilon}{1!}\left(\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}\right) \\ &+ \frac{(\delta\epsilon)^2}{2!}\left(\xi^2 \frac{\partial^2 f}{\partial x^2} + 2\xi\eta \frac{\partial^2 f}{\partial x \partial y} + \eta^2 \frac{\partial^2 f}{\partial y^2}\right) \\ &+ \dots \\ &+ \frac{(\delta\epsilon)^n}{n!}\left(\xi^n \frac{\partial^n f}{\partial x^n} + C_1^n \xi^{n-1} \eta \frac{\partial^n f}{\partial x^{n-1} \partial y} + \dots \right. \\ &\quad \left. \dots + \eta^n \frac{\partial^n f}{\partial y^n}\right) \\ &= f(x,y) + \frac{\delta\epsilon}{1!} Uf + \frac{(\delta\epsilon)^2}{2!} U^2 f + \dots \end{aligned} \tag{2.6}$$

where

$$Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} \tag{2.7}$$

is called the group representation and $U^n f$ means repeating the operator U for n times.

2.1.3. Invariant Function

If $f(x_1,y_1) = f(x,y)$, then f is invariant under the infinitesimal transformation.

Theorem. The necessary and sufficient condition that $f(x,y)$ be invariant under the group represented by Uf is Uf = 0; i.e.,

$$\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} = 0 \tag{2.8}$$

To solve for the invariant function, we solve the related differential equation

$$\frac{dx}{\xi} = \frac{dy}{\eta} \tag{2.9}$$

If the solution is

$$\Omega(x,y) = \text{constant} \tag{2.10}$$

then this function is the invariant function for the infinitesimal transformation represented by Uf. Since Eq. (2.9) has only one independent solution depending on a simple arbitrary constant, a one-parameter group in two variables has one and only one independent invariant.

2.1.4. Extension to n Variables

The condition for $f(x_1,\ldots,x_n)$ to be invariant under an infinitesimal transformation is

$$Uf = \xi_1(x_1,\ldots,x_n) \frac{\partial f}{\partial x_1} + \ldots$$

$$\ldots + \xi_n(x_1,\ldots,x_n) \frac{\partial f}{\partial x_n} = 0 \tag{2.11}$$

To get the invariant functions, we solve

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \ldots = \frac{dx_n}{\xi_n} \tag{2.12}$$

Since there exists (n - 1) independent solutions, a one-parameter group in n variables has (n - 1) independent invariants.

2.1.5. Invariance of an Ordinary Differential Equation

Consider a kth-order ordinary differential equation

$$F(x,y,y',y'',\dots,y^{(k)}) = 0 \tag{2.13}$$

The equation is invariant under the infinitesimal transformation defined by

$$\begin{aligned} \bar{x} &= x + (\delta\epsilon)\ \xi(x,y,y') \\ \bar{y} &= y + (\delta\epsilon)\ \eta(x,y,y') \\ \bar{y}' &= y' + (\delta\epsilon)\ \pi_1(x,y,y') \\ &\dots\dots \\ &\dots\dots \\ \bar{y}^{(k)} &= y^{(k)} + (\delta\epsilon)\ \pi_k(x,y,y',\dots,y^{(k)}) \end{aligned} \tag{2.14}$$

if the following condition is satisfied:

$$UF = 0 \tag{2.15a}$$

or, in expanded form,

$$\xi \frac{\partial F}{\partial x} + \eta \frac{\partial F}{\partial y} + \pi_1 \frac{\partial F}{\partial y'} + \dots + \pi_k \frac{\partial F}{\partial y^{(k)}} = 0 \tag{2.15b}$$

If Eq. (2.13) is invariant under the group of transformations defined in Eq. (2.14), then Eq. (2.13) can be expressed in terms of the (k - 1) functionally independent invariants of the transformation which can be found as solutions of the system of equations [8,9]:

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \dots = \frac{dy^{(k)}}{\pi_k} \tag{2.16}$$

For example, if a second-order differential equation

$$F(x,y,y',y") = 0 \tag{2.17}$$

is invariant under the group of transformation represented by

$$U = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \pi_1 \frac{\partial}{\partial y'} + \pi_2 \frac{\partial}{\partial y"} \tag{2.18}$$

where

$$\begin{aligned} \xi &= -y \\ \eta &= x \\ \pi_1 &= 1 + y'^2 \\ \pi_2 &= 3y'y" \end{aligned} \tag{2.19}$$

then it can be expressed in terms of the three functionally independent solutions of the system of equations

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dy'}{1 + y'^2} = \frac{dy"}{3y'y"} \tag{2.20}$$

The three independent solutions of Eq. (2.20) are

$$\begin{aligned} x^2 + y^2 &= C_1 \\ \frac{y - xy'}{x + yy'} &= C_2 \\ \frac{y"^2}{(1 + y'^2)^3} &= C_3 \end{aligned} \tag{2.21}$$

Based on the theory, Eq. (2.17) can be expressed in terms of these three invariants, i.e., if we designate

$$\begin{aligned} \eta_1 &= x^2 + y^2 \\ \eta_2 &= \frac{y - xy'}{x + yy'} \\ \eta_3 &= \frac{y"^2}{(1+y'^2)^3} \end{aligned} \tag{2.22}$$

Then, Eq. (2.17) can be transformed to

$$E(\eta_1, \eta_2, \eta_3) = 0 \tag{2.23}$$

The concepts described will serve as a basis for the method developed in this report.

For a given group of transformations, the functions $\xi, \eta, \pi_1, \ldots, \pi_k$ are known. Equation (2.15) gives the condition which the given differential equation, Eq. (2.13), must satisfy if it is to be invariant under the given transformation group. For example, in the illustration just presented, the invariants, Eq. (2.22), are fixed if the group of transformations is given, as in Eqs. (2.18) and (2.19). Then the condition (2.15) will serve only as a test as to whether a given equation, Eq. (2.17), can actually be transformed to (2.23). However, if the transformation is not given, Eq. (2.15) alone will not be enough to search for possible groups under which a given differential equation will be invariant. At this point, the theories developed in Ref. 8 on the concept of an infinitesimal contact transformation must be introduced which will ultimately make it possible to express these functions in terms of the so-called "characteristic function."

2.1.6. Definition of a Contact Transformation [5]

The definition of a contact transformation was given by Lie as follows [5]: When $Z, X_1, X_2, \ldots, X_n$, $P_1, \ldots, P_n$ are n + 1 independent functions of the 2n + 1 independent quantities $z, x_1, \ldots, x_n$, $p_1, \ldots, p_n$ such that the relation

$$dZ - P_i dX_i = \rho(dz - p_i dx_i) \tag{2.24}$$

(where ρ does not vanish) is identically satisfied, then the transformation defined by the equations

$$z' = Z, \quad x' = X, \quad p' = P \tag{2.25}$$

is called a contact transformation.

For a detailed discussion on the meaning of this transformation, the reader is referred to Refs. 5, 8, and 9.

2.1.7. Infinitesimal Contact Transformation

From Eq. (2.24),

$$\frac{\partial Z}{\partial z} dz + \frac{\partial Z}{\partial x_i} dx_i + \frac{\partial Z}{\partial p_i} dp_i - P_i\left(\frac{\partial X_i}{\partial z} dz + \frac{\partial X_i}{\partial x_r} dx_r + \frac{\partial X_i}{\partial p_r} dp_r\right)$$

$$= \rho(dz - p_i dx_i) \qquad (2.26)$$

For the infinitesimal transformation

$$Z = z + (\delta\epsilon)\zeta; \quad X_i = x_i + (\delta\epsilon)\xi_i; \quad P_i = p_i + (\delta\epsilon)\pi_i \qquad (2.27)$$

we get

$$\begin{aligned} \frac{\partial\zeta}{\partial z} - p_i \frac{\partial\xi_i}{\partial z} &= \sigma \\ \frac{\partial\zeta}{\partial p_r} - p_i \frac{\partial\xi_i}{\partial p_r} &= 0 \\ \frac{\partial\zeta}{\partial x_r} - p_i \frac{\partial\xi_i}{\partial x_r} &= -\sigma p_r \end{aligned} \qquad (2.28)$$

If a characteristic function is defined as

$$\begin{aligned} \xi_r &= \frac{\partial W}{\partial p_r}, \quad \zeta = p_i \frac{\partial W}{\partial p_i} - W, \\ \pi_r &= -\frac{\partial W}{\partial x_r} - p_r \frac{\partial W}{\partial z} \end{aligned} \qquad (2.29)$$

Higher-order transformation functions, π_{ij}, π_{ijk}, etc., can also be expressed in terms of W. However, due to the complexity in their derivation, they will not be included here (see Ref. 8). However, a special case with one independent and one dependent variable will be given here since it will be needed later. For this case, we consider an infinitesimal transformation

$$\begin{aligned}
x' &= x + (\delta\epsilon)\ \xi(x,y,p) \\
y' &= y + (\delta\epsilon)\ \Theta(x,y,p) \\
p' &= p + (\delta\epsilon)\ \pi(x,y,p) \\
q' &= q + (\delta\epsilon)\ k(x,y,p,q) \\
r' &= r + (\delta\epsilon)\ \rho(x,y,p,q,r)
\end{aligned} \tag{2.30}$$

where $p = dy/dx$, $q = d^2y/dx^2$, and $r = d^3y/dx^3$. The transformation functions can be expressed in terms of a characteristic function W as [8,9]:

$$\xi = \frac{\partial W}{\partial p}$$

$$\Theta = p\,\frac{\partial W}{\partial p} - W$$

$$-\pi = XW$$

$$-k = (X^2 + 2qX\,\frac{\partial}{\partial p} + q^2\,\frac{\partial^2}{\partial p^2} + q\,\frac{\partial}{\partial f})W$$

$$\begin{aligned}
-\rho = {} & (X^3 + 3qX^2\,\frac{\partial}{\partial p} + 3q^2X\,\frac{\partial^2}{\partial p^2} + q^3\,\frac{\partial^3}{\partial p^3} + 3qX\,\frac{\partial}{\partial y} + 3q^2\,\frac{\partial^2}{\partial f\,\partial p})W \\
& + r(3q\,\frac{\partial^2}{\partial p^2} + 3X\,\frac{\partial}{\partial y})W
\end{aligned} \tag{2.31}$$

where

$$X = \frac{\partial}{\partial x} + p\,\frac{\partial}{\partial y} \quad .$$

2.2. THE GENERAL METHOD

Consider again the differential equation

$$F_1(x,y,y',\ldots,y^{(k)}) = 0 \tag{2.32}$$

and the infinitesimal transformation defined in Eq. (2.14). In this case, the transformation is not given, i.e., $\xi, \eta, \pi_1, \ldots, \pi_k$ are not known functions of the variables, and the problem is to seek possible groups of transformations under which Eq. (2.32) is invariant.

To this end, we again employ the condition that Eq. (2.32) will be invariant under any group of transformations if

$$Uf = 0 \tag{2.33}$$

or, in its expanded form

$$\xi \frac{\partial F_1}{\partial x} + \eta \frac{\partial F_1}{\partial y} + \pi_1 \frac{\partial F_1}{\partial y'} + \ldots + \pi_k \frac{\partial F_1}{\partial y^{(k)}} = 0 \tag{2.34}$$

Now, the expression for F_1 is substituted into Eq. (2.34) and also the transformation functions, $\xi, \eta, \pi_1, \ldots, \pi_k$ (which are expressible as functions of a single unknown function, namely, the characteristic function W) as given in Eq. (2.31), are replaced by the expressions given in Eq. (2.31). The resulting equation will be an equation for the characteristic function W. Any solution of W from this equation will satisfy the condition that the differential equation is invariant.

Once the characteristic function is determined, the transformation functions can be determined from Eq. (2.31) and the invariants of this group can be found from the solution of Eq. (2.16). The (k + 1) functionally independent solutions of Eq. (2.16) will be the new variables, i.e., if the (k + 1) solutions to Eq. (2.16) are

$$u_i(x,y,y',\ldots,y^{(k)}) = C_i = \text{constant}$$

$$(i = 1,\ldots,k+1)$$

then the transformation is defined by

$$\eta_i = u_i(x,y,y',\ldots,y^{(k)})$$

$$(i = 1,\ldots,k+1)$$

Generally, there are three approaches to finding classes of transformations that will transform a nonlinear differential equation to linear equation. We may start from a nonlinear differential equation and search for a particular transformation which will reduce the equation to a linear form. A second approach is to start from a linear differential equation and search for all possible groups of transformations, each of which can be used to transform the linear equation to a class of nonlinear equations. The answers thus obtained will define classes of nonlinear equations reducible to the particular linear equation. Finally a nonlinear equation may be reduced to another whose solution is known, or which can itself be reduced to a linear form. All three cases will be considered in this report.

3. APPLICATIONS OF THE GENERAL GROUP-THEORETIC METHOD

In this section the general theories given in section 2 will be applied to nonlinear ordinary differential equations of the first-second-, and third-orders with variable coefficients. The transformations thus obtained will be compared with available transformations in the literature.

3.1. FIRST-ORDER DIFFERENTIAL EQUATIONS

There are very few ordinary nonlinear differential equations which can be reduced to linear equations by special transformations. Among the few are the equations of Bernoulli, Jacobs, and Riccati.

The Bernoulli's equation

$$y' + B(x)y = R(x)y^{k} \tag{3-1}$$

provides a good example for the definition of a class of transformations from nonlinear to linear differential equations, with the order of differentiation remaining the same. The transformation discovered by Leibnitz [4] transforming Eq. (3-1) to a linear form is:

$$Z(x) = y^{1-k} \quad (k \neq 0,\ k \neq 1) \tag{3-2}$$

Under this transformation, Eq. (3-1) becomes

$$Z' + B(1 - k)Z = R(1 - k) \tag{3-3}$$

which is a first-order linear differential equation.

The transformation defined in Eq. (3-2) is not the only one which can reduce the Bernoulli's equation to a linear form. Sugai, in a recent paper [10], introduced the transformation

$$y = \left\{\left[\frac{Rg(1 - k)}{g'}\right]\right\}^{1/1-k} \tag{3-4}$$

which transforms Eq. (3-1) to a second-order linear ordinary differential equation

$$g'' + \left[B(1 - k) + \left(\frac{R'}{R}\right)\right] g' = 0 \tag{3-5}$$

This class of transformations in which linearization is achieved by raising the order of the equation can be applied to nonlinear differential equations of higher-orders.

Another example of this class of transformation is associated with the well-known generalized Riccati's equation

$$y' + B(x)y + Q(x)y^2 = R(x) \tag{3-6}$$

By introducing the Riccati transformation [10]

$$y = \frac{Z'}{QZ} \tag{3-7}$$

where $Z = Z(x)$, Eq. (3-6) becomes

$$Z'' + \left[B - \left(\frac{Q'}{Q}\right)\right] Z' - QRZ = 0 \tag{3-8}$$

Most recently, Mason [7] introduced a new transformation which can be applied to a class of nonlinear first-order ordinary differential equations which are reducible either to the Bernoulli's equation or the Riccati's equation. Linearization is achieved by further transforming this equation to the first-order linear ordinary differential equation using Eq. (3-2) or Eq. (3-7).

According to Mason [7], equations of the form

$$y'f(y) + B(x) \int f(y)dy = R(x) \{\int f(y)dy\}^k \tag{3-9}$$

under the transformation defined by

$$x = X \ , \quad \frac{dY}{dy} = f(y) \tag{3-10}$$

will be transformed to the Bernoulli's equations

$$Y' + B(X)Y = R(X)Y^k \tag{3-11}$$

If the transformation defined in Eq. (3-2) is introduced,

$$Z = Y^{1-k} \tag{3-12}$$

Eq. (3-11) is reduced to

$$Z' + B(X)Z = R(X) \tag{3-13}$$

Combining the two transformations, we conclude that Eq. (3-9) is transformed to the linear equation, Eq. (3-13), by

$$x = X \quad , \quad Z = \{\int f(y)dy\}^{1-k} \tag{3-14}$$

The other class of nonlinear equations treated by Mason [7] is

$$y'f(y) + B(x)\ \{\int f(y)dy\} + Q(x)\ \{\int f(y)dy\}^2 = R(x) \tag{3-15}$$

Under the transformation defined by Eq. (3-10), Eq. (3-15) becomes

$$Y' + B(X)Y + Q(X)Y^2 = R(X) \tag{3-16}$$

which is seen to be the Riccati's equation. Equation (3-16) can be linearized by using the transformation defined by Eq. (3-7), i.e.,

$$Y = \frac{Z'}{QZ} \tag{3-17}$$

which leads to Eq. (3-8). Combining the two transformations, Eqs. (3-10) and (3-7), we conclude that Eq. (3-15) can be transformed to the linear differential equation, Eq. (3-8), by the following transformation

$$\int f(y)dy = \frac{Z'}{QZ} \quad , \quad X = x \tag{3-18}$$

While the utility of this type of transformation is obvious, the main difficulty one faces is the arbitrariness in the definition of a proper transformation. In other words, for a given nonlinear differential equation, there is no general way to derive a transformation which will linearize the equation. To overcome this difficulty, an attempt is made in the present report to develop a method which will serve such a purpose. We now treat, as a first example, the Bernoulli's equation, given in Eq. (3-1),

$$Y' + B(X)Y = R(X)Y^k \tag{3-19}$$

Introducing the notation $p = Y'$, Eq. (3-19) becomes

$$p + B(X)Y = R(X)Y^k \tag{3-20}$$

Next, an infinitesimal transformation is defined as

$$\begin{aligned} X' &= X + (\delta\epsilon)\xi(X,Y,p) \\ Y' &= Y + (\delta\epsilon)\theta(X,Y,p) \\ p' &= p + (\delta\epsilon)\pi(X,Y,p) \end{aligned} \tag{3-21}$$

where, in terms of the characteristic function W, [8],

$$\xi = \frac{\partial W}{\partial p}$$

$$\theta = p \frac{\partial W}{\partial p} - W \tag{3-22}$$

$$\pi = -\frac{\partial W}{\partial X} - p \frac{\partial W}{\partial Y}$$

According to the theories discussed previously, we impose the condition on the differential equation that it be invariant under the transformation defined by Eq. (3-21), i.e.,

$$\frac{\delta F}{\delta \epsilon} = 0 \tag{3-23}$$

where from Eq. (3-20),

$$F = p + B(X)Y - R(X)Y^k \tag{3-24}$$

The condition (3-23) can be written in expanded form as

$$\xi \frac{\partial F}{\partial X} + \theta \frac{\partial F}{\partial Y} + \pi \frac{\partial F}{\partial p} = 0 \tag{3-25}$$

Upon substitution of F from Eq. (3-24),

$$(B'Y - R'Y^k) \frac{\partial W}{\partial p} + (B - kRY^{k-1})(p \frac{\partial W}{\partial p} - W) - (\frac{\partial W}{\partial X} + p \frac{\partial W}{\partial Y}) = 0 \tag{3-26}$$

Eq. (3-26) is a first-order linear partial differential equation in W, which can be used to determine the functional form of W. We note that any function W which satisfies Eq. (3-26) will satisfy the condition of invariance for Eq. (3-20) under the infinitesimal contact transformation defined by Eq. (3-21). Once W is determined, the invariants can be found by solving the following system of equations

$$\frac{\partial X}{\xi} = \frac{dY}{\theta} = \frac{dp}{\pi} \tag{3-27}$$

In terms of the characteristic function W,

$$\frac{dX}{\frac{\partial W}{\partial p}} = \frac{dY}{p \frac{\partial W}{\partial p} - W} = \frac{dp}{-(\frac{\partial W}{\partial X} + p \frac{\partial W}{\partial Y})} \tag{3-28}$$

If the two functionally independent solutions of Eq. (3-28) are denoted by

$$u(X,Y,p) = \text{constant}$$
$$v(X,Y,p) = \text{constant}$$

the new independent and dependent variables can then be taken as

$$\begin{aligned} \eta &= u(X,Y,p) \\ f(\eta) &= v(X,Y,p) \end{aligned} \tag{3-29}$$

All the examples mentioned in this section deal with the type of transformation in which the independent variable X remains the same, i.e., X = X'. Making this assumption, we have, from Eq. (3-22),

$$\xi = \frac{\partial W}{\partial p} \equiv 0 \tag{3-30}$$

or, W is independent of p. Equation (3-26) now becomes

$$(B - kRY^{k-1})W + \frac{\partial W}{\partial X} + p\,\frac{\partial W}{\partial Y} = 0 \tag{3-31}$$

which will be used to determine W.

Since Eq. (3-31) is linear in W, we assume a solution

$$W(X,Y) = \phi(X)\psi(Y) \tag{3-32}$$

Upon substitution of W from Eq. (3-32) into Eq. (3-31), we get,

$$B - kRY^{k-1} + \frac{\phi'}{\phi} + p\,\frac{\psi'}{\psi} = 0 \tag{3-33}$$

Substituting p from Eq. (3-20) into Eq. (3-33) and rearranging the terms, Eq. (3-33) becomes

$$\left\{(1 - k)B + \frac{\phi'}{\phi}\right\} - p\left\{\frac{k}{Y} - \frac{\psi'}{\psi}\right\} = 0 \tag{3-34}$$

from which

$$\phi' = -(1 - k)B\phi$$

$$\psi' = \frac{k\psi}{Y}$$

Thus,

$$\phi = \exp\{-\int (1 - k)BdX\} \tag{3-35}$$
$$\psi = Y^k$$

and the characteristic function becomes

$$W(X,Y) = Y^k \exp\{-\int (1 - k)BdX\} \tag{3-36}$$

Equation (3-28) now takes the form

$$\frac{dX}{0} = \frac{dY}{-Y^k \exp\{-\int (1-k)BdX\}} = \frac{dp}{\{Y^k(1-k)B - kpY^{k-1}\} \exp\{-\int (1-k)BdX\}} \tag{3-37}$$

The first solution from Eq. (3-37) is obviously

$$X = \text{constant} \tag{3-38}$$

and the second can be found from

$$\frac{dY}{-Y^k} = \frac{dp}{(1 - k)BY^k - kpY^{k-1}} \tag{3-39}$$

However, if p is solved from Eq. (3-20) and substituted into Eq. (3-39), we have

$$\frac{dY}{-Y^k} = \frac{dY}{-Y^k} \tag{3-39a}$$

which means Eq. (3-39) is not an independent equation. As a result, the second solution becomes

$$F(Y) = \text{constant} \tag{3-40}$$

since any function, F, of Y will satisfy Eq. (3-39a). Thus, from Eq. (3-29), we put

$$\eta = X \tag{3-41}$$
$$f(\eta) = F(Y)$$

In particular, if $F(Y) = Y^{1-k}$, the transformation (3-2) is obtained. If F(Y) is written in the form of

$$Y = \int f(\eta)d\eta \tag{3-42}$$

the transformation, (3-10), is obtained.

As a result, the transformation defined in Eq. (3-2) serves as a connection between the Bernoulli's equation and the general linear, first-order ordinary differential equation, Eq. (3-3), whereas transformations of type (3-10) connect one class of equations, Eq. (3-9) to the Bernoulli's equation. Combination of these two transformations, (3-2) and (3-10), leads to the transformation required to linearize Eq. (3-9), as given in Eq. (3-14).

The class of equations given in Eq. (3-9) is not the only class of equations reducible to the Bernoulli's equation. In order to show the general nature of the present method, consider the case in which W is chosen arbitrarily as:

$$W = p \tag{3-43}$$

The form of W will satisfy the condition of invariance of the differential equation (3-1) if Eq. (3-26) is satisfied. Upon substituting W into that equation, we get

$$(B'Y - R'Y^k) = 0 \tag{3-44}$$

Eq. (3-44) is satisfied if both B and R are constants.

The invariants for this transformation can be solved from Eq. (3-28) which now is

$$\frac{dX}{1} = \frac{dY}{0} = \frac{dp}{0} \tag{3-45}$$

The solutions to Eq. (3-45) are

$$Y = \text{constant} \quad , \quad p = \text{constant}$$

From (3-29), we get

$$\eta = Y \quad , \quad \frac{df}{d\eta} = \frac{dY}{dX} \tag{3-46}$$

Eq. (3-1) then becomes

$$\frac{df}{d\eta} + (B\eta - R\eta^k) = 0 \tag{3-47}$$

which is again seen to be a linear differential equation.

Next, we consider the class of transformation in which linearization is achieved by raising the order of differentiation, as given in Eq. (3-4). In order to illustrate another approach, we start from the linear second-order ordinary differential equation

$$F = f'' + A(\eta)f' + B(\eta)f + C(\eta) = 0 \tag{3-48}$$

In terms of p and q where

$$p = f' \quad , \quad q = f''$$

Eq. (3-48) becomes

$$q + A(\eta)p + B(\eta)f + C(\eta) = 0 \tag{3-49}$$

As before, an infinitesimal contact transformation is introduced as

$$\begin{aligned} \eta' &= \eta + (\delta\epsilon)\xi(\eta,f,p) \\ f' &= f + (\delta\epsilon)\theta(\eta,f,p) \\ p' &= p + (\delta\epsilon)\pi(\eta,f,p) \\ q' &= q + (\delta\epsilon)k(\eta,f,p,q) \end{aligned} \tag{3-50}$$

where, in terms of the characteristic function W,

$$\begin{aligned} \xi &= \frac{\partial W}{\partial p} \\ \theta &= p\frac{\partial W}{\partial p} - W \\ -\pi &= XW \\ -k &= (X^2 + 2qX\frac{\partial}{\partial p} + q^2\frac{\partial^2}{\partial p^2} + q\frac{\partial}{\partial f})W \end{aligned} \tag{3-51}$$

The operator X in Eq. (3-51) is defined as

$$X = \frac{\partial}{\partial \eta} + p\frac{\partial}{\partial f}$$

Again, the invariance of Eq. (3-48) requires that

$$\frac{\delta F}{\delta \epsilon} = 0 \tag{3-52a}$$

or, in its expanded form,

$$\xi\frac{\partial F}{\partial \eta} + \theta\frac{\partial F}{\partial f} + \pi\frac{\partial F}{\partial p} + k\frac{\partial F}{\partial q} = 0 \tag{3-52b}$$

Replacing F by the expression given in Eq. (3-48), Eq. (3-52b) gives

$$(A'p + B'f + C')\xi + B\theta + A\pi + k = 0$$

In terms of the characteristic function W, we have,

$$(A'p + B'f + C')\frac{\partial W}{\partial p} + B(p\frac{\partial W}{\partial p} - W) - A(\frac{\partial W}{\partial \eta} + p\frac{\partial W}{\partial f})$$

$$-\left\{\frac{\partial^2 W}{\partial \eta^2} + 2p\frac{\partial^2 W}{\partial \eta \partial f} + p^2\frac{\partial^2 W}{\partial f^2} + 2q\frac{\partial^2 W}{\partial p \partial \eta}\right. \qquad (3\text{-}53a)$$

$$\left. + 2pq\frac{\partial^2 W}{\partial p \partial f} + q^2\frac{\partial^2 W}{\partial p^2} + q\frac{\partial W}{\partial f}\right\} = 0$$

Solving for q from Eq. (3-49) and substituting the result into Eq. (3-53), we finally obtain

$$\frac{\partial^2 W}{\partial \eta^2} + 2p\frac{\partial^2 W}{\partial \eta \partial f} + p^2\frac{\partial^2 W}{\partial f^2} + A(\frac{\partial W}{\partial \eta} + p\frac{\partial W}{\partial f})$$

$$- B(p\frac{\partial W}{\partial p} - W) - (A'p + B'f + C')\frac{\partial W}{\partial p}$$

$$(3\text{-}54)$$

$$- (2\frac{\partial^2 W}{\partial p \partial \eta} + 2p\frac{\partial^2 W}{\partial p \partial f} + \frac{\partial W}{\partial f})(Ap + Bf + C)$$

$$+ \frac{\partial^2 W}{\partial p^2}(Ap + Bf + C)^2 = 0$$

Equation (3-54) is the equation which determines the functional form of the characteristic function W.

Consider first the case in which the independent variable η is not transformed, i.e.,

$$x \equiv \eta$$

From Eqs. (3-50) and (3-51), this means

$$\frac{\partial W}{\partial p} = 0$$

or, W is independent of p.

Equation (3-54) then becomes

$$\frac{\partial^2 W}{\partial f^2} p^2 + 2 \frac{\partial^2 W}{\partial \eta \partial f} p + \frac{\partial^2 W}{\partial \eta^2} + A \frac{\partial W}{\partial \eta} + BW - (\frac{\partial W}{\partial f}) (Bf + C) = 0 \qquad (3\text{-}55)$$

Since W is independent of p, the coefficients of terms with various powers of p in Eq. (3-55) must be zero. We then have

$$\frac{\partial^2 W}{\partial f^2} = 0 \qquad (3\text{-}56a)$$

$$\frac{\partial^2 W}{\partial \eta \partial f} = 0 \qquad (3\text{-}56b)$$

$$\frac{\partial^2 W}{\partial \eta^2} + A \frac{\partial W}{\partial \eta} + BW - (\frac{\partial W}{\partial f}) (Bf + C) = 0 \qquad (3\text{-}56c)$$

Equations (3-56a,b) show that W should be of the form

$$W(\eta, f) = W_1(\eta) + fW_2 \qquad (3\text{-}57)$$

where W_2 is a constant. Substituting W given in Eq. (3-57) into Eq. (3-56c), we get

$$\frac{d^2 W_1}{d\eta^2} + A(\eta) \frac{dW_1}{d\eta} + B(\eta) W_1 = W_2 C(\eta) \qquad (3\text{-}58)$$

This is an ordinary linear differential equation for the solution of W_1. It can be obtained if A, B, and C are given.

Consider first the case in which

$$W_1 = 0 \ , \quad C = 0$$

For this case, Eq. (3-58) is obviously satisfied and the characteristic function becomes

$$W = fW_2$$

The invariances can be solved by

$$\frac{d\eta}{0} = \frac{df}{-W_2 f} = \frac{dp}{-W_2 p}$$

which gives

$$\eta = \text{constant}$$
$$\frac{p}{f} = \text{constant} \tag{3-59}$$

Thus, the new variables are

$$x = \eta$$
$$y = \frac{p}{f} = \frac{f'}{f} \tag{3-60}$$

As an example, the well-known generalized Riccat's equation

$$y' + P(x)y + Q(x)y^2 = R(x) \tag{3-61}$$

can be cited. It was shown by Riccati [4] that a transformation

$$y = \frac{f'}{Qf} \quad , \quad x = \eta \tag{3-62}$$

will reduce Eq. (3-61) to the linear form

$$f'' + (P - \frac{Q'}{Q})f' - QRf = 0 \tag{3-63}$$

Thus, by putting

$$A = P - \frac{Q'}{Q} \quad , \quad B = - QR \quad , \quad C = 0$$

into Eq. (3-48) and using transformation (3-60),* Eq. (3-48) is seen to be reducible to the generalized Riccati's equation.

Sugai [10] considered the extended Riccati's equation

$$y' + P(x)y + Q(x)y^{k} = R(x) \tag{3-64}$$

A transformation

$$y = \left[\frac{f'}{Qf(k - 1)}\right]^{1/k-1} \tag{3-65}$$

*Without losing its generality, we may divide the right-hand side of Eq. (3-60) by Q, as in (3-62).

was introduced and Eq. (3-64) was transformed to the form:

$$f'' + \left[P(k - 1) - \frac{Q'}{Q}\right] f' - R[Qf(f')^{k-2}(k - 1)^k]^{1/k-1} = 0 \tag{3-66}$$

Linearization is achieved in either of the following cases:

a. $k = 2 \ , \quad R \neq 0$

b. $k \neq 2 \ , \quad k \neq 1 \ , \quad R = 0$

For case a, Eq. (3-66) becomes

$$f'' + \left[P - \left(\frac{Q'}{Q}\right)\right] f' - RQf = 0$$

which is the case just treated [cf. Eq. (3-61)]. For case b, Eq. (3-66) gives

$$f'' + \left[P(k - 1) - \frac{Q'}{Q}\right] f' = 0 \tag{3-67}$$

Equation (3-67) is a special case of Eq. (3-48) with B = C = 0. Equation (3-59) is still valid. Thus, Eq. (3-67) can be expressed in terms of the following variables:

$$\begin{aligned} x &= F_1(\eta) \\ y &= F_2\left(\frac{f'}{f}\right) \end{aligned} \tag{3-68a}$$

where F_1 and F_2 represent any function. In particular, let us choose:

$$\begin{aligned} x &= \eta \\ y &= \left(\frac{f'}{Qfm}\right)^{1/n} \end{aligned} \tag{3-68b}$$

in order to obtain Eq. (3-65), where m and n are constants to be determined. The quantities Q and m are included in the denominator for the purpose of reducing Eq. (3-66b) to exactly the same form of Eq. (3-65). With or without Q and m, the transformed equation will always be in the form of Eq. (3-64). The difference lies only in the coefficients. Thus, the second transformation in (3-68) merely means that a power-law form of (3-59) is chosen.

Substituting transformation (3-68) into Eq. (3-67) then gives

$$y' + Py\left(\frac{k - 1}{n}\right) + \frac{m}{N} Qy^{n+1} = 0 \tag{3-69}$$

In order to reduce Eq. (3-69) to Eq. (3-64) (with $R = 0$), it is seen that

$$m = n = k - 1$$

Thus, Eq. (3-68) becomes the same transformation as given in Eq. (3-65).

As another example, if F_1 and F_2 in Eq. (3-68a) are chosen as

$$\begin{aligned} x &= \eta \\ y &= \frac{R(\eta)}{(\frac{f'}{f}) + B(\eta)} \end{aligned} \tag{3-70}$$

then the generalized Riccati's equation, Eq. (3-6), can be linearized to become

$$f'' - (\frac{R'}{R} - B)\, f' - R\left[Q - (\frac{B}{R})'\right] f = 0 \tag{3-71}$$

In order to show the general nature of the group-theoretic method, we consider the case in which $W_1 \neq 0$ in Eq. (3-58). If we now choose

$$A = \frac{1}{\eta}\ , \quad B = 1 - \frac{n}{\eta^2}\ , \quad C = 0$$

in Eqs. (3-48) and (3-58), we then let

$$f'' + \frac{1}{\eta} f' + (1 - \frac{n}{\eta^2})\, f = 0 \tag{3-72}$$

and the equation for W_1 as

$$\frac{d^2W_1}{d\eta^2} + \frac{1}{\eta}\frac{dW_1}{d\eta} + (1 - \frac{n}{\eta^2})\, W_1 = 0 \tag{3-73}$$

W_1 then becomes

$$W_1 = J_n(\eta) \tag{3-74}$$

and the characteristic function, Eq. (3-57) becomes

$$W = J_n(\eta) + f$$

where W_2 is taken to be 1 without loss of generality. The invariants can be solved by

$$\frac{d\eta}{0} = \frac{df}{- J_n - f} = \frac{dp}{- J_n' - f'}$$

which gives the transformation

$$s = \eta \quad , \quad y = \frac{f + J_n(\eta)}{p + J_n'(\eta)} \tag{3-75}$$

The class of nonlinear differential equations reducible to Bessel equation is thus

$$y' - Ay + \left[\frac{J_n'' + AJ_n' - Bf + C}{f + J_n}\right] y^2 - 1 = 0 \tag{3-76a}$$

where f is related to y through Eq. (3-75), i.e.,

$$\frac{df}{dx} = \frac{f + J_n}{y} \tag{3-76b}$$

3.2. SECOND-ORDER DIFFERENTIAL EQUATIONS

In the last example of article 3.1, a linear second-order differential equation, Eq. (3-48), was analyzed. The problem considered was the search for certain transformations under which a first-order nonlinear ordinary differential equations can be transformed to Eq. (3-48). In other words, we are considering linearization by raising the order of equations. In the present section, however, we will consider the class of transformation in which a second-order nonlinear ordinary differential equation is reduced to a second-order linear equation.

Suppose that the characteristic function in Eq. (3-51) is a linear function of p, i.e.

$$W(\eta, f, p) = W_1(\eta, f)p \tag{3-77}$$

then the condition for the invariance of Eq. (3-48) requires that Eq. (3-54) be satisfied identically. For W given in the form of Eq. (3-77), Eq. (3-54) becomes

$$\begin{aligned} &\frac{\partial^2 W_1}{\partial \eta^2} p + 2p^2 \frac{\partial^2 W_1}{\partial \eta \partial f} + p^3 \frac{\partial^2 W_1}{\partial f^2} \\ &+ Ap \left(\frac{\partial W_1}{\partial \eta} + p \frac{\partial W_1}{\partial f}\right) - (A'p + B'f + C')W_1 \\ &- \left(2 \frac{\partial W_1}{\partial \eta} + 3p \frac{\partial W_1}{\partial f}\right) (Ap + Bf + C) = 0 \end{aligned} \tag{3-78}$$

If we further simplify the problem by considering a linear differential equation with constant coefficients, i.e., A, B, and C in Eq. (3-48) are constants, Eq. (3-78) then becomes

$$f_o + f_1 p + f_2 p^2 + f_3 p^3 = 0 \tag{3-79}$$

where

$$f_o = -2 \frac{\partial W_1}{\partial \eta} (Bf + C) \tag{3-80a}$$

$$f_1 = \frac{\partial^2 W_1}{\partial \eta^2} - A \frac{\partial W_1}{\partial \eta} - 3 \frac{\partial W_1}{\partial f} (Bf + C) \tag{3-80b}$$

$$f_2 = 2 \frac{\partial^2 W_1}{\partial \eta \partial f} - 2A \frac{\partial W_1}{\partial f} \tag{3-80c}$$

$$f_3 = \frac{\partial^2 W_1}{\partial f^2} \tag{3-80d}$$

Since all the f's are independent of p, Eq. (3-79) is satisfied if all the coefficients are zero, i.e.,

$$f_o = f_1 = f_2 = f_3 = 0$$

The condition $f_o = 0$ shows that W_1 is independent of η. The second condition $f_1 = 0$ then leads to the conclusion that W_1 is independent of f. Therefore, W_1 has to be a constant. The remaining two conditions, namely, $f_2 = f_3 = 0$, are satisfied automatically. Thus, the characteristic function becomes

$$W = W_1 p \tag{3-81}$$

The invariances can be solved from the equations

$$\frac{d\eta}{W_1} = \frac{df}{0} = \frac{dp}{0} \tag{3-82}$$

The two independent solutions of this system of equations are:

$$f = \text{constant and } p = \text{constant} \tag{3-83}$$

Thus, the new variables (x,y) are obtained as follows:

$$y = F_1(f) \tag{3-84a}$$

$$\frac{dy}{dx} = \frac{df}{d\eta} \tag{3-84b}$$

If the transformation (3-84a) is written as

$$\frac{df}{dy} = F_2(y) \tag{3-85}$$

then Eq. (3-84) becomes

$$\frac{df}{dy} = \frac{d\eta}{dx} = F_2(y) \tag{3-86}$$

Then Eq. (3-48) is transformed to

$$\frac{d^2y}{dx^2} + AF_2(y)\,\frac{dy}{dx} + BF_2(y) \int F_2(y)dy + CF_2(y) = 0 \tag{3-87}$$

In other words, any nonlinear second-order differential equation of the type given in Eq. (3-87) can be reduced to the linear differential equation, Eq. (3-48), through the transformation defined by Eq. (3-86).

As an example, if $F_2(y)$ is expressed in powers of y, then the transformation

$$\frac{df}{dy} = \frac{d\eta}{dx} = y^n \tag{3-88}$$

will reduce the nonlinear equation

$$\frac{d^2y}{dx^2} + Ay^n\,\frac{dy}{dx} + \frac{B}{n+1}\,y^{2n+1} + Cy^n = 0 \tag{3-89}$$

to the linear equation

$$\frac{d^2f}{d\eta^2} + A\,\frac{df}{d\eta} + Bf + C = 0 \tag{3-90}$$

This is the transformation proposed most recently by Dasarathy and Srinivasan [2] As an application, the differential equation involved in the problem of the free oscillations of a surge tank [6]

$$\frac{d^2y}{dx^2} + a\,|y|\,\frac{dy}{dx} + bxy = 0$$

belong to this class, where a and b are constants dependent upon various fixed physical parameters. This is the special case of Eq. (3-89) with n = 1 and B = 0. Thus, a transformation

$$\frac{df}{dy} = \frac{d\eta}{dx} = y$$

will linearize the equation to the form

$$\frac{d^2f}{d\eta^2} + A\frac{df}{d\eta} + C = 0$$

Next, we consider the class of transformation in which linearization is achieved by raising the order of equation. Following the same reasoning as that given in the preceding article, we would expect a nonlinear ordinary differential equation of the second-order to be transformed to a third-order linear differential equation.

Let us consider the linear third-order differential equation

$$f''' + Af'' + Bf' = 0 \tag{3-91}$$

where A and B are functions of η. Using the notation

$$p = f' \quad , \quad q = f'' \quad , \quad r = f'''$$

Eq. (3-91) becomes

$$F = r + Aq + Bp = 0 \tag{3-92}$$

Next, an infinitesimal transformation is defined as

$$\begin{aligned} \eta' &= \eta + (\delta\epsilon)\xi(\eta,f,p) \\ f' &= f + (\delta\epsilon)\theta(\eta,f,p) \\ p' &= p + (\delta\epsilon)\pi(\eta,f,p) \\ q' &= q + (\delta\epsilon)k(\eta,f,p,q) \\ r' &= r + (\delta\epsilon)\rho(\eta,f,p,q,r) \end{aligned} \tag{3-93}$$

where, in terms of the characteristic function W,

$$\begin{aligned} \xi &= \frac{\partial W}{\partial p} \\ \theta &= p\frac{\partial W}{\partial p} - W \\ -\pi &= XW \\ -k &= (X^2 + 2qX\frac{\partial}{\partial p} + q^2\frac{\partial^2}{\partial p^2} + q\frac{\partial}{\partial f})W \\ -\rho &= (X^3 + 3qX^2\frac{\partial}{\partial p} + 3q^2X\frac{\partial^2}{\partial p^2} + q^3\frac{\partial^3}{\partial p^3} + 3qX\frac{\partial}{\partial f} + 3q^2\frac{\partial^2}{\partial f\partial p})W \\ &\quad + r(3q\frac{\partial^2}{\partial p^2} + 3X\frac{\partial}{\partial p} + \frac{\partial}{\partial f})W \end{aligned} \tag{3-94}$$

The operator X is defined as

$$X = \frac{\partial}{\partial \eta} + p \frac{\partial}{\partial f} \tag{3-95}$$

Again, the invariance of the differential equation requires that

$$\frac{\delta F}{\delta \epsilon} = 0$$

or, in its expanded form

$$\xi \frac{\partial F}{\partial \eta} + \theta \frac{\partial F}{\partial f} + \pi \frac{\partial F}{\partial p} + k \frac{\partial F}{\partial q} + \rho \frac{\partial F}{\partial r} = 0 \tag{3-96}$$

Replacing F by the differential equation, Eq. (3-91), will yield a general equation for the determination of the characteristic function W. However, since the independent variable η is unchanged in this class of transformation, it means, from Eqs. (3-93) and (3-94), that W is independent of p. The transformation functions in Eq. (3-94) now become

$$\begin{aligned} \xi &= 0 \\ \theta &= -W \\ -\pi &= XW \\ -k &= X^2W + q\frac{\partial W}{\partial f} \\ -\rho &= X^3W + 3qX\frac{\partial W}{\partial f} + r\frac{\partial W}{\partial f} \end{aligned} \tag{3-97}$$

Substituting F from Eq. (3-91) and the transformation functions from Eq. (3-97) into Eq. (3-96), we get:

$$f_0 + f_1 p + f_2 q + f_3 p^2 + f_4 pq + f_5 p^3 = 0 \tag{3-98}$$

where

$$\begin{aligned} f_0 &= -\frac{\partial^3 W}{\partial \eta^3} - A\frac{\partial^2 W}{\partial \eta^2} - B\frac{\partial W}{\partial \eta} \\ f_1 &= -3\frac{\partial^3 W}{\partial \eta^2 \partial f} - 2A\frac{\partial^2 W}{\partial \eta \partial f} \\ f_2 &= -3\frac{\partial^2 W}{\partial \eta \partial f} \\ f_3 &= -3\frac{\partial^3 W}{\partial \eta \partial f^2} - A\frac{\partial^2 W}{\partial f^2} \\ f_4 &= -3\frac{\partial^2 W}{\partial f^2} \\ f_5 &= -\frac{\partial^3 W}{\partial f^3} \end{aligned} \tag{3-99}$$

Since the f's are independent of p and q, Eq. (3-98) is satisfied if

$$f_o = f_1 = f_2 = f_3 = f_4 = f_5 = 0 \qquad (3\text{-}100)$$

The last three conditions in Eq. (3-100), namely, $f_3 = f_4 = f_5 = 0$ indicates a linear function of the characteristic function W in f, i.e.,

$$W(\eta, f) = W_1(\eta) + fW_2 \qquad (3\text{-}101)$$

where W_2 should be a constant based on the equation $f_2 = 0$. The condition $f_1 = 0$ is then satisfied. The last condition $f_o = 0$ gives

$$\frac{d^3W_1}{d\eta^3} + A\frac{d^2W_1}{d\eta^2} + B\frac{dW_1}{d\eta} = 0 \qquad (3\text{-}102)$$

The invariants can be solved from the following equations:

$$\frac{d\eta}{\frac{\partial W}{\partial p}} = \frac{df}{p\frac{\partial W}{\partial p} - W} = \frac{dp}{-\frac{\partial W}{\partial \eta} - p\frac{\partial W}{\partial f}} \qquad (3\text{-}103)$$

or, for W given by Eqs. (3-101) and (3-102),

$$\frac{d\eta}{0} = \frac{df}{-W_1 - fW_2} = \frac{dp}{-\frac{dW_1}{d\eta} - pW_2} \qquad (3\text{-}104)$$

The two solutions are

$$\eta = \text{constant} \qquad (3\text{-}105a)$$

$$\frac{W_2p + \frac{dW_1}{d\eta}}{W_2f + W_1} = \text{constant} \qquad (3\text{-}105b)$$

Thus, the new variables are

$$\begin{aligned} x &= \eta \\ y &= F\left(\frac{W_2p + \frac{dW_1}{d\eta}}{W_2f + W_1}\right) \end{aligned} \qquad (3\text{-}106)$$

where F represents any function of the argument.

A special form of this transformation was given by Sugai [10] in which a transformation

$$x = \eta \quad , \quad y = \frac{f'}{\kappa(\eta) f}$$

reduced the nonlinear equation

$$yy'' - 2y'^2 + 3\kappa y' - \kappa^2 - \kappa y = 0$$

to the linear equation

$$f''' - 2\frac{\kappa'}{\kappa} f'' - \left(\frac{\kappa''}{\kappa} - 2\frac{\kappa'^2}{\kappa^2}\right) f' = 0$$

3.3. HIGHER-ORDER DIFFERENTIAL EQUATIONS

The examples discussed in the previous sections have been cited to show the general nature of the present method. The method can be extended to high-order differential equations with equal success. For example, in a recent paper by Dasarathy and Srinivasan [3], a nonlinear third-order ordinary differential equation

$$y''' - \left[\frac{F^*(y)}{F(y)}\right] y'y'' + aF(y)y'' + bF^2(y)y' + CF^2(y) + dF^2(y) \int F(y)dy = 0 \quad (3\text{-}107)$$

where $F^*(y) = (d/dy)[F(y)]$, was linearized by the transformation

$$\frac{df}{dy} = \frac{d\eta}{dx} = F(y) \quad (3\text{-}108)$$

leading to

$$f''' + af'' + bf' + df + C = 0 \quad (3\text{-}109)$$

Refering to the analysis leading to the derivation of transformation (3-86), the transformation defined in (3-108) is the special case in which

$$\theta = p\frac{\partial W}{\partial p} - W = 0; - \pi = \frac{\partial W}{\partial \eta} + p\frac{\partial W}{\partial f} = 0 \quad (3\text{-}110\ a,b)$$

i.e., the characteristic function is

$$W = W_1 p \quad (3\text{-}111)$$

where W_1 is constant. The invariants are then determined by

$$\frac{d\eta}{W_1} = \frac{df}{0} = \frac{dp}{0} \quad (3\text{-}112)$$

and the transformation is derived in exactly the same way as in the analysis following from Eq. (3-82) to ending with Eq. (3-86). It should be mentioned that the form of W given in Eq. (3-111) satisfies the condition

$$\frac{\delta F}{\delta \epsilon} = 0$$

identically.

An application of this class of transformation was given by Dasarathy and Srinivasan [3]. The equation arises in gyroscopic theory where the inertial motion of a gyro rotating about its mass of center is considered. The Euler's equations of motion, are three simultaneous nonlinear first-order equations. They are combined to give

$$y''' - \frac{(y'y'')}{y} + by^2y' = 0$$

which belong to the class of Eq. (3-107).

3.4. CONCLUDING REMARKS

The method developed in this report is presented as being very general in nature. Basically, the method establishes the connection between one equation and all other equations under the contact group of transformation, by the procedure of choosing different forms of the characteristic function W. It is therefore not merely restricted to finding the class of transformations for reducing nonlinear differential equations to linear differential equations.

In review, the method follows the following steps: A given differential equation is required to be invariant under an infinitesimal contact transformation group, i.e.,

$$\frac{\delta F}{\delta \epsilon} = 0 \tag{3-113}$$

where F is the differential equation. Equation (3-113) is used to search for all possible forms of W. With W known, the invariants are obtained by integrating the system of equations

$$\frac{d\eta}{\xi} = \frac{df}{\theta} = \frac{dp}{\pi}$$

where ξ, θ and π are related to the characteristic function W as shown in Eq. (3-51). If the solutions are

$$u(\eta, f, p) = C_1 \quad , \quad v(\eta, f, p) = C_2$$

then the new variables are

$$x = u(\eta, f, p) \quad , \quad y = v(\eta, f, p)$$

or any function of u and v.

This method may be applied to a nonlinear equation with the purpose of reducing it to a linear form. It may also be applied to a linear equation where the classes of nonlinear equations reducible to this linear form are to be determined. It may also be applied to a transformation from one nonlinear equation to another whose solution is known or which is known to be reducible to a linear equation.

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