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Technical Report

SIMILARITY ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS BY THE MULTIPARAMETER LIE GROUP

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FOREWORD

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TABLE OF CONTENTS

ABSTRACT

A systematic method using S. Lie's multiparameter infinitesimal transformation groups is developed in this report which enables the reduction of the number of variables by more than one in a single step. Details of the method are presented through two examples, namely, the unsteady boundary layer equations and the nonlinear diffusion equation.

1. INTRODUCTION

The investigations to be presented in this report are an outgrowth of a continuing study of the application of Lie's continuous transformation groups to problems of physical or engineering interest, and the present report is the fourth in a series of reports summarizing the results of investigation. In the first report,[7] a systematic way of reducing the number of variables was developed based on Lie's infinitesimal contact transformation groups. The second report[8] treated the class of transformations from boundary value to initial value problems, and a method was introduced to search for possible groups. In the third report,[9] a deductive method was developed in transforming from nonlinear to linear differential equations.

In this report, we again consider the class of transformations in which the number of variables are reduced. In the original method developed in the first report,[7] the number of variables that can be reduced in each step is one. If more than one variable needs to be reduced, such as in the transformation of the three-dimensional boundary layer equations to ordinary differential equations, the method has to be repeated for as many times as the number of variables to be reduced. Clearly, this is a tedius process. The question which naturally arises is whether it is possible to reduce the number of variables by more than one in a single step.

Manohar[4] was the first to propose such a method. Reduction of the number of variables by more than one was achieved by a one-parameter group of transformations and in another case by a two-parameter group of transformations. While the two-parameter method can be explained by the theories given by Eisenhart,[3] the one-parameter method is without foundation, since a one-parameter group can only reduce the number of variables by one. This was pointed out by Moran and Gagioli,[5,6] who also discovered that the reason Manohar did get the <u>correct</u> transformation using the <u>incorrect</u> one-parameter method is that the transformations obtained happened to be the transformations derived from another two-parameter group. Moran and Gagioli[5] further illustrated by an example in showing that the one-parameter method by Manohar[4] cannot, in general, reduce the number of variables by more than one.

The purpose of this report is twofold. Firstly, a systematic method following the same general approach as given in the previous three reports will be developed. Secondly, the important point raised by Moran and Gagioli[5] on Manohar's one-parameter method will be supported by the present method.

2. THE CONCEPT OF MULTIPARAMETER INFINITESIMAL CONTACT TRANSFORMATION GROUPS AND THE GENERAL METHOD

In this section, a brief review of the concept of multiparameter infinitesimal contact transformation groups will be presented. Only those concepts closely related to the present analysis will be given. For a detailed treatment, the reader is referred to the first report in the series[7] and the book by Cohen.[2] Application of a r-parameter group to eliminate r-variables in a partial differential equation will be discussed at the end of this section.

2.1. THE INFINITESIMAL CONTACT TRANSFORMATION

2.1.1. Infinitesimal Transformation

Consider the r-parameter group of transformations

$$\begin{aligned} x_1 &= \phi(x,y,a_1,\ldots,a_r) \\ y_1 &= \psi(x,y,a_1,\ldots,a_r) \end{aligned} \tag{2.1}$$

where the a's are the parameters of transformation. If the identical transformation is

$$\begin{aligned} \phi(x,y,a_1^o,\ldots,a_r^o) &= x \\ \psi(x,y,a_1^o,\ldots,a_r^o) &= y \end{aligned} \tag{2.2}$$

then the infinitesimal transformation

$$\begin{aligned} x_1 &= \phi(x,y,a_1^o + \delta a_1,\ldots,a_r^o + \delta a_r) \\ y_1 &= \psi(x,y,a_1^o + \delta a_1,\ldots,a_r^o + \delta a_r) \end{aligned} \tag{2.3}$$

can be expanded as

$$x_1 = x + \sum_{i=1}^{r} \frac{\partial\phi(x,y,a^o)}{\partial a_i^o}\,\delta a_i + \ldots \tag{2.4a}$$

$$y_1 = y + \sum_{i=1}^{r} \frac{\partial\psi(x,y,a^o)}{\partial a_i^o} \delta a_i + \ldots \qquad (2.4b)$$

where the expressions

$$\frac{\partial\phi(x,y,a^o)}{\partial a_i^o} \text{ and } \frac{\partial\psi(x,y,a^o)}{\partial a_i^o}$$

stand for what

$$\frac{\partial\phi(x,y,a_1,a_2,\ldots,a_r)}{\partial a_i} \text{ and } \frac{\partial\psi(x,y,a_1,a_2,\ldots,a_r)}{\partial a_i}$$

respectively become when $a_1 = a_1^o, a_2 = a_2^o, \ldots, a_r = a_r^o$, and the unexpressed term in Eq. (2.4) are of higher degrees than the first in $\delta a_1, \delta a_2, \ldots, \delta a_r$. Neglecting higher order terms in Eq. (2.4) and introducing the notation

$$\xi_i(x,y) = \frac{\partial\phi(x,y,a^o)}{\partial a_i^o} \qquad (2.5a)$$

and

$$\eta_i(x,y) = \frac{\partial\psi(x,y,a^o)}{\partial a_i^o} \qquad (2.5b)$$

Eq. (2.4) can be written as

$$x_1 = x + \sum_{i=1}^{r} \xi_i(x,y)\delta a_i \qquad (2.6a)$$

$$y_1 = y + \sum_{i=1}^{r} \eta_i(x,y)\delta a_i \qquad (2.6b)$$

Since $\delta a_1, \delta a_2, \ldots, \delta a_r$ are any infinitesimal increments of the first order, they can be written as:

$$\delta a_1 = e_1\, \delta a,\ \delta a_2 = e_2\, \delta a, \ldots, \delta a_r = e_r\, \delta a$$

Eq. (2.6) can therefore be written as

$$x_1 = x + \xi\, \delta a$$
$$y_1 = y + \eta\, \delta a \tag{2.7a,b}$$

where

$$\xi = e_1\, \xi_1 + e_2\, \xi_2 + \ldots + e_r\, \xi_r \tag{2.8a}$$

and

$$\eta = e_1\, \eta_1 + e_2\, \eta_2 + \ldots + e_r\, \eta_r \tag{2.8b}$$

2.1.2. Notation of Infinitesimal Transformation

Introducing the notation

$$Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} = \sum_{i=1}^{r} e_i \left(\xi_i \frac{\partial f}{\partial x} + \eta_i \frac{\partial f}{\partial y} \right) \tag{2.9}$$

and similarly,

$$U_i f = \xi_i \frac{\partial f}{\partial x} + \eta_i \frac{\partial f}{\partial y} \tag{2.10}$$

we have

$$Uf = e_1\, U_1 f + e_2\, U_2 f + \ldots + e_r\, U_r f \tag{2.11}$$

The above is for two variables, x and y. If n variables are involved, namely, $x_1, \ldots, x_n$, the notations become

$$Uf = \xi^1(x_1, x_2, \ldots, x_n) \frac{\partial f}{\partial x_1} + \ldots$$
$$\ldots + \xi^n(x_1, x_2, \ldots, x_n) \frac{\partial f}{\partial x_n} \tag{2.12a}$$

or

$$Uf = e_1\, U_1 f + \ldots + e_r\, U_r f \tag{2.12b}$$

where

$$U_i f = \xi_i^1 \frac{\partial f}{\partial x_1} + \xi_i^2 \frac{\partial f}{\partial x_2} + \ldots + \xi_i^n \frac{\partial f}{\partial x_n} \qquad (2.12c)$$

$$(i = 1,2,\ldots,r)$$

2.1.3. Invariant Function

If $f(x_1', x_2', \ldots, x_n') = f(x_1, x_2, \ldots, x_n)$, then f is invariant under the infinitesimal transformation

$$x_i' = x_i + \sum_{j=1}^{r} \xi_j^i \, \delta a_j \qquad (2.13)$$

By following the same reasoning as in the one-parameter group method, the following theorem can be proved.

Theorem. The necessary and sufficient condition for $f(x_1, x_2, \ldots, x_n)$ to be invariant under the group of transformation represented by Uf is Uf = 0; i.e.,

$$Uf = (e_1\xi_1^1 + e_2\xi_2^1 + \ldots + e_r\xi_r^1) \frac{\partial f}{\partial x_1} + \ldots + (e_1\xi_1^n + e_2\xi_2^n + \ldots + e_r\xi_r^n) \frac{\partial f}{\partial x_n} = 0 \qquad (2.14)$$

To get the invariant functions, it is necessary to solve

$$\frac{dx_1}{e_1\,\xi_1^1 + e_2\,\xi_2^1 + \ldots + e_r\,\xi_r^1} = \ldots = \frac{dx_n}{e_1\,\xi_1^n + e_2\,\xi_2^n + \ldots + e_r\,\xi_r^n} = \delta a \qquad (2.15a)$$

or,

$$\frac{dx_1}{\xi_1^1 + d_2\,\xi_2^1 + \ldots + d_r\,\xi_r^1} = \ldots = \frac{dx_n}{\xi_1^n + d_2\,\xi_2^n + \ldots + d_r\,\xi_r^n} = e_1\delta a \qquad (2.15b)$$

where $d_2 = e_2/e_1, \ldots, d_r = e_r/e_1$. Equation (2.15) is seen to give (n - 1) invariants. However, in these invariant functions, there are (r - 1) arbitrary constants, namely, $d_2, d_3, \ldots, d_r$. Elimination of the (r - 1) constants from the (n - 1) invariants therefore leads to (n - r) functionally independent absolute invariants. Thus, a r-parameter group of continuous transformations has (n - r) functionally independent absolute invariants.

2.1.4. Invariance of a Partial Differential Equation

To illustrate the manner in which the above theories can be applied to reduce the number of variables by r using a r-parameter transformation group, let us consider the case of a partial differential equation with one dependent variable and three independent variables. To reduce this equation to an ordinary differential equation, a two-parameter group is therefore needed.

Consider now a function

$$F = F\left(x_1, x_2, x_2;\ y,\ \frac{\partial y}{\partial x_1},\ \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial^2 y}{\partial x_3^2}\right) \tag{2.16}$$

the arguments of which, 13 in number, contain derivatives of y up to the second order. Such a function is known as a differential form of the second order in three independent variables. Designate the arguments by $z_1, \ldots, z_p$, i.e.,

$$z_1 = x_1$$

$$z_2 = x_2$$

$$z_{p-1} = \frac{\partial^2 y}{\partial x_2\, \partial x_3}$$

$$z_p = \frac{\partial^2 y}{\partial x_3^2} \tag{2.17}$$

where p = 13. Thus, Eq. (2.16) can be written in a simpler form as

$$F = F(z_1, \ldots, z_p) \tag{2.18}$$

The function F is said to admit of a given two-parameter group represented by [cf. Eq. (2.12)]:

$$Uf = \xi^1(z_1, \ldots, z_p)\, \frac{\partial f}{\partial z_1} + \ldots + \xi^p(z_1, \ldots, z_p)\, \frac{\partial f}{\partial z_p} \tag{2.19}$$

if it is invariant under this group of transformation. Therefore, the function, F admits of a group if

$$UF = 0 \tag{2.20a}$$

or,

$$\xi^1(z_1,\ldots,z_p)\frac{\partial F}{\partial z_1} + \ldots + \xi^p(z_1,\ldots,z_p)\frac{\partial F}{\partial z_p} = 0 \tag{2.20b}$$

or,

$$e_1\, U_1F + e_2\, U_2F = 0 \tag{2.20c}$$

where

$$U_iF = \xi_i^1 \frac{\partial F}{\partial z_1} + \xi_i^2 \frac{\partial F}{\partial z_2} + \ldots + \xi_i^p \frac{\partial F}{\partial z_p} \tag{2.20d}$$

$$(i = 1,2)$$

Since e_1 and e_2 are arbitrary constants, Eq. (2.20c) gives

$$U_1F = 0 \text{ and } U_2F = 0 \tag{2.21}$$

where U_1F and U_2F are given by Eq. (2.20d).

Based on a theorem given by Eisenhart[3] and later proved in detail by Moran, Gagioli, and Scholten,[6] a function F in p variables can be expressed in terms of the (p - r) functionally independent invariants if it is invariant under the r-parameter group of transformations. For the present example in which a two-parameter group of transformations is introduced, the three independent variables can therefore be reduced to a single independent variable.

To solve for the invariants, Eq. (2.15) is used. Thus, it is necessary to solve:

$$\frac{dz_1}{\xi_1^1 + d_2\, \xi_2^1} = \ldots = \frac{dz_p}{\xi_1^p + d_2\, \xi_2^p} = e_1\, \delta a \tag{2.22}$$

Eq. (2.22) gives (p - 1) functionally independent solutions containing the parameter d_2. Elimination of d_2 then leads to (p - 2) invariants which are denoted by:

$$\eta_1, \eta_2, \ldots, \eta_{p-2}$$

Thus, the differential equation

$$F(z_1, z_2, \ldots, z_p) = 0 \tag{2.23a}$$

can be transformed into the differential equation

$$\psi(\eta_1, \eta_2, \ldots, \eta_{p-2}) = 0 \tag{2.23b}$$

2.1.5. Infinitesimal Contact Transformation

For a given r-parameter group of transformations, the transformation functions, ξ^i_j, are known, the above procedure is adequate in seeking the similarity transformations. However, if the group is not given, Eq. (2.14) alone will not be enough to search for possible groups. At this point, the theories developed in Reference 7 on the concept of an infinitesimal contact transformation must be introduced so that the transformation functions ξ^i_j can be expressed in terms of r characteristic functions. The details of the derivation is given in great detail in Reference 7. Here, only the final form is summarized as follows:

For the extended infinitesimal contact transformation

$$\begin{aligned}
Z_i &= z_i + \delta\epsilon\, m_i(z_\nu, x_\mu, p^\nu_\mu) \\
X_k &= x_k + \delta\epsilon\, \alpha_k(z_\nu, x_\mu, p^\nu_\mu) \\
P^i_k &= p^i_k + \delta\epsilon\, \pi^i_k(z_\nu, x_\mu, p^\nu_\mu) \\
P^i_{jk} &= p^i_{jk} + \delta\epsilon\, \pi^i_{jk}(z_\nu, x_\mu, p^\nu_\mu, p^\nu_{\mu s}) \\
P^i_{jk\ell} &= p^i_{jk\ell} + \delta\epsilon\, \pi^i_{jk\ell}(z_\nu, x_\mu, p^\nu_\mu, p^\nu_{\mu s}, p^\nu_{\mu st})
\end{aligned} \tag{2.24}$$

the transformation functions can be expressed in terms of characteristic functions W_i as follows:

$$m_i = \frac{\partial W_\nu}{\partial p^\nu_\mu} p^i_\mu - W_i \tag{2.25}$$

(no sum on ν)

$$\alpha_\mu = \frac{\partial W_i}{\partial p^i_\mu} \tag{2.26}$$

(no sum on i)

$$\pi^i_\mu = -\frac{\partial W_i}{\partial z_\nu} p^\nu_\mu - \frac{\partial W_i}{\partial x_\mu} \tag{2.27}$$

$$\pi^i_{jk} = \frac{\partial \pi^i_k}{\partial x_j} + \frac{\partial \pi^i_k}{\partial z_s} p^s_j + \frac{\partial \pi^i_k}{\partial p^\nu_\mu} p^\nu_{\mu j} - p^i_{\lambda k}\left(\frac{\partial \alpha_\lambda}{\partial x_j} + \frac{\partial \alpha_\lambda}{\partial z_s} p^s_j + \frac{\partial \alpha_\lambda}{\partial p^\nu_\mu} p^\nu_{\mu j}\right) \tag{2.28}$$

$$\pi^i_{jk\ell} = \frac{\partial \pi^i_{jk}}{\partial x_\ell} + \frac{\partial \pi^i_{jk}}{\partial z_s} p^s_\ell + \frac{\partial \pi^i_{jk}}{\partial p^\nu_\mu} p^\nu_{\mu\ell} + \frac{\partial \pi^i_{jk}}{\partial p^a_{bc}} p^a_{bc\ell} - p^i_{jkt}\left(\frac{\partial \alpha_t}{\partial x_\ell} + \frac{\partial \alpha_t}{\partial z_s} p^s_\ell + \frac{\partial \alpha_t}{\partial p^\nu_\mu} P^\nu_{\mu\ell}\right) \tag{2.29}$$

2.2. THE GENERAL METHOD

With the background discussed in Section 2.1.5 in mind, the partial differential equation given in Eq. (2.16) is again used to illustrate the steps necessary for reducing it to an ordinary differential equation. The method proceeds as follows:

1. An infinitesimal is defined, as in Eq. (2.24). The differential equation, F = 0, as given in (2.23a), is required to be invariant under this group of transformation, i.e., it must satisfy Eq. (2.20) or Eq. (2.21).

2. The transformation functions in U_1F and U_2F, defined by Eq. (2.20d), can be expressed as functions of W and $\bar{W}$, respectively, using Eqs. (2.25) through (2.29). Since W should not be equal to $\bar{W}$, some properties may be imposed on W and/or $\bar{W}$. For example, we may require W independent of p_1, while $\bar{W}$ be independent of p_2, etc. W and $\bar{W}$ can then be determined by Eq. (2.21).

3. Solve the independent invariants using Eq. (2.15). (Note that ξ^i_j are known functions since W and $\bar{W}$ are known.)

4. The differential equation can then be expressed in terms of the p - 2 invariants.

Two examples will be given in the next section.

3. APPLICATIONS OF THE METHOD

In this section the general theories given in Section 2 will be applied to two examples, namely, the unsteady, two-dimensional laminar boundary layer equations originally treated by Shuh[10] and the nonlinear diffusion equation discussed recently by Ames.[1] Although the method is presented through two specific examples in transport processes, the general nature of the method makes it possible to be applied to equations in other fields.

3.1. APPLICATION TO UNSTEADY, TWO-DIMENSIONAL, LAMINAR BOUNDARY LAYER EQUATIONS

Consider the unsteady, two-dimensional, laminar boundary layer equations, expressed in terms of the stream function ψ,

$$\frac{\partial^2\psi}{\partial t\ \partial y} + \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\ \partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x} + \frac{\partial^3\psi}{\partial y^3} \tag{3.1}$$

subject to the boundary conditions

$$y = 0: \quad \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial x} = 0$$

$$y = \infty: \quad \frac{\partial\psi}{\partial y} = U_e(x,t)$$

Equation (3.1) can be written in a shorthand form as:

$$F = p_{333} + \phi - p_{13} - p_3\, p_{23} + p_2\, p_{33} = 0 \tag{3.2}$$

where

$$p_1 = \frac{\partial\psi}{\partial t}, \quad p_2 = \frac{\partial\psi}{\partial x}, \quad p_3 = \frac{\partial\psi}{\partial y}, \quad p_{13} = \frac{\partial^2\psi}{\partial t\ \partial y}, \text{ etc.,}$$

and

$$\phi = \frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x} \tag{3.3}$$

The differential equation F = 0, as given in Eq. (3.2), will be invariant under the two-parameter infinitesimal transformation

$$
\begin{aligned}
t' &= t + \delta\epsilon_1\, \alpha_1(t,x,y,\psi) + \delta\epsilon_2\, \bar{\alpha}_1(t,x,y,\psi) \\
x' &= x + \delta\epsilon_1\, \alpha_2(t,x,y,\psi) + \delta\epsilon_2\, \bar{\alpha}_2(t,x,y,\psi) \\
y' &= y + \delta\epsilon_1\, \alpha_3(t,x,y,\psi) + \delta\epsilon_2\, \bar{\alpha}_3(t,x,y,\psi) \\
\psi' &= \psi + \delta\epsilon_1\, \zeta(t,x,y,\psi) + \delta\epsilon_2\, \bar{\zeta}(t,x,y,\psi) \\
p_2' &= p_2 + \delta\epsilon_1\, \pi_2(t,x,y,\psi,p_1,p_2,p_3) + \delta\epsilon_2\, \bar{\pi}_2(t,x,y,p_1,p_2,p_3) \\
p_3' &= p_3 + \delta\epsilon_1\, \pi_3(t,x,y,\psi,p_1,p_2,p_3) + \delta\epsilon_2\, \pi_3(t,x,y,\psi,p_1,p_2,p_3) \qquad (3.4) \\
p_{13}' &= p_{13} + \delta\epsilon_1\, \pi_{13}(t,x,y,\psi,p_1,p_2,p_3,p_{11},\ldots,p_{33}) + \delta\epsilon_2\, \bar{\pi}_{13}(t,x,y,\psi,\ldots,p_{33}) \\
p_{23}' &= p_{23} + \delta\epsilon_1\, \pi_{23}(t,x,y,\psi,\ldots,p_{33}) + \delta\epsilon_2\, \bar{\pi}_{23}(t,x,y,\psi,\ldots,p_{33}) \\
p_{33}' &= p_{33} + \delta\epsilon_1\, \pi_{33}(t,x,y,\psi,\ldots,p_{33}) + \delta\epsilon_2\, \bar{\pi}_{33}(t,x,y,\psi,\ldots,p_{33}) \\
p_{333}' &= p_{333} + \delta\epsilon_1\, \pi_{333}(t,x,y,\psi,p_1,\ldots,p_{333}) + \delta\epsilon_2\, \bar{\pi}_{333}(t,x,y,\psi,p_1,\ldots,p_{333})
\end{aligned}
$$

if

$$UF = 0 \qquad (3.5)$$

or, from Eq. (2.20c),

$$e_1\, U_1F + e_2\, U_2F = 0 \qquad (3.6)$$

Since e_1 and e_2 are arbitrary, Eq. (3.6) means

$$U_1F = 0 \text{ and } U_2F = 0 \qquad (3.7a,b)$$

simultaneously.

In their expanded form, Eqs. (3.7a,b) can be written as:

$$\alpha_1 \frac{\partial F}{\partial t} + \alpha_2 \frac{\partial F}{\partial x} + \alpha_3 \frac{\partial F}{\partial y} + \zeta \frac{\partial F}{\partial \psi} + \pi_i \frac{\partial F}{\partial p_i} + \pi_{ij} \frac{\partial F}{\partial p_{ij}} + \pi_{ijk} \frac{\partial F}{\partial p_{ijk}} = 0 \qquad (3.8a)$$

and

$$\bar{\alpha}_1 \frac{\partial F}{\partial t} + \bar{\alpha}_2 \frac{\partial F}{\partial x} + \bar{\alpha}_3 \frac{\partial F}{\partial y} + \bar{\zeta} \frac{\partial F}{\partial \psi} + \bar{\pi}_i \frac{\partial F}{\partial p_i} + \bar{\pi}_{ij} \frac{\partial F}{\partial p_{ij}} + \bar{\pi}_{ijk} \frac{\partial F}{\partial p_{ijk}} = 0 \qquad (3.8b)$$

Putting F from Eq. (3.2) into Eq. (3.8), we get

$$\pi_{333} + \frac{\partial\phi}{\partial t}\alpha_1 + \frac{\partial\phi}{\partial x}\alpha_2 - \pi_{13} - p_3\,\pi_{23} - p_{23}\,\pi_3 + \pi_2\,p_{33} + p_2\,\pi_{33} = 0 \qquad (3.9a)$$

and

$$\bar{\pi}_{333} + \frac{\partial\phi}{\partial t}\bar{\alpha}_1 + \frac{\partial\phi}{\partial x}\bar{\alpha}_2 - \bar{\pi}_{13} - p_3\,\bar{\pi}_{23} - p_{23}\,\bar{\pi}_3 + \bar{\pi}_2\,p_{33} + p_2\,\bar{\pi}_{33} = 0 \qquad (3.9b)$$

The next step is to express the transformation functions, α_1, α_2, etc., in Eq. (3.9a) in terms of a characteristic function W; and the transformation functions, $\bar{\alpha}_1$, $\bar{\alpha}_2$, etc., in Eq. (3.9b) be expressed in terms of a second characteristic function $\bar{W}$. This differs from the one-parameter method in that two characteristic functions, instead of one, have to be determined. The functional form of the W's can be determined from Eqs. (3.9a) and (3.9b). Since the transformation groups in Eqs. (3.9a) and (3.9b) should be different (otherwise, the two-parameter groups will be reduced to a one-parameter group), we choose, as an example, two groups as follows:

G_1: α_1 in Eq. (3.4) is zero

G_2: α_2 in Eq. (3.4) is zero.

For the first group, G_1, the transformation functions, α_2, α_3, etc., can be expressed in terms of the characteristic functions as

$$\alpha_2 = \frac{\partial W_1}{\partial p_2}, \quad \alpha_3 = \frac{\partial W_1}{\partial p_3}, \quad \zeta = p_2\frac{\partial W_1}{\partial p_2} + p_3\frac{\partial W_1}{\partial p_3} - W_1$$

$$\pi_1 = -\frac{\partial W_1}{\partial t} - p_1\frac{\partial W_1}{\partial \psi}, \quad \pi_2 = -\frac{\partial W_1}{\partial x} - p_2\frac{\partial W_1}{\partial \psi}, \quad \pi_3 = -\frac{\partial W_1}{\partial y} - p_3\frac{\partial W_1}{\partial \psi} \qquad (3.10a\text{-}f)$$

$$-\pi_{13} = \frac{\partial^2 W_1}{\partial t\,\partial y} + p_3\frac{\partial^2 W_1}{\partial t\,\partial \psi} + p_1\frac{\partial^2 W_1}{\partial \psi\,\partial y} + p_1\,p_3\frac{\partial^2 W_1}{\partial \psi^2} + p_{21}\left(\frac{\partial^2 W_1}{\partial p_2\,\partial y} + p_3\frac{\partial^2 W_1}{\partial p_2\,\partial \psi}\right)$$

$$+ p_{31}\left(\frac{\partial^2 W_1}{\partial p_3\,\partial y} + p_3\frac{\partial^2 W_1}{\partial p_3\,\partial \psi} + \frac{\partial W_1}{\partial \psi}\right) + p_{23}\left(\frac{\partial^2 W_1}{\partial t\,\partial p_2} + p_1\frac{\partial^2 W_1}{\partial \psi\,\partial p_2}\right)$$

$$+ p_{33}\left(\frac{\partial^2 W_1}{\partial t\,\partial p_3} + p_1\frac{\partial^2 W_1}{\partial \psi\,\partial p_3}\right) \qquad (3.10g)$$

$$-\pi_{23} = \left(\frac{\partial^2 W_1}{\partial x\,\partial y} + p_3 \frac{\partial^2 W_1}{\partial x\,\partial\psi}\right) + p_2\left(\frac{\partial^2 W_1}{\partial\psi\,\partial y} + p_3 \frac{\partial^2 W_1}{\partial\psi^2}\right) + p_{22}\left(\frac{\partial^2 W_1}{\partial p_2\,\partial y} + p_3 \frac{\partial^2 W_1}{\partial p_2\,\partial\psi}\right)$$

$$+ p_{32}\left(\frac{\partial^2 W_1}{\partial p_3\,\partial y} + p_3 \frac{\partial^2 W_1}{\partial p_3\,\partial\psi} + \frac{\partial W_1}{\partial\psi}\right) + p_{23}\left(\frac{\partial^2 W_1}{\partial x\,\partial p_2} + \frac{\partial^2 W}{\partial\psi\,\partial p_2}\,p_2\right)$$

$$+ p_{33}\left(\frac{\partial^2 W_1}{\partial x\,\partial p_3} + p_2 \frac{\partial^2 W_1}{\partial\psi\,\partial p_3}\right) \qquad (3.10h)$$

$$-\pi_{33} = \left(\frac{\partial^2 W_1}{\partial y^2} + p_3 \frac{\partial^2 W_1}{\partial y\,\partial\psi}\right) + p_3\left(\frac{\partial^2 W_1}{\partial\psi\,\partial y} + p_3 \frac{\partial^2 W_1}{\partial\psi^2}\right) + p_{33}\frac{\partial W_1}{\partial\psi}$$

$$+ 2p_{23}\left(\frac{\partial^2 W_1}{\partial p_2\,\partial y} + p_3 \frac{\partial^2 W_1}{\partial p_2\,\partial\psi}\right) + 2p_{33}\left(\frac{\partial^2 W_1}{\partial p_3\,\partial y} + p_3 \frac{\partial^2 W_1}{\partial p_3\,\partial\psi}\right) \qquad (3.10i)$$

$$-\pi_{333} = \frac{\partial^3 W_1}{\partial y^3} + 2p_3 \frac{\partial^3 W_1}{\partial y^2\,\partial\psi} + p_3^2 \frac{\partial^3 W_1}{\partial y\,\partial\psi^2} + p_{33}\frac{\partial^2 W_1}{\partial y\,\partial\psi}$$

$$+ p_3\left\{\left(\frac{\partial^3 W_1}{\partial\psi\,\partial y^2} + 2p_3 \frac{\partial^3 W_1}{\partial\psi^2\,\partial y} + p_3^2 \frac{\partial^3 W_1}{\partial\psi^3}\right) + 2p_{23}\left(\frac{\partial^3 W_1}{\partial\psi\,\partial p_2\,\partial y} + p_3 \frac{\partial^3 W_1}{\partial p_2\,\partial\psi^2}\right)\right.$$

$$\left. + 2p_{33}\left(\frac{\partial^3 W_1}{\partial p_3\,\partial y\,\partial\psi} + p_3 \frac{\partial^3 W_1}{\partial p_3\,\partial\psi^2}\right)\right\} + p_3\,p_{33}\frac{\partial^2 W}{\partial\psi^2}$$

$$+ p_{23}\left(3\frac{\partial^3 W_1}{\partial y^2\,\partial p_2} + 4p_3 \frac{\partial^3 W_1}{\partial p_2\,\partial y\,\partial\psi} + p_3^2 \frac{\partial^3 W_1}{\partial p_2\,\partial\psi^2} + p_{33}\frac{\partial^2 W_1}{\partial p_2\,\partial\psi}\right)$$

$$+ p_{33}\left(3\frac{\partial^3 W_1}{\partial y^2\,\partial p_3} + 4p_3 \frac{\partial^3 W_1}{\partial p_3\,\partial y\,\partial\psi} + p_3^2 \frac{\partial^3 W_1}{\partial p_3\,\partial\psi^2} + p_{33}\frac{\partial^2 W_1}{\partial p_3\,\partial\psi}\right)$$

$$+ 3p_{233}\left(\frac{\partial^2 W_1}{\partial p_2\,\partial y} + p_3 \frac{\partial^2 W_1}{\partial p_2\,\partial\psi}\right) + 3p_{13}\frac{\partial^2 W_1}{\partial p_3\,\partial y} + 3p_3\,p_{13}\frac{\partial^2 W_1}{\partial p_3\,\partial\psi}$$

$$+ 3p_3\,p_{23}\frac{\partial^2 W_1}{\partial p_3\,\partial y} + 3p_3^2\,p_{23}\frac{\partial^2 W_1}{\partial p_3\,\partial\psi} - 3p_2\,p_{33}\frac{\partial^2 W_1}{\partial p_3\,\partial y} - 3p_2\,p_3\,p_{33}\frac{\partial^2 W_1}{\partial p_3\,\partial\psi}$$

$$- 3\phi\frac{\partial^2 W_1}{\partial p_3\,\partial y} - 3p_3\,\phi\frac{\partial^2 W_1}{\partial p_3\,\partial\psi} + \frac{\partial W_1}{\partial\psi}\left(p_{13} + p_3\,p_{23} - p_2\,p_{33} - \phi\right) \qquad (3.10j)$$

Now, substituting the transformation functions from Eqs. (3.10a) to (3.10j) into Eq. (3.9a) and eliminating p_{333} from Eq. (3.2), we get:

$$f_o + f_1 p_{11} + f_2 p_{21} + f_3 p_{31} + f_4 p_{22} + f_5 p_{23} + f_6 p_{33} + f_7 p_{133} + f_8 p_{233} + f_9 p_{13} p_{33} + f_{10} p_{23} p_{33} + f_{11} p_{33}^2 = 0 \qquad (3.11)$$

where

$$f_o = -\left(\frac{\partial^3 W_1}{\partial y^3} + 3p_3 \frac{\partial^3 W_1}{\partial y^2 \partial\psi} + 3p_3^2 \frac{\partial^3 W_1}{\partial y \partial\psi^2} + p_3^3 \frac{\partial^3 W_1}{\partial\psi^3}\right) + 3\phi\left(\frac{\partial^2 W_1}{\partial p_3 \partial y} + p_3 \frac{\partial^2 W_1}{\partial p_3 \partial\psi}\right)$$
$$+ \frac{\partial\phi}{\partial x}\frac{\partial W_1}{\partial p_2} - \phi \frac{\partial W_1}{\partial\psi} + \left(\frac{\partial^2 W_1}{\partial t \partial y} + p_3 \frac{\partial^2 W_1}{\partial t \partial\psi}\right) + p_1\left(\frac{\partial^2 W_1}{\partial\psi \partial y} + p_3 \frac{\partial^2 W_1}{\partial\psi^2}\right)$$
$$+ p_3\left(\frac{\partial^2 W_1}{\partial x \partial y} + p_3 \frac{\partial^2 W_1}{\partial x \partial\psi}\right) - p_2\left(\frac{\partial^2 W_1}{\partial y^2} + p_3 \frac{\partial^2 W_1}{\partial y \partial\psi}\right) \qquad (3.12)$$

$$f_1 = 0 \qquad (3.13)$$

$$f_2 = \frac{\partial^2 W_1}{\partial p_2 \partial y} + p_3 \frac{\partial^2 W_1}{\partial p_2 \partial\psi} \qquad (3.14)$$

$$f_3 = -2 \frac{\partial^2 W_1}{\partial\psi \partial p_3} p_3 - 2 \frac{\partial^2 W_1}{\partial p_3 \partial y} \qquad (3.15)$$

$$f_4 = p_3\left(\frac{\partial^2 W_1}{\partial p_2 \partial y} + p_3 \frac{\partial^2 W_1}{\partial p_2 \partial\psi}\right) \qquad (3.16)$$

$$f_5 = -\left(3 \frac{\partial^3 W_1}{\partial y^2 \partial p_2} + 4p_3 \frac{\partial^2 W_1}{\partial y \partial p_2 \partial\psi}\right) - 2p_3\left(\frac{\partial^3 W_1}{\partial\psi \partial p_2 \partial y} + p_3 \frac{\partial^3 W_1}{\partial p_2 \partial\psi^2}\right) - p_3^2 \frac{\partial^3 W_1}{\partial p_2 \partial\psi^2}$$
$$- 2p_3\left(\frac{\partial^2 W_1}{\partial p_3 \partial y} + p_3 \frac{\partial^2 W_1}{\partial p_3 \partial\psi}\right) + \left(\frac{\partial^2 W_1}{\partial t \partial p_2} + \frac{\partial^2 W_1}{\partial\psi \partial p_2} p_1\right) + p_3\left(\frac{\partial^2 W_1}{\partial x \partial p_2} + p_2 \frac{\partial^2 W_1}{\partial\psi \partial p_2}\right)$$
$$+ \left(\frac{\partial W_1}{\partial y} + p_3 \frac{\partial W_1}{\partial\psi}\right) - 2\left(\frac{\partial^2 W_1}{\partial p_2 \partial y} + p_3 \frac{\partial^2 W_1}{\partial p_2 \partial\psi}\right) p_2 \qquad (3.17)$$

$$f_6 = \left(\frac{\partial^2 W_1}{\partial t\ \partial p_3} + \frac{\partial^2 W_1}{\partial \psi\ \partial p_3} p_1\right) + p_3 \left(\frac{\partial^2 W_1}{\partial x\ \partial p_3} + \frac{\partial^2 W_1}{\partial \psi\ \partial p_3} p_2\right) - \left(\frac{\partial W_1}{\partial x} + p_2 \frac{\partial W_1}{\partial \psi}\right)$$

$$- \left(3 \frac{\partial^3 W_1}{\partial p_3\ \partial y^2} + 4p_3 \frac{\partial^3 W_1}{\partial y\ \partial p_3\ \partial \psi}\right) - 2p_3 \left(\frac{\partial^3 W_1}{\partial p_3\ \partial y\ \partial \psi} + p_3 \frac{\partial^3 W_1}{\partial p_3\ \partial \psi^2}\right) - p_3^2 \frac{\partial^3 W_1}{\partial p_3\ \partial \psi^2}$$

$$- \frac{\partial^2 W_1}{\partial y\ \partial \psi} - p_3 \frac{\partial^2 W_1}{\partial \psi^2} + p_2 \left(\frac{\partial^2 W_1}{\partial p_3\ \partial y} + p_3 \frac{\partial^2 W_1}{\partial p_3\ \partial \psi}\right) \qquad (3.18)$$

$$f_7 = 0 \qquad (3.19)$$

$$f_8 = -3 \left(\frac{\partial^2 W_1}{\partial p_2\ \partial y} + p_3 \frac{\partial^2 W_1}{\partial p_2\ \partial \psi}\right) \qquad (3.20)$$

$$f_9 = 0 \qquad (3.21)$$

$$f_{10} = - \frac{\partial^2 W_1}{\partial p_2\ \partial \psi} \qquad (3.22)$$

$$f_{11} = - \frac{\partial^2 W_1}{\partial p_3\ \partial \psi} \qquad (3.23)$$

For Eq. (3.11) to be satisfied, all the f's should be zero since the characteristic function is only a function of t, x, y, ψ, p_1, p_2, and p_3. Therefore, we get 11 equations:

$$f_o = f_1 = f_2 = \ldots = f_{11} = 0 \qquad (3.24)$$

Three of these, namely, f_1, f_7, and f_9, are zero identically.

For point transformations, as in Eq. (3.4), the characteristic function is linear with respect to the p's. For the group under consideration, G_1, $\alpha_1 = 0$, which means

$$\frac{\partial W_1}{\partial p_1} = 0$$

i.e., W_1 is independent of p_1. Thus, we write

$$W_1 = W_{11}(t,x,y,\psi)p_2 + W_{12}(t,x,y,\psi)p_3 + W_{13}(t,x,y,\psi) \tag{3.25}$$

From $f_{10} = 0$ and $f_{11} = 0$, we get

$$\frac{\partial W_{11}}{\partial \psi} = 0 \text{ and } \frac{\partial W_{12}}{\partial \psi} = 0$$

which means W_{11} and W_{12} are independent of ψ. The three conditions $f_2 = f_4 = f_8 = 0$ gives

$$\frac{\partial W_{11}}{\partial y} + p_3 \frac{\partial W_{11}}{\partial \psi} = 0 \tag{3.26}$$

Since W_{11} is independent of p_3, Eq. (3.26) shows that W_{11} is independent of both y and ψ.

Following the same reasoning, the condition $f_3 = 0$ leads to the conclusion that W_{12} is independent of both y and ψ.

The condition $f_5 = 0$ gives

$$\frac{\partial W_{11}}{\partial t} + p_3 \frac{\partial W_{11}}{\partial x} + \frac{\partial W_{13}}{\partial y} + p_3 \frac{\partial W_{13}}{\partial \psi} = 0$$

Again, due to the independence of W_{11} and W_{13} on p_3, the following two equations result:

$$\frac{\partial W_{11}}{\partial t} + \frac{\partial W_{13}}{\partial y} = 0 \tag{3.27}$$

$$\frac{\partial W_{11}}{\partial x} + \frac{\partial W_{13}}{\partial \psi} = 0 \tag{3.28}$$

Equations (3.27) and (3.28) will be used later.

From the condition $f_6 = 0$, three equations are obtained, namely,

$$-\frac{\partial^2 W_{13}}{\partial y\, \partial \psi} + \frac{\partial W_{12}}{\partial t} - \frac{\partial W_{13}}{\partial x} = 0 \tag{3.29}$$

$$\frac{\partial W_{11}}{\partial x} + \frac{\partial W_{13}}{\partial \psi} = 0 \tag{3.30}$$

$$\frac{\partial^2 W_{13}}{\partial \psi^2} = 0 \tag{3.31}$$

Equation (3.30) is the same as Eq. (3.28). From Eq. (3.31), we get

$$W_{13} = W_{131}(t,x,y)\psi + W_{132}(t,x,y) \tag{3.32}$$

Substituting W_{13} from Eq. (3.30) into Eq. (3.29), we get

$$\frac{\partial W_{131}}{\partial y} + \frac{\partial W_{12}}{\partial t} + \frac{\partial W_{131}}{\partial x}\psi + \frac{\partial W_{132}}{\partial x} = 0 \tag{3.33}$$

Since the W's in Eq. (3.33) are all independent of ψ, we get

$$\frac{\partial W_{131}}{\partial x} = 0 \tag{3.34}$$

$$\frac{\partial W_{131}}{\partial y} + \frac{\partial W_{12}}{\partial t} + \frac{\partial W_{132}}{\partial x} = 0 \tag{3.35}$$

Equation (3.34) shows that W_{131} is independent of x.

Thus, the characteristic function now becomes

$$W_1 = W_{11}(t,x)p_2 + W_{12}(t,x)p_3 + W_{131}(t,y)\psi + W_{132}(t,x,y) \tag{3.36}$$

where the W's on the right-hand side have to satisfy Eqs. (3.27), (3.28), and (3.35).

From Eq. (3.30), we conclude that

$$W_{11} = W_{111}(t)x + W_{112}(t) \tag{3.37}$$

since W_{131} is independent of x. Equation (3.30) now becomes

$$W_{111}(t) + W_{131}(t,y) = 0 \tag{3.38}$$

which means W_{131} is independent of y.

From Eq. (3.27), we get

$$\frac{dW_{111}}{dt}x + \frac{dW_{112}}{dt} + \frac{\partial W_{132}}{\partial y} = 0 \tag{3.39}$$

which shows that W_{132} is linear in y, i.e.,

$$W_{132} = W_{1321}(t,x)y + W_{1322}(t,x) \tag{3.40}$$

Equation (3.39) now becomes

$$\frac{dW_{111}}{dt} x + \frac{dW_{112}}{dt} + W_{1321} = 0 \tag{3.41}$$

From Eq. (3.35), we get

$$\frac{\partial W_{12}}{\partial t} + \frac{\partial W_{1321}}{\partial x} y + \frac{\partial W_{1322}}{\partial x} = 0 \tag{3.42}$$

Since the W's in Eq. (3.42) are independent of y, we get

$$\frac{\partial W_{1321}}{\partial x} = 0 \tag{3.43}$$

$$\frac{\partial W_{12}}{\partial t} + \frac{\partial W_{1322}}{\partial x} = 0 \tag{3.44}$$

Equation (3.43) shows that W_{1321} is independent of x. Equation (3.44) suggests that a function $\theta(t,x)$ can be defined such that

$$W_{12} = \frac{\partial \theta}{\partial x}, \quad W_{1322} = -\frac{\partial \theta}{\partial t} \tag{3.45}$$

Since W_{1321} is independent of x, Eq. (3.41) gives:

$$\frac{dW_{111}}{dt} = 0 \tag{3.46}$$

$$\frac{dW_{112}}{dt} + W_{1321} = 0 \tag{3.47}$$

Equation (3.46) shows that W_{111} is independent of t. Equation (3.47) establishes the relation between W_{112} and W_{1321}.

The final form of the characteristic function is therefore:

$$\boxed{W_1 = [W_{111}x + W_{112}(t)]p_2 + \frac{\partial \theta}{\partial x} p_3 - \left\{W_{111}\psi + \frac{dW_{112}}{dt} y - \frac{\partial \theta}{\partial t}\right\}} \tag{3.48}$$

where W_{111} is a constant, W_{112} is an arbitrary function of t and θ an arbitrary function of t and x.

The condition $f_o = 0$ has to be checked after the boundary conditions are considered.

The transformation functions, α_1, α_2, α_3 and ζ for G_1 are therefore given by

$$\alpha_1 = 0$$

$$\alpha_2 = W_{111}x + W_{112}(t)$$

$$\alpha_3 = \frac{\partial\theta}{\partial x} \qquad (3.49a\text{-}d)$$

$$\zeta = W_{111}\psi + \frac{dW_{112}}{dt}\, y + \frac{\partial\theta}{\partial t}$$

which are obtained by substituting the characteristic function W_1, Eq. (3.48), into Eqs. (3.10a-c).

For the second group, G_2,

$$\alpha_2 = \frac{\partial W_2}{\partial p_2} = 0 \qquad (3.50)$$

which means the characteristic function W_2 is independent of p_2. The other transformations, α_1, α_3, ζ, etc. can be expressed in terms of the characteristic function W_2 as follows:

$$\alpha_1 = \frac{\partial W_2}{\partial p_1}, \quad \alpha_3 = \frac{\partial W_2}{\partial p_3}, \quad \zeta = p_1\frac{\partial W_2}{\partial p_1} + p_3\frac{\partial W_2}{\partial p_3} - W_2$$

(3.51a-f)

$$\pi_1 = -\frac{\partial W_2}{\partial t} - p_1\frac{\partial W_2}{\partial \psi}, \quad \pi_2 = -\frac{\partial W_2}{\partial x} - p_2\frac{\partial W_2}{\partial \psi}, \quad \pi_3 = -\frac{\partial W_2}{\partial y} - p_3\frac{\partial W_2}{\partial \psi}$$

$$-\pi_{13} = \frac{\partial^2 W_2}{\partial t\,\partial y} + p_3\frac{\partial^2 W_2}{\partial t\,\partial \psi} + p_1\left(\frac{\partial^2 W_2}{\partial \psi\,\partial y} + p_3\frac{\partial^2 W_2}{\partial \psi^2}\right) + p_{11}\left(\frac{\partial^2 W_2}{\partial p_1\,\partial y} + p_3\frac{\partial^2 W_2}{\partial p_1\,\partial \psi}\right)$$

$$+ p_{31}\left(\frac{\partial^2 W_2}{\partial p_3\,\partial y} + p_3\frac{\partial^2 W_2}{\partial p_3\,\partial \psi} + \frac{\partial W_2}{\partial \psi}\right) + p_{13}\left(\frac{\partial^2 W_2}{\partial p_1\,\partial t} + \frac{\partial^2 W_2}{\partial p_1\,\partial \psi}\, p_1\right)$$

$$+ p_{33}\left(\frac{\partial^2 W_2}{\partial t\,\partial p_3} + \frac{\partial^2 W_2}{\partial \psi\,\partial p_3}\, p_1\right) \qquad (3.51g)$$

$$- \pi_{23} = \frac{\partial^2 W_2}{\partial x\, \partial y} + p_3 \frac{\partial^2 W_2}{\partial x\, \partial \psi} + p_2 \left(\frac{\partial^2 W_2}{\partial \psi\, \partial y} + p_3 \frac{\partial^2 W_2}{\partial \psi^2} \right) + p_{12} \left(\frac{\partial^2 W_2}{\partial p_1\, \partial y} + p_3 \frac{\partial^2 W_2}{\partial p_1\, \partial \psi} \right)$$

$$+ p_{32} \left(\frac{\partial^2 W_2}{\partial p_3\, \partial y} + p_3 \frac{\partial^2 W_2}{\partial p_3\, \partial \psi} + \frac{\partial W_2}{\partial \psi} \right) + p_{13} \left(\frac{\partial^2 W_2}{\partial x\, \partial p_1} + \frac{\partial^2 W_2}{\partial \psi\, \partial p_1} p_2 \right)$$

$$+ p_{33} \left(\frac{\partial^2 W_2}{\partial x\, \partial p_3} + p_2 \frac{\partial^2 W_2}{\partial \psi\, \partial p_3} \right) \qquad (3.51h)$$

$$- \pi_{33} = \frac{\partial^2 W_2}{\partial y^2} + p_3 \frac{\partial^2 W_2}{\partial y\, \partial \psi} + p_3 \left(\frac{\partial^2 W_2}{\partial \psi\, \partial y} + p_3 \frac{\partial^2 W_2}{\partial \psi^2} \right) + 2p_{13} \left(\frac{\partial^2 W_2}{\partial p_1\, \partial y} + p_3 \frac{\partial^2 W_2}{\partial p_1\, \partial \psi} \right)$$

$$+ 2p_{33} \left(\frac{\partial^2 W_2}{\partial p_3\, \partial y} + p_3 \frac{\partial^2 W_2}{\partial p_3\, \partial \psi} \right) + p_{33} \frac{\partial W_2}{\partial \psi} \qquad (3.51i)$$

$$- \pi_{333} = \frac{\partial^3 W_2}{\partial y^3} + 2p_3 \frac{\partial^3 W_2}{\partial y^2\, \partial \psi} + p_3^2 \frac{\partial^3 W_2}{\partial y\, \partial \psi^2} + 2p_{13} \left(\frac{\partial^3 W_2}{\partial p_1\, \partial y^2} + p_3 \frac{\partial^3 W_2}{\partial y\, \partial p_1\, \partial \psi} \right)$$

$$+ 2p_{33} \left(\frac{\partial^3 W_2}{\partial p_3\, \partial y^2} + p_3 \frac{\partial^3 W_2}{\partial y\, \partial p_3\, \partial \psi} \right) + p_{33} \frac{\partial^2 W_2}{\partial y\, \partial \psi} + p_3 \left\{ \frac{\partial^3 W_2}{\partial \psi\, \partial y^2} + 2p_3 \frac{\partial^3 W_2}{\partial \psi^2\, \partial y} \right.$$

$$+ p_3^2 \frac{\partial^3 W_2}{\partial \psi^3} + 2p_{13} \left(\frac{\partial^3 W_2}{\partial \psi\, \partial p_1\, \partial y} + p_3 \frac{\partial^3 W_2}{\partial p_1\, \partial \psi^2} \right) + 2p_{33} \left(\frac{\partial^3 W_2}{\partial p_3\, \partial y\, \partial \psi} + p_3 \frac{\partial^3 W_2}{\partial p_3\, \partial \psi^2} \right)$$

$$\left. + p_{33} \frac{\partial^2 W_2}{\partial \psi^2} \right\} + p_{13} \left(\frac{\partial^3 W_2}{\partial y^2\, \partial p_1} + 2p_3 \frac{\partial^3 W_2}{\partial p_1\, \partial y\, \partial \psi} + p_3^2 \frac{\partial^3 W_2}{\partial p_1\, \partial \psi^2} + p_{33} \frac{\partial^3 W_2}{\partial p_1\, \partial \psi} \right)$$

$$+ p_{33} \left(\frac{\partial^3 W_2}{\partial y^2\, \partial p_3} + 2p_3 \frac{\partial^3 W_2}{\partial p_3\, \partial y\, \partial \psi} + p_3^2 \frac{\partial^3 W_2}{\partial p_3\, \partial \psi^2} + p_{33} \frac{\partial^2 W_2}{\partial p_3\, \partial \psi} \right)$$

$$+ 3p_{133} \left(\frac{\partial^2 W_2}{\partial p_1\, \partial y} + p_3 \frac{\partial^2 W_2}{\partial p_1\, \partial \psi} \right)$$

$$+ \left(3 \frac{\partial^2 W_2}{\partial p_3\, \partial y} + 3p_3 \frac{\partial^2 W_2}{\partial p_3\, \partial \psi} + \frac{\partial W_2}{\partial \psi} \right) (p_{13} + p_3\, p_{23} - p_2\, p_{33} - \phi) \qquad (3.51j)$$

As before, the transformation functions given in Eqs. (3.51a-j) are substituted into Eq. (3.9b) and p_{333} is eliminated from Eq. (3.2). We then get

$$f_o + f_1 p_{11} + f_2 p_{21} + f_3 p_{31} + f_4 p_{22} + f_5 p_{23} + f_6 p_{33} + f_7 p_{133} + f_8 p_{233} + f_9 p_{13} p_{33} + f_{10} p_{23} p_{33} + f_{11} p_{33}^2 = 0 \quad (3.52)$$

where

$$f_o = -\left(\frac{\partial^3 W_2}{\partial y^3} + 2p_3 \frac{\partial^3 W_2}{\partial y^2 \partial\psi} + p_3^2 \frac{\partial^3 W_2}{\partial y \partial\psi^2}\right) - p_3\left(\frac{\partial^3 W_2}{\partial\psi \partial y^2} + 2p_3 \frac{\partial^3 W_2}{\partial\psi^2 \partial y} + p_3^2 \frac{\partial^3 W_2}{\partial\psi^3}\right)$$
$$+ 3\phi\left(\frac{\partial^2 W_2}{\partial p_3 \partial y} + p_3 \frac{\partial^2 W_2}{\partial p_3 \partial\psi}\right) + \frac{\partial\phi}{\partial t}\frac{\partial W_2}{\partial p_1} + \frac{\partial\phi}{\partial x}\frac{\partial W_2}{\partial p_2} - \phi\frac{\partial W_2}{\partial\psi} + \left(\frac{\partial^2 W_2}{\partial t \partial y} + p_3 \frac{\partial^2 W_2}{\partial t \partial\psi}\right)$$
$$+ p_1\left(\frac{\partial^2 W_2}{\partial\psi \partial y} + p_3 \frac{\partial^2 W_2}{\partial\psi^2}\right) + p_3\left(\frac{\partial^2 W_2}{\partial x \partial y} + p_3 \frac{\partial^2 W_2}{\partial x \partial\psi}\right) - p_2\left(\frac{\partial^2 W_2}{\partial y^2} + p_3 \frac{\partial^2 W_2}{\partial y \partial\psi}\right) \quad (3.53)$$

$$f_1 = \frac{\partial^2 W_2}{\partial p_1 \partial y} + p_3 \frac{\partial^2 W_2}{\partial p_1 \partial\psi} \quad (3.54)$$

$$f_2 = p_3\left(\frac{\partial^2 W_2}{\partial p_1 \partial y} + p_3 \frac{\partial^2 W_2}{\partial p_1 \partial\psi}\right) \quad (3.55)$$

$$f_3 = -\left(3\frac{\partial^3 W_2}{\partial p_1 \partial y^2} + 4p_3 \frac{\partial^3 W_2}{\partial y \partial p_1 \partial\psi}\right) - 2p_3\left(\frac{\partial^3 W_2}{\partial\psi \partial p_1 \partial y} + p_3 \frac{\partial^3 W_2}{\partial p_1 \partial\psi^2}\right) - p_3^2 \frac{\partial^3 W_2}{\partial p_1 \partial\psi^2}$$
$$- 2\left(\frac{\partial^2 W_2}{\partial p_3 \partial y} + p_3 \frac{\partial^2 W_2}{\partial p_3 \partial\psi}\right) + \left(\frac{\partial^2 W_2}{\partial p_1 \partial t} + \frac{\partial^2 W_2}{\partial p_1 \partial\psi} p_1\right) + p_3\left(\frac{\partial^2 W_2}{\partial x \partial p_1} + \frac{\partial^2 W_2}{\partial\psi \partial p_1} p_2\right)$$
$$- 2\left(\frac{\partial^2 W_2}{\partial p_1 \partial y} + p_3 \frac{\partial^2 W_2}{\partial p_1 \partial\psi}\right) p_2 \quad (3.56)$$

$$f_4 = 0 \quad (3.57)$$

$$f_5 = -2p_3\left(\frac{\partial^2 W_2}{\partial p_3 \partial y} + p_3 \frac{\partial^2 W_2}{\partial p_3 \partial\psi}\right) + \left(\frac{\partial W_2}{\partial y} + p_3 \frac{\partial W_2}{\partial\psi}\right) \quad (3.58)$$

$$f_6 = -\left(3\frac{\partial^3 W_2}{\partial p_3\,\partial y^2} + 4p_3\frac{\partial^3 W_2}{\partial y\,\partial p_3\,\partial\psi}\right) - 2p_3\left(\frac{\partial^3 W_2}{\partial p_3\,\partial y\,\partial\psi} + p_3\frac{\partial^3 W_2}{\partial p_3\,\partial\psi^2}\right) - p_3^2\frac{\partial^3 W_2}{\partial p_3\,\partial\psi^2}$$

$$+ p_2\left(\frac{\partial^2 W_2}{\partial p_3\,\partial y} + p_3\frac{\partial^2 W_2}{\partial p_3\,\partial\psi}\right) - \frac{\partial^2 W_2}{\partial y\,\partial\psi} - p_3\frac{\partial^2 W_2}{\partial\psi^2} + \left(\frac{\partial^2 W_2}{\partial t\,\partial p_3} + p_1\frac{\partial^2 W_2}{\partial\psi\,\partial p_3}\right)$$

$$+ p_3\left(\frac{\partial^2 W_2}{\partial x\,\partial p_3} + p_2\frac{\partial^2 W_2}{\partial\psi\,\partial p_3}\right) - \left(\frac{\partial W_2}{\partial x} + p_2\frac{\partial W_2}{\partial\psi}\right) \tag{3.59}$$

$$f_7 = -3\left(\frac{\partial^2 W_2}{\partial p_1\,\partial y} + p_3\frac{\partial^2 W_2}{\partial p_1\,\partial\psi}\right) \tag{3.60}$$

$$f_8 = 0 \tag{3.61}$$

$$f_9 = -\frac{\partial^2 W_2}{\partial p_1\,\partial\psi} \tag{3.62}$$

$$f_{10} = 0 \tag{3.63}$$

$$f_{11} = -\frac{\partial^2 W_2}{\partial p_3\,\partial\psi} \tag{3.64}$$

For Eq. (3.52) to be satisfied, all the f's should be zero since the characteristic function W_2 is only a function of $t,x,y,\psi,p_1,p_2,$ and p_3. Therefore, we get 12 equations:

$$f_o = f_1 = f_2 = \ldots = f_{11} = 0 \tag{3.65}$$

Three of these equations are satisfied identically which are f_4, f_8, and f_{10}.

For the point transformation under consideration, with $\alpha_2 = 0$, the characteristic function can be written as

$$W_2 = W_{21}(t,x,y,\psi)p_1 + W_{22}(t,x,y,\psi)p_3 + W_{23}(t,x,y,\psi) \tag{3.66}$$

From the conditions

$$f_1 = 0\,, \quad f_2 = 0\,, \quad f_7 = 0\,, \quad f_9 = 0\,, \quad f_{11} = 0$$

we conclude that W_{21} is independent of y and ψ and that W_{22} is independent of ψ.

The condition $f_5 = 0$ gives

$$- 2p_3 \frac{\partial W_{22}}{\partial y} + \frac{\partial W_{22}}{\partial y} p_3 + \frac{\partial W_{23}}{\partial y} + p_3 \frac{\partial W_{23}}{\partial \psi} = 0 \tag{3.67}$$

which gives

$$\frac{\partial W_{23}}{\partial y} = 0 \tag{3.68}$$

$$\frac{\partial W_{23}}{\partial \psi} - \frac{\partial W_{22}}{\partial y} = 0 \tag{3.69}$$

From Eq. (3.68), W_{23} is independent of y. Equation (3.69) will be needed later.

The condition $f_3 + 0$ gives

$$- 2 \frac{\partial W_{22}}{\partial y} + \frac{\partial W_{21}}{\partial t} + p_3 \frac{\partial W_{21}}{\partial x} = 0 \tag{3.70}$$

which can be separated into two equations:

$$\frac{\partial W_{21}}{\partial x} = 0 \tag{3.71}$$

$$\frac{\partial W_{21}}{\partial t} - 2 \frac{\partial W_{22}}{\partial y} = 0 \tag{3.72}$$

Equation (3.71) shows that W_{21} is independent of x.

Since W_{23} is independent of y, Eq. (3.69) shows that W_{22} should be linear in y. Thus, we write

$$W_{22} = W_{221}(t,x)y + W_{222}(t,x) \tag{3.73}$$

Substitution of W_{22} from Eq. (3.73) into Eqs. (3.69) and (3.72) leads to the following relations:

$$W_{221} = \frac{\partial W_{23}}{\partial \psi} \tag{3.74}$$

$$2W_{221} = \frac{dW_{21}}{dt} \tag{3.75}$$

Thus, W_{221} is independent of x. So,

$$W_{22} = \frac{1}{2} \frac{dW_{21}}{dt} y + W_{222}(t,x) \tag{3.76}$$

The characteristic function W_2 now becomes

$$W_2 \quad = \quad W_{21}(t)p_1 + \left\{ \left[\frac{1}{2} \frac{dW_{21}}{dt} y + W_{222}(t,x) \right] \right\} p_3 + W_{23}(t,x,\psi) \qquad (3.77)$$

From Eq. (3.74), W_{23} is linear in ψ, i.e.,

$$W_{23} \quad = \quad W_{231}(t,x)\psi + W_{232}(t,x) \qquad (3.78)$$

Equations (3.74) and (3.75) then give

$$W_{231} \quad = \quad \frac{1}{2} \frac{dW_{21}}{dt} \qquad (3.79)$$

which also shows that W_{231} is independent of x. Thus,

$$W_{23} \quad = \quad \frac{1}{2} \frac{dW_{21}}{dt} \psi + W_{232}(t,x) \qquad (3.80)$$

The condition $f_6 = 0$ gives:

$$\frac{1}{2} \frac{d^2W_{21}}{dt^2} y + \frac{\partial W_{222}}{\partial t} + \frac{\partial W_{232}}{\partial x} \quad = \quad 0 \qquad (3.81)$$

which can be separated into two equations as follows:

$$\frac{d^2W_{21}}{dt^2} \quad = \quad 0 \qquad (3.82)$$

$$- \frac{\partial W_{222}}{\partial t} + \frac{\partial W_{232}}{\partial x} \quad = \quad 0 \qquad (3.83)$$

Equation (3.82) shows that W_{21} is a linear function of t, i.e.,

$$W_{21}(t) \quad = \quad W_{211}t + W_{212} \qquad (3.84)$$

where W_{211} and W_{212} are constants. From Eq. (3.83), a function $\bar{\theta}(t,x)$ can be introduced such that

$$W_{222} \quad = \quad \frac{\partial \bar{\theta}}{\partial x} , \qquad W_{232} \quad = \quad + \frac{\partial \bar{\theta}}{\partial t} \qquad (3.85)$$

The final form of the characteristic function, W_2, is therefore

$$\boxed{W_2 \quad = \quad (W_{211}t + W_{212})p_1 + \left(\frac{1}{2} W_{211}y + \frac{\partial \bar{\theta}}{\partial x} \right) p_3 + \frac{1}{2} W_{211}\psi + \frac{\partial \bar{\theta}}{\partial t}} \qquad (3.86)$$

where W_{211} and W_{212} are constants and $\bar{\theta}$ is an arbitrary function of t and x.

Again, the condition $f_o = 0$ has to be checked after the boundary conditions are considered.

The transformation functions, $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\alpha}_3$, and $\bar{\zeta}$ for G_2 are therefore given by

$$
\begin{aligned}
\bar{\alpha}_1 &= W_{211}t + W_{212} \\
\bar{\alpha}_2 &= 0 \\
\bar{\alpha}_3 &= \frac{1}{2} W_{211}y + \frac{\partial\bar{\theta}}{\partial x} \\
\bar{\zeta} &= \frac{\partial\bar{\theta}}{\partial t} - \frac{1}{2} W_{211}\psi
\end{aligned} \qquad (3.87a\text{-}d)
$$

With the characteristic functions W_1 and W_2 for the two groups, G_1 and G_2 known, the next step is to find the absolute invariants. For the combined two-parameter group of transformations defined in Eqs. (3.4) to (3.9), the absolute invariants can be solved from the following system of equations:

$$
\frac{dt}{\alpha_1 + a\bar{\alpha}_1} = \frac{dx}{\alpha_2 + a\bar{\alpha}_2} = \frac{dy}{\alpha_3 + a\bar{\alpha}_3} = \frac{d\psi}{\zeta + a\bar{\zeta}} = e_1 d\epsilon_1
$$

where $a = e_2/e_1$. Substituting the transformation functions from Eqs. (3.49) and (3.87) into Eq. (3.88), we get

$$
\begin{aligned}
\frac{dt}{a(W_{211}t + W_{212})} &= \frac{dx}{W_{111}x + W_{112}(t)} \\
&= \frac{dy}{\frac{\partial\theta}{\partial x} + a\left(\frac{1}{2} W_{211}y + \frac{\partial\bar{\theta}}{\partial x}\right)} \\
&= \frac{d\psi}{\left(W_{111}\psi + \frac{dW_{112}}{dt} y + \frac{\partial\theta}{\partial t}\right) + a\left(\frac{\partial\bar{\theta}}{\partial t} - \frac{1}{2} W_{211}\psi\right)} = e_1 d\epsilon_1 \qquad (3.89)
\end{aligned}
$$

As an example, consider the case in which θ, $\bar{\theta}$, W_{112}, and W_{212} are all aero. Eq. (3.89) then becomes:

$$
\frac{dt}{aW_{211}t} = \frac{dx}{W_{111}x} = \frac{dy}{\frac{a}{2} W_{211}y} = \frac{d\psi}{W_{111}\psi - \frac{a}{2} W_{211}\psi} = e_1 d\epsilon_1 \qquad (3.90)
$$

The three independent solutions are

$$\frac{x}{t^{\frac{W_{111}}{aW_{211}}}} = c_1 \tag{3.91a}$$

$$\frac{y}{t^{\frac{1}{2}}} = c_2 \tag{3.91b}$$

and

$$\frac{\psi}{t^{\frac{W_{111}}{aW_{211}} - \frac{1}{2}}} = c_3 \tag{3.91c}$$

As a final step, the parameter a has to be eliminated from Eq. (3.91). We then get

$$\frac{y}{t^{\frac{1}{2}}} = c_1 \text{ and } \frac{\psi}{xt^{-\frac{1}{2}}} = \frac{c_3}{c_1} \tag{3.92a,b}$$

where Eq. (3.92b) is obtained by eliminating a from Eqs. (3.91a) and (3.91c).

The similarity variables are therefore

$$\eta = \frac{y}{t^{\frac{1}{2}}} \text{ and } f(\eta) = \frac{\psi}{xt^{-\frac{1}{2}}} \tag{3.93a,b}$$

which are the same as the variables defined by Shuh.[10]

The boundary condition at the edge of the boundary layer, namely,

$$y = \infty: \ \frac{\partial\psi}{\partial y} = U_e(x,t)$$

is then transformed to be

$$\eta = \infty: \ \frac{x}{t} f'(\infty) = U_e(x,t)$$

which gives

$$U_e(x,t) = \frac{x}{t} \tag{3.94}$$

The function Φ, as defined in Eq. (3.3), therefore becomes $\Phi = 0$. It can be shown by simple substitution that this function of Φ satisfies the condition $f_o = 0$ for both G_1 and G_2.

Other special cases may be considered. For example, if we considered the case in which θ, W_{112}, and W_{212} are zero; and $\bar{\theta} = f_2(t)(x)$ where f_2 is a constant, Eq. (3.89) becomes

$$\frac{dt}{aW_{211}t} = \frac{dx}{W_{111}x} = \frac{dy}{\frac{a}{2} W_{211}y + af_2} = \frac{d\psi}{W_{111}\psi - \frac{a}{2} W_{211}\psi} \tag{3.95}$$

The three independent solutions are:

$$\frac{x}{t^{\frac{W_{111}}{aW_{211}}}} = c_1 \tag{3.96a}$$

$$yt^{-\frac{1}{2}} + 2\frac{f_2}{W_{211}} t^{-\frac{1}{2}} = c_2 \tag{3.96b}$$

and

$$\frac{\psi}{t^{\frac{W_{111}}{aW_{211}} - \frac{1}{2}}} = c_3 \tag{3.96c}$$

which, upon elimination of a, give

$$\eta = yt^{-\frac{1}{2}} + 2\frac{f_2}{W_{2111}} t^{-\frac{1}{2}} \tag{3.97a}$$

and

$$f(\eta) = \frac{\psi}{xt^{-\frac{1}{2}}} \tag{3.97b}$$

However, since the new form of η cannot transform $y = 0$ to $\eta = 0$, we conclude that f_2 must be zero. Equation (3.97) then becomes the transformations given in Eq. (3.93).

Consider now another pair of groups, namely,

G_3: a general point transformation

G_4: a general point transformation with $\alpha_3 = 0$.

The transformation function, α_1, α_2, α_3, ζ, π_1, π_2, π_3, π_{13}, π_{23}, π_{33}, and π_{333} can be expressed in terms of the characteristic function, W_3, as follows:

$$\alpha_i = \frac{\partial W_3}{\partial p_i}$$

$$\zeta = p_i \frac{\partial W_3}{\partial p_i} - W_3$$

$$\pi_1 = -\frac{\partial W_3}{\partial t} - p_i \frac{\partial W_3}{\partial \psi}$$

$$\pi_2 = -\frac{\partial W_3}{\partial x} - p_2 \frac{\partial W_3}{\partial \psi}$$

$$\pi_3 = -\frac{\partial W_3}{\partial y} - p_3 \frac{\partial W_3}{\partial \psi}$$

$$\pi_{13} = \frac{\partial \pi_3}{\partial t} + \frac{\partial \pi_3}{\partial \psi} p_1 + \frac{\partial \pi_3}{\partial p_i} p_{i1} - p_{i3}\left(\frac{\partial \alpha_i}{\partial t} + \frac{\partial \alpha_i}{\partial \psi} p_1\right)$$

$$\pi_{23} = \frac{\partial \pi_3}{\partial x} + \frac{\partial \pi_3}{\partial \psi} p_2 + \frac{\partial \pi_3}{\partial p_i} p_{i2} - p_{i3}\left(\frac{\partial \alpha_i}{\partial x} + \frac{\partial \alpha_i}{\partial \psi} p_2\right)$$

$$\pi_{33} = \frac{\partial \pi_3}{\partial y} + \frac{\partial \pi_3}{\partial \psi} p_3 + \frac{\partial \pi_3}{\pi p_i} p_{i3} - p_{i3}\left(\frac{\partial \alpha_i}{\partial y} + \frac{\partial \alpha_i}{\partial \psi} p_3\right)$$

$$\pi_{333} = \frac{\partial \pi_{33}}{\partial y} + \frac{\partial \pi_{33}}{\partial \psi} p_3 + \frac{\partial \pi_{33}}{\partial p_i} p_{i3} + \frac{\partial \pi_{33}}{\partial p_{ij}} p_{ij3} - p_{33i}\left(\frac{\partial \alpha_i}{\partial y} + \frac{\partial \alpha_i}{\partial \psi} p_3\right) \qquad (3.98)$$

By following the same steps as in the cases of groups G_1 and G_2, the characteristic function W_3 for group G_3 is found to be:

$$\begin{aligned} W_3 = {} & (W_{311}t + W_{312})p_1 + [W_{321}x + W_{322}(t)]p_2 \\ & + \left(\frac{1}{2} W_{311}y + \frac{\partial B}{\partial x}\right) p_3 + \left[\frac{1}{2} W_{311} - W_{321}\right] \psi \\ & - \frac{dW_{322}}{dt} y + \frac{\partial B}{\partial t} \end{aligned} \qquad (3.99)$$

where W_{311}, W_{312}, and W_{321} are constants, W_{322} is a function of t, and B(t,x) is an arbitrary function of t and x. In addition, an equation which comes from $f_o = 0$ has to be satisfied. This equation is:

$$\frac{3}{2}\,\phi\, W_{311} + \frac{\partial\phi}{\partial t}\,(W_{311}t + W_{312}) + \frac{\partial\phi}{\partial x}\,[W_{321}(t)x + W_{322}(t)]$$

$$- \phi\left[\frac{1}{2}\,W_{311} - W_{321}(t)\right] - p_3\,\frac{dW_{32}}{dt} = 0 \qquad (3.100)$$

Similarly, for G_4 where $\alpha_3 = 0$, the transformation functions can be written as:

$$\alpha_1 = \frac{\partial W_4}{\partial p_1}, \quad \alpha_2 = \frac{\partial W_4}{\partial p_2}, \quad \alpha_3 = 0,$$

$$\zeta = p_1\frac{\partial W_4}{\partial p_1} + p_2\frac{\partial W_4}{\partial p_2} - W_4$$

$$\pi_1 = -\frac{\partial W_4}{\partial t} - p_1\frac{\partial W_4}{\partial \psi}$$

$$\pi_2 = -\frac{\partial W_4}{\partial x} - p_2\frac{\partial W_4}{\partial \psi}$$

$$\pi_3 = -\frac{\partial W_4}{\partial y} - p_3\frac{\partial W_4}{\partial \psi}$$

$$\pi_{13} = \frac{\partial\pi_3}{\partial t} + \frac{\partial\pi_3}{\partial\psi}\,p_1 + \frac{\partial\pi_3}{\partial p_i}\,p_{i1} - p_{i3}\left(\frac{\partial\alpha_i}{\partial t} + \frac{\partial\alpha_i}{\partial\psi}\,p_1\right)$$

$$\pi_{23} = \frac{\partial\pi_3}{\partial x} + \frac{\partial\pi_3}{\partial\psi}\,p_2 + \frac{\partial\pi_3}{\partial p_i}\,p_{i3} - p_{i3}\left(\frac{\partial\alpha_i}{\partial x} + \frac{\partial\alpha_i}{\partial\psi}\,p_2\right)$$

$$\pi_{33} = \frac{\partial\pi_3}{\partial y} + \frac{\partial\pi_3}{\partial\psi}\,p_3 + \frac{\partial\pi_3}{\partial p_i}\,p_{i3} - p_{i3}\left(\frac{\partial\alpha_i}{\partial y} + \frac{\partial\alpha_i}{\partial\psi}\,p_3\right)$$

$$\pi_{333} = \frac{\partial\pi_{33}}{\partial y} + \frac{\partial\pi_{33}}{\partial\psi}\,p_3 + \frac{\partial\pi_{33}}{\partial p_i}\,p_{i3} + \frac{\partial\pi_{33}}{\partial p_{ij}}\,p_{ij3} - p_{33i}\left(\frac{\partial\alpha_i}{\partial y} + \frac{\partial\alpha_i}{\partial\psi}\,p_3\right) \qquad (3.101)$$

where i = 1,2 in the expressions of π_{13}, π_{23}, π_{33}, and π_{333}. By following the same steps as in the cases of G_1 and G_2 where W_1 and W_2 are formed, the characteristic function W_4 for group G_4 can be determined, which is:

$$W_4 = W_{41}\, p_1 + [W_{421}x + W_{422}(t)]p_2 - W_{421}\psi - \frac{dW_{422}}{dt}\, y + W_{4322}(t)$$

(3.102)

where W_{41}, W_{421}, are constants, and W_{422} and W_{4322} are functions of t. In addition, the condition $f_o = 0$ will lead to the following equation:

$$W_{41}\frac{\partial\phi}{\partial t} + [W_{421}x + W_{422}(t)]\frac{\partial\phi}{\partial x} + W_{421}\phi = 0 \qquad (3.103)$$

The transformation functions for the two-parameter group are therefore:

$$G_3:\ \alpha_1 = W_{311}t + W_{312}\ ,\quad \alpha_2 = W_{321}x + W_{322}(t)\ ,\quad \alpha_3 = \frac{1}{2}W_{311}y + \frac{\partial B}{\partial x}$$

$$\zeta = -\left[\frac{1}{2}W_{311} - W_{321}\right]\psi + \frac{dW_{322}}{dt}\, y - \frac{\partial B}{\partial t} \qquad (3.104)$$

$$G_4:\ \bar{\alpha}_1 = W_{41}\ ,\quad \bar{\alpha}_2 = W_{421}x + W_{432}(t)\ ,\quad \bar{\alpha}_3 = 0\ ,$$

$$\bar{\zeta} = W_{421}\psi + \frac{dW_{422}}{dt}\, y = W_{4322}(t) \qquad (3.105)$$

According to the theorems presented in Section 2. The absolute invariants can be solved from the following system of equations:

$$\frac{dt}{(W_{311}t + W_{312}) + aW_{41}} = \frac{dx}{[W_{321}x + W_{322}(t)] + a[W_{421}x + W_{422}(t)]} = \frac{dy}{\frac{1}{2}W_{311}y + \frac{\partial B}{\partial x}}$$

$$= \frac{d\psi}{-\left[\frac{1}{2}W_{311} - W_{321}\right]\psi + \frac{dW_{322}}{dt}\, y - \frac{\partial B}{\partial t} + a\left[W_{421}\psi + \frac{dW_{422}}{dt}\, y - W_{4322}(t)\right]} = e_1 d\epsilon_1$$

(3.106)

As an example, we consider the case in which B, W_{322}, W_{421}, and W_{4322} are all zero and W_{321} and W_{422} are constants. For this case, Eq. (3.106) becomes:

$$\frac{dt}{W_{311}t + W_{312} + aW_{41}} = \frac{dx}{W_{321}x + aW_{432}} = \frac{dy}{\frac{1}{2}W_{311}y} = \frac{d\psi}{\left(-\frac{1}{2}W_{311} + W_{321}\right)\psi} = e_1 d\epsilon_1$$

(3.107)

The three independent solutions to Eq. (3.107) are:

$$\frac{W_{321}x + aW_{422}}{(W_{311}t + W_{312} + a\,W_{41})^{n}} = c_1 \qquad (3.108a)$$

$$\frac{y}{\sqrt{W_{311}t + W_{312} + a\,W_{41}}} = c_2 \qquad (3.108b)$$

and

$$\frac{\psi}{(W_{311}t + W_{312} + aW_{41})^{n - \frac{1}{2}}} = c_3 \qquad (3.108c)$$

where $n = W_{321}/W_{311}$. For the special case in which $n = 1$ and $W_{312} = 0$, elimination of a among Eqs. (3.108a), (3.108b) and (3.108c) then gives

$$\frac{y}{\sqrt{W_{432}t - W_{41}x}} = c_4 \qquad (3.109a)$$

and

$$\frac{\psi}{\sqrt{W_{422}t - W_{41}x}} = c_5 \qquad (3.109b)$$

The similarity variables are therefore:

$$\eta = \frac{y}{\sqrt{W_{422}t - W_{41}x}}, \qquad f(\eta) = \frac{\psi}{\sqrt{W_{423}t - W_{41}x}} \qquad (3.110)$$

which is the second transformation obtained by Shuh.[10] It demonstrates again that this transformation should be obtained from a two-parameter group of transformation instead of the one-parameter group of transformation given in Manohar's work. This important point was shown by Moran and Gagioli[5] in a recent report, which is supported here by the analysis using the method developed by the present authors.

The boundary condition at the edge of the boundary layer is transformed to the form:

$$\eta = \infty: \quad f(\infty) = U_e(x,t)$$

which means U_e has to be a constant. It can be shown that this form of U_e satisfies Eqs. (3.100) and (3.103).

For other values of n in Eq. (3.108), it can be shown that Eqs. (3.100) and (3.103) will not be satisfied even though transformations can be obtained.

3.2. APPLICATION TO THE NONLINEAR DIFFUSION EQUATION

As a second example, the nonlinear diffusion equation is considered. This equation is:

$$\frac{\partial}{\partial x}\left(\psi^n \frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\psi^n \frac{\partial \psi}{\partial y}\right) = \frac{\partial \psi}{\partial t} \tag{3.111}$$

which can be written as:

$$F = \psi^n(p_{22} + p_{33}) + n\,\psi^{n-1}(p_2^2 + p_3^2) - p_1 = 0 \tag{3.112}$$

where

$$p_1 = \frac{\partial \psi}{\partial t}, \quad p_2 = \frac{\partial \psi}{\partial x}, \quad p_{22} = \frac{\partial^2 \psi}{\partial x^2}, \text{ etc.}$$

The differential equation F = 0, given in Eq. (3.112), will be invariant under the two-parameter group of transformation

$$\begin{aligned}
t' &= t + \delta\epsilon_1\alpha_1(t,x,y,\psi) + \delta\epsilon_2\bar{\alpha}_1(t,x,y,\psi) \\
x' &= x + \delta\epsilon_1\alpha_2(t,x,y,\psi) + \delta\epsilon_2\bar{\alpha}_2(t,x,y,\psi) \\
y' &= y + \delta\epsilon_1\alpha_3(t,x,y,\psi) + \delta\epsilon_2\bar{\alpha}_3(t,x,y,\psi) \\
\psi' &= \psi + \delta\epsilon_1\zeta(t,x,y,\psi) + \delta\epsilon_2\bar{\zeta}(t,x,y,\psi) \\
p_i' &= p_i + \delta\epsilon_1\pi_i(t,x,y,\psi,p_1,p_2,p_3) + \delta\epsilon_2\bar{\pi}_i(t,x,y,\psi,p_1,p_2,p_3) \\
p_{ii}' &= p_{ii} + \delta\epsilon_1\pi_{ii}(t,x,y,\psi,p_1,\ldots,p_{33}) + \delta\epsilon_2\bar{\pi}_{ii}(t,x,y,\psi,p_1,\ldots,p_{33})
\end{aligned} \tag{3.113}$$

if

$$UF = 0 \tag{3.114}$$

or, from Eq. (2.20c),

$$e_1 \, U_1 F + e_2 \, U_2 F = 0 \tag{3.115}$$

Since e_1 and e_2 are arbitrary constants, Eq. (3.115) gives

$$U_1 F = 0 \text{ and } U_2 F = 0 \tag{3.116}$$

In their expanded form, Eq. (3.116) can be written as

$$\alpha_1 \frac{\partial F}{\partial t} + \alpha_2 \frac{\partial F}{\partial x} + \alpha_3 \frac{\partial F}{\partial y} + \zeta \frac{\partial F}{\partial \psi} + \pi_i \frac{\partial F}{\partial p_i} + \pi_{ij} \frac{\partial F}{\partial p_{ij}} = 0 \tag{3.117a}$$

and

$$\bar{\alpha}_1 \frac{\partial F}{\partial t} + \bar{\alpha}_2 \frac{\partial F}{\partial x} + \bar{\alpha}_3 \frac{\partial F}{\partial y} + \bar{\zeta} \frac{\partial F}{\partial \psi} + \bar{\pi}_i \frac{\partial F}{\partial p_i} + \bar{\pi}_{ij} \frac{\partial F}{\partial p_{ij}} = 0 \tag{3.117b}$$

The second step is to introduce two groups. Let us consider two groups as follows:

G_1: general point transformation (whose characteristic function W_1 is therefore linear with respect to the p's,

G_2: same as G_1 except the characteristic function be independent of p_3.

For group G_1, substitution of F from Eq. (3.112) and the transformations in terms of W_1 from Eqs. (2.25) to (2.28) into Eq. (3.117a) then gives:

$$f_o + f_1 \, p_{12} + f_2 \, p_{13} + f_3 \, p_{22} + f_4 \, p_{32} = 0 \tag{3.118}$$

where

$$\begin{aligned} f_o = {} & \left[n\psi^{-1}p_1 - n\psi^{n-2}(p_2^2 + p_3^2)\right] \left(p_1 \frac{\partial W_1}{\partial p_1} + p_2 \frac{\partial W_1}{\partial p_2} + p_3 \frac{\partial W_1}{\partial p_3} - W_1 \right) + \frac{\partial W_1}{\partial t} + p_1 \frac{\partial W_1}{\partial \psi} \\ & - 2n\psi^{n-1} p_2 \left(\frac{\partial W_1}{\partial x} + p_2 \frac{\partial W_1}{\partial \psi} \right) - 2n\psi^{n-1} p_3 \left(\frac{\partial W_1}{\partial y} + p_3 \frac{\partial W_1}{\partial \psi} \right) - \psi^n \left(\frac{\partial^2 W_1}{\partial x^2} + 2p_2 \frac{\partial^2 W_1}{\partial x \, \partial \psi} \right. \\ & \left. + p_2^2 \frac{\partial^2 W_\mu}{\partial \psi^2} + \frac{\partial^2 W_1}{\partial y^2} + 2p_2 \frac{\partial^2 W_1}{\partial \psi^2} + p_3^2 \frac{\partial^2 W_1}{\partial \psi^2} \right) + \left[\psi^{-n} p_1 - n\psi^{-1}(p_2^2 + p_3^2)\right] \\ & \left[2 \frac{\partial^2 W_1}{\partial p_3 \, \partial y} + 2p_3 \frac{\partial^2 W_1}{\partial p_3 \, \partial \psi} + \frac{\partial W_1}{\partial \psi} \right] \end{aligned} \tag{3.119}$$

$$f_1 = 2\left(\frac{\partial^2 W_1}{\partial p_1\, \partial x} + p_2 \frac{\partial^2 W_1}{\partial p_1\, \partial \psi}\right) \tag{3.120}$$

$$f_2 = 2\left(\frac{\partial^2 W_1}{\partial p_1\, \partial y} + p_3 \frac{\partial^2 W_1}{\partial p_1\, \partial \psi}\right) \tag{3.121}$$

$$f_3 = 2\frac{\partial^2 W_1}{\partial p_2\, \partial x} + 2p_2 \frac{\partial^2 W_1}{\partial p_2\, \partial \psi} - 2\frac{\partial^2 W_1}{\partial p_3\, \partial y} - 2p_3 \frac{\partial^2 W_1}{\partial p_3\, \partial \psi} \tag{3.122}$$

$$f_4 = 2\left(\frac{\partial^2 W_1}{\partial p_3\, \partial x} + p_2 \frac{\partial^2 W_1}{\partial p_3\, \partial \psi} + \frac{\partial^2 W_1}{\partial p_2\, \partial y} + p_3 \frac{\partial^2 W_1}{\partial p_2\, \partial \psi}\right) \tag{3.123}$$

Since the f's are independent of the f's, Eq. (3.118) is satisfied if the following equations are true:

$$f_o = f_1 = f_2 = f_3 = f_4 = 0 \tag{3.124}$$

Now, the characteristic function W_1 can be written as:

$$W_1 = W_{11}(t,x,y,\psi)p_1 + W_{12}(t,x,y,\psi)p_2 + W_{13}(t,x,y,\psi)p_3 + W_{14}(t,x,y,\psi) \tag{3.125}$$

The conditions $f_1 = 0$ and $f_2 = 0$, Eqs. (3.120) and (3.121), show that W_{11} is independent of x, y, and ψ. The condition $f_3 = 0$ gives

$$\frac{\partial W_{12}}{\partial x} - \frac{\partial W_{13}}{\partial y} + p_2 \frac{\partial W_{12}}{\partial \psi} - p_3 \frac{\partial W_{13}}{\partial \psi} = 0 \tag{3.125}$$

which can be separated into three equations, namely,

$$\frac{\partial W_{12}}{\partial \psi} = 0$$

$$\frac{\partial W_{13}}{\partial \psi} = 0 \tag{3.127a-c}$$

$$\frac{\partial W_{12}}{\partial x} - \frac{\partial W_{13}}{\partial y} = 0$$

Equations (3.127a,b) show that both W_{12} and W_{13} are independent of ψ. Equation (3.127c) will be used later.

Similarly, the condition $f_4 = 0$ gives

$$\frac{\partial W_{13}}{\partial x} + \frac{\partial W_{12}}{\partial y} = 0 \tag{3.128}$$

Finally, the condition $f_o = 0$ gives

$$g_o + g_1 p_1 + 2g_2 p_2^2 + g_3 p_2 + g_4 p_3 = 0 \qquad (3.129)$$

Since the functions g's are independent of p_1, p_2, and p_3, Eq. (3.129) gives

$$g_o = g_1 = g_2 = g_3 = g_4 = 0 \qquad (3.130)$$

which, in complete form, are:

$$\frac{\partial W_{14}}{\partial x} - \psi^n \left(\frac{\partial^2 W_{14}}{\partial x^2} + \frac{\partial^2 W_{14}}{\partial y^2} \right) = 0 \qquad (3.131)$$

$$- n\psi^{-1} W_{14} + \frac{dW_{11}}{dt} + \frac{\partial W_{14}}{\partial \psi} + \psi^{-n} \left(2 \frac{\partial W_{13}}{\partial y} + \frac{\partial W_{14}}{\partial \psi} \right) = 0 \qquad (3.132)$$

$$- n\psi^{-1} \left(2 \frac{\partial W_{13}}{\partial y} + \frac{\partial W_{14}}{\partial \psi} \right) + n\psi^{n-2} W_{14} - \psi^n \frac{\partial^2 W_{14}}{\partial \psi^2} - 2n\psi^{n-1} \left(\frac{\partial W_{13}}{\partial y} + \frac{\partial W_{14}}{\partial \psi} \right) = 0 \qquad (3.133)$$

$$\frac{\partial W_{12}}{\partial t} - 2n\psi^{n-1} \frac{\partial W_{14}}{\partial x} - 2\psi^n \frac{\partial^2 W_{14}}{\partial x \, \partial \psi} - 2\psi^n \frac{\partial^2 W_{14}}{\partial \psi^2} = 0 \qquad (3.134)$$

$$- 2n\psi^{n-1} \frac{\partial W_{14}}{\partial y} + \frac{\partial W_{13}}{\partial t} = 0 \qquad (3.135)$$

Equations (3.127c), (3.128), (3.131) through (3.135) are the conditions to be satisfied by the functions W_{11}, W_{12}, W_{13}, and W_{14} in the characteristic function. Although the general solutions to these functions are difficult to obtain, special cases can easily be investigated. For example, if we consider the special case in which W_{14} is linear in ψ, then

$$W_{14}(t,x,y,\psi) = W_{141}(t,x,y)\psi + W_{142}(t,x,y) \qquad (3.136)$$

Equation (3.135) gives three equations, namely,

$$\frac{\partial W_{141}}{\partial y} = \frac{\partial W_{142}}{\partial y} = \frac{\partial W_{13}}{\partial t} = 0$$

from which we conclude that W_{141} and W_{142} are independent of y and W_{13} is independent of t.

Equation (3.134) gives

$$\frac{\partial W_{12}}{\partial t} + 2(n - 1)\psi^{n} \frac{\partial W_{141}}{\partial x} + 2n\psi^{n-1} \frac{\partial W_{142}}{\partial x} = 0$$

which can be separated into three equations (for $n \neq 1$) as follows

$$\frac{\partial W_{12}}{\partial t} = \frac{\partial W_{141}}{\partial x} = \frac{\partial W_{142}}{\partial x} = 0 \qquad (3.137)$$

Equation (3.137) shows that W_{12} is independent of t, and that W_{141} and W_{142} are independent of x.

Equation (3.131) is therefore satisfied automatically and Eqs. (3.133) and (3.132) become

$$- n\psi^{-1}\left(2\frac{\partial W_{13}}{\partial y} + W_{141}\right) + n\psi^{n-2}(W_{141}\psi + W_{142}) - 2n\psi^{n-1}\left(\frac{\partial W_{13}}{\partial y} + W_{141}\right) = 0 \qquad (3.138)$$

$$- n^2\psi^{n-2}(W_{141}\psi + W_{142}) + n\psi^{n-1}\frac{dW_{11}}{dt} + n\psi^{n-1}W_{141} + n\psi^{-1}\left(2\frac{\partial W_{13}}{\partial y} + W_{141}\right) = 0 \qquad (3.139)$$

Summation of Eqs. (3.138) and (3.139) then gives

$$n(1 - n)\psi^{n-2}W_{142} + n\psi^{n-1}\left\{nW_{141} + 2\frac{\partial W_{13}}{\partial y} - \frac{dW_{11}}{dt}\right\} = 0 \qquad (3.140a)$$

from which,

$$W_{142} = 0$$

$$nW_{141} + 2\frac{\partial W_{13}}{\partial y} - \frac{dW_{11}}{dt} = 0 \qquad (3.140b)$$

Finally, conditions (3.127c) and (3.128) show that a function $\phi_1(x,y)$ can be introduced such that

$$W_{12} = \frac{\partial \phi_1}{\partial y}, \quad W_{13} = \frac{\partial \phi_1}{\partial x} \qquad (3.141)$$

and

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0 \qquad (3.142)$$

The final form of the characteristic function for group G_1 is

$$\boxed{W_1 = W_{11}(t)p_1 + \frac{\partial \phi_1}{\partial y} p_2 + \frac{\partial \phi_1}{\partial x} p_3 + \frac{1}{n}\left(\frac{dW_{11}}{dt} - 2\frac{\partial^2 \phi_1}{\partial x\, \partial y}\right)\psi} \qquad (3.143)$$

The transformation functions for group G_1 are therefore

$$\begin{aligned} \alpha_1 &= W_{11}(t) \\ \alpha_2 &= \frac{\partial \phi_1}{\partial y} \\ \alpha_3 &= \frac{\partial \phi_1}{\partial x} \\ \zeta &= \frac{1}{n}\left(2\frac{\partial^2 \phi_1}{\partial x\, \partial y} - \frac{dW_{11}}{dt}\right)\psi \end{aligned} \qquad (3.144)$$

For group G_2, the characteristic function W_2 can be found by following the same steps. If, in addition, W_2 is assumed to be linear in ψ, we then get

$$\boxed{W_2 = W_{21}(t)p_1 + W_{22}p_2 + \frac{1}{n}\frac{dW_{21}}{dt}\psi} \qquad (3.145)$$

The transformation functions for group G_2 are therefore:

$$\begin{aligned} \bar{\alpha}_1 &= W_{21}(t) \\ \bar{\alpha}_2 &= W_{22} \\ \bar{\alpha}_3 &= 0 \\ \bar{\zeta} &= -\frac{1}{n}\frac{dW_{21}}{dt}\psi \end{aligned} \qquad (3.146)$$

The next step is to find the absolute invariants of the two-parameter group G_{12} by solving the following system of equations:

$$\frac{dt}{W_{11}(t) + aW_{21}(t)} = \frac{dx}{\frac{\partial\phi_1}{\partial y} + aW_{22}} = \frac{dy}{\frac{\partial\phi_1}{\partial x}}$$

$$= \frac{d\psi}{-\left[\frac{1}{n}\left(\frac{dW_{11}}{dt} - 2\frac{\partial^2\phi_1}{\partial x\,\partial y}\right) + \frac{a}{n}\frac{dW_{21}}{dt}\right]\psi} = e_1 d\epsilon \qquad (3.147)$$

As an illustration, consider the case in which $W_{11}(t) = t$, $\phi_1 = xy$, and W_{21} is a constant. Equation (3.147) then becomes

$$\frac{dt}{t + aW_{21}} = \frac{dx}{x + aW_{22}} = \frac{dy}{y} = \frac{d\psi}{\frac{1}{n}\psi} = e_1 d\epsilon \qquad (3.148)$$

Solutions to Eq. (3.148) are

$$\frac{x + aW_{22}}{t + aW_{21}} = c_1$$

$$\frac{y}{t + aW_{21}} = c_2 \qquad (3.149a\text{-}c)$$

$$\frac{\psi}{(t + aW_{21})^{\frac{1}{n}}} = c_3$$

To illiminate a, Eq. (3.149a) is first solved for a as follows:

$$a = \frac{c_1 t - x}{W_{22} - c_1 W_{21}} \qquad (3.150)$$

The parameter "a" from Eq. (3.150) is then substituted into Eqs. (3.149b,c) and we get

$$\frac{y}{(W_{22}t - W_{21}x)} = c_4 \qquad (3.151a)$$

$$\frac{\psi}{(W_{22}t - W_{21}x)^{\frac{1}{n}}} = c_5 \qquad (3.151b)$$

The similarity transformation is therefore given by

$$\eta = \frac{y}{(W_{22}t - W_{21}x)}, \qquad f(\eta) = \frac{\psi}{(W_{22}t - W_{21}x)^{\frac{1}{n}}} \tag{3.152}$$

which is a form not given in the literature.

Consider now a third group G_3 defined as:

G_3: general point transformation with W_3 independent of p_1.

By following the same steps as before, the characteristic function is found to be:

$$\boxed{W_3 = \frac{\partial \phi_3}{\partial y} p_2 + \frac{\partial \phi_3}{\partial x} p_3 - \frac{2}{n} \frac{\partial^2 \phi_3}{\partial x \, \partial y} \psi} \tag{3.153}$$

where ϕ_3 is any function of x and y satisfying the Laplace equation:

$$\frac{\partial^2 \phi_3}{\partial x^2} + \frac{\partial^2 \phi_3}{\partial y^2} = 0 \tag{3.154}$$

The transformation functions for group G_3 are therefore:

$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_2 &= \frac{\partial \phi_3}{\partial y} \\ \alpha_3 &= \frac{\partial \phi_3}{\partial x} \\ \zeta &= \frac{2}{n} \frac{\partial^2 \phi_3}{\partial x \, \partial y} \psi \end{aligned} \tag{3.155}$$

For a two-parameter group G_{32} whose transformation functions are given by Eqs. (3.146) and (3.155), the absolute invariants can be solved from

$$\frac{dt}{aW_{21}(t)} = \frac{dx}{\frac{\partial\phi_3}{\partial y} + aW_{22}} = \frac{dy}{\frac{\partial\phi_3}{\partial x}} = \frac{d\psi}{\left(\frac{2}{n}\frac{\partial^2\phi_3}{\partial x\,\partial y} - \frac{a}{n}\frac{dW_{21}}{dt}\right)\psi} = e_1 d\epsilon \qquad (3.156)$$

As an example, consider the case in which $W_{21} = t$, $W_{22} = 0$, and $\phi_3 = xy$. Equation (3.156) becomes:

$$\frac{dt}{at} = \frac{dx}{x} = \frac{dy}{y} = \frac{d\psi}{\left(\frac{2}{n} - \frac{a}{n}\right)\psi} = e_1 d\epsilon \qquad (3.157)$$

Solutions of Eq. (3.157) are:

$$\frac{x}{t^{\frac{1}{a}}} = c_1\ , \quad \frac{y}{x} = c_2\ , \quad \frac{\psi}{t^{\frac{2-a}{na}}} = c_3 \qquad (3.158)$$

Elimination of "a" then gives:

$$\frac{y}{x} = c_2\ , \quad \frac{\psi}{t^{-\frac{1}{n}}\,x^{\frac{2}{n}}} = c_4$$

The similarity transformation is therefore

$$\eta = \frac{y}{x}\ , \quad f(\eta) = \frac{\psi}{t^{-\frac{1}{n}}\,x^{\frac{2}{n}}}$$

which is given in the literature.[1]

Other groups can be defined and transformations found, by following exactly the same steps as in the above examples.

3.3. CONCLUDING REMARKS

The method developed in this section is seen to be very general and, like other reports in the series,[7,8,9] involves mostly algebraic manipulations. It is systematic and does not require specification of an arbitrarily defined group at the beginning. On the contrary, the group is systematically determined.

The method also supports the important point that reduction of r variables can only be achieved by introducing a r-parameter group.[5]

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