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DEFORMATION OF ELLIPSOIDAL SHELLS OF REVOLUTION

By

P. M. NAGHDI

C. NEVIN DeSILVA

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SUMMARY

A single complex differential equation is deduced, and its solution by means of asymptotic integration is obtained, for thin elastic shells of revolution of both uniform and nonuniform thickness. Specifically, the stress distribution is determined for ellipsoidal shells of revolution under edge loadings. It is found that the stresses in such shells are essentially the same whether the thickness is uniform or varies in a certain specified manner.

1. Introduction

The formulation of the theory of small deflection of shells of revolution and its application, since H. Reissner's work[1] on spherical shells of uniform thickness, has been the subject of numerous investigations. A more recent formulation of finite deformation theory of shells of revolution, which also contains the theory of small deflection (linear theory) and where an account of the historical development of the subject may be found, was given by E. Reissner[2].

The present paper considers the smaller deformation of thin elastic ellipsoidal shells of revolution, with both uniform and nonuniform thickness, under axisymmetric loading. The solutions obtained are by means of the method of asymptotic integration of a complex differential equation involving a large parameter. An interesting feature of the results, which may be of practical interest, is that the stress distribution in an ellipsoidal shell of uniform thickness is found to be essentially the same as that of a shell of nonuniform thickness (see Fig. 4) under the same loading.

While attention is devoted mainly to ellipsoidal shells of revolution, both the complex differential equation mentioned and its solution are entirely general and are applicable to all shells of revolution of uniform thickness, as well as to a large class of shells of nonuniform thickness.

2. The Basic Equations of Shells of Revolution

In this section, we discuss briefly the basic equations of small deformation of elastic shells of revolution, with reference to a more recent formulation of the theory given by Eric Reissner[2]. Our discussion is general and pertains to shells of variable, as well as uniform, thickness.

Using cylindrical coordinates r, Θ, z, the parametric equation of the middle surface of the shell (Fig. 1) may be represented by

$$r = r(\xi), \qquad z = z(\xi). \tag{2.1}$$

Denoting by ϕ the slope of the tangent to the meridian of the shell, then

$$r' = \alpha \cos \phi, \qquad z' = \alpha \sin \phi, \tag{2.2}$$

where

$$\alpha = \left[(r')^2 + (z')^2\right]^{1/2} \tag{2.3}$$

and prime denotes differentiation with respect to ξ.

We note for future reference that the principal radii of curvature r_1 and r_2 are, respectively, the radius of curvature of the curve generating the middle surface and the length of the normal intercepted between this curve (generating curve) and the axis of rotation. It follows from the geometry of the middle surface that

$$r = r_2 \sin \phi. \tag{2.4}$$

The stress resultants N_ξ, N_Θ, and Q, and the stress couples M_ξ and M_Θ acting on an element of the shell are shown in Fig. 1. Also, as in Ref. 2, it is convenient to introduce "horizontal" and "vertical" stress resultants H and V, given by

$$\alpha N_\xi = r'H + z'V, \qquad \alpha Q = -z'H + r'V. \tag{2.5}$$

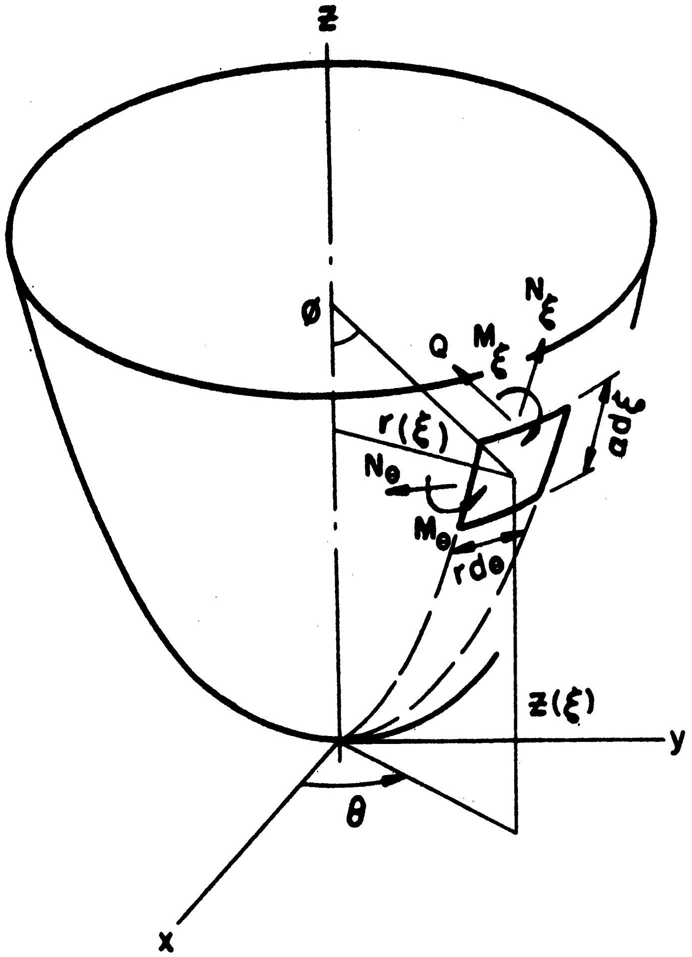


Fig. 1

We now record the basic equations of the small deflection theory of elastic shells of revolution with axisymmetric loading.

$$\left.\begin{aligned}
r\,V &= -\int r\,\alpha\,p_V\,d\xi \\
\alpha\,N_\Theta &= (rH)' + r\,\alpha\,p_H \\
r\,N_\xi &= (rH)\cos\phi + (rV)\sin\phi \\
r\,Q &= -rH\sin\phi + rV\cos\phi \\
M_\xi &= \frac{D}{\alpha}\left[\beta' + \nu\frac{r'}{r}\beta\right] \\
M_\Theta &= \frac{D}{\alpha}\left[\frac{r'}{r}\beta + \nu\beta'\right] \\
u &= \frac{r}{Eh}\left[N_\Theta - \nu N_\xi\right] \\
w &= \int\left[\frac{z'}{c}(N_\xi - \nu N_\Theta) - r'\beta\right]d\xi,
\end{aligned}\right\} \qquad (2.6)$$

where β is the negative change in ϕ due to deformation; u and w are the components of displacement in the radial and axial directions; p_H and p_V denote the components of load intensity in the r and z directions; h is the thickness of the shell, and

$$C = \frac{1}{Eh}, \qquad D = \frac{Eh^3}{12\,(1-\nu^2)}, \qquad (2.7)$$

E and ν being Young's modulus and Poisson's ratio, respectively.

The components of stress, due to stress couples (bending) and due to stress resultants N_ξ and N_Θ (membrane), as well as the shearing stress τ, are defined in the usual manner by

$$\left.\begin{aligned}
&\bar{\sigma}_{\xi b} \equiv (\sigma_{\xi b})_{max} = \frac{6M_\xi}{h^2}, \qquad \bar{\sigma}_{\Theta b} \equiv (\sigma_{\Theta b})_{max} = \frac{6M_\Theta}{h^2} \\
&\sigma_{\xi m} = \frac{N_\xi}{h}, \qquad \sigma_{\Theta m} = \frac{N_\Theta}{h}, \qquad \tau = \frac{3Q}{2h},
\end{aligned}\right\} \qquad (2.8)$$

where subscripts b and m refer to bending and membrane stresses respectively.

With β and rH as basic variables, proper elimination between equations (2.6), differential equations of equilibrium and compatibility, leads to the following two second-order differential equations:

$$\beta'' + \frac{(rD/\alpha)'}{(rD/\alpha)}\beta' - [(\frac{r'}{r})^2 - \nu\frac{(r'D/\alpha)'}{(rD/\alpha)}]\beta + \frac{z'}{(rD/\alpha)}(rH) = \frac{r'}{(rD/\alpha)}(rV) \quad (2.9)$$

$$(rH)'' + \frac{(r/C\alpha)'}{(r/C\alpha)}(rH)' - [(\frac{r'}{r})^2 + \nu\frac{(r'/C\alpha)'}{(r/C\alpha)}](rH) - \frac{z'}{(r/C\alpha)}\beta$$

$$= [\frac{z'r'}{r^2} + \nu\frac{(z'/C\alpha)'}{(r/C\alpha)}](rV) + \nu\frac{z'}{r}(rV)' \quad (2.10)$$

$$- [\frac{(r/C\alpha)'}{(r/C\alpha)} + \nu\frac{r'}{r}](r\alpha p_H) - (r\alpha p_H)' .$$

3. Normal Form of the Differential Equations

Substitution of the quantities C and D from (2.7) into (2.9) and (2.10) and rearrangement of terms result in

$$L_1(\beta) + \nu[\frac{(r'/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h}\frac{r'}{r}]\beta + \frac{\alpha^2 m}{r_2 h_o}(\frac{h_o}{h})\psi$$

$$= (\frac{\alpha^2 m}{r_2 h_o})(\frac{h_o}{h})\frac{mrV}{Eh^2}\cot\phi \quad (3.1)$$

$$L_1(\psi) - \nu[\frac{(r'/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h}\frac{r'}{r}]\psi - \frac{\alpha^2 m}{r_2 h_o}(\frac{h_o}{h})\beta$$

$$+ 2[\frac{h''}{h} + 2\nu\frac{h'}{h}\frac{r'}{r} + \frac{h'}{h}\frac{(r/\alpha)'}{(r/\alpha)}]\psi = Z\frac{m}{Eh^2} . \quad (3.2)$$

where Z denotes the right-hand side of (2.10), h_o is the value of h at some reference section (say $\xi = \xi_o$), and

$$L_1(\quad) = (\quad)'' + \left[\frac{(r/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h}\right](\quad)' - \left(\frac{r'}{r}\right)^2(\quad)$$

$$\psi = \frac{mrH}{Eh^2}, \qquad m = [12(1-\nu^2)]^{1/2}. \tag{3.3}$$

In (3.1) and (3.2), let

$$\frac{\alpha^2 m}{r_2 h_o} = 2\mu^2 f(\xi), \tag{3.4}$$

where μ is constant and it is to be noted that $f(\xi)$ is independent of the thickness $h(\xi)$. Then multiplication throughout (3.1) and (3.2) by $\{h[h_o f(\xi)]^{-1}\}$ results in

$$L(\beta) + \nu\lambda\beta + 2\mu^2\psi = F \tag{3.5}$$

$$L(\psi) - (\nu\lambda-\delta)\psi - 2\mu^2\beta = G, \tag{3.6}$$

where

$$\left.\begin{aligned}
L(\quad) &= \left[\frac{h_o}{h} f(\xi)\right]^{-1} L_1(\quad) \\
\lambda &= \left[\frac{h_o}{h} f(\xi)\right]^{-1} \left\{\frac{(r'/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h}\frac{r'}{r}\right\} \\
\delta &= 2\left[\frac{h_o}{h} f(\xi)\right]^{-1} \left\{\frac{h''}{h} + 2\nu\frac{h'}{h}\frac{r'}{r} + \frac{h'}{h}\frac{(r/\alpha)'}{(r/\alpha)}\right\} \\
F &= 2\mu^2 \frac{mrV}{Eh^2}\cot\phi \\
G &= \frac{m}{Eh^2}\left[\frac{h_o}{h} f(\xi)\right]^{-1} Z
\end{aligned}\right\} \tag{3.7}$$

Introducing the complex function

$$U = \beta + ik\psi; \qquad i = \sqrt{-1}. \tag{3.8}$$

where k is an arbitrary function of ξ to be determined, the differential equations (3.5) and (3.6) may be combined to read

$$L\,(U) = 2\mu^2 \left(ik - \frac{\nu\lambda}{2\mu^2}\right) \left\{\beta + \frac{\left[ik\left(\frac{\nu\lambda}{2\mu^2} - \frac{\delta}{2\mu^2}\right) - 1\right]}{\left(ik - \frac{\nu\lambda}{2\mu^2}\right)} \psi\right\} \tag{3.9}$$

$$+ i\psi\left[\frac{h_o}{h} f(\xi)\right]^{-1} \left\{k'' + \left[2\frac{\psi'}{\psi} + \frac{(r/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h}\right] k'\right\}$$

$$+ (F + ikG)\ .$$

Taking k in the form

$$k = -i\frac{1}{2\mu^2}\left(\nu\lambda - \frac{\delta}{2}\right) + \left\{1 - \left[\frac{1}{2\mu^2}\left(\nu\lambda - \frac{\delta}{2}\right)\right]^2\right\}^{1/2} \tag{3.10}$$

with the restriction (the implication of this restriction will be discussed later) that

$$k' = k'' = 0\ , \tag{3.11}$$

(3.9) transforms into

$$L\,(U) = 2\mu^2\left(ik - \frac{\nu\lambda}{2\mu^2}\right) U + (F + ikG)\ . \tag{3.12}$$

By putting the last complex differential equation in the form

$$L_1\,(U) - i\,\frac{2\mu^2\, h_o\left(k + i\,\frac{\nu\lambda}{2\mu^2}\right)}{h} f(\xi)\, U = \frac{h_o\, f(\xi)}{h}(F + ikG)$$

and observing that the coefficient of U', resulting from the application of the operator L_1 defined by (3.3) is

$$R = \left[\frac{(r/\alpha)'}{(r/\alpha)} + \frac{3h'}{h}\right]$$

and that

$$\exp\left[1/2 \int R d\xi\right] = h^{3/2} (r/\alpha)^{1/2}$$

then, with the aid of the transformation

$$W = \left(\frac{h}{h_o}\right)^{3/2} \left(\frac{r}{\alpha}\right)^{1/2} U \qquad (3.13)$$

we obtain

$$W'' + [2\, i^3 \mu^2 \Psi^2 (\xi) + \Lambda(\xi)]\, W = \left[\frac{h}{h_o} \frac{r}{\alpha}\right]^{1/2} f(\xi)\, (F + ikG), \qquad (3.14)$$

which is the normal form of (3.11) and where

$$\left.\begin{aligned} \Psi^2 &= \left(k + i \frac{\nu\lambda}{2\mu^2}\right) \left(\frac{h_o}{h}\right) f(\xi) \\ \Lambda &= -1/2 \frac{(r/\alpha)''}{(r/\alpha)} + 1/4 \left[\frac{(r/\alpha)'}{(r/\alpha)}\right]^2 - \left(\frac{r'}{r}\right)^2 \\ &\quad - 3/2 \frac{(r/\alpha)'}{(r/\alpha)} \frac{h'}{h} - 3/2 \frac{h''}{h} - 3/4 \left(\frac{h'}{h}\right)^2 . \end{aligned}\right\} (3.15)$$

We now return to (3.12) and observe that condition (3.11) is fulfilled only if k is a constant, and this may be achieved by proper choice of λ and δ. In particular, we note the following two cases:

(a). For shells of variable thickness, and with reference to differential equation (3.12), the condition (3.11) is satisfied, provided $(\nu\lambda - \frac{\delta}{2})$ is constant. Thus, by (3.7)

$$\left\{ h'' + \left[\frac{(r/\alpha)'}{(r/\alpha)} - \nu \frac{r'}{r}\right] h' - \nu \frac{(r'/\alpha)'}{(r/\alpha)} h \right\} = K f(\xi) , \qquad (3.16)$$

where K is any constant. A particular choice of thickness $h(\xi)$ may be obtained from (3.16) by setting K = 0. This corresponds to the vanishing value of $(\nu\lambda - \frac{\delta}{2})$, or by (3.10) to k = 1.

(b). For shells of uniform thickness, δ vanishes identically and we have

$$\lambda = f^{-1}(\xi)\left[\frac{(r'/\alpha)'}{(r/\alpha)}\right], \qquad \delta = 0 \tag{3.17}$$

Clearly, λ is a function of ξ and its form is determined by the geometry of the middle surface. However, for numerous shell configurations λ, and by (3.10) k, are either exactly or very nearly constant.

It should be mentioned that whenever the radius of curvature of the generating curve r_1 is a constant (r_2 may be a function of ξ), then with proper choice of ξ ($\xi = \phi$) and by (2.2), (3.4), and (3.17), λ and thus k are in fact constant. The cases of conical shell and toroidal shell treated recently by Clark[3] are included in this class.

4. Solution of Differential Equation by Asymptotic Integration

Let us first consider the homogeneous differential equation associated with (3.14), namely

$$W'' + [2\, i^3\mu^2\, \Psi^2(\xi) + \Lambda(\xi)]\, W = 0. \tag{4.1}$$

According to Langer[4], the solution of (4.1) admits asymptotic representation with respect to μ^2 (as a large parameter) as dictated by the coefficient of W; i.e., both Ψ^2 and Λ are suitably regular and bounded over a finite interval of the ξ-axis. Also, there exists a related differential equation (Langer's related differential equation) of the form

$$Y'' + [2\, i^3\mu^2\, \Psi^2(\xi) + \omega(\xi)]\, Y = 0 \tag{4.2}$$

whose solution

$$Y = \left[\Psi^{-1}\int_{\xi_0}^{\xi}\Psi\, d\xi\right]^{1/2}\left\{AJ_{-\frac{1}{n+2}}(\eta) + BJ_{\frac{1}{n+2}}(\eta)\right\} \tag{4.3}$$

is dominant in the asymptotic representation of W. Thus, (4.3) is an approximate solution of (4.1).

In (4.3), A and B are arbitrary constants, n is the order to which Ψ^2 vanishes at ξ_0, and ω and η are defined by

$$\left.\begin{aligned} \omega &= \frac{1}{4\chi^2}\left[(\Omega^2)'' - 3(\Omega')^2 + \left(\frac{2}{n+2}\right)^2\right] \\ \eta &= (2i^3)^{1/2}\,\mu\int_{\xi_0}^{\xi} \Psi\,d\xi , \end{aligned}\right\} \quad (4.4)$$

where

$$\left.\begin{aligned} \Omega &= \Psi^{-1}\int_{\xi_0}^{\xi} \Psi\,d\xi \\ \chi &= \Psi^{-1/2}\left[\int_{\xi_0}^{\xi} \Psi\,d\xi\right]^{\frac{n}{n+2}} . \end{aligned}\right\} \quad (4.5)$$

For the sake of clarity and completeness, it is expedient to describe briefly the so-called Stokes' phenomenon[4,5] which often arises in the solution of the differential equations of the type (4.2).

With n = 0, Ψ^2 is bounded from zero everywhere in the interval of convergence (ξ-axis) and (4.3) becomes

$$Y = \left[\Psi^{-1}\int \Psi\,d\xi\right]^{1/2}\left\{AJ_{-1/2}(\bar{\eta}) + BJ_{+1/2}(\bar{\eta})\right\}, \quad (4.6)$$

which, by means of well-known relations, may be written as

$$Y = \Psi^{-1/2}\left\{A_0 \exp(-i\bar{\eta}) + B_0 \exp(i\bar{\eta})\right\}, \quad (4.7)$$

where A_0 and B_0 are constants and

$$\bar{\eta} = (2i^3)^{1/2}\,\mu\int \Psi\,d\xi . \quad (4.8)$$

Clearly, (4.7) is an asymptotic solution of (4.1), and with the necessary provision that Ψ^2 vanishes nowhere in the interval of convergence, the constants A_o and B_o will be single-valued. If, on the other hand, Ψ^2 vanishes at some point ξ_o within the interval, then (4.7) tends to infinity at ξ_o and is multi-valued in the neighborhood of ξ_o. This multi-valued character of the solution (or of the constants A_o and B_o) in the interval excluding ξ_o is Stokes' phenomenon.

By reversing the procedure that led from (4.3) to (4.6) and to (4.7), we may represent solution (4.7) in terms of Bessel functions and then generalize to obtain (4.3) with single-valued coefficients A and B, when Ψ^2 vanishes to the degree n at some point ξ_o and only ξ_o, within the interval of convergence. Thus, solution (4.3) is free from the difficulties associated with Stokes' phenomenon.

If in (4.1) the restriction on Λ is relaxed so that it is no longer bounded everywhere in the interval, but has a pole of, at most, order 2 at ξ_o, then the Stokes' phenomenon is present and quantitatively depends on the nature of the pole. In such cases the representation of W in terms of Bessel functions is still possible, and furthermore is valid at ξ_o.[5]

We now return to the inhomogeneous equation (3.14) and note that in the presence of load intensity p_H and p_V, the right-hand side of (3.14) does not vanish. In addition to solution (4.3), it then becomes necessary to obtain an appropriate particular integral of (3.14). Such particular integrals are relatively easy to obtain and it will suffice to state that they may be determined approximately by the membrane theory of shells[6] or use may be made of a more recent method developed by Clark and Reissner[7].

5. Ellipsoidal Shells of Uniform Thickness.

Let us consider an ellipsoidal shell of revolution whose middle surface in rectangular cartesian coordinates is specified by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1 , \tag{5.1}$$

a and c being the semi-major axes of the ellipsoid.

Choosing the independent variable ξ as ϕ, it follows from the geometry of the middle surface and (2.2) that $\alpha = r_1$. The radii of curvature are

$$r_1 = \alpha = \frac{c^2}{a\,[1 + \rho^2 \cos^2\phi]^{3/2}} , \qquad r_2 = \frac{a}{[1 + \rho^2 \cos^2\phi]^{1/2}} \tag{5.2}$$

and by (2.4)

$$r = \frac{a \sin \phi}{[1 + \rho^2 \cos^2\phi]^{1/2}} \tag{5.3}$$

where

$$\rho^2 = \frac{c^2 - a^2}{a^2} \tag{5.4}$$

From (3.4) and (3.7) by (5.2), we have

$$2\mu^2 = (\frac{c}{a})^4 (\frac{a}{h_o})\, m , \qquad f(\xi) = (1 + \rho^2 \cos^2\phi)^{-5/2} \tag{5.5}$$

and

$$\delta = 0 , \qquad \lambda = -\frac{c^2}{a^2}(1 + \rho^2 \cos^2\phi)^{3/2} , \tag{5.6}$$

where m is given by (3.3). Since we are concerned here with shells of uniform thickness, in the remainder of this section h_o will be replaced by h.

With a view toward approximating k to a constant, so that condition (3.11) is fulfilled, we note that restriction of (c/a) to $O(1)$ is consistent with $\nu\lambda << 2\mu^2$, and by (3.10), $[1 - (\nu\lambda/2\mu^2)^2]^{1/2} \approx 1$ or $k \approx 1$. Thus by this approximation, equation (4.1) is valid and the functions Ψ^2 and Λ as well as transformation (3.13) read

$$\Psi^2 = (1 + \rho^2 \cos^2\phi)^{-5/2}$$
$$\Lambda = -\frac{3}{4}\left[\frac{\lambda^2 \cot^2\phi}{(1 + \rho^2 \cos^2\phi)^5}\right] + \frac{3}{16}\left[\frac{\rho^2 \sin^2\phi}{(1 + \rho^2 \cos^2\phi}\right]^2 + \frac{3}{2}\rho^2\left[\frac{\cos^2\phi}{(1 + \rho^2 \cos^2\phi)}\right] + \frac{1}{2}(\frac{c}{a})^2\left[\frac{\sin^2\phi + \rho^2 \cos^2\phi}{(1 + \rho^2 \cos^2\phi)^2}\right] \tag{5.7a}$$

and

$$W = [\sin\phi\,(1+\rho^2\cos^2\phi)]^{1/2}\,U\,. \tag{5.7b}$$

Differential equation (4.1), with Ψ^2 and Λ given by (5.7a), has regular singular points at $\phi = n\pi$ ($n = 0, \pm 1, \ldots$). Thus by section 4, Stokes' phenomenon is present in the interval $0 < |\phi| < \pi$. However, since geometrically ϕ is never negative, we may restrict the solution to the subinterval $0 < \phi < \pi$, and adopt the form of (4.7) as the general solution for ellipsoidal shells of revolution (with uniform thickness) under edge loadings. Hence

$$W = \Psi^{-1/2}\left\{A\exp(-i\eta) + B\exp(i\eta)\right\}, \tag{5.8}$$

where A and B are complex constants, and

$$\eta = (2i^3)^{1/2}\,\mu\,\Phi \tag{5.9a}$$

$$\begin{aligned}\Phi = \int \Psi d\phi &= \phi\,[1 - \frac{5}{8}\rho^2 + \frac{135}{256}\rho^4 + \ldots] \\ &- \frac{1}{2}\sin 2\phi\,[\frac{5}{8}\rho^2 - \frac{135}{256}\rho^4 + \ldots] \\ &+ \frac{1}{2}\sin 2\phi\cos^2\phi\,[\frac{45}{128}\rho^4 + \ldots.] + \ldots.\end{aligned} \tag{5.9b}$$

The function $\Phi\,(\phi;\frac{c}{a})$ is tabulated in Table I for various values of c/a, in 2° increments of the angle ϕ. For c/a = 1, which is the case of the spherical shell, Φ reduces to ϕ as may be seen from (5.9b).

With $\pm\, i\eta = \mp\,(1+i)\,\Phi$, (5.8) becomes

$$\begin{aligned}W = \Psi^{-1/2}\Big\{&A\,[\cos\mu\Phi + i\sin\mu\Phi]\,e^{\mu\Phi} \\ &+ B\,[\cos\mu\Phi - i\sin\mu\Phi]\,e^{-\mu\Phi}\Big\},\end{aligned} \tag{5.10}$$

and by (5.7), (5.10), and (3.8) the quantities β, ψ, β', and ψ', which will

TABLE I

Values of $\Phi\ (\phi;c/a) = \int (1 + \rho^2 \cos^2\phi)^{-5/4}\ d\phi$

for Shells of Uniform Thickness

ϕ°	$\frac{c}{a}$ = 0.7	$\frac{c}{a}$ = 0.8	$\frac{c}{a}$ = 0.9	$\frac{c}{a}$ = 1.10	$\frac{c}{a}$ = 1.20
0	0	0	0	0	0
2	0.069908	0.056965	0.044965	0.027910	0.025211
4	0.139698	0.113860	0.089899	0.055834	0.050423
6	0.209260	0.170619	0.134777	0.083778	0.075637
8	0.278475	0.227170	0.179570	0.111758	0.100853
10	0.347237	0.283453	0.224248	0.139787	0.126074
12	0.415436	0.339399	0.268787	0.167875	0.151305
14	0.482974	0.394950	0.313162	0.196034	0.176552
16	0.549754	0.450049	0.357345	0.224278	0.201818
18	0.615685	0.504640	0.401314	0.252614	0.227116
20	0.680688	0.558671	0.445049	0.281060	0.252455
22	0.744584	0.612099	0.488524	0.309625	0.277848
24	0.807612	0.664882	0.531728	0.338323	0.303311
26	0.869410	0.716982	0.574640	0.367164	0.328861
28	0.930027	0.768368	0.617243	0.396160	0.354515
30	0.989427	0.819013	0.659530	0.425325	0.380299
32	1.047574	0.868893	0.701483	0.454669	0.406231
34	1.104450	0.917995	0.743097	0.484202	0.432336
36	1.160036	0.966304	0.784364	0.513934	0.458639
38	1.214330	1.013819	0.825279	0.543879	0.485168
40	1.267335	1.060534	0.865840	0.574041	0.511948
42	1.319058	1.106451	0.906042	0.604431	0.539003
44	1.369521	1.151580	0.945890	0.635057	0.566362
46	1.418749	1.195932	0.985385	0.665923	0.594046
48	1.466768	1.239522	1.024532	0.697036	0.622078
50	1.513620	1.282369	1.063336	0.728399	0.650483
52	1.559345	1.324494	1.101803	0.760017	0.679278
54	1.603987	1.365926	1.139946	0.791889	0.708478
56	1.647598	1.406691	1.177772	0.824015	0.738097
58	1.690232	1.446818	1.215295	0.856395	0.768146
60	1.731943	1.486341	1.252526	0.889023	0.798628
62	1.772786	1.525293	1.289478	0.921893	0.829547
64	1.812826	1.563708	1.326171	0.955004	0.860900
66	1.852115	1.601624	1.362614	0.988342	0.892681
68	1.890719	1.639080	1.398829	1.021901	0.924877
70	1.928696	1.676109	1.434830	1.055664	0.957474
72	1.966108	1.712755	1.470635	1.089624	0.990451
74	2.002722	1.749052	1.506265	1.123762	1.023785
76	2.039469	1.785040	1.541736	1.158064	1.057447
78	2.075538	1.820760	1.577066	1.192512	1.091404
80	2.111273	1.856248	1.612278	1.227086	1.125623
82	2.146735	1.891545	1.647391	1.261769	1.160064
84	2.181978	1.926689	1.682421	1.296540	1.194686
86	2.217055	1.961715	1.717391	1.331377	1.229445
88	2.252026	1.996667	1.752323	1.366257	1.264297
90	2.286941	2.031580	1.787231	1.401159	1.299196

be needed in the solution of examples that follow immediately, are

$$\left.\begin{aligned}\beta &= \frac{(1+\rho^2\cos^2\phi)^{1/8}}{\sqrt{\sin\phi}}\begin{Bmatrix} e^{\mu\Phi}(A_o\cos\mu\Phi - A_1\sin\mu\Phi) \\ + e^{-\mu\Phi}(B_o\cos\mu\Phi + B_1\sin\mu\Phi)\end{Bmatrix} \\ \psi &= \frac{(1+\rho^2\cos^2\phi)^{1/8}}{\sqrt{\sin\phi}}\begin{Bmatrix} e^{\mu\Phi}(A_o\sin\mu\Phi + A_1\cos\mu\Phi) \\ - e^{-\mu\Phi}(B_o\sin\mu\Phi - B_1\cos\mu\Phi)\end{Bmatrix}\end{aligned}\right\} \quad (5.11a)$$

$$\left.\begin{aligned}\beta' &= -L\beta + \mu\sin^{-1/2}\phi\,(1+\rho^2\cos^2\phi)^{-9/8}M \\ \psi' &= -L\psi + \mu\sin^{-1/2}\phi\,(1+\rho^2\cos^2\phi)^{-9/8}N,\end{aligned}\right\} \quad (5.11b)$$

where

$$\left.\begin{aligned}L &= \frac{1}{4}\cot\phi\,(1+\rho^2\cos^2\phi)^{-1}[2(1+\rho^2) - \rho^2\sin^2\phi] \\ M &= \Big\{[(A_o - A_1)e^{\mu\Phi} - (B_o - B_1)e^{-\mu\Phi}]\cos\mu\Phi \\ &\quad - [(A_o + A_1)e^{\mu\Phi} + (B_o + B_1)e^{-\mu\Phi}]\sin\mu\Phi\Big\} \\ N &= \Big\{[(A_o + A_1)e^{\mu\Phi} - (B_o + B_1)e^{-\mu\Phi}]\cos\mu\Phi \\ &\quad + [(A_o - A_1)e^{\mu\Phi} + (B_o - B_1)e^{-\mu\Phi}]\sin\mu\Phi\Big\}\end{aligned}\right\} \quad (5.12)$$

and

$$A = A_o + i A_1, \quad B = B_o + i B_1. \quad (5.13)$$

To illustrate the nature of the solution just obtained, two examples for ellipsoidal shells of revolution (closed at the apex $\phi = 0$) with uniform thickness, under edge loadings only will be considered: (1) uniform stress couple M_o applied around the edge $\phi = \pi/2$, and (2) uniform radial stress resultant H_o applied around the edge $\phi = \pi/2$. In both these examples, the transition conditions at the apex $\phi = 0$ require that

$$\phi = 0; \quad \beta, \beta', \psi, \psi' \text{ remain finite.} \quad (5.14)$$

Since $\phi = 0$ is a regular singular point, (5.11a,b) are not valid there. However, guided by the solution of corresponding examples of spherical shells[8], and on account of physical requirements, evidently instead of (5.14) we may require that quantities β, β', ψ, ψ' be finite in the neighborhood of the pole $\phi = 0$. This is achieved by setting at the outset the constants B_o and B_1 equal to zero in (5.11a, b) and (5.12).

Example (1): The boundary conditions in this case are

$$\phi = \frac{\pi}{2}; \quad M_\phi = M_o, \quad Q = 0 \tag{5.15a}$$

or equivalently

$$\phi = \frac{\pi}{2}; \quad \beta' = \frac{c^2}{aD} M_o, \quad \psi = 0. \tag{5.15b}$$

Applying the above conditions to equations (5.11), the constants A_o and A_1 are determined as follows:

$$\left.\begin{aligned} A_o &= \left[\frac{2m}{Eh^2} \frac{\mu}{(c/a)^2} e^{-\mu\Phi_1} \cos \mu \Phi_1\right] M_o \\ A_1 &= -\left[\frac{2m}{Eh^2} \frac{\mu}{(c/a)^2} e^{-\mu\Phi_1} \sin \mu \Phi_1\right] M_o, \end{aligned}\right\} \tag{5.16}$$

where $\Phi_1 = \Phi(\frac{\pi}{2} ; \frac{c}{a})$.

Using (2.10), the components of stress were obtained for the case of $(c/a) = 0.8$ (see Table I), $(a/h) = 20$, and $\nu = 0.3$ (i.e., $\mu = 3.679$). Figure 2 shows the ratio of these stresses to that of σ_{M_o}, defined by $\sigma_{M_o} = 6M_o/h^2$.

Example (2): With boundary conditions

$$\phi = \frac{\pi}{2}, \quad Q = -H_o, \quad M_\phi = 0 \tag{5.17a}$$

or

$$\phi = \frac{\pi}{2}, \quad \psi = \frac{ma}{Eh^2} H_o, \quad \beta' = 0 \tag{5.17b}$$

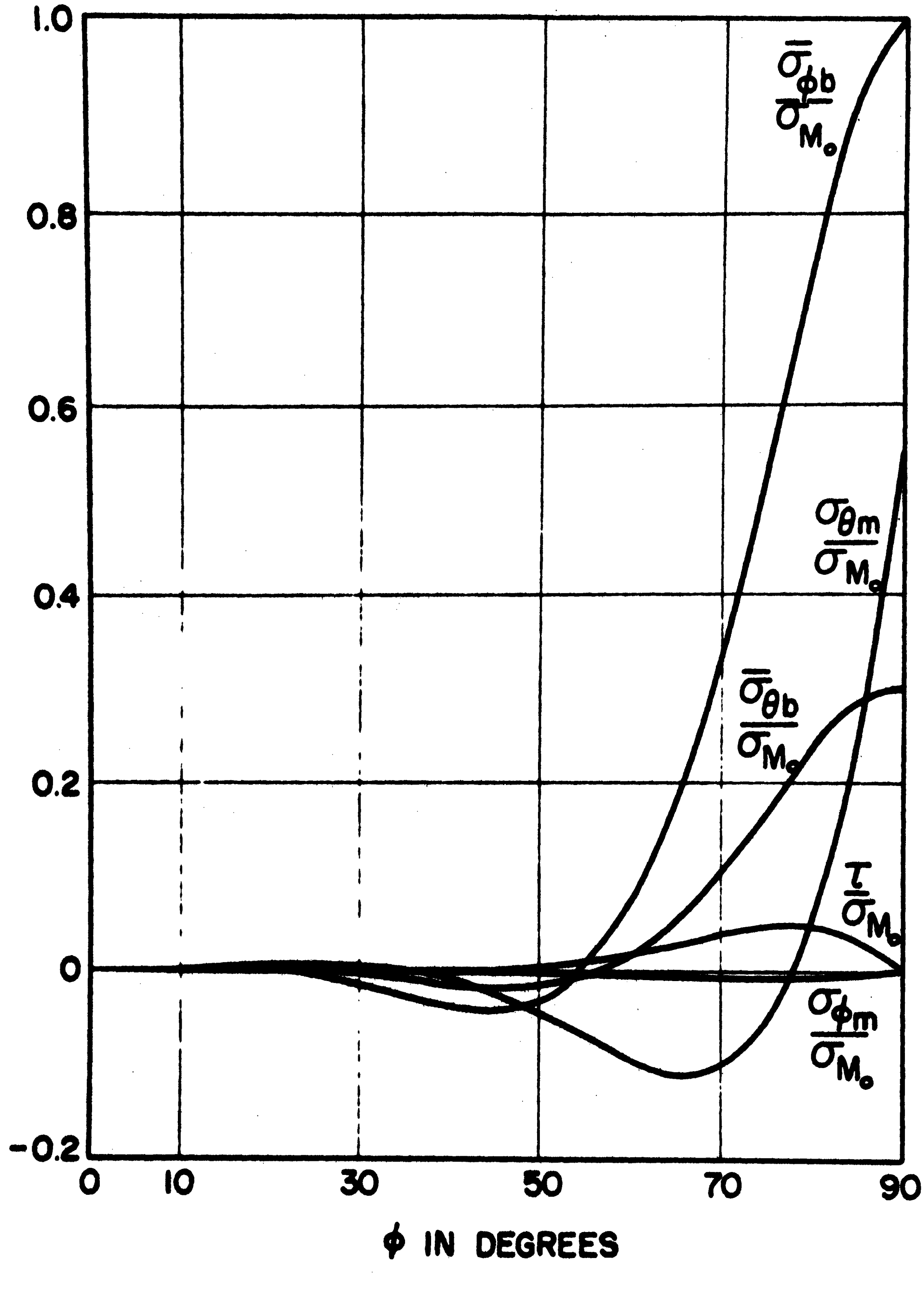

1.0
0.8
0.6
0.4
0.2
0
-0.2
0
10
30
50
70
90
φ IN DEGREES
$\bar{\sigma}_{\phi b} / \sigma_{M_o}$
$\sigma_{\theta m} / \sigma_{M_o}$
$\bar{\sigma}_{\theta b} / \sigma_{M_o}$
τ / σ_{M_o}
$\sigma_{\phi m} / \sigma_{M_o}$

Fig. 2

and proceeding as in the previous example, the constants A_o and A_1 are

$$A_o = [\frac{ma}{Eh^2} e^{-\mu\Phi_1} (\cos \mu \Phi_1 + \sin \mu \Phi_1)] H_o \tag{5.18}$$

$$A_1 = [\frac{ma}{Eh^2} e^{-\mu\Phi_1} (\cos \mu \Phi_1 - \sin \mu \Phi_1)] H_o .$$

As in example (1), the ratio of the stresses to that of $\sigma_{H_o} = 3 H_o/2h$ are plotted in Fig. 3; again these results are for the case of $c/a = 0.8$, $a/h = 20$, and $\nu = 0.3$.

6. Ellipsoidal Shells of Nonuniform Thickness

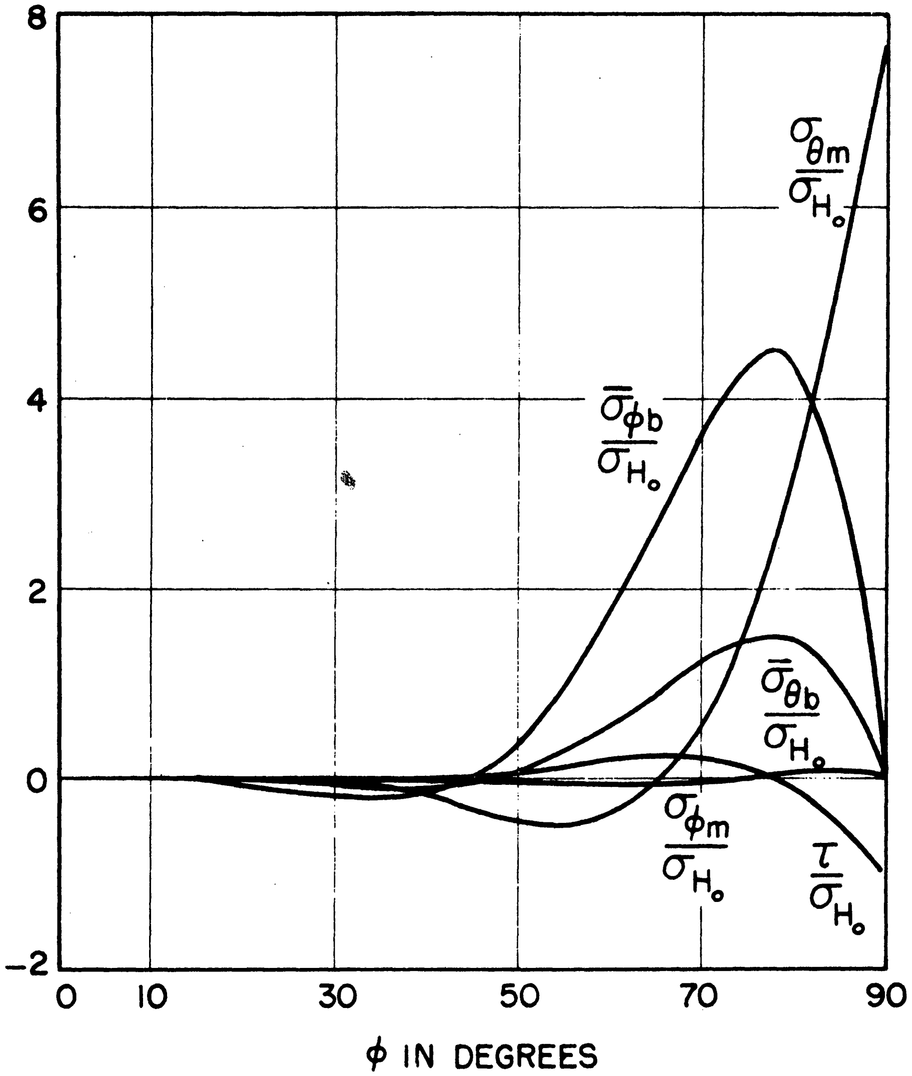
It was shown in section 3, that condition (3.11) is satisfied identically if the thickness of the shell is such as to satisfy equation (3.16). Again, we choose the variable ξ as ϕ and recall that relations (5.1) to (5.5), which were obtained from consideration of the middle surface only, are valid here. Thus on substitutions from (5.2) and (5.3), (3.16) reads as follows:

$$h'' + \frac{\cot \phi}{(1 + \rho^2 \cos^2\phi)} [\frac{c^2}{a^2} (1-\nu) - 3 \rho^2 \sin^2\phi] h' + \nu \frac{c^2}{a^2} \frac{h}{(1 + \rho^2 \cos^2\phi)} = K (1 + \rho^2\cos^2\phi)^{-5/2} , \tag{6.1}$$

where K is any constant.

While any choice of $h(\phi)$ which satisfies (6.1) falls within the scope of the application of differential equation (3.14), when adapted to ellipsoidal shells of revolution, we shall confine our attention to the case where $K = 0$, or by (3.10), $k = 1$. With $K = 0$, the solution of (6.1) about the analytic point $\phi = \pi/2$ in the interval of convergence $0 < \phi < \pi$ may be taken in the form

$$\frac{h}{h_o} = 1 - \sum_{n=1}^{\infty} b_{2n} \cos^{2n} \phi , \tag{6.2}$$



8
6
4
2
0
-2
0
10
30
50
70
90
ϕ IN DEGREES
$\frac{\sigma_{\theta m}}{\sigma_{H_o}}$
$\frac{\bar{\sigma}_{\phi b}}{\sigma_{H_o}}$
$\frac{\bar{\sigma}_{\theta b}}{\sigma_{H_o}}$
$\frac{\sigma_{\phi m}}{\sigma_{H_o}}$
$\frac{\tau}{\sigma_{H_o}}$

Fig. 3

where $\phi = \pi/2$ is the reference section, so that $h_o = h(\frac{\pi}{2})$ and

$$\left.\begin{aligned} b_2 &= \frac{\nu}{2}\frac{c^2}{a^2} \\ b_4 &= \frac{\nu}{8}\frac{c^2}{a^2}\left[4 - \frac{c^2}{a^2}(2+\nu)\right] \\ b_n &= \frac{n-4}{n}\rho^2 b_{n-4} + \frac{1}{n}\left\{2(n-2) - \frac{c^2}{a^2}\left[(n-2)+\nu\right]\right\} b_{n-2}\ ;\ n \geq 4. \end{aligned}\right\} \qquad (6.3)$$

Since, irrespective of the thickness variation, the solution of (4.1) does not hold at $\phi = 0$, the form of $h(\phi)$ given by (6.2), which is also invalid at this singular point, should not be disturbing. In fact, as may be seen from Fig. 4, this particular choice of variation in the thickness is of practical interest [In plotting this curve, 12 terms in the solution of (6.2) were used].

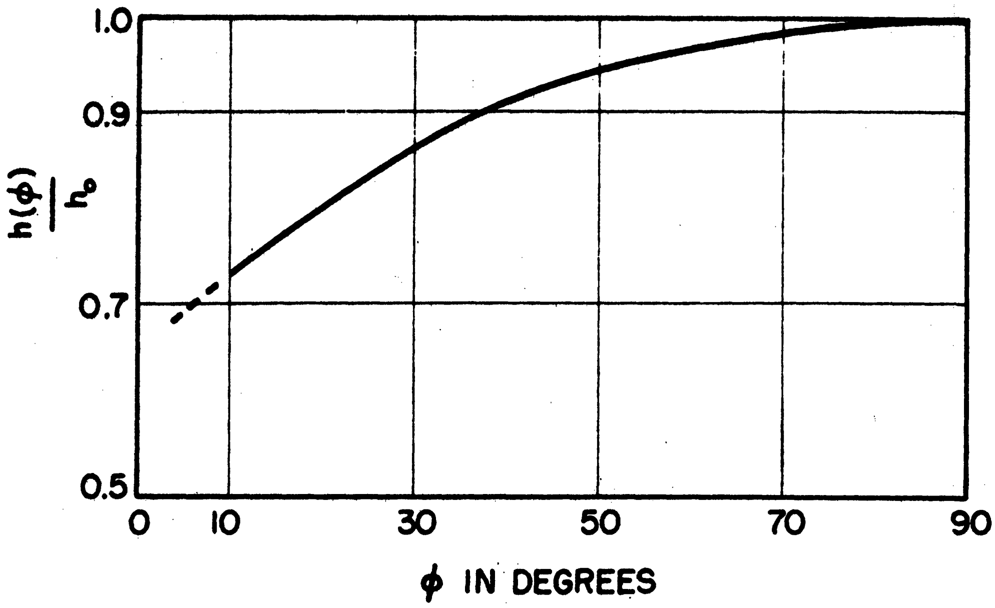


Fig. 4

With (6.2), the transformation (3.13) is

$$W = [\sin\phi\,(1 + \rho^2\cos^2\phi)]^{1/2}\left(\frac{h}{h_o}\right)^{3/2} U \qquad (6.4)$$

and the functions Ψ^2 and Λ in (4.1) become

$$\Psi^2 = (1 + i\frac{\nu\lambda}{2\mu^2})\left(\frac{h_o}{h}\right)(1 + \rho^2\cos^2\phi)^{-5/2} \tag{6.5}$$

$$\Lambda = \Lambda_o - \left[\frac{3}{2}\frac{(r/\alpha)'}{(r/\alpha)}\frac{h'}{h} + \frac{3}{2}\frac{h''}{h} + \frac{3}{4}\left(\frac{h'}{h}\right)^2\right],$$

where

$$\left.\begin{aligned}
\lambda &= -\left(\frac{h}{h_o}\right)(1+\rho^2\cos^2\phi)^{3/2}\frac{c^2}{a^2}\left\{1 - \frac{3\cos^2\phi\,(2b_2 + 4b_4\cos^2\phi + \ldots)}{1 - b_2\cos^2\phi - b_4\cos^4\phi + \ldots}\right\}\\
\frac{h'}{h} &= +\sin 2\phi\,\frac{[b_2 + 2b_4\cos^2\phi + 3b_6\cos^4\phi + \ldots]}{[1 - b_2\cos^2\phi - b_4\cos^4\phi + \ldots]}\\
\frac{h''}{h} &= \left[\frac{-\sin^2\phi\,(2b_2 + 12b_4\cos^2\phi + \ldots) + \cos^2\phi\,(2b_2 + 4b_4\cos^2\phi + \ldots)}{1 - b_2\cos^2\phi - b_4\cos^4\phi + \ldots}\right]\\
\frac{(r/\alpha)'}{(r/\alpha)} &= \frac{\cos\phi}{\sin\phi\,(1 + \rho^2\cos^2\phi)}\left[1 + \rho^2(\cos^2\phi - 2\sin^2\phi)\right]
\end{aligned}\right\} \tag{6.6}$$

and $\Lambda_o = \Lambda(\frac{\pi}{2})$ is given by (5.7a).

As in section 5, again Ψ^2 is bounded from zero everywhere within the interval of convergence and the solution of (4.1) may be written as

$$W = \Psi^{-1/2}\left\{A e^{-i\eta} + B e^{i\eta}\right\}, \tag{6.7}$$

where A and B are constants and

$$\left.\begin{aligned}
\eta &= (2i^3)^{1/2}\mu\,\Phi\\
\Phi &= \int(1 + i\frac{\nu\lambda}{2\mu^2})^{1/2}(1 + \rho^2\cos^2\phi)^{-5/4}\left(\frac{h_o}{h}\right)^{1/2}d\phi\,.
\end{aligned}\right\} \tag{6.8}$$

In combination, (6.7) and (6.8) constitute an "exact" asymptotic solution of (4.1) for ellipsoidal shells of revolution, when the thickness is specified by (6.2).

If now c/a is restricted to O(1), then $(1 + \frac{i\nu\lambda}{2\mu^2})^{1/2} \approx 1$ and the function Φ of (6.8) becomes

$$\Phi = \int (1 + \rho^2 \cos^2\phi)^{-5/4} \left(\frac{h_o}{h}\right)^{1/2} d\phi \quad (6.9a)$$

$$= \phi \left[1 - \frac{5}{8}\rho^2 + \frac{135}{256}\rho^4 + \dots + \frac{b_2}{4} - \frac{15}{64} b_2 \rho^2 + \frac{3}{16}(b_4 + \frac{3}{4} b_2^2) + \dots\right]$$

$$- \frac{1}{2}\sin 2\phi \left[\frac{5}{8}\rho^2 - \frac{135}{256}\rho^4 + \dots - \frac{b_2}{4} + \frac{15}{64} b_2 \rho^2 - \frac{3}{16}(b_4 + \frac{3}{4} b_2^2) + \dots\right]$$

$$+ \frac{1}{2}\sin 2\phi \cos^2\phi \left[\frac{45}{128}\rho^4 + \dots - \frac{5}{32} b_2\rho^2 + \frac{1}{8}(b_4 + \frac{3}{4} b_2^2) + \dots\right]$$

$$+ \dots$$

and

$$\Psi = \left(\frac{h_o}{h}\right)^{1/2} (1 + \rho^2 \cos^2\phi)^{-5/4} . \quad (6.9b)$$

Expressed in terms of trigonometric functions, W will have the form of (5.10), except that Φ and Ψ are now given by (6.9). With the aid of (6.4), the quantities β, ψ, β', and ψ' are

$$\left.\begin{aligned} \beta &= \left(\frac{h}{h_o}\right)^{-5/4} \sin^{-1/2}\phi \,(1 + \rho^2 \cos^2\phi)^{1/8} \Big\{\cos \mu\Phi \,[A_o \exp(\mu\Phi) + \\ &\quad B_o \exp(-\mu\Phi)] + \sin \mu\Phi \,[B_1 \exp(-\mu\Phi) - A_1 \exp(\mu\Phi)]\Big\} \\ \psi &= \left(\frac{h}{h_o}\right)^{-5/4} \sin^{-1/2}\phi \,(1 + \rho^2 \cos^2\phi)^{1/8} \Big\{\cos \mu\Phi \,[A_1 \exp(\mu\Phi) + \\ &\quad B_1 \exp(-\mu\Phi)] + \sin \mu\Phi \,[A_o \exp(\mu\Phi) - B_o \exp(-\mu\Phi)]\Big\} \end{aligned}\right\} (6.10a)$$

$$\left.\begin{aligned} \beta' &= \mu \left(\frac{h}{h_o}\right)^{-7/4} \sin^{-1/2}\phi \,(1 + \rho^2 \cos^2\phi)^{-9/8} M - \left[L + \frac{5}{4}\frac{h'}{h}\right] \beta \\ \psi' &= \mu \left(\frac{h}{h_o}\right)^{-7/4} \sin^{-1/2}\phi \,(1 + \rho^2 \cos^2\phi)^{-9/8} N - \left[L + \frac{5}{4}\frac{h'}{h}\right] \psi . \end{aligned}\right\} (6.10b)$$

where L, M, and N are defined as in (5.12), with Φ and Ψ given by (6.9), and A_o, A_1, B_o, and B_1 are related to A and B by (5.13).

In order to study the effect of the variable thickness on the stress distribution, let us consider the case which corresponds to example (1) of the previous section, namely: an ellipsoidal shell of revolution of nonuniform thickness [specified by (6.2)] under the action of a uniform stress couple M_o applied around the edge $\phi = \pi/2$.

Again, the transition conditions at the apex $\phi = 0$ require that $B_o = B_1 = 0$ (see section 5), and the boundary conditions

$$\phi = \frac{\pi}{2} ; \qquad M_\phi = M_o , \qquad Q = 0$$

result in

$$\begin{aligned} A_o &= [\frac{2m}{Eh_o^2} \; \frac{\mu}{(c/a)^2} \; e^{-\mu\Phi_1} \cos \mu\Phi_1] \, M_o \\ A_1 &= [\frac{2m}{Eh_o^2} \; \frac{\mu}{(c/a)^2} \; e^{-\mu\Phi_1} \sin \mu\Phi_1] \, M_o , \end{aligned} \qquad (6.11)$$

where $\Phi_1 = \Phi(\frac{\pi}{2} ; \frac{c}{a})$.

Table II provides values of $\Phi(\phi ; c/a)$ appropriate to (6.2) for the case $c/a = 0.8$, in 2° increments of the angle ϕ.

TABLE II

Values of $\Phi(\phi ; c/a) = \int (1 + \rho^2 \cos^2\phi)^{-5/4} (\frac{h_o}{h})^{1/2} d\phi$ for $h(\phi)$ given by (6.2) and $c/a = 0.8$

ϕ°	Φ	ϕ°	Φ	ϕ°	Φ	ϕ°	Φ
0	0	24	0.704565	48	1.302374	72	1.782562
2	0.060573	26	0.759346	50	1.346302	74	1.819017
4	0.121060	28	0.813279	52	1.389395	76	1.855127
6	0.181384	30	0.866338	54	1.431687	78	1.890936
8	0.241463	32	0.918493	56	1.473211	80	1.926487
10	0.301219	34	0.969729	58	1.514007	82	1.961826
12	0.360574	36	1.020035	60	1.554110	84	1.996996
14	0.419460	38	1.069404	62	1.593561	86	2.032036
16	0.477765	40	1.117836	64	1.632406	88	2.066992
18	0.535544	42	1.165335	66	1.670683	90	2.101900
20	0.592618	44	1.211914	68	1.708440		
22	0.648974	46	1.257587	70	1.745776		

The stresses which were obtained for the case $c/a = 0.8$, $a/h_o = 20$, and $\nu = 0.3$ are not plotted because for all practical purposes they are the same as those (for ellipsoidal shell of uniform thickness) shown in Fig. 2. To substantiate this, a comparison is made in Table III between the stress distribution of the present case with the corresponding quantities for the shell of uniform thickness [Example (1), section 5].

TABLE III

$\phi°$		20	40	60	70	80	90
$\frac{\sigma_{\phi b}}{\sigma_{M_o}}$	$h=h_o$	-0.00111	-0.03975	+0.06863	+0.33327	+0.73274	1.00000
	$h=h(\phi)$	-0.00017	-0.03937	+0.06996	+0.33329	0.72856	1.00000
$\frac{\sigma_{\Theta b}}{\sigma_{M_o}}$	$h=h_o$	0.00143	0.01723	0.01527	0.10406	0.23156	0.30000
	$h=h(\phi)$	0.00178	-0.01688	+0.01558	0.10404	0.23030	0.30000
$\frac{\sigma_{\phi m}}{\sigma_{M_o}}$	$h=h_o$	0.00125	0.00148	-0.00692	-0.00920	-0.00535	0.00000
	$h=h(\phi)$	0.00099	0.00159	-0.00687	-0.00921	-0.00535	0.00000
$\frac{\sigma_{\Theta m}}{\sigma_{M_o}}$	$h=h_o$	0.00421	-0.00924	-0.09672	-0.10054	+0.06062	0.55075
	$h=h(\phi)$	0.00409	-0.00873	-0.09573	-0.09807	+0.06263	0.55075
$\frac{\tau}{\sigma_{M_o}}$	$h=h_o$	-0.00068	-0.00187	0.01797	0.03792	0.04550	0.00000
	$h=h(\phi)$	-0.00054	-0.00200	0.01785	0.03794	0.04548	0.00000

7. Concluding Remarks

The very close agreement between the results for shells of uniform and nonuniform thickness is, at least at first glance, somewhat surprising and warrants the following comments.

The results for the case of nonuniform thickness (communicated in Table III) should not be considered as affected by the approximation introduced immediately following (6.8), since this very same approximation was also used for the case of uniform thickness. Returning, however, to (6.5), an examination of $\Delta(\phi)$ reveals that the quantity in the bracket, which is due to the variation of thickness, is finite and small with respect to μ^2 everywhere within the interval of convergence. Hence, this quantity will have but a small effect on the asymptotic integration of the differential equation (4.1) with respect to μ^2.

Furthermore, as may be seen from Fig. 4, the variation of h/h_o should not lead to much deviation of Φ in (6.9) from that of (5.9b). That this is indeed the case is apparent from proper comparison of Tables I and II.

It is reasonable to expect that other forms of variation in the thickness, as solutions of (6.1) with $K \neq 0$, should result in a more appreciable deviation of the stress distribution from that of the case of uniform thickness. For, in such cases the quantity k will no longer be unity, and this may result in a more pronounced effect on the stresses.

In conclusion, we reiterate that although attention has been given specifically to ellipsoidal shells of revolution, the results obtained in sections 2 - 4 are quite general and applicable to all shells of revolution.

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