

THE UNIVERSITY OF MICHIGAN
INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

ON EVALUATION OF THE GRAPH TREES AND
THE DRIVING POINT ADMITTANCE

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I. INTRODUCTION

Recently much interest has been aroused in topological treatment of networks. Ku¹ called the attention of electrical engineers to original rules of Maxwell and Kirchhoff, which are the fundamentals of topological characteristics of the electrical network. So far few have paid attention but Percival,² who developed his theory from the graph of topology and did not, rather naturally, reach to such systematic and algebraic method as conventional matrix or determinant. Trent,⁸ on the other hand, showed how to calculate the characteristic numbers concerning the tree by making use of a "primitive node-pair connection matrix" but it is not as simple as Percival's. Synge⁶ investigated the matrix theory of electrical networks from the viewpoint of topology, Saltzer⁷ furthered the relationship between node analysis and mesh analysis from the same point of view, and Reed⁴ systematized these approaches. Cederbaum^{9,10} investigated fundamental characteristics of the network determinant. These investigators did not try to find an efficient method of obtaining characteristic numbers concerning the tree. Kron¹¹ has been working on a large system but his approach to a small network was adopted by Synge.⁶ On the field of switching theory, however, Hohn³ proposed a simple, clear idea of a "connection matrix" and showed an interesting technique of the design of relay circuits. Why cannot we make use of the same matrix in the electrical circuit? Strangely enough, few have realized that it would still remain to be a powerful tool. Some work has been done in this thesis concerning this point.

The purpose of this thesis, therefore, would deal with three subjects: the first is the introduction of a new algorithm of a

"Foldant"; the second to show that by making use of the foldant, such characteristic numbers concerning the tree as the total number, the tree summation of all possible trees of a network and the driving point admittance are performed systematically and algebraically without setting up a "primitive node-pair connection matrix", and the third to show that the algorithm of the foldant is so straightforward that the programming of the method may be realized.

Throughout this paper, networks are supposed to be linear, time invariant and passive and to contain no mutual inductances. Although some of the restrictions can be removed, they are adopted for immediate convenience.

II. DEFINITIONS CONCERNING A GRAPH

Several definitions are given below. Let us consider a graph of n nodes. The definition of a tree has been well accepted. For a matter of convenience, it will be rewritten below.

Definition 1:⁵ A "tree" of a graph is a connected graph containing all the nodes of the graph and containing no loops.

The following definition is the same as the one of a tree product in reference 5, but the present name is preferred for avoiding confusion in later development of the theory.

Definition 2: The "tree designation" is the product of all branch designations of a tree.

The following definition is set up for convenience of handling the later theories, although the idea has been used for years without definite nomenclature.

Definition 3: The "tree summation" of a graph is the summation of all tree designations of a graph.

The following two definitions of matrices are newly proposed here.³

These, however, will not be used directly in the development of our theory but only for deriving the two matrices given after them.

Definition 4: The "primary branch matrix" (B) is an $n \times n$ upper diagonal matrix such that

$$(B) = (b_{ij}) = \begin{cases} \text{sum of the branch designations between} \\ \text{nodes } i \text{ and } j, i < j \\ 1, i = j \\ 0, i > j \end{cases}$$

Definition 5: The "primary branch number matrix" (N) is an nxn upper diagonal matrix such that

$$(N) = (n_{ij}) = \begin{cases} \text{number of the branches between} \\ \text{nodes } i \text{ and } j, i < j \\ 1, i = j \\ 0, i > j \end{cases}$$

With the following two matrices, our theory will be developed.

Definition 6: The "branch matrix" (B°) is the (n-1)x(n-1) upper diagonal matrix derived from the primary branch matrix by deleting all diagonal entries of the latter.

Thus the branch matrix has the following form:

$$(B^\circ) = \begin{pmatrix} b_{12} & b_{13} & \dots & b_{1,n-1} & b_{1n} \\ 0 & 0 & \dots & b_{2,n-1} & b_{2n} \\ & & \dots & & \\ 0 & 0 & \dots & 0 & b_{n-1,n} \end{pmatrix} \quad (1)$$

Definition 7: The "branch number matrix" (N) is the (n-1)x(n-1) upper diagonal matrix derived from the primary branch number matrix by deleting all diagonal entries of the latter.

A couple of examples will be given, concerning the definitions.

Example 1. With branch designations A, B, and C of the network, Figure 1.

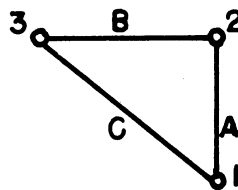


Figure 1

a tree designation = AB

the tree summation = AB + AC + BC

the branch matrix = $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$

the branch number matrix = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Example 2. With branch designations A_1 , A_2 , B and C of the network, Figure 2.

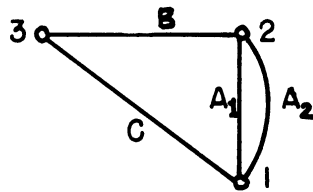


Figure 2

a tree designation = A_1B

the tree summation = $A_1B + A_2B + A_1C + A_2C + BC$

the branch matrix = $\begin{bmatrix} A_1 + A_2 & C \\ 0 & B \end{bmatrix}$

the branch number matrix = $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

III. ALGORITHM OF THE FOLDANT

The following new algorithm is proposed here to make a direct computation of the tree summation and the driving point admittance. Other application is anticipated.

Let (A) be an nxn upper diagonal matrix. Then the "foldant"

|A| is defined by the following recursive rules:

|A| = |a₁₁| = a₁₁ when n = 1, and

$$\begin{array}{l}
 |A| = |a_{ij}| = a_{1n} \\
 + a_{2n} \\
 + a_{3n}
 \end{array}
 \left|
 \begin{array}{cccccc}
 a_{11} + a_{2n} & a_{12} + a_{3n} & a_{13} + a_{4n} & \dots & a_{1,n-1} + a_{nn} \\
 0 & a_{22} & a_{23} & \dots & a_{2,n-1} \\
 0 & 0 & a_{33} & \dots & a_{3,n-1} \\
 & & & \dots & \\
 0 & 0 & 0 & \dots & a_{n-1,n-1} \\
 \\
 a_{11} & a_{12} & a_{13} & \dots & a_{1,n-1} \\
 0 & a_{22} + a_{3n} & a_{23} + a_{4n} & \dots & a_{2,n-1} + a_{nn} \\
 0 & 0 & a_{33} & \dots & a_{3,n-1} \\
 & & & \dots & \\
 0 & 0 & 0 & \dots & a_{n-1,n-1} \\
 \\
 a_{11} & a_{12} & a_{13} & \dots & a_{1,n-1} \\
 0 & a_{22} & a_{23} & \dots & a_{2,n-1} \\
 0 & 0 & a_{33} + a_{4n} & \dots & a_{3,n-1} + a_{nn} \\
 & & & \dots & \\
 0 & 0 & 0 & \dots & a_{n-1,n-1}
 \end{array}
 \right|$$

+ ...

$$+ a_{nn} \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,n-1} \\ 0 & a_{22} & a_{23} & \dots & a_{2,n-1} \\ 0 & 0 & a_{33} & \dots & a_{3,n-1} \\ & & & \dots & \\ 0 & 0 & 0 & \dots & a_{n-1,n-1} \end{vmatrix}$$

IV. THE TREE SUMMATION AND THE NUMBER OF TREES

Several theorems will be given. The following theorem 1 has been proved as Theorem 6 in Percival's paper.²

Theorem 1: Let a graph be given with the tree summation T . Then, with reference to any branch designation b ,

$$T = T_{\perp}^{\circ} + b T_{\perp}^{\prime} ,$$

where T_{\perp}° is the tree summation of a graph derived from the given graph by deleting the branch b , and T_{\perp}^{\prime} is the tree summation of a graph derived from the given graph by identifying the nodes of the branch b .

Proof: The tree summation T can be partitioned into two terms, one a subset of those tree designations not containing the branch b and the other a subset of those tree designations containing the branch b . Let the first one be called T_{\perp}° . Then it is clear that T_{\perp}° is the tree summation of a graph derived from the given graph by deleting the branch b .

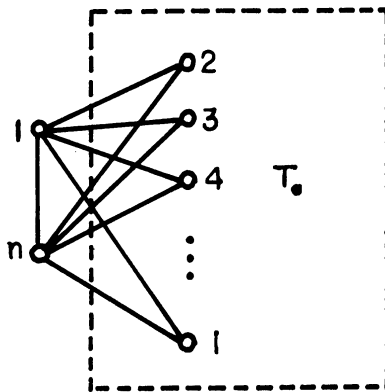


Figure 3

Now, with Figure 3, let us consider those tree summations containing branch b . Let the nodes of the branch b be 1 and n . Let those nodes of the given graph, connected to the nodes 1 or n , be denoted as 2, 3, . . . , and i . And imagine a graph derived from the given graph by

deleting the nodes 1 and n and those branches which were connected to the nodes 1 or n from the nodes 2, 3, . . . , and i. Let the tree summation of this graph be denoted as T_0 . Then those tree summation containing the branch b of the given graph may be expressed as follows:

$$b \sum_{v=2}^i b_{1v} T_0 + b \sum_{v=2}^i b_{nv} T_0$$

or

$$b \sum_{v=2}^i (b_{1v} + b_{nv}) T_0 .$$

$\sum_{v=2}^i (b_{1v} + b_{nv}) T_0$ is certainly the tree summation T'_1 of a graph derived

from the given graph by identifying the nodes of the branch b, 1 and n. Thus the Theorem is proved.

Corollary: Let a graph of n nodes be given with the tree summation T. Then, with reference to b_{1n} , the sum of the branch designations between nodes 1 and n,

$$T = T_1 + b_{1n} T'_1 ,$$

where T_1 is the tree summation of a graph derived from the given graph by deleting all the branches between the nodes 1 and n, and T'_1 is the tree summation of a graph derived from the given graph by identifying the nodes 1 and n.

Proof: Let $b_{1n} = b_1 + b_2 + . . . + b_j$. By Theorem 1,

$$T = T_1^0 + b_1 T'_1 ,$$

where T_1^0 is the tree summation of a graph derived from the given graph by deleting the branch b_1 . Similarly,

$$T_1^\circ = T_2^\circ + b_2 T_1', \quad ,$$

where T_2° is the tree summation of a graph derived from the given graph by deleting the branches b_1 and b_2 . Thus,

$$T = T_2^\circ + (b_1 + b_2)T_1' \quad .$$

Repeating the process,

$$T = T_1 + (b_1 + b_2 + \dots + b_j)T_1' = T_1 + b_{1n} T_1' \quad .$$

Thus the corollary is proved.

By making use of the Corollary, the following new Theorem concerning the evaluation of the tree summation is derived.

Theorem 2: The tree summation of a graph is equal to the foldant $|B^\circ|$ of the branch matrix (B°) of the graph.

Proof: The proof is given by induction. When the number of nodes $n = 1$, the graph is not defined. When $n = 2$, the theorem is obvious. Let the theorem be true when the number of nodes is $(n-1)$. Now, let us consider a graph of n nodes with the tree summation T . Let the graph have the branch matrix (B°) of the equation (1) above.

From the Corollary,

$$T = T_1 + b_{1n} T_1' \quad .$$

It is noticed that T_1' is a graph with $(n-1)$ nodes. Hence the induction hypothesis can be applied to T_1' . Thus,

$$T = T_1 + b_{1n} \left| \begin{array}{cccc} b_{12} + b_{2n} & b_{13} + b_{3n} & \dots & b_{1,n-1} + b_{n-1,n} \\ 0 & b_{23} & \dots & b_{2,n-1} \\ & & \dots & \\ 0 & 0 & \dots & b_{n-2,n-1} \end{array} \right|, \quad (2)$$

where the graph with the tree summation T has the following branch matrix:

$$\begin{vmatrix} b_{12} & b_{13} & \dots & b_{1,n-2} & b_{1,n-1} & 0 \\ 0 & b_{23} & \dots & b_{2,n-2} & b_{2,n-1} & b_{2n} \\ & & \dots & & & \\ 0 & 0 & \dots & 0 & 0 & b_{n-1,n} \end{vmatrix}$$

Likewise, the Corollary is applied to the graph T, concerning the entry b. Repeating the process, T is developed into the form:

$$\begin{aligned} T = b_{1n} & \begin{vmatrix} b_{12} + b_{2n} & b_{13} + b_{3n} & \dots & b_{1,n-1} + b_{n-1,n} \\ 0 & b_{23} & \dots & b_{2,n-1} \\ & & \dots & \\ 0 & 0 & \dots & b_{n-2,n-1} \end{vmatrix} \\ & + b_{2n} \begin{vmatrix} b_{12} & b_{13} & \dots & b_{1,n-1} \\ 0 & b_{23} + b_{3n} & \dots & b_{2,n-1} + b_{n-1,n} \\ & & \dots & \\ 0 & 0 & \dots & b_{n-2,n} \end{vmatrix} \\ & + \dots \\ & + b_{n-1,n} \begin{vmatrix} b_{12} & b_{13} & \dots & b_{1,n-1} \\ 0 & b_{23} & \dots & b_{2,n-1} \\ & & \dots & \\ 0 & 0 & \dots & b_{n-2,n-1} \end{vmatrix} \end{aligned}$$

The right hand side is the foldant defined in Sec.II.

Thus $T = |B^\circ|$, and the Theorem is proved.

A Theorem concerning the enumeration of the number of trees will also newly be given as follows:

Theorem 3: The number of trees of a graph is equal to the foldant of the branch number matrix of the graph.

Proof: The proof follows the same procedure as the proof above. That is, it is clear from the Corollary that the number N of trees of a graph is expressed as

$$N = N_1 + N'_1 ,$$

where N_1 is the number of the trees contained in the tree summation T_1 and N'_1 is the number of the trees contained in the tree summation T'_1 . Then we can apply such induction procedure for the branch number matrix of the graph as used in Theorem 2. Thus Theorem 3 can be proved.

A couple of examples will be given, illustrating Theorems 2 and 3.

Example 3: On Figure 1,

$$\text{The tree summation} = \begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = C(A + B) + BA$$

$$\text{The number of trees} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1.2 + 1.1 = 3$$

Example 4: With the branch designations A, B, C, D, E and F of the graph, Figure 4,

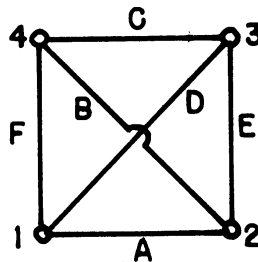


Figure 4

$$\begin{aligned} \text{The tree summation} &= \begin{vmatrix} A & D & F \\ 0 & E & B \\ 0 & 0 & C \end{vmatrix} \\ &= F \begin{vmatrix} A+B & D+C \\ 0 & E \end{vmatrix} + B \begin{vmatrix} A & D \\ 0 & E+C \end{vmatrix} + C \begin{vmatrix} A & D \\ 0 & E \end{vmatrix} \\ &= F(D+C)(A+B+E) + FE(A+B) + BD(A+E+C) \\ &\quad + B(E+C)A + CD(A+E) + CEA \end{aligned}$$

$$\begin{aligned} \text{The number of trees} &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \\ &= 2.3 + 1.2 + 1.3 + 2.1 + 3 \\ &= 16 \end{aligned}$$

The result agrees with those of Figure 1 of Ku's paper.¹

V. DRIVING POINT ADMITTANCE

As is shown in Ku's paper,¹ the driving point admittance of a passive network without mutual inductance is obtained as $|B^\circ|/\Delta_{1n}$, where Δ_{1n} is the sum of the products of y 's taken $(n-2)$ at a time, omitting all terms containing y_{1n} or forming a closed circuit with it and all other terms forming closed circuits themselves. We have the following new theorem with respect to Δ_{1n} :

Theorem 4: Let a network of n nodes with the branch matrix of the equation (1) be given. Set the entry b_{1n} as 1. Then Δ_{1n} is equal to the foldant appearing as the coefficient of b_{1n} in the equation (2) above

Proof: It is seen from the derivation of the equation (2) that $y_{1n}T'_{11}$ is the tree summation of the set of all trees containing y_{1n} . It is also seen from the rule for obtaining the driving point admittance that $y_{1n}\Delta_{1n}$ is the tree summation of the set of all trees containing y_{1n} . Hence the theorem is proved.

A couple of examples will be given.

Example 5: With the admittances A , B , and C of the network, Figure 5,

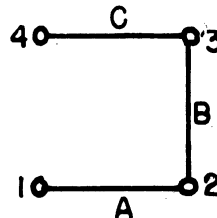


Figure 5

$$|B^\circ| = \begin{vmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{vmatrix} = C \cdot B \cdot A.$$

$$\Delta_{14} = \begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = CA + CB + AB$$

the driving point admittance at 14

$$= \frac{ABC}{AB + BC + CA} = \frac{1}{\frac{1}{A} + \frac{1}{B} + \frac{1}{C}}$$

Example 6: With the admittances A, B, C, D, E and F of the network, Figure 6,

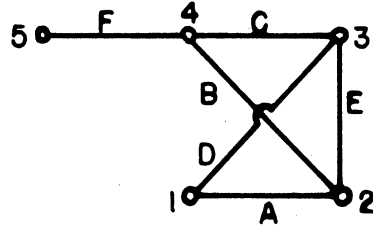


Figure 6

$$|B^\circ| = \begin{vmatrix} A & D & 0 & 0 \\ 0 & E & B & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & F \end{vmatrix} = F \begin{vmatrix} A & D & 0 \\ 0 & E & B \\ 0 & 0 & C \end{vmatrix}$$

$$= FB \begin{vmatrix} A & D \\ 0 & E+C \end{vmatrix} + FC \begin{vmatrix} A & D \\ 0 & E \end{vmatrix}$$

$$= FBD(A+E+C) + FB(E+C)A + FCD(A+E) + FCEA$$

$$\Delta_{15} = \begin{vmatrix} A & D & F \\ 0 & E & B \\ 0 & 0 & C \end{vmatrix} = F \begin{vmatrix} A+B & C+D \\ 0 & E \end{vmatrix} + B \begin{vmatrix} A & D \\ 0 & E+C \end{vmatrix} + C \begin{vmatrix} A & D \\ 0 & E \end{vmatrix}$$

$$= F(C+D)(A+B+E) + FE(A+B) + BD(A+E+C) + B(E+C)A + CD(A+E) + CEA$$

the driving point admittance at 15

$$= \frac{FBD(A+E+C) + FB(E+C)A + FCD(A+E) + FCEA}{F(C+D)(A+B+E) + FE(A+B) + BD(A+E+C) + B(E+C)A + CD(A+E) + CEA}$$

The result agrees with i_p of Figure 1 of Ku's paper.¹

VI. MIDAC PROGRAMMING FOR THE EVALUATION OF THE TREE SUMMATION

The straightforward coding of the alphanumeric representation of the tree summation of any arbitrary graph having seven nodes or fewer was tried. If the number of nodes is eight, the number of total instructions of our program would exceed the capacity of the high-speed memory of MIDAC which is 512 as described below. Furthermore, if the number of nodes is nine or more, even if we use the tape memory of MIDAC, the result would be so bulky that it would hardly be worthwhile. That is, suppose we make a program so that each of the two terms of the tree summation of the three-nodes graph shown in Example 3 is printed on each row. Then it is known from Theorem 2 that the number of rows of the four-nodes graph shown in Example 4 is $3 \cdot 2 = 3! = 6$. Thus the number of rows of a nine-nodes graph with the branches connected between any nodes in any possible way would be $8! = 40,320$. It would need 611 sheets of paper to be printed, since 66 rows can be printed on a sheet of paper. As a more elaborate program, we may incorporate Kron's method of tearing with ours, and the computation time would be considerably reduced.¹¹ However, the number of result sheets discussed above still would be too many for us to get significant information. Thus it is conceivable that our way of the alphanumeric representation of the tree summation is not appropriate for the analysis of a graph of many nodes. But we will see what result we get. A general flow diagram of the coding is shown on Figure 7.

Before going into some detail of the coding, a brief description will be given about the MIDAC computer. Patterned after SEAC of the National Bureau of Standards, MIDAC (Michigan Digital Automatic

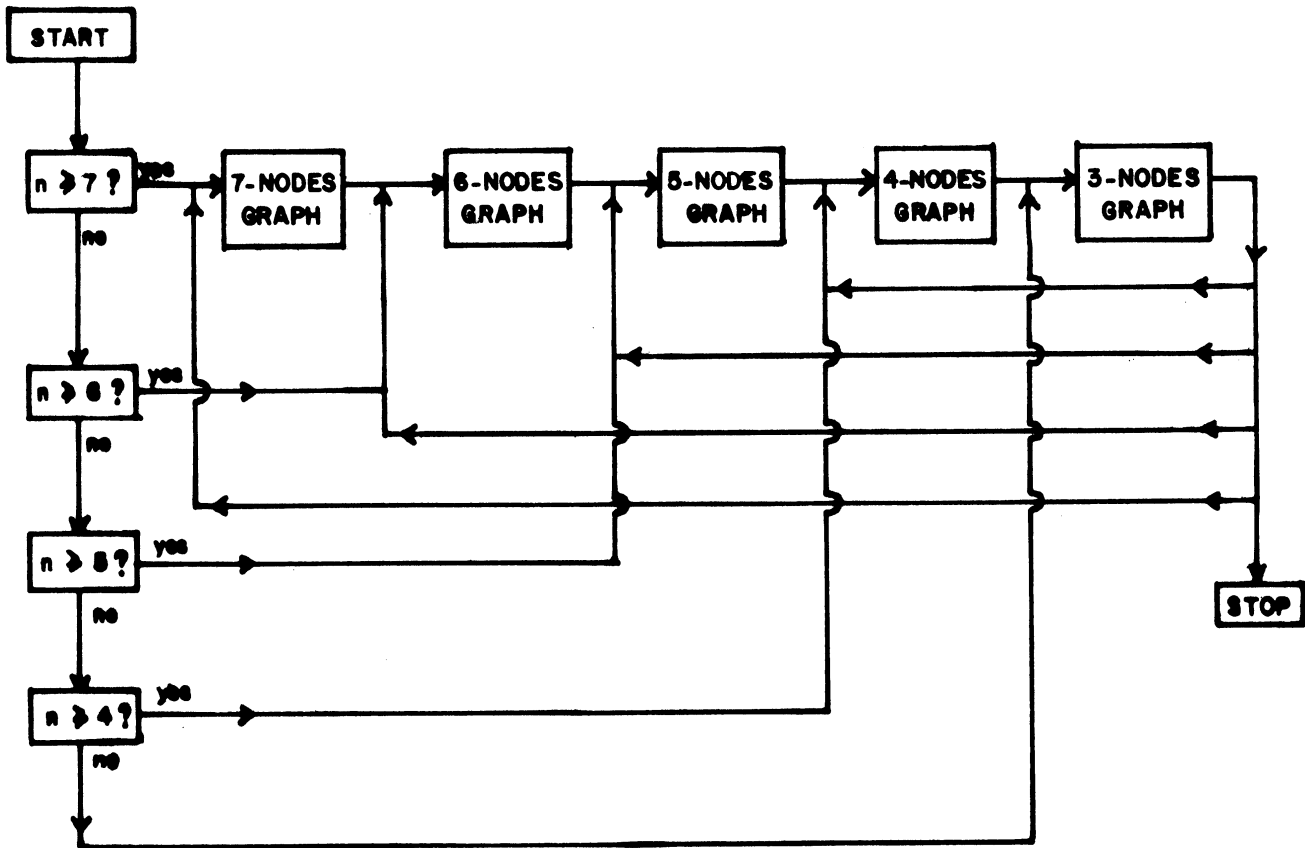


Figure 7 General Flow Diagram of Coding

Computer) is essentially a serial computer and has electric circuitry based on the so-called "dynamic flip-flop". The mercury acoustic delay storage has the capacity of 512 words. The drum system has a capacity of 6,144 words. Each word contains 44 binary digits plus a sign digit. The MIDAC speed is, on the average, 1000 additions per second or 300 multiplications per second, with corresponding times for other operations. The 45 binary digits of an instruction word are interpreted as consisted of three addresses and an operation code. There are nineteen operations such as 1) Add, 2) Subtract, 3) Multiply, Low Order, 4) Multiply, High Order, 5) Multiply, Rounded, 6) Divide, 7) Power Extract,

8) Shift Number, 9) Extract, 10) Decimal to Binary Conversion, 11) Binary to Decimal Conversion, 12) Compare Numbers, 13) Compare Magnitude, 14) Base, 15) File, 16) Read In, 17) Read Out, 18) Move Tape Forward, and 19) Move Tape Backward.

By making use of these instructions, a detailed program can be prepared for the general flow diagram of Figure 7. At first, coding of a graph with three nodes will be explained. The result form of the Example 3 will be

A.B

(A+B)C.

However, any case where either one or more of A, B, or C may be zero should be taken care of. Furthermore, in order that the routine can be used as a subroutine for a graph of more nodes, it is preferable that the coding can produce the following form:

A.B.V.W.X.Y

(A+B)C.V.W.X.Y ,

where V is the entry decided by a four-nodes graph, W by a five-nodes graph, X by a six-nodes graph and Y by a seven-nodes graph. The flow diagram of a three-nodes graph is shown on Figure 8. In Figure 8, "Print" instruction is special subroutine, by which a content of a word is checked five binary digits as a group and, if it is not zero, an alphabetic character corresponding to those five binary digits is read out into the output of Flexowriter as a printed form. Likewise, a flow diagram of a four-nodes graph is shown on Figure 9. On the figure, the symbol \rightarrow shows a specific subroutine, by which two or more alphabetic characters are put into a word memory so that they may be read out by the subroutine mentioned above in connection with Figure 8.

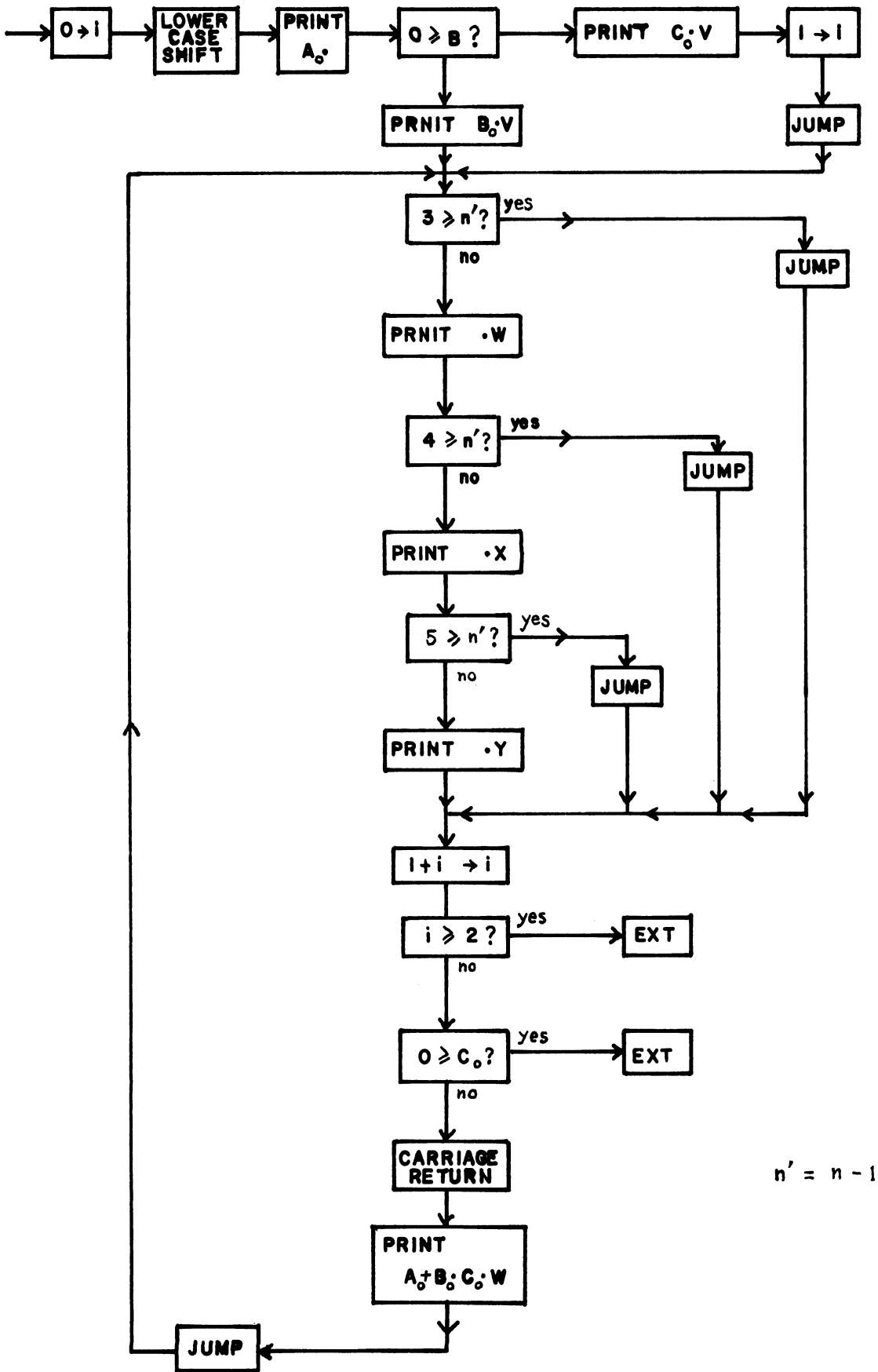


Figure 8 Three-Nodes Graph Subroutine

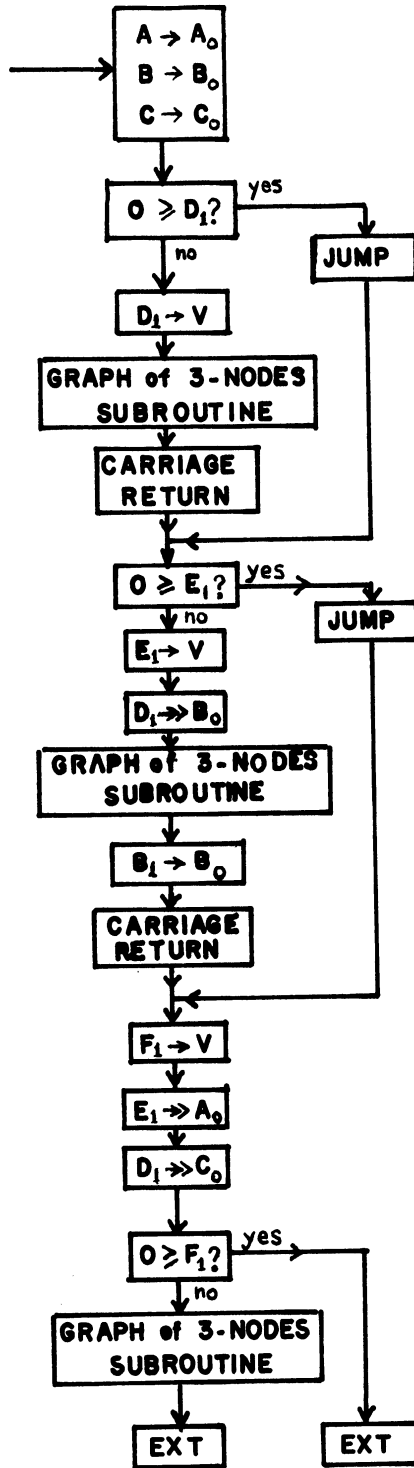


Figure 9 Four-Nodes Graph Subroutine

Similar flow are drawn for graphs with more nodes and the result of running is shown in the next sections.

A. The number of nodes $n = 3$

This is trivial and the result obtained agrees with Example 3 above.

B. The number of nodes $n = 4$

The result obtained agrees with Example 4.

C. The number of nodes $n = 5$

With branch designations, A, B, C, D, E, F, G, H, I and J of the graph, Figure 10, the result obtained is presented below. The tree summation is the addition of the following 24 rows. (As for the symbols used, "." is stronger than "+" so that A.B+C.D+E, for instance, means A(B+C)(D+E).)

- A.B.D.G.
- A+B.C.D.G.
- A.B+D.E.G
- A+B+D.C.E.G
- A+E.B.F.G
- A+E+B.C+D.F.G
- A.B.D+G.H
- A+B.C.D+G.H
- A.B+D+G.E.H
- A+B+D+G.C.E.H.
- A+E.B.F.H.
- A+E+B.C+D+G.F.H
- A.B+H.D.I
- A+B+H.C.D.I
- A.B+H+D.E+G.I
- A+B+H+D.C.E+G.I
- A+E+G.B+H.F.I
- A+E+G+B+H.C+D.F.I
- A+I.B.D.J
- A+I+B.C+H.D.J
- A+I.B+D.E.J
- A+I+B+D.C+H.E.J
- A+I+E.B.F+G.J
- A+I+E+B.C+H+D.F+G.J

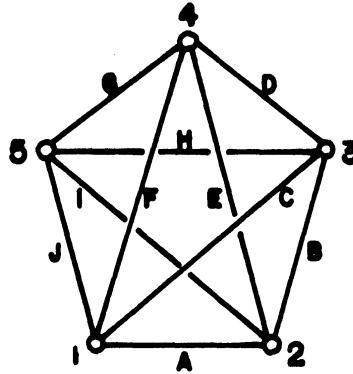


Figure 10

D. The number of nodes $n = 6$, Figure 11

- A.B.D.G.K
- A+B.C.D.G.K
- A.B+D.E.G.K
- A+B+D.C.E.G.K
- A+E.B.F.G.K
- A+E+B.C+D.F.G.K
- A.B.D+G.H.K
- A+B.C.D+G.H.K
- A.B+D+G.E.H.K
- A+B+D+G.C.E.H.K
- A+E.B.F.H.K
- A+E+B.C+D+G.F.H.K
- A.B+H.D.I.K
- A+B+H.C.D.I.K
- A.B+H+D.E+G.I.K
- A+B+H+D.C.E+G.I.K
- A+E+G.B+H.F.I.K
- A+E+G+B+H.C+D.F.I.K
- A+I.B.D.J.K
- A+I+B.C+H.D.J.K
- A+I.B+D.E.J.K
- A+I+B+D.C+H.E.J.K
- A+I+E.B.F+G.J.K
- A+I+E+B.C+H+D.F+G.J.K
- A.B.D.G+K.L
- A+B.C.D.G+K.L
- A.B+D.E.G+K.L
- A+B+D.C.E.G+K.L
- A+E.B.F.G+K.L
- A+E+B.C+D.F.G+K.L
- A.B.D+G+K.H.L
- A+B.C.D+G+K.H.L
- A.B+D+G+K.E.H.L
- A+B+D+G+K.C.E.H.L
- A+E.B.F.H.L
- A+E+B.C+D+G+K.F.H.L
- A.B+H.D.I.L
- A+B+H.C.D.I.L
- A.B+H+D.E+G+K.I.L
- A+B+H+D.C.E+G+K.I.L
- A+E+G+K.B+H.F.I.L
- A+E+G+K+B+H.C+D.F.I.L
- A+I.B.D.J.L
- A+I+B.C+H.D.J.L
- A+I,B+D.E.J.L
- A+I+B+D.C+H.E.J.L
- A+I+E.B.F+G+K.J.L
- A+I+E+B.C+H+D.F+G+K.J.L
- A.B.D+L.G.M.
- A+B.C.D+L.G.M

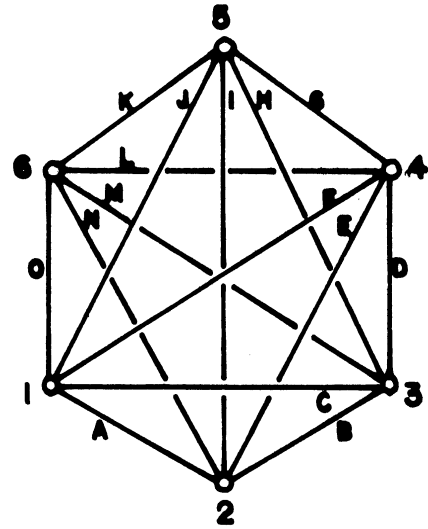


Figure 11

A.B+D+L.E.G.M
A+B+D+L.C.E.G.M
A+E.B.F.G.M
A+E+B,C+D+L.F.G.M
A.B.D+L+G.H+K.M
A+B.C.D+L+G.H+K.M
A.B+D+L+G.E.H+K.M
A+B+D+L+G.C.E.H+K.M
A+E.B.F.H+K.M
A+E+B.C+D+L+G.F.H+K.M
A.B+H+K.D+L.I.M
A+B+H+K.C.D+L.I.M
A.B+H+K+D+L.E+G.I.M
A+B+H+K+D+L.C.E+G.I.M
A+E+G.B+H+K.F.I.M
A+E+G+B+H+K.C+D+L.F.I.M
A+I.B.D+L.J.M
A+I+B.C+H+K.D+L.J.M
A+I.B+D+L.E.J.M
A+I+B+D+L.C+H+K.E.J.M
A+I+E.B.F+G.J.M
A+I+E+B.C+H+K+D+L.F+G.J.M
A.B+M.D.G.N
A+B+M.C.D.G.N
A.B+M+D.E+L.G.N
A+B+M+D.C.E+L.G.N
A+E+L.B+M.F.G.N
A+E+L+B+M.C+D.F.G.N
A.B+M.D+G.H.N
A+B+M.C.D+G.H.N
A.B+M+D+G.E+L.H.N
A+B+M+D+G.C.E+L.H.N
A+E+L.B+M.F.H.N
A+E+J+B+M.C+D+G.F.H.N
A.B+M+H.D.I+K.N
A+B+M+H.C.D.I+K.N
A.B+M+H+D.E+L+G.I+K.N
A+B+M+H+D.C.E+L+G.I+K.N
A+E+L+G.B+M+H.F.I+K.N
A+E+J+G+B+M+J.C+D.F.I+K.N
A+I+K.B+M.D.J.N
A+I+K+B+M.C+H.D.J.N
A+I+K.B+M+D.E+L.J.N
A+I+K+B+M+D.C+H.E+L.J.N
A+I+K+E+L.B+M.F+G.J.N
A+I+K+E+L+B+M.C+H+D.F+G.J.N
A+N.B.D.G.O
A+N+B.C+M.D.G.O
A+N.B+D.E.G.O
A+N+B+D.C+M.E.G.O
A+N+E.B.F+L.G.O
A+N+E+B.C+M+D.F+L.G.O
A+N.B.D+G.H.O

A+N+B.C+M.D+G.H.O
 A+N.B+D+G.E.H.O
 A+N+B+D+G.C+M.E.H.O
 A+N+E.B.F+L.H.O
 A+N+E+B.C+M+D+G.F+L.H.O
 A+N.B+H.D.I.O
 A+N+B+H.C+M.D.I.O
 A+N.B+H+D.E+G.I.O
 A+N+B+H+D.C+M.E+G.I.O
 A+N+E+G.B+H.F+L.I.O
 A+N+E+G+B+H.C+M+D.F+L.I.O
 A+N+I.B.D.J+K.O
 A+N+I+B.C+M+H.D.J+K.O
 A+N+I.B+D.E.J+K.O
 A+N+I+B+D.C+M+H.E.J+K.O
 A+N+I+E.B.F+L+G.J+K.O
 A+N+I+E+B.C+M+H+D.F+L+G.J+K.O

E. The number of nodes $n = 7$, Figure 12

A.B.D.G.K.P
 A+B.C.D.G.K.P
 A.B+D.E.G.K.P
 A+B+D.C.E.G.K.P
 A+E.B.F.G.K.P
 A+E+B.C+D.F.G.K.P
 A.B.D+G.H.K.P
 A+B.C.D+G.H.K.P
 A.B+D+G.E.H.K.P
 A+B+D+G.C.E.H.K.P
 A+E.B.F.H.K.P
 A+E+B.C+D+G.F.H.K.P
 A.B+H.D.I.K.P
 A+B+H.C.D.I.K.P
 A.B+H+D.E+G.I.K.P
 A+B+H+D.C.E+G.I.K.P
 A+E+G.B+H.F.I.K.P
 A+E+G+B+H.C+D.F.I.K.P
 A+I.B.D.J.K.P
 A+I+B.C+H.D.J.K.P
 A+I,B+D.E.J.K.P
 A+I+B+D.C+H.E.J.K.P
 A+I+E.B.F+G.J.K.P
 A+I+E+B.C+H+D.F+G.J.K.P
 A.B.D.G+K.L.P
 A+B.C.D.G+K.L.P
 A.B+D.E.G+K.L.P

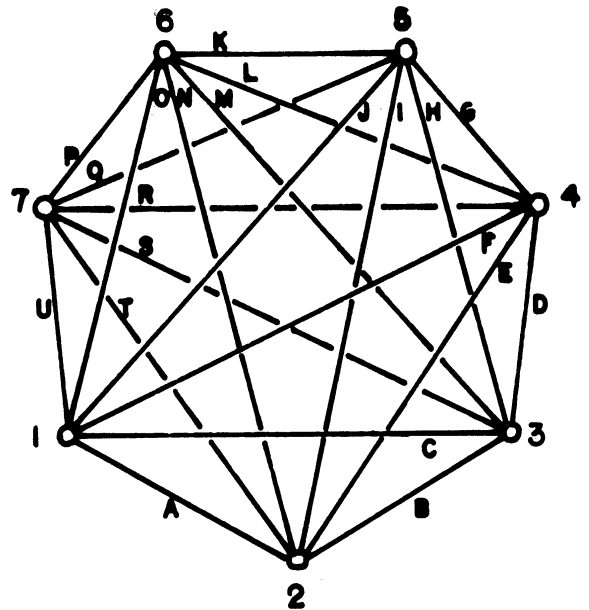


Figure 12

A+B+D.C.E.G+K.L.P
A+E.B.F.G+K.L.P
A+E+B.C+D.F.G+K.L.P
A.B.D+G+K.H.L.P
A+B.C.D+G+K.H.L.P
A.B+D+G+K.E.H.L.P
A+B+D+G+K.C.E.H.L.P
A+E.B.F.H.L.P
A+E+B.C+D+G+K.F.H.L.P
A.B+H.D.I.L.P
A+B+H.C.D.I.L.P
A.B+H+D.E+G+K.I.L.P
A+B+H+D.C.E+G+K.I.L.P
A+E+G+K.B+H.F.I.L.P
A+E+G+K+B+H.C+D.F.I.L.P
A+I.B.D.J.L.P
A+I+B.C+H.D.J.L.P
A+I.B+D.E.J.L.P
A+I+B+D.C+H.E.J.L.P
A+I+E.B.F+G+K.J.L.P
A+I+E+B.C+H+D.F+G+K.J.L.P
A.B.D+L.G.M.P
A+B.C.D+L.G.M.P
A.B+D+L.E.G.M.P
A+B+D+L.C.E.G.M.P
A+E.B.F.G.M.P
A+E+B.C+D+L.F.G.M.P
A.B.D+L+G.H+K.M.P
A+B.C.D+L+G.H+K.M.P
A.B+D+L+G.E.H+K.M.P
A+B+D+L+G.C.E.H+K.M.P
A+E.B.F.H+K.M.P
A+E+B.C+D+L+G.F.H+K.M.P
A.B+H+K.D+L.I.M.P
A+B+H+K.C.D+L.I.M.P
A.B+H+K+D+L.E+G.I.M.P
A+B+H+K+D+L.C.E+G.I.M.P
A+E+G.B+H+K.F.I.M.P
A+E+G+B+H+K.C+D+L.F.I.M.P
A+I.B.D+L.J.M.P
A+I+B.C+H+K.D+L.J.M.P
A+I.B+D+L.E.J.M.P
A+I+B+D+L.C+H+K.E.J.M.P
A+I+E.B.F+G.J.M.P
A+I+E+B.C+H+K+D+L.F+G.J.M.P
A.B+M.D.G.N.P
A+B+M.C.D.G.N.P
A.B+M+D.E+L.G.N.P
A+B+M+D.C.E+L.G.N.P
A+E+L.B+M.F.G.N.P
A+E+L+B+M.C+D.F.G.N.P
A.B+M.D+G.H.N.P
A+B+M.C.D+G.H.N.P

A.B+M+D+G.E+L.H.N.P
A+B+M+D+G.C+E+L.H.N.P
A+E+L.B+M.F.H.N.P
A+E+L+B+M.C+D+G.F.H.N.P
A.B+M+H.D.I+K.N.P
A+B+M+H.C.D.I+K.N.P
A.B+M+H+D.E+L+G.I+K.N.P
A+B+M+H+D.C.E+L+G.I+K.N.P
A+E+L+G.B+M+H.F.I+K.N.P
A+E+L+G+B+M+H.C+D.F.I+K.N.P
A+I+K.B+M.D.J.N.P
A+I+K+B+M.C+H.D.J.N.P
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A+I+K+B+M+D.C+H.E+L.J.N.P
A+I+K+E+L.B+M.F+G.J.N.P
A+I+K+E+L+B+M.C+H+D.F+G.J.N.P
A+N.B.D.G.O.P
A+N+B.C+M.D.G.O.P
A+N.B+D.E.G.O.P
A+N+B+D.C+M.E.G.O.P
A+N+E.B.F+L.G.O.P
A+N+E+B.C+M+D.F+L.G.O.P
A+N.B.D+G.H.O.P
A+N+B.C+M.D+G.H.O.P
A+N.B+D+G.E.H.O.P
A+N+B+D+G.C+M.E.H.O.P
A+N+E.B.F+L.H.O.P
A+N+E+B.C+M+D+G.F+L.H.O.P
A+N.B+H.D.I.O.P
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A+N.B+H+D.E+G.I.O.P
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A+N+E+G.B+H.F+L.I.O.P
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A+N+I.B.D.J+K.O.P
A+N+I+B.C+M+H.D.J+K.O.P
A+N+I.B+D.E.J+K.O.P
A+N+I+B+D.C+M+H.E.J+K.O.P
A+N+I+E.B.F+L+G.J+K.O.P
A+N+I+E+B.C+M+H+D.F+L+G.J+K.O.P
A.B.D.G.K+P.Q
A+B.C.D.G.K+P.Q
A.B+D.E.G.K+P.Q
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A+E.B.F.G.K+P.Q
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A.B.D+G.H.K+P.Q
A+B.C.D+G.H.K+P.Q
A.B+D+G.E.H.K+P.Q
A+B+D+G.C.E.H.K+P.Q
A+E.B.F.K+P.Q
A+E+B.C+D+G.F.K+P.Q

A.B+H.D.I.K+P.Q
A+B+H.C.D.I.K+P.Q
A.B+H+D.E+G.I.K+P.Q
A+B+H+D.C.E+G.I.K+P.Q
A+E+G.B+H.F.I.K+P.Q
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A.B.D.G+K+P.L.Q
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A.B+D.E.G+K+P.L.Q
A+B+D.C.E.G+K+P.L.Q
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A+E+B.C+D.F.G+K+P.L.Q
A.B.D+G+K+P.H.L.Q
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A+B+M.C.D.G.N.Q
A.B+M+D.E+L.G.N.Q
A+B+M+D.C.E+L.G.N.Q
A+E+L.B+M.F.G.N.Q
A+E+L+B+M.C+D.F.G.N.Q
A.B+M.D+G.H.N.Q
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A.B+M+D+G.E+L.H.N.Q
A+B+M+D+G.C.E+L.H.N.Q
A+E+L.B+M.F.H.N.Q
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A.B+M+H+D.E+L+G.I+K+P.N.Q
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A+N+I.B.D.J+K+P.O.Q
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A+N+I.B+D.E.J+K+P.O.Q
A+N+I+B+D.C+M+H.E.J+K+P.O.Q
A+N+I+E.B.F+L+G.J+K+P.O.Q

A+N+I+E+B.C+M+H+D.F+L+G.J+K+P.O.Q
A.B.D.G+Q.K.R
A+B.C.D.G+Q.K.R
A.B+D.E.G+Q.K.R
A+B+D.C.E.G+Q.K.R
A+E.B.F.G+Q.K.R
A+E+B.C+D.F.G+Q.K.R
A.B.D+G+Q.H.K.R
A+B.C.D+G+Q.H.K.R
A.B+D+G+Q.E.H.K.R
A+B+D+G+Q.C.E.H.K.R
A+E.B.F.H.K.R
A+E+B.C+D+G+Q.F.H.K.R
A.B+H.D.I.K.R
A+B+H.C.D.I.K.R
A.B+H+D.E+G+Q.I.K.R
A+B+H+D.C.E+G+Q.I.K.R
A+E+G+Q.B+H.F.I.K.R
A+E+G+Q+B+H.C+D.F.I.K.R
A+I.B.D.J.K.R
A+I+B.C+H.D.J.K.R
A+I.B+D.E.J.K.R
A+I+B+D.C+H.E.J.K.R
A+I+E.B.F+G+Q.J.K.R
A+I+E+B.C+H+D.F+G+Q.J.K.R
A.B.D.G+Q+K.L+P.R
A+B.C.D.G+Q+K.L+P.R
A.B+D.E.G+Q+K.L+P.R
A+B+D.C.E.G+Q+K.L+P.R
A+E.B.F.G+Q+K.L+P.R
A+E+B.C+D.F.G+Q+K.L+P.R
A.B.D+G+Q+K.H.L+P.R
A+B.C.D+G+Q+K.H.L+P.R
A.B+D+G+Q+K.E.H.L+P.R
A+B+D+G+Q+K.C.E.H.L+P.R
A+E.B.F.H.L+P.R
A+E+B.C+D+G+Q+K.F.H.L+P.R
A.B+H.D.I.L+P.R
A+B+H.C.D.I.L+P.R
A.B+H+D.E+G+Q+K.I.L+P.R
A+B+H+D.C.E+G+Q+K.I.L+P.R
A+E+G+Q+K.B+H.F.I.L+P.R
A+E+G+Q+K+B+H.C+D.F.I.L+P.R
A+I.B.D.J.L+P.R
A+I+B.C+H.D.J.L+P.R
A+I.B+D.E.J.L+P.R
A+I+B+D.C+H.E.J.L+P.R
A+I+E.B.F+G+Q+K.J.L+P.R
A+I+E+B.C+H+D.F+G+Q+K.J.L+P.R
A.B.D+L+P.G+Q.M.R
A+B.C.D+L+P.G+Q.M.R
A.B+D+L+P.E.G+Q.M.R
A+B+D+L+P.C.E.G+Q.M.R
A+E.B.F.G+Q.M.R
A+E+B.C+D+L+P.F.F+Q.M.R

A.B.D+L+P+G+Q.H+K.M.R
A+B.C.D+L+P+G+Q.H+K.M.R
A.B+D+L+P+G+Q.E.H+K.M.R
A+B+D+L+P+G+Q.C.E.H+K.M.R
A+E.B.F.H+K.M.R
A+E+B.C+D+L+P+G+Q.F.H+K.M.R
A.B+H+K.D+L+P.I.M.R
A+B+H+K.C.D+L+P.I.M.R
A.B+H+K+D+L+P.E+G+Q.I.M.R
A+B+H+K+D+L+P.C.E+G+Q.I.M.R
A+E+G+Q.B+H+K.F.I.M.R
A+E+G+Q+B+H+K.C+D+L+P.F.I.M.R
A+I.B.D+L+P.J.M.R
A+I+B.C+H+K.D+L+P.J.M.R
A+I.B+D+L+P.E.J.M.R
A+I+B+D+L+P.C+H+K.E.J.M.R
A+I+E.B.F+G+Q.J.M.R
A+I+E+B.C+H+K+D+L+P.F+G+Q.J.M.R
A.B+M.D.G+Q.N.R
A+B+M.C.D.G+Q.N.R
A.B+M+D.E+L+P.G+Q.N.R
A+B+M+D.C.E+L+P.G+Q.N.R
A+E+L+P.B+M.F.G+Q.N.R
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A+B+M.C.D+G+Q.H.N.R
A.B+M+D+G+Q.E+L+P.H.N.R
A+B+M+D+G+Q.C.E+L+P.H.N.R
A+E+L+P.B+M.F.H.N.R
A+E+L+P+B+M.C+D+G+Q.F.H.N.R
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A+I+K+B+M.C+H.D.J.N.R
A+I+K.B+M+D.E+L+P.J.N.R
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A+N+B.C+M.D.G+Q.O.R
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A+T+N+I+B+D.C+S+M+H.E.J+Q+K.O+P.U
A+T+N+I+E.B.F+R+L+G.J+Q+K.O+P.U
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VII. CONCLUSION

With the introduction of the new algorithm of a "Foldant", characteristic numbers concerning the tree are found to be obtained in a more economical and simple way than with a conventional matrix or determinant or a "primitive-node-pair connection matrix". The algorithm is algebraic as contrasted with the topological method dealing with the tree. It is so close to the fundamental idea of the tree that future investigation concerning its relationship with the other characteristic matrices of the tree derived thus far can be expected to systematize a customary matrix theory of a linear graph from a new point of view.

The direct application of the algorithm is for the computation of the driving point admittance as it was discussed substantially in the thesis. If the computation of the transfer admittance by means of the foldant is as simple as the one of the driving point admittance, the algorithm would certainly compete with these prevailing methods of matrix or determinant.

By incorporating Kron's method, the computation of the numerical values of the driving point admittance by the algorithm might be as efficient as by the matrix method, and the programming of the numerical computation can be developed directly from the one of the thesis.

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