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RESPONSE OF SHALLOW VISCOELASTIC SPHERICAL SHELLS TO TIME-DEPENDENT
AXISYMMETRIC LOADS

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1. INTRODUCTION

The quasi-static treatment of problems in the linear theory of viscoelasticity has received increasing attention in recent years. The method of solution employed in such problems rests on the use of Laplace transform (to eliminate dependence on time), and the correspondence principle - between the field equations and boundary conditions in the linear theories of homogeneous and isotropic elasticity and viscoelasticity - which has been established for incompressible media by Alfrey [1], and in general form by Lee [2]¹. Evidently, the solution of viscoelastic problems in which the inertia effects are taken into account have been so far limited to the one-dimensional wave propagation in viscoelastic rods, such as those discussed in [4,5,6,7].

Closely related to the scope of the present investigation is the recent work on vibrations of thin shallow elastic shells by E. Reissner [8], who, by utilizing the linear differential equations due to Marguerre [9], has shown that for transverse vibrations of shallow shells the longitudinal inertia terms (with negligible error) may be omitted; and hence the formulation of the elastokinetic problems of shallow shells, as in the case of elastostatics, may be reduced to the determination of axial displacement and an Airy stress function. Subsequently, E. Reissner [10] dealt with transverse vibrations of axisymmetric shallow elastic spherical shells, and in particular obtained the solution for an unlimited shell due to an oscillating point load (varying harmonically in time) at the apex.

The present paper is concerned with the response of shallow viscoelastic spherical shells to arbitrary time-dependent axisymmetric loads; the medium is assumed homogeneous and isotropic. Although emphasis is placed on unlimited shallow spherical shells, shallow spherical shell segments are also considered and discussed in Sec. 7. The solutions, employing the differential equations governing the transverse motion of thin shallow elastic shells, are obtained with the joint use of the Laplace and the Hankel transforms which, by interchanging the order of the inversions, avoids an otherwise intricate task of contour integration in the complex Laplace transform-plane. Explicit results in integral form are deduced for viscoelastic shells under instantaneous pulse loading (including those uniformly distributed about and concentrated at the apex), and are particularized to the cases of Maxwell and Kelvin solids. The solutions for a shallow elastic shell and for the case of a flat plate are

1. The extension of Lee's analogy to problems involving time-dependent temperature fields has been very recently given by Sternberg [3], where an exposition of the subject may be found. For other references, see [2,3,4,5,6,7].

also given as by-products of the general solution and comparison is made with known results [10]. It may be further noted that the transform technique employed here appears to be useful also in connection with other axisymmetric problems of stress wave propagation in viscoelastic solids.

2. PRELIMINARY BACKGROUND

With reference to rectangular Cartesian coordinates x_i , the stress-strain law for an isotropic and homogeneous viscoelastic medium may be written as²

$$\begin{aligned} P_1(\theta) s_{ij} &= P_2(\theta) e_{ij} \\ P_3(\theta) \sigma_{ii} &= P_4(\theta) \epsilon_{ii} \end{aligned} \quad (2.1)$$

where σ_{ij} and ϵ_{ij} are the components of the stress and the strain tensor, s_{ij} and e_{ij} designate the deviatoric components of stress and strain, the operators $P_m(\theta)$ involving the constant coefficients $c_m^{(n)}$ ($m = 1, 2, 3, 4$) are defined by

$$P_m(\theta) = \sum_{n=0}^{N_m} c_m^{(n)} \theta^n \quad ; \quad [c_m^{(N_m)} \neq 0] \quad (2.2)$$

$$\theta \equiv \frac{\partial}{\partial t} \quad ,$$

and t denotes time.

For future reference, we also recall that the Laplace transform with respect to t of a (suitably restricted) function $U(x, t)$ is given by³

$$\hat{U}(x, s) = \mathcal{L}\{U(x, t); s\} \equiv \int_0^\infty e^{-st} U(x, t) dt \quad (2.3)$$

where s is the transform parameter, and that the Hankel transform of order

2. The Latin indices have, unless otherwise stated, the range of $i, j = 1, 2, 3$, and the repeated indices imply the summation convention.

3. See, for example, Churchill [11]; the argument x in U refers to the space variable.

zero of the function $\hat{U}(x,s)$ is defined by⁴

$$U^*(\xi, s) = \int_0^\infty x J_0(\xi x) \hat{U}(x, s) dx \quad (2.4)$$

provided that (i) the integral $\int_0^\infty \hat{U}(x, s) dx$ is absolutely convergent, and (ii) the function \hat{U} is of bounded variation over the region of interest. Furthermore, in connection with the Hankel transform of $\partial \hat{U} / \partial x (x, s)$, we need the property that (iii) $x \hat{U}(x, s)$ vanishes at $x = 0$ and as $x \rightarrow \infty$; this ensures the existence of the inverse transform of $U^*(\xi, s)$.

As the operators $P_m(\theta)$ in the physical plane become $P_m(s)$ in the Laplace transform-plane, the Laplace transform of (2.1) may alternatively (in a form more suitable to the theory of shells) be written as⁵

$$\begin{aligned} \hat{\sigma}_{ij} &= P_2(s) P_1^{-1}(s) \hat{\epsilon}_{ij} - \frac{1}{3} [P_2(s) P_1^{-1}(s) \\ &\quad - P_4(s) P_3^{-1}(s)] \hat{\epsilon}_{kk} \delta_{ij} \\ \hat{\sigma}_{ii} &= P_4(s) P_3^{-1}(s) \hat{\epsilon}_{ii} \end{aligned} \quad (2.5)$$

where δ_{ij} is the Kronecker delta.

It follows from the correspondence principle⁶ that the field equations and the boundary conditions governing the original viscoelastic problem are reducible to the field equations and boundary conditions of an associated problem in the linear theory of elasticity, with Young's modulus E and Poisson's ratio ν of the elastic solid replaced by

$$\begin{Bmatrix} E(s) \\ \nu(s) \end{Bmatrix} = [P_2(s) P_3(s) + 2 P_1(s) P_4(s)]^{-1} \begin{Bmatrix} 3 P_4(s) P_2(s) \\ [P_1(s) P_4(s) - P_2(s) P_3(s)] \end{Bmatrix} \quad (2.6)$$

4. See, for example, Sneddon [12]. It may be noted here that the Hankel transform defined by (2.4) formally differs from that defined in [17]: however, with the transformation $\hat{U}(s, s) = x^{-1/2} f$ (f being the function corresponding to \hat{U} in [17], multiplying both sides of (2.4) by $\xi^{1/2}$ and setting $f^* = \xi^{1/2} U^*$, the two definitions are brought together.

5. It should be noted that unlike $P_m^{-1}(s)$ the operators $P_m^{-1}(\theta)$ are in general noncommutative [13].

6. The correspondence principle as stated by Sternberg [3] holds also in the presence of inertia forces.

It is easily verified that the correspondence principle and the results (2.6) are also valid for any of the various (consistent) theories of thin shells.

We also note that in (2.6) the linear, homogeneous, and isotropic elastic medium may be identified by allowing $E(s) \rightarrow E$ and $\nu(s) \rightarrow \nu$ (corresponding to $P_1(s) = P_2 = 1$, $P_3(s) = 2\mu$, $P_4(s) = 3K$, μ and K being the shear and the bulk moduli of the elastic solid, respectively). For the Maxwell solid

$$\begin{aligned} P_1(s) &= s + \tau^{-1}, & P_2(s) &= 2/\mu \\ P_3(s) &= s, & P_4(s) &= 3Ks \end{aligned} \quad (2.7a)$$

and (2.6) becomes

$$\begin{aligned} E(s) &= \frac{s}{s + \frac{2}{3}(1+\nu)\tau^{-1}} E \\ \nu(s) &= \frac{s + \frac{1+\nu}{3\tau}\tau^{-1}}{s + \frac{2}{3}(1+\nu)\tau^{-1}} \nu \end{aligned} \quad (2.7b)$$

where $\tau = \eta/\mu$ is the relaxation time, η being the viscosity. Similarly, for the Kelvin solid

$$\begin{aligned} P_1(s) &= 1, & P_2(s) &= 2/\mu(1+\tau s) \\ P_3(s) &= \tau s, & P_4(s) &= 3K\tau s \end{aligned} \quad (2.8a)$$

and

$$\begin{aligned} E(s) &= \frac{1 + \tau s}{1 + \frac{1-2\nu}{3}\tau s} E \\ \nu(s) &= \frac{1 - \frac{1-2\nu}{3}\tau s}{1 + \frac{1-2\nu}{3}\tau s} \nu \end{aligned} \quad (2.8b)$$

where in (2.8b) τ denotes the retardation time.

3. DIFFERENTIAL EQUATIONS FOR TRANSVERSE VIBRATIONS OF SHALLOW ELASTIC SPHERICAL SHELLS

Let H denote the rise of the shell segment, L the characteristic length (which for spherical shells may be conveniently taken as the radius of curvature R), h the shell thickness, ρ the density, and γ (defined by $\gamma^{-2} = \rho\omega^2/E$, ω being the circular frequency) the representative wavelength. Then, for transverse vibrations of thin shallow elastic shells, with the stipulation that $(H/L)^2 \ll 1$ and $(h/\gamma)^2 \ll 1$, it has been shown by E. Reissner [8] that the effect of longitudinal inertia (in comparison with transverse inertia) may be omitted (with negligible error) from the differential equations governing the motion of the shell as long as⁷

$$\left(\frac{\gamma}{L}\right) \gtrless 1, \quad \left[\left(\frac{\gamma}{L}\right) \neq 1\right] \quad (3.1)$$

Thus, with references to polar coordinates (r designating the polar radius) and with omission of the longitudinal inertia term, the differential equations for the axisymmetric transverse vibrations of shallow elastic spherical shells are characterized by [10]

$$\begin{aligned} D \nabla^2 \nabla^2 w + \frac{1}{R} \nabla^2 F &= -\rho h \frac{\partial^2 w}{\partial t^2} + p(r, t) \\ \nabla^2 \nabla^2 F - \frac{h E}{R} \nabla^2 w &= 0 \end{aligned} \quad (3.2)$$

and the various stress resultants and stress couples are given by

$$N_r = \frac{1}{2} F', \quad N_\theta = F'' \equiv \nabla^2 F - \frac{1}{2} F' \quad (3.3a)$$

$$M_r = -D \left[w'' + \frac{\nu}{2} w' \right], \quad M_\theta = -D \left[\frac{1}{2} w' + \nu w'' \right] \quad (3.3b)$$

$$Q = -D (\nabla^2 w)' \quad (3.3c)$$

where w is the axial displacement, F is the Airy stress function, p is the axial component of the surface load, $D = Eh^3/12(1-\nu^2)$, $\nabla^2 () \equiv ()'' + 1/r ()'$, and prime denotes differentiation with respect to r .

⁷ In [8], E. Reissner has further remarked that the neglect of the effect of longitudinal inertia is permissible when $(\gamma/L) \ll 1$ but not when $(\gamma/L) \gg 1$.

It is also relevant to recall here that the steady state solution (for axisymmetric transverse vibrations of shallow elastic spherical shells) given by E. Reissner [10], where w and F are assumed to have the form $w = W(r) e^{i\omega t}$, $F = f(r) e^{i\omega t}$, [$i = (-1)^{1/2}$], involves Bessel functions (J_0 , Y_0 , I_0 , K_0) of argument λr where

$$\lambda^4 = - \frac{12(1-\nu^2)}{(Rh)^2} \left[1 - \left(\frac{R}{\gamma} \right)^2 \right] \quad (3.4)$$

Indeed, since by (3.1) the case of $\left(\frac{\gamma}{R}\right) = 1$ (corresponding to $\lambda = 0$) is not admissible, for axisymmetric vibrations treated in [10] two sets of solutions of (3.2) associated with the two frequency ranges $\left(\frac{R}{\gamma}\right) \gtrless 1$ exist.

The integration of the second of (3.2), together with the condition of vanishing circumferential displacement (which as in the elastostatic solution of shallow spherical shells [14] demands the vanishing of the coefficient of the logarithmic term) leads to

$$\nabla^2 F = \frac{hE}{R} w + C(t) \quad (3.5)$$

If attention is confined to unlimited shallow shells, then since $w(\infty, t) = 0$ and $\nabla^2 F(\infty, t) = 0$ [the latter condition, by (3.3a), is due to independent vanishing of N_r and N_θ at $r = \infty$], it follows that in (3.5) $C(t) = 0$ and for unlimited shallow shells the system of differential equations (3.2) reduces to

$$\begin{aligned} D \nabla^2 \nabla^2 w + \frac{1}{R} \nabla^2 F &= -\rho h \frac{\partial^2 w}{\partial t^2} + p(r, t) \\ \nabla^2 F &= \frac{hE}{R} w \end{aligned} \quad (3.6)$$

Furthermore, for unlimited spherical shells, the remaining boundary conditions associated with (3.6) are

$$w'(\infty, t) = \nabla^2 w(\infty, t) = 0 \quad (3.7)$$

and the regularity requirements for oscillating distributed load are specified by

$$w(0, t), w'(0, t), N_r(0, t), N_\theta(0, t) ; \text{finite} \quad (3.8a)$$

$$M_r(0, t), M_\theta(0, t) ; \text{finite} \quad (3.8b)$$

and those appropriate for an oscillating point load are given by (3.8a):

4. UNLIMITED SHALLOW VISCOELASTIC SPHERICAL SHELLS

The differential equations and the boundary conditions (as well as the regularity requirements) appropriate for an unlimited shallow viscoelastic spherical shell ($0 \leq r \leq \infty$), subjected to an arbitrary time-dependent axisymmetric load, following the application of the Laplace transform, and with an appeal to the correspondence principle (Sec. 2), are obtained from (3.6), (3.7), and (3.8) with the moduli E and ν (of the elastic solid) replaced by $E(s)$ and $\nu(s)$, respectively. In particular with zero initial conditions, i.e.,

$$w(r, 0) = \frac{\partial w}{\partial t}(r, 0) = F(r, 0) = 0 \quad (4.1)$$

the differential equations of motion in the Laplace transform-plane read

$$\begin{aligned} D(s) \nabla^2 \nabla^2 \hat{w} + \frac{1}{R} \nabla^2 \hat{F} + \rho h s^2 \hat{w} &= \hat{p} \\ \nabla^2 \hat{F} &= \frac{h E(s)}{R} \hat{w} \end{aligned} \quad (4.2)$$

As the solution of the system of differential equations (4.2) involves Kelvin functions whose arguments are polynomials (of fourth degree for compressible and second degree for incompressible media) in s , thus prohibiting simple inversions, the determination of w and F will in general require cumbersome contour integration in the complex Laplace transform-plane. To overcome this difficulty, we consider the application of Hankel transform of order zero to (4.2), and require that w and its derivatives up to the fourth and F' and its derivatives up to the first, vanish to a suitable order at infinity such that all integrals employed in the following analysis exist.

Recalling the formula for the derivative of Hankel transform of order zero [12, p. 62], i.e.,

$$\int_0^\infty (\nabla^2 \hat{w}) r J_0(r \xi) dr = -\xi^2 w^*(\xi, s) \quad (4.3a)$$

and by iteration

$$\int_0^\infty (\nabla^2 \nabla^2 \hat{w}) r J_0(r \xi) dr = \xi^4 w^*(\xi, s) \quad (4.3b)$$

then, with the aid of (4.3), and application of the Hankel transform of zero order to (4.2), we reach

$$D(s) \xi^4 w^* + \frac{1}{R} (\nabla^2 F)^* + \rho h s^2 w^* = \beta^* \quad (4.4)$$

$$(\nabla^2 F)^* = \frac{h E(s)}{R} w^*$$

Elimination of $(\nabla^2 F)^*$ from (4.4) results in

$$w^* = D^{-1}(s) \left[\xi^4 + \lambda_0^4(s) \right]^{-1} \beta^*(\xi, s) \quad (4.5)$$

where

$$\lambda_0^4 = \frac{12[1-\nu^2(s)]}{(R h)^2} \left[1 + R^2 \left(\frac{\rho s^2}{E(s)} \right) \right] \quad (4.6a)$$

$$= \ell^{-4} + \frac{12[1-\nu^2(s)]}{h^2 E(s)} \rho s^2$$

$$\ell^4 = \frac{(R h)^2}{12[1-\nu^2(s)]} \quad (4.6b)$$

It is now clear that since the right-hand side of (4.5) involves only polynomials in ξ and s , the inversion of w^* in s is easily carried out, and its inversion in the Hankel transform parameter ξ (being a real variable) is also possible. Thus by first taking the inverse Laplace transform of (4.5), followed by the inverse Hankel transform (which is a self-reciprocating transform), we obtain

$$w(r, t) = \int_0^\infty \xi J_0(r \xi) d\xi \int_0^t \mathcal{L}^{-1} \{ \beta^*(\xi, s); s \} \times \mathcal{L}^{-1} \{ \psi; t-s \} ds \quad (4.7a)$$

where

$$\psi = D^{-1}(s) \left[\xi^4 + \lambda_0^4(s) \right]^{-1} \quad (4.7b)$$

It remains to determine the function F . However, since F does not necessarily conform to the requirements for the validity of its Hankel transform, while N_r and N_θ as given by (3.3a) are independently finite at $r = 0$ and vanish at $r = \infty$, we proceed instead to establish $\nabla^2 F$ and $1/r F'$. To this end we turn to the second of (4.4) and, with the aid of (4.5), write

$$(\nabla^2 F)^* = \frac{h E(s)}{R} \beta^* \left\{ D(s) \xi^4 + \rho h s^2 + \frac{h E(s)}{R^2} \right\}^{-1} \quad (4.8)$$

Again, taking the inverse Laplace transform of (4.8) followed by the inverse Hankel transform, we obtain

$$\nabla^2 F(r, t) = \frac{h}{R} \int_0^\infty \xi J_0(r\xi) d\xi \int_0^t \mathcal{L}^{-1}\{\beta^*(\xi, s); \mathfrak{s}\} \mathcal{L}^{-1}\{E(s) \psi(\xi, s); t-\mathfrak{s}\} d\mathfrak{s} \quad (4.9a)$$

and by intergration, since $N_r(o, t) = N_\theta(o, t) = \frac{1}{2} \nabla^2 F(o, t)$,

$$N_r = \frac{1}{r} F' = \frac{1}{r^2} \int_0^r x [\nabla^2 F(x, t) dx] \quad (4.9b)$$

which completes the desired solution.

Before closing this section, we record some special types of loading as well as their Hankel transforms, which will be of interest presently. For a pulse instantaneously applied and removed at $t = t_0$

$$\beta(r, t) = g(r) \delta(t - t_0) \quad (4.10)$$

where the Dirac delta function is defined by

$$\begin{aligned} \delta(t - t_0) &= 0, \quad (t \neq t_0) \\ \int_{-\infty}^{\infty} \delta(t - t_0) dt &= 1 \end{aligned} \quad (4.11)$$

If, on the other hand, the pulse applied at $t = t_0$ continues to act indefinitely, then it is only necessary to replace $\delta(t - t_0)$ in (4.10) with the Heaviside step function $H(t - t_0)$ defined by

$$H(t - t_0) = \begin{cases} 1 & (t - t_0 > 0) \\ \frac{1}{2} & (t = t_0) \\ 0 & (t < t_0) \end{cases} \quad (4.12)^8$$

Among the various forms which the function $q(r)$ may assume we specifically cite those associated with (a) uniformly distributed pulse over $0 \leq r \leq a$ ($a/R \ll 1$),

⁸ The use of the letter H with argument $(t - t_0)$ for the Heaviside step function should not be confused with the notation for the shell rise in Sec. 2.

and (b) point load pulse applied at $r = 0$, each given, respectively, by

$$g(r) = -p_0 H(a-r) \quad (4.13a)$$

$$g(r) = -\frac{P}{\pi r} \delta(r) \quad (4.13b)$$

Also, for future reference, the Hankel transform of g specified by (4.13a) and (4.13b) are given, respectively, by [12, p. 88]

$$g^*(\xi) = -\frac{p_0 a}{\xi} J_1(a\xi) \quad (4.14a)$$

and by [15, p. 67]

$$g^*(\xi) = -\frac{P}{2\pi} \quad (4.14b)$$

5. THE ELASTIC SOLUTION AS A BY-PRODUCT. REDUCTION TO KNOWN RESULTS

By allowing $E(s) \rightarrow E$ and $\nu(s) \rightarrow \nu$ in (4.5) and (4.8), and by taking their inverse transforms, we deduce the complete solution for the unlimited shallow elastic spherical shell under arbitrary time-dependent axisymmetric load. In particular, when the load is specified by (4.10) and (4.13a), then with the aid of (4.14a) and tables of integral transforms [16, Table 5.2] and [9, p. 323], there follows

$$w(r,t) = -\frac{p_0 a}{\rho h} \int_0^\infty J_0(r\xi) J_1(a\xi) \phi d\xi \quad (5.1)$$

$$\begin{Bmatrix} \nabla^2 F(r,t) \\ F'(r,t) \end{Bmatrix} = - \begin{Bmatrix} \frac{p_0 a E}{\rho R} \\ \frac{1}{2} \int_0^r x \frac{p_0 a E}{\rho R} dx \end{Bmatrix} \int_0^\infty J_0(r\xi) J_1(a\xi) \phi d\xi \quad (5.2)$$

where

$$\phi = \frac{\sin \left\{ \left(\frac{E}{\rho} \right)^{1/2} \frac{1}{R} [1 + (\ell \xi)^4]^{1/2} (t - t_0) \right\}}{\left(\frac{E}{\rho} \right)^{1/2} \frac{1}{R} [1 + (\ell \xi)^4]^{1/2}} \quad (5.3)$$

As further specialization of the above solution, consider the case of a flat plate by letting $R \rightarrow \infty$ and setting $F = 0$. Thus, for an infinite elas-

tic circular plate subjected to a pulse uniformly distributed over $0 \leq r \leq a$, (5.1) reduces to

$$w(r,t) = -\frac{p_0}{2\pi\rho h} \left(\frac{\rho h}{D}\right)^{1/2} \int_0^\infty \xi J_0(r\xi) \frac{\sin\left\{\left(\frac{D}{\rho h}\right)^{1/2} \xi^2 (t-t_0)\right\}}{\xi^2} d\xi \quad (5.4)$$

which, with $t_0 = 0$, the notation $r_0 = (E/\rho)^{1/2} t$, and the use of [17, Table 8.2] becomes

$$w(r,t) = \frac{-p_0}{4\pi\rho h^2} \left[12(1-\nu^2) \frac{\rho}{E}\right]^{1/2} \text{Si}\left\{\frac{[12(1-\nu^2)]^{1/2}}{4} \left(\frac{r}{h}\right)\left(\frac{r}{r_0}\right)\right\} \quad (5.5a)$$

$$\text{Si}(r) = -\int_r^\infty \frac{\sin x}{x} dx = -\frac{1}{2}\pi + \text{Si}(r), \quad (5.5b)$$

Si(r) being the sine integral.

In the remainder of this section, we confine attention to shallow elastic spherical shells subjected to loading of the types

$$p(r,t) = -p_0 H(a-r) e^{i\omega t} \quad (5.6a)$$

$$p(r,t) = -\frac{P}{\pi a} \delta(r) e^{i\omega t} \quad (5.6b)$$

which vary harmonically in time. For such steady state solutions, it is more convenient to return to the viscoelastic solution (4.5) in the Laplace-Hankel transform-plane. Following Lee [6] we replace s by $i\omega$ in all quantities except in $p^*(\xi, s)$ which is replaced by⁹ $p^*(\xi)$, and then take the inverse Hankel transform leading to the (real) steady state amplitude $W(r)$ for the axial displacement. When $p(r,t)$ is specified by (5.6a), through the process just described and with the aid of (4.14a), we obtain

$$W(r) = -p_0 a D^{-1} \int_0^\infty \frac{J_0(r\xi) J_1(a\xi)}{[\xi^4 + \lambda_0^4]} d\xi \quad (5.7)$$

along with similar expressions for $\nabla^2 f$ and $f'(r)$, $f(r)$ being the amplitude of $F(r,t)$. Unfortunately, the solution (5.7) does not admit a closed representation; and probably for this case a more direct approach, involving the use of the elastic solution of the problem [10], and the Laplace transform, will (from a practical point of view) prove fruitful.

9. Here, this process is equivalent to taking the inverse Laplace transform.

For the steady state solution due to an oscillating point load specified by (5.6b), with the aid of (4.14b), W and $\nabla^2 f$ are given by

$$W(r) = -\frac{P}{2\pi D} \int_0^\infty \frac{\xi J_0(\lambda_0 \xi)}{[\xi^4 + \lambda_0^4]} d\xi \quad (5.8a)$$

$$\nabla^2 f(r) = -\frac{hE}{R} W(r) \quad (5.8b)$$

The integral in (5.8a) with λ_0^4 as positive, admits a closed representation [17, Table 8.2]. Hence

$$W(r) = \frac{P}{2\pi D \lambda_0^2} \text{kei}(\lambda_0 r) \quad (5.9)$$

which is in exact agreement with the results given in [10] for $(R/\gamma)^2 > 1$ or $\omega^2 > (E/\rho)/R^2$ (corresponding to $\lambda_0^4 > 0$); in (5.9) $\text{kei } x$ (together with $\text{ber } x$, $\text{bei } x$, and $\text{ker } x$ to be introduced presently) are the Kelvin functions.

On the other hand, if in (5.8) $\lambda_0^4 < 0$, i.e., for $(R/\gamma)^2 < 1$ or $\omega^2 < (E/\rho)/R^2$, we introduce the quantity λ through

$$\lambda_0^4 = e^{-i\frac{\pi}{4}} \lambda^4 \quad (5.10)$$

[where, by (4.6a) with s replaced by $i\omega$, λ is defined by (3.4)] and also employ the relations [18]

$$\text{ker}(z) = \frac{\pi}{4} \left[-i J_0(\bar{e}^{-i\frac{\pi}{4}} z) - Y_0(\bar{e}^{-i\frac{\pi}{4}} z) + i J_0(e^{i\frac{\pi}{4}} z) - Y_0(e^{i\frac{\pi}{4}} z) \right] \quad (5.11a)$$

$$\text{kei}(z) = \frac{\pi}{4} \left[-J_0(\bar{e}^{-i\frac{\pi}{4}} z) + i Y_0(\bar{e}^{-i\frac{\pi}{4}} z) - J_0(e^{i\frac{\pi}{4}} z) - i Y_0(e^{i\frac{\pi}{4}} z) \right] \quad (5.11b)$$

whose argument $z = x + iy$ is complex. Thus, for $\lambda_0^4 < 0$, substitution of (5.11) and (5.10) into (5.8) and the use of [15, Table 8.2] lead to expressions whose real parts¹⁰, i.e.

$$W = \text{Re} \left\{ \frac{P}{4\pi D \lambda_0^2} \left[K_0(\lambda_0 r) + \frac{\pi}{2} Y_0(\lambda_0 r) - i \frac{\pi}{2} J_0(\lambda_0 r) \right] \right\} \quad (5.12a)$$

10. It may be recalled that on account of the assumed form of the exponential time-dependence of the solution, the various quantities are in general complex, and their real parts give the desired results.

$$f = R_e \left\{ \frac{P}{4\pi D \lambda_0^2} \left(\frac{hE}{R} \right) \left[K_0(\lambda_0 r) - \frac{\pi}{2} Y_0(\lambda_0 r) + i \frac{\pi}{2} J_0(\lambda_0 r) + 2 \ln \frac{r}{\ell} \right] \right\} \quad (5.12b)$$

agree with the corresponding results in [8]¹¹ for $(R/\gamma) < 1$.

6. SOLUTIONS FOR SPECIAL VISCOELASTIC MATERIALS

As $\lambda(s)$, specified by (4.6), contains fourth-degree polynomials in s , for simplicity's sake we limit ourselves to incompressible media ($\nu = 1/2$), in which case (2.6) and (4.6) become

$$\begin{aligned} E(s) &= \frac{3}{2} P_2(s) P_1^{-1}(s) \\ \lambda^4(s) &= \frac{9}{\rho^2} \left[\frac{1}{R^2} + \frac{\rho s^2}{E(s)} \right] \end{aligned} \quad (6.1)$$

and deduce explicit solutions for the viscoelastic Maxwell and Kelvin solids.

(a) For the incompressible Maxwell solid with the aid of (2.7b), (6.1) reduces to

$$\begin{aligned} E(s) &= \frac{s}{s + \tau^{-1}} E \\ \lambda_0^4(s) &= \frac{9\rho}{E\rho^2} \left[s^2 + s\tau^{-1} + \frac{E}{\rho R^2} \right] \end{aligned} \quad (6.2)$$

As tables for inverse transforms of the functions involved here are available, we turn to (4.5) and write

$$\begin{aligned} w^*(\xi, s) &= \frac{9}{E\rho^3} \frac{s + \tau^{-1}}{s} \frac{\beta^*(\xi, s)}{\xi^4 + \frac{9\rho}{E\rho^2} \left[s^2 + s\tau^{-1} + \frac{E}{\rho R^2} \right]} \\ &= \frac{1}{\rho h} \beta^*(\xi, s) \frac{s + \tau^{-1}}{s \left[(s + (\tau^{-1}))^2 + (\beta^2 - (\tau^{-1})^2) \right]} \end{aligned} \quad (6.3a)$$

11. With (5.10) and with the use of known relations of the type (see e.g., [18, p. 20])

$$Y_0(i^{1/2} x) = [\text{bei } x + i \text{ber } x] + \frac{2}{\pi} [-\text{ker } x + i \text{kei } x]$$

(5.12) for the range $R/\gamma > 1$ may be reduced to the form (5.9); in this connection compare with [10, Eqs. (42) to (45)].

where

$$\beta = \left\{ \left(\frac{E}{\rho} \right) \left(\frac{\tau}{R} \right)^2 \left[1 + (\ell \xi)^4 \right] \right\}^{1/2} \tau^{-1} \quad (6.3b)$$

Then, for instantaneous pulse specified by (4.10), recalling the Laplace transform of the δ -function [11, p. 27], as well as its inverse transform [11, p. 323], and using [16, Table 5.2], we take the inverse (Laplace followed by Hankel) transforms of (6.3a) to obtain

$$\begin{aligned} w(r, t) = & \frac{1}{\rho h} \int_0^\infty \left[\xi g^*(\xi) J_0(r\xi) \right] \frac{\tau^{-1}}{\beta^2} \left\{ 1 - \right. \\ & e^{-\frac{t-t_0}{2}} \tau^{-1} \left\langle \cos \left[\left(\beta^2 - \frac{1}{4\tau^2} \right)^{1/2} (t-t_0) \right] - \right. \\ & \left. \left. \frac{\tau \beta^2 - (r\tau)^{-1}}{\left(\beta^2 - \frac{1}{4\tau^2} \right)^{1/2}} \sin \left[\left(\beta^2 - \frac{1}{4\tau^2} \right)^{1/2} (t-t_0) \right] \right\rangle \right\} d\xi \quad (6.4) \end{aligned}$$

In a similar manner

$$\begin{aligned} \nabla^2 F(r, t) = & \frac{E}{\rho R} \int_0^\infty \left[\xi g^*(\xi) J_0(r\xi) \right] \left\{ \right. \\ & e^{-\frac{t-t_0}{2\tau}} \left. \frac{\ell}{\left(\beta^2 - \frac{1}{4\tau^2} \right)^{1/2}} \sin \left[\left(\beta^2 - \frac{1}{4\tau^2} \right)^{1/2} (t-t_0) \right] \right\} d\xi \quad (6.5) \end{aligned}$$

and F' , as in Sec. 4, is obtained by integration.

(b) For the incompressible Kelvin solid with the aid of (2.8b), (6.1) reads as

$$\begin{aligned} E(s) &= (1 + \tau s) E \\ \lambda_0^4(s) &= \frac{g\rho}{E h^2} \frac{s^2 + \left[\frac{E}{(\rho R^2)} \right] (1 + \tau s)}{(1 + \tau s)} \quad (6.6) \end{aligned}$$

and again to avoid the convolution integral in (4.7) and (4.9), we return to (4.5) and (4.8) and obtain

$$w(r,t) = \frac{1}{\rho h} \int_0^\infty \left[\xi g^*(\xi) J_0(r\xi) \right] e^{-\frac{\tau \beta^2}{2}(t-t_0)} \times$$

$$\frac{\sin \left\{ \beta \left[1 - \left(\frac{\tau \beta}{2} \right)^2 \right]^{1/2} (t-t_0) \right\}}{\beta \left[1 - \left(\frac{\tau \beta}{2} \right)^2 \right]^{1/2}} d\xi \quad (6.7)$$

$$\nabla^2 F(r,t) = \frac{E\tau}{\rho R} \int_0^\infty \left[\xi g^*(\xi) J_0(r\xi) \right] e^{-\frac{\tau \beta^2}{2}(t-t_0)} \times$$

$$\left\{ \cos \left\langle \beta \left[1 - \left(\frac{\tau \beta}{2} \right)^2 \right]^{1/2} (t-t_0) \right\rangle + \right.$$

$$\left. \frac{1 - \left(\frac{\tau \beta}{2} \right)^2}{\tau \beta \left[1 - \left(\frac{\tau \beta}{2} \right)^2 \right]^{1/2}} \sin \left\langle \beta \left[1 - \left(\frac{\tau \beta}{2} \right)^2 \right]^{1/2} (t-t_0) \right\rangle \right\} d\xi \quad (6.8)$$

In the above solutions, it is to be noted that as $t \rightarrow \infty$, the second term in the integrand of (6.4) diminishes exponentially while the first remains finite. Hence, under instantaneous loading, the shell medium for the Maxwell solid will assume a permanent deformation; no such effect is present in the solution (6.7) for the Kelvin solid. That this observation is not unexpected becomes evident by merely recalling the absence and presence of a restoring force in the one-dimensional Maxwell and Kelvin models, respectively. On the other hand, for an oscillatory load which itself (unlike an instantaneous load) supplies the restoring force, no permanent deformation takes place in the Maxwell medium.

As the integrals appearing in the solution (6.4) and (6.5), as well as (6.7) and (6.8), converge rapidly, numerical evaluation of the results is feasible. By way of example, with $R/h = 30$ and for a pulse of the type (4.10) and (4.13a) uniformly distributed over the circular region $0 \leq r/l \leq a/l = 0.257$, the plots of the axial displacement observed at $t/\tau = 12.05 \times 10^{-6}$ for the incompressible Maxwell and Kelvin solids (both media being characterized by

$$\frac{(E/\rho)^{1/2}}{(R/\tau)} = 2 \times 10^4$$

in the range of $0 \leq r/l \leq 3$ are shown respectively in¹² Figs. 1 and 2. Also shown in Fig. 1 is the axial displacement for an elastic shell due to the same

12. The numerical work resulting in Figs. 1 and 2 was carried out on the I.B.M 704 electronic computer.

pulse and observed at the same time as that of the Maxwell (and Kelvin) solid; the apparent coincidence of the two solutions (which are not the same) in Fig. 1 is due to the fact that in the range plotted, their numerical difference occurs only in the fourth significant figure. For further comparison, the axial displacement of an infinite elastic plate (for both compressible and incompressible media) of corresponding thickness, subjected to the same identical pulse, and observed at the same corresponding time, i.e.

$$\left(\frac{E}{\rho}\right)^{1/2} \frac{t}{h} = 7.23$$

is shown in Fig. 3.

The displacement contours in Figs. 1 and 2 are reminiscent of those of a viscoelastic rod under impact given by Lee and Kanter [4], who attributed this behavior to the fact that the differential equations (of the rod) for the Maxwell and elastic solids are hyperbolic and therefore admit wave phenomena, while the differential equation for the Kelvin solid is parabolic and represents a diffusion phenomenon. It is furthermore of interest to note that for all three solutions presented in Figs. 1 and 2, as well as for the case of the elastic plate in Fig. 3, at any instant $t > 0$ and at any finite distance from the origin, there is some small disturbance resulting from the instantaneous load at $t = 0$. This implies that some components of the response propagate with very high velocities and that the shell (or plate) acts as a dispersive conductor for transverse waves. It is this dispersive effect which makes it difficult to clearly locate the wave front in either the elastic or the viscoelastic shell. If, however, for purposes of discussion we consider the innermost zero value of the disturbance in Figs. 1, 2, and 3, as defining a pseudo wave front, then it is seen that such a front moves with a smaller velocity than that of a sound wave. The position of this latter wave front, considered as originating at the outer edge of the loaded region ($a/l = 0.257$) at the time of impact and moving outward with the velocity $(E/\rho)^{1/2}$, is indicated by the vertical dashed line in Figs. 1 to 3.

7. SHALLOW VISCOELASTIC SHELL SEGMENTS

The solutions for shallow viscoelastic spherical shell segments, in principle, may be obtained in a manner similar to those for unlimited shallow shells (Sec. 4); the chief difference, however, is the use of finite Hankel transform (in place of Hankel transform) together with some manipulations necessary in order to accommodate the edge boundary conditions of the shell segment. Although the method of solution (to be discussed presently) permits the treatment of shell segments with various boundary conditions (such as those considered in [14], i.e., simply supported, clamped, etc.), for simplicity's sake we confine attention to the case of shallow spherical shell segments ($0 \leq r \leq r_0$) with simply supported edge at $r = r_0$. Here, the regularity requirements at $r = 0$

are identical with (3.8), and the boundary conditions are given by

$$w(r_0, t) = N_r(r_0, t) = M_r(r_0, t) = 0 \quad (7.1)$$

Before proceeding further, we recall that the finite Hankel transform of a (suitably restricted) function $\hat{U}(r, s)$, as well as its inverse transform in the interval $0 \leq r \leq r_0$ are defined respectively by [12, p. 83]

$$U^*(\xi_i, s) = \int_0^{r_0} r J_0(r \xi_i) \hat{U}(r, s) dr \quad (7.2a)$$

and

$$\hat{U}(r, s) = \frac{2}{r_0^2} \sum_i U^*(\xi_i, s) \frac{J_0(r \xi_i)}{[J_1(r_0 \xi_i)]^2} \quad (7.2b)$$

where ξ_i are the roots of the transcendental equation

$$J_0(r_0 \xi_i) = 0 \quad (7.2c)$$

and in (7.2b), the summation is intended over all positive roots of (7.2c).

After introducing the new variables v and G

$$\begin{aligned} v &= w - \frac{r^2 - r_0^2}{4} v_0, & v_0 &= \nabla^2 w(r_0, t) \\ G &= \nabla^2 F - G_0, & G_0 &= \nabla^2 F(r_0, t), \end{aligned} \quad (7.3)$$

defined to satisfy

$$v(r_0, t) = \nabla^2 v(r_0, t) = G(r_0, t) = 0, \quad (7.4)$$

we return to (3.2) and, following the application of Laplace transform (with zero initial conditions) and with an appeal to the correspondence principle, we deduce

$$\begin{aligned} D(s) \nabla^2 \nabla^2 \hat{v} + \frac{1}{R} (\hat{G} + \hat{G}_0) &= \hat{\beta}(r, s) \\ \nabla^2 \hat{G} &= \frac{\hbar E(s)}{R} (\nabla^2 \hat{v} + \hat{v}_0) \end{aligned} \quad (7.5)$$

From the finite Hankel transform of (7.5), there eventually follows

$$\begin{aligned} \psi^*(\xi_i, s) = & \left[\xi_i^4 + \lambda_o^4(s) \right]^{-1} \left\{ \frac{\lambda_o}{\xi_i^3} \lambda_o^4(s) J_1(\lambda_o \xi_i) \hat{\psi}_o(s) \right. \\ & \left. - \frac{\lambda_o}{R D(s)} \frac{J_1(\lambda_o \xi_i)}{\xi_i^4} \hat{G}_o(s) + \frac{\rho^*(\xi_i, s)}{D(s)} \right\} \end{aligned} \quad (7.6a)$$

and

$$\begin{aligned} G^*(\xi_i, s) = & \frac{h E(s)}{R} J_1(\lambda_o \xi_i) \left[\xi_i^4 + \lambda_o^4(s) \right]^{-1} \left\{ \frac{\lambda_o}{\xi_i^3} \lambda_o^4(s) \hat{\psi}_o(s) \right. \\ & \left. - \frac{1}{R D(s)} \frac{\hat{G}_o(s)}{\xi_i} \right\} + h E(s) \left\{ D^{-1}(s) \left[\xi_i^4 + \lambda_o^4(s) \right]^{-1} \rho^*(\xi_i, s) \right. \\ & \left. - \frac{1}{R} \frac{\lambda_o}{\xi_i^3} J_1(\lambda_o \xi_i) \hat{\psi}_o(s) \right\} \end{aligned} \quad (7.6b)$$

The above two equations may be put in the form

$$\psi^*(\xi_i, s) = \hat{\psi}_o f_1^*(\xi_i, s) - \hat{G}_o f_2^*(\xi_i, s) + \psi_1^*(\xi_i, s) \quad (7.7a)$$

$$G^*(\xi_i, s) = \hat{\psi}_o g_1^*(\xi_i, s) - \hat{G}_o g_2^*(\xi_i, s) + G_1^*(\xi_i, s) \quad (7.7b)$$

where the functions f_1^* , f_2^* , g_1^* , g_2^* , ψ_1^* , and G_1^* are defined by direct comparison with (7.6). It may be noted here that except for ψ_1^* and G_1^* all other functions involved in (7.7) are polynomials in the Laplace parameter s .

In order to eliminate the functions $\hat{\psi}_o$ and \hat{G}_o , we take the inverse Hankel transform of (7.7) and obtain

$$\hat{\psi}(\lambda, s) = \hat{\psi}_o \hat{f}_1(\lambda, s) - \hat{G}_o \hat{f}_2(\lambda, s) + \hat{\psi}_1(\lambda, s) \quad (7.8a)$$

$$\hat{G}(\lambda, s) = \hat{\psi}_o \hat{g}_1(\lambda, s) - \hat{G}_o \hat{g}_2(\lambda, s) + \hat{G}_1(\lambda, s) \quad , \quad (7.8b)$$

and also by the second of (7.3)

$$\begin{aligned} \lambda \frac{\partial \hat{F}}{\partial \lambda} = & \hat{G}_o \int_0^\lambda \eta [1 - \hat{g}_2(\eta, s)] d\eta \\ & + \hat{\psi}_o \int_0^\lambda \eta \hat{g}_1(\eta, s) d\eta + \int_0^\lambda \eta \hat{G}_1(\eta, s) d\eta \end{aligned} \quad (7.8c)$$

In view of the functional form of M_r in (3.3), we next introduce the operator

$$\Lambda \equiv \frac{\partial^2}{\partial r^2} + \frac{\nu}{r} \frac{\partial}{\partial r} \quad (7.9)$$

and with the aid of (7.3), by virtue of the last two of the boundary conditions (7.1) when applied to (7.8), determine \hat{v}_0 and \hat{G}_0 as functions of $\hat{f}_1, \hat{f}_2, \hat{g}_1, \hat{g}_2, \hat{v}_1$, and \hat{G}_1 i.e.,

$$\begin{aligned} \hat{v}_0 & \left\{ 1 + [\Lambda \hat{f}_2(r_0, s)] [\Lambda \hat{f}_1(r_0, s)]^{-1} \frac{\int_0^{r_0} \eta \hat{g}_1 d\eta}{\int_0^{r_0} \eta [1 - \hat{g}_2] d\eta} \right\} \\ & = [\Lambda \hat{f}_1(r_0, s)]^{-1} \left\{ \Lambda \hat{v}_1(r_0, s) - \Lambda \hat{f}_2(r_0, s) \frac{\int_0^{r_0} \eta \hat{G}_1 d\eta}{\int_0^{r_0} \eta [1 - \hat{g}_2] d\eta} \right\} \end{aligned} \quad (7.10a)$$

and

$$\hat{G}_0 = - \left\{ \int_0^{r_0} \eta [1 - \hat{g}_2] d\eta \right\}^{-1} \left\{ \hat{v}_0 \int_0^{r_0} \eta \hat{g}_1 d\eta + \int_0^{r_0} \eta \hat{G}_1 d\eta \right\}. \quad (7.10b)$$

Next, substituting (7.10) into (7.8), and taking the inverse Laplace transform followed by inverse Hankel transform, we obtain

$$\begin{aligned} v(r, t) = \frac{2}{r^2} \sum_i \left\{ \mathcal{L}^{-1} [\hat{v}_0 \hat{f}_1^*(\xi_i, s); t] - \right. \\ \left. \mathcal{L}^{-1} [\hat{G}_0 \hat{f}_2^*(\xi_i, s); t] + \mathcal{L}^{-1} [\hat{v}_1^*(\xi_i, s); t] \right\} \frac{J_0(r \xi_i)}{[J_1(r_0 \xi_i)]^2} \end{aligned} \quad (7.11)$$

$$\begin{aligned} G(r, t) = \frac{2}{r^2} \sum_i \left\{ \mathcal{L}^{-1} [\hat{v}_0 \hat{g}_1^*(\xi_i, s); t] - \right. \\ \left. \mathcal{L}^{-1} [\hat{G}_0 \hat{g}_2^*(\xi_i, s); t] + \mathcal{L}^{-1} [\hat{G}_1^*(\xi_i, s); t] \right\} \frac{J_0(r \xi_i)}{[J_1(r_0 \xi_i)]^2} \end{aligned} \quad (7.12)$$

which together with (7.3) and (7.10) formally complete the solution for simply supported shallow viscoelastic spherical shell segments subjected to arbitrary time-dependent axisymmetric loads.

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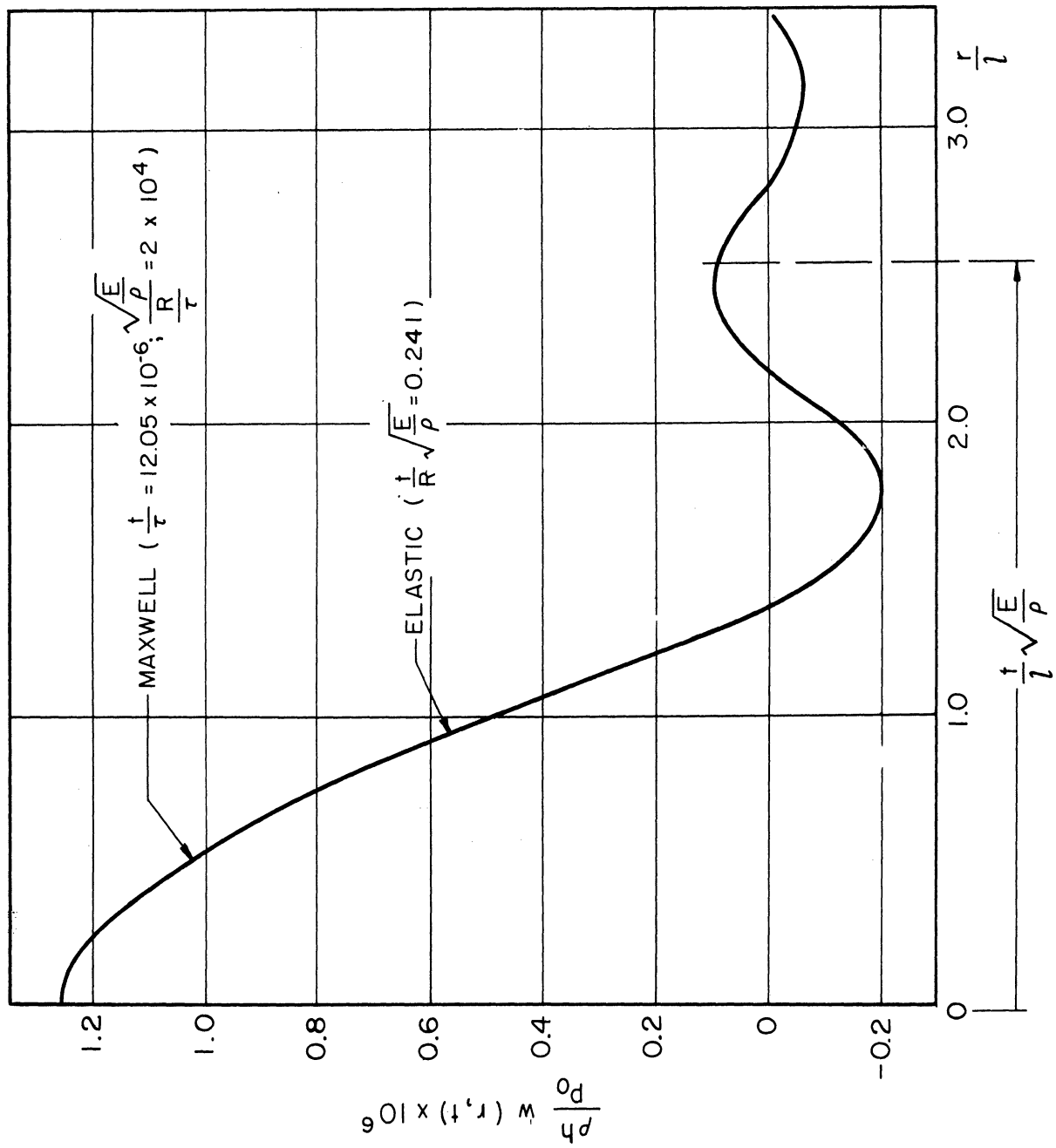


Fig. 1

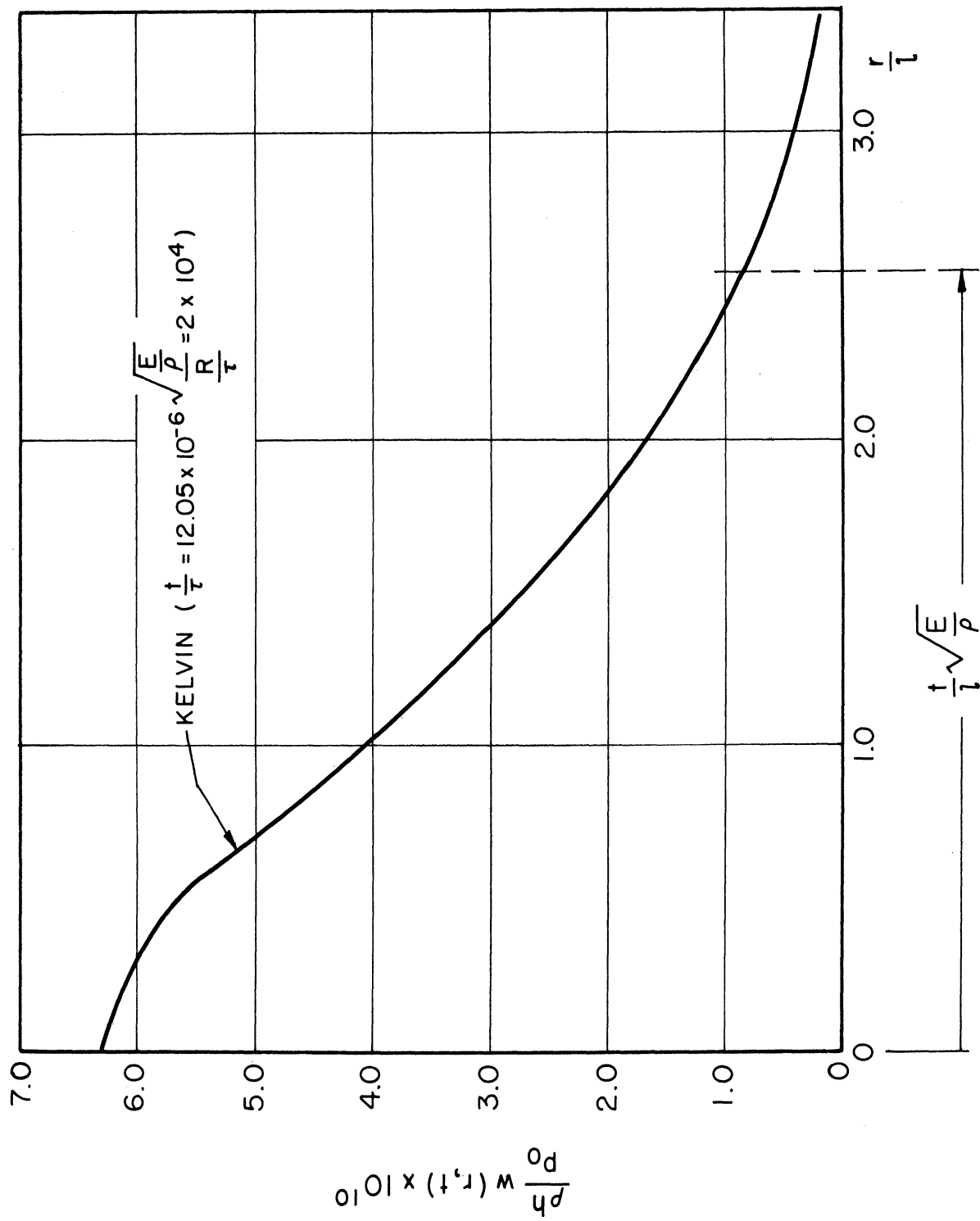


Fig. 2

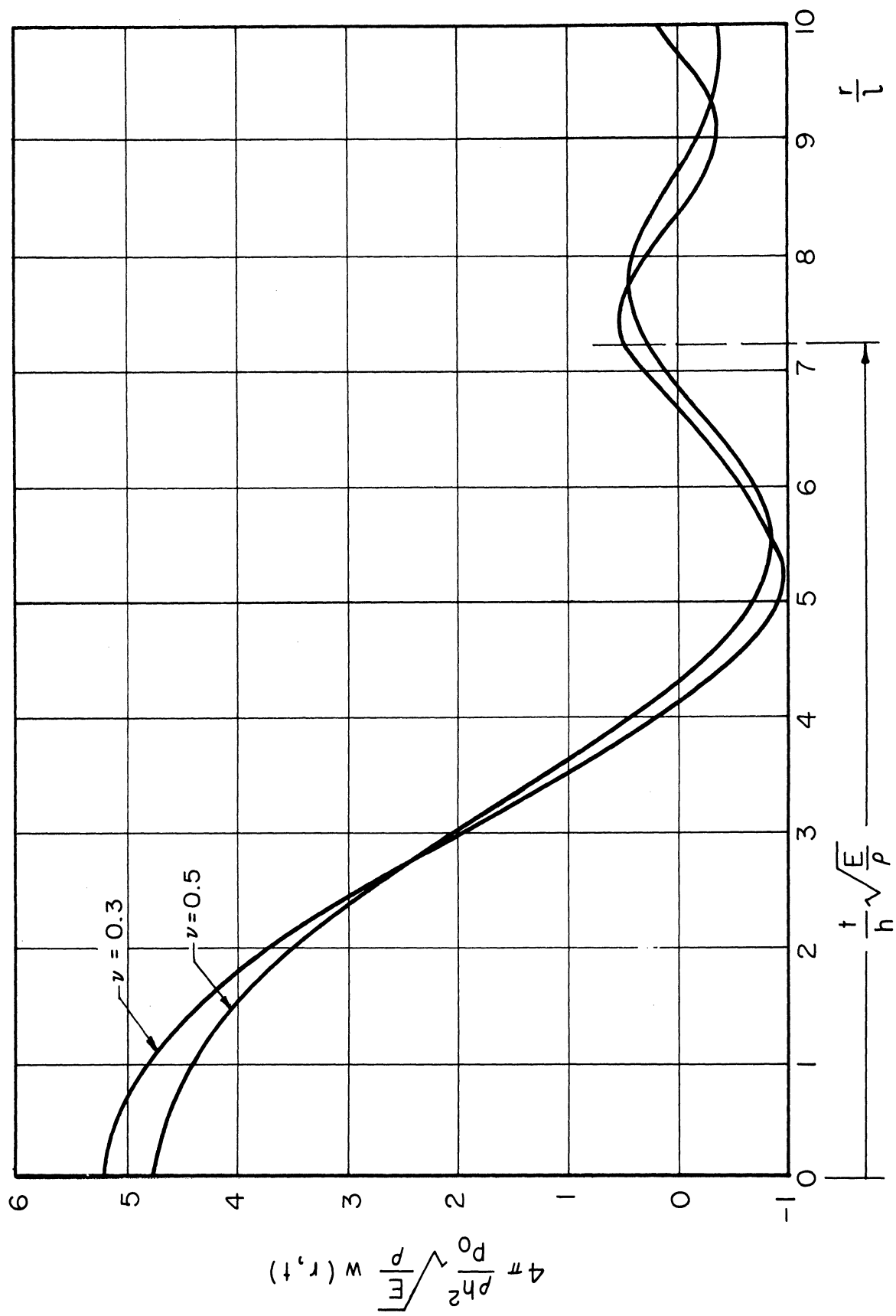


Fig. 3

