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A GENERAL THEORY OF MINIMUM-FUEL SPACE TRAJECTORIES

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by
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Introduction: This paper is concerned with the trajectories of vehicles moving in free space, i.e., of vehicles that are subject only to gravitational and propulsive forces. The following problem is fundamental in the control of such trajectories: Given the vehicle position, velocity, and mass at a specified initial time, find a propulsion program that brings the vehicle to a prescribed terminal state (in a terminal time which may be free or fixed) with a minimum expenditure of fuel. Such a program will be called optimal.

The mathematical treatment of this problem depends very strongly on the model used for the fuel expenditure. In the case of a rocket engine, an excellent approximation is that the rate of fuel consumption is proportional to the magnitude of the thrust vector, and this article will deal exclusively with this representation.

We shall assume throughout that no constraints are imposed on the vehicle position and velocity. This assumption is physically reasonable in many (but by no means all) space problems. Further, except for a brief discussion in section 9, we shall always suppose that there is no constraint on the allowed value of the thrust vector. Minimum-fuel thrust programs in the absence of any such constraints generally consist of a finite number of impulses. Although impulsive corrections can never be realized by an actual rocket engine, a knowledge of the optimum impulses will often make it possible to compute the optimum, or near optimum, thrust program in the presence of the thrust amplitude limits which must exist in actual engines.

The problem described above is clearly a variational one. In order to permit impulses, and yet have a precise mathematical formulation, it is necessary to place the problem in a somewhat unorthodox framework, and thereby arrive at a non-classical variational problem. This development is carried out in section 2. In section 8 we show that this framework is a reasonable one by proving both an existence theorem for solutions of the resultant variational problem and an approximation theorem which states that solutions of the unorthodox variational problem can be approximated by conventional thrust programs to any desired degree of accuracy.

In sections 3-6, necessary conditions that an optimum thrust program and associated trajectory must satisfy are derived. Many of these conditions have been previously obtained by examining the necessary conditions in the presence of a thrust amplitude constraint, and then passing to the limit formally as the maximum allowed amplitude tends to ∞ (see, e.g., Lawden [1]). In section 9 we show that this limiting argument is, in a sense, justified, and also prove an existence theorem for optimum trajectories in the presence of thrust amplitude constraints.

Some specific examples of contemporary interest are discussed in section 7.

Ewing [2] in 1961 adopted a viewpoint very similar to the one taken in this paper in his investigation of the same problem for the particular case where the gravitational field in which the vehicle moves is uniform.

The case where the gravitational field is linear in the space coordinates has been previously treated by the author [3]. While preparing this manuscript, it has come to the author's attention that the problem discussed in this paper has recently been studied, but from a slightly different viewpoint, also by Rishel [4].

2. <u>Problem formulation:</u> The motion of a vehicle that is subject only to gravitational and propulsive forces can be described by the following differential equations:

$$\ddot{r}_i = G_i(r_1,r_2,r_3,t) + \frac{T_i(t)}{M(t)}, \quad i = 1,2,3,$$

where r_1, r_2, and r_3 are the coordinates of the vehicle center of gravity in some inertial, Cartesian coordinate system; G_1, G_2, and G_3 are the components of the vehicle acceleration due to the action of the gravitational force field; T_1, T_2, and T_3 are the components of the vehicle's thrust vector; and M is the vehicle mass. Denoting the vectors (r_1,r_2,r_3), (G_1, G_2,G_3), and (T_1,T_2,T_3) by $\underset{\sim}{r}$, $\underset{\sim}{G}$, and $\underset{\sim}{T}$, respectively, we may write down the single vector differential equation

$$\ddot{\underset{\sim}{r}} = \underset{\sim}{G}(\underset{\sim}{r},t) + \frac{\underset{\sim}{T}(t)}{M(t)}. \tag{2.1}$$

The rate of change of mass, $\dot{M}$, is the negative of the fuel expenditure rate and, for a single rocket engine, is given by

$$\dot{M} = -\frac{\|\underset{\sim}{T}\|}{g\, I_{sp}}, \tag{2.2}$$

where $\|\ \|$ denoted the Euclidean norm, and g and I_{sp} (the nominal acceleration due to gravity at the earth's surface and the specific impulse of the fuel, respectively) will be assumed to be known positive constants. We denote $(g\ I_{sp})$ by A.

We shall also suppose that $\underset{\sim}{G}(\cdot,\cdot)$ is a continuous, bounded function from $E_3 \times E_1$ to E_3 (E_m denotes Euclidean m-space) possessing continuous bounded first partial derivatives with respect to all of its arguments. This assumption is consistent with the conventional models of gravitational fields.

Throughout this paper we shall assume that an initial time t_o (without loss of generality, and for ease of notation, we shall set $t_o = 0$) and initial values for eqs. (2.1) and (2.2) have been given: $M(0) = M_o > 0$, $\underset{\sim}{r}(0) = \underset{\sim}{r}_o$, $\dot{\underset{\sim}{r}}(0) = \underset{\sim}{v}_o$. If $\underset{\sim}{F}(\cdot)$ is a summable function from $[0,\infty)$ to E_3 satisfying the inequality $\int_0^\infty \|\underset{\sim}{F}(t)\|dt < AM_o$, then it follows from standard existence theorems that eqs. (2.1) and (2.2) with $\underset{\sim}{T}(\cdot) = \underset{\sim}{F}(\cdot)$ have a unique solution[1] for $0 \le t < \infty$ that satisfies the above initial conditions. This solution will be denoted by $\underset{\sim}{r}(t;\underset{\sim}{F})$, $M(t;\underset{\sim}{F})$; $\dot{\underset{\sim}{r}}(t;\underset{\sim}{F})$ denotes the time derivative of $\underset{\sim}{r}(t;\underset{\sim}{F})$.

Finally, we shall suppose that there are given functions $h_i(\cdot,\cdot,\cdot)$ from $E_3 \times E_3 \times [0,\infty)$ to E_1, where $i = 1,\ldots\nu$ and $\nu \le 6$, with the following two properties: 1. The h_i are continuous and have continuous first partial derivatives with respect to all of their arguments. 2. If $H(t) = \{(\underset{\sim}{x},\underset{\sim}{y}):\underset{\sim}{x}\in E_3,\ \underset{\sim}{y}\in E_3,\ h_i(\underset{\sim}{x},\underset{\sim}{y},t) = 0 \text{ for } i = 1,\ldots,\nu\}$, then H(t) is a smooth manifold in E_6 for each $t \ge 0$. For each $\bar{t} > 0$, let $\mathcal{F}(\bar{t})$ denote the class of all summable functions $\underset{\sim}{F}(\cdot)$ from $[0,\bar{t}]$ to E_3 that satisfy the relations $\int_0^{\bar{t}} \|\underset{\sim}{F}(t)\| dt < AM_0$ and $h_i(\underset{\sim}{r}(\bar{t};\underset{\sim}{F}), \dot{\underset{\sim}{r}}(\bar{t};\underset{\sim}{F}),\bar{t}) = 0$ for $i = 1,\ldots,\nu$. Physically speaking, $\mathcal{F}(\bar{t})$ consists of all thrust programs that "transfer" the vehicle from the position $\underset{\sim}{r}_0$ and velocity $\underset{\sim}{v}_0$ at $t = 0$ to a new state (at the time $\bar{t}$) that satisfies the boundary conditions $h_i(\underset{\sim}{r},\dot{\underset{\sim}{r}},\bar{t}) = 0$, $i = 1,\ldots,\nu$.

In the sequel, we shall consider two variational problems. The first, the <u>fixed terminal time problem</u>, consists in finding, for a given $t_1 > 0$, an element $\hat{\underset{\sim}{F}}(\cdot)\in\mathcal{F}(t_1)$ such that $M(t_1;\hat{\underset{\sim}{F}}) \geq M(t_1;\underset{\sim}{F})$ for every $\underset{\sim}{F}\in\mathcal{F}(t_1)$. The second, the <u>variable terminal time problem</u>, consists in finding a time $t_1 \geq 0$ and an element $\hat{\underset{\sim}{F}}(\cdot)\in\mathcal{F}(t_1)$ such that $M(t_1;\hat{\underset{\sim}{F}}) \geq M(t;\underset{\sim}{F})$ for every pair $(t,\underset{\sim}{F})$ with $t > 0$ and $\underset{\sim}{F}\in\mathcal{F}(t)$. In concrete terms, the basic problem is to find a thrust program that, for the given initial values, achieves prescribed boundary conditions, and that, in so doing, maximizes the terminal mass.

If $\nu = 6$ and $H(t)$ consists of a single point for each t, we shall say that the variational problem is a <u>fixed endpoint problem</u>; if $\nu < 6$, the problem will be called a <u>variable endpoint problem</u>.

Let us now consider variational problems that are derived from, and equivalent to the above problems. Namely, replace eqs. (2.1) and (2.2) by the equations

$$\dot{\underset{\sim}{z}} = \underset{\sim}{G}(\underset{\sim}{\rho},t), \qquad \dot{\underset{\sim}{\rho}} = \underset{\sim}{z} + \underset{\sim}{u}(t), \tag{2.3}$$

where $\underset{\sim}{z}$, $\underset{\sim}{\rho}$, $\underset{\sim}{G}$, and $\underset{\sim}{u}$ are 3-vectors, $\underset{\sim}{G}(\cdot,\cdot)$ is the same function that appears in eq. (2.1), and $\underset{\sim}{u}(\cdot)$ is assumed to be an absolutely continuous, bounded function from $[0,\infty)$ to E_3. We shall consider solutions $\underset{\sim}{z}(\cdot)$, $\underset{\sim}{\rho}(\cdot)$ of eqs. (2.3) that satisfy the initial conditions $\underset{\sim}{z}(0) = \underset{\sim}{v}_0$ and $\underset{\sim}{\rho}(0) = \underset{\sim}{r}_0$. For a given bounded, absolutely continuous function $\underset{\sim}{w}(\cdot)$ from $[0,\infty)$ to E_3, we shall denote the solution of (2.3) with $\underset{\sim}{u}(\cdot) = \underset{\sim}{w}(\cdot)$ that satisfies the given initial conditions (it is easily seen that this solution exists and is unique for $0 \leq t < \infty$) by $\underset{\sim}{z}(t;\underset{\sim}{w})$, $\underset{\sim}{\rho}(t;\underset{\sim}{w})$. We shall also say that $\underset{\sim}{\rho}(t;\underset{\sim}{w})$ is the trajectory that corresponds to $\underset{\sim}{w}$.

For every $\bar{t} > 0$, we denote by $\mathscr{U}(\bar{t})$ the class of absolutely continuous functions $\underset{\sim}{w}(\cdot)$ from $[0,\bar{t}]$ to E_3 for which the relations $\underset{\sim}{w}(0) = 0$ and $h_i(\underset{\sim}{\rho}(\bar{t};\underset{\sim}{w}),\underset{\sim}{z}(\bar{t};\underset{\sim}{w}) + \underset{\sim}{w}(\bar{t}),\bar{t}) = 0$ for $i = 1,\ldots,\nu$ are satisfied.

Now the <u>derived</u> fixed terminal time problem consists in finding, for a given $t_1 > 0$, an element $\hat{\underset{\sim}{u}}(\cdot)\in\mathscr{U}(t_1)$ such that

$$\int_0^{t_1} \left\|\frac{d\hat{\underset{\sim}{u}}(t)}{dt}\right\| dt \leq \int_0^{t_1} \left\|\frac{d\underset{\sim}{u}(t)}{dt}\right\| dt$$

for every $\underset{\sim}{u}\in\mathscr{U}(t_1)$; the derived variable terminal time problem consists in finding a time $t_1 > 0$ and an element $\hat{\underset{\sim}{u}}(\cdot)\in\mathscr{U}(t_1)$ such that

$$\int_0^{t_1} \left\|\frac{d\hat{\underset{\sim}{u}}(t)}{dt}\right\| dt \leq \int_0^{\tau} \left\|\frac{d\underset{\sim}{u}(t)}{dt}\right\| dt$$

for every pair $(\tau,\underset{\sim}{u})$ with $\tau > 0$ and $\underset{\sim}{u}\in\mathscr{U}(\tau)$.

We shall show that the original and derived variational problems are equivalent. Namely, we shall exhibit a mapping Φ that, for each $t_1 > 0$, is one-to-one from $\mathscr{F}(t_1)$ onto $\mathscr{U}(t_1)$ (if we identify elements in $\mathscr{F}(t_1)$

that differ only on a set of measure zero), and shall prove that $\underset{\sim}{F} \in \mathscr{F}(t_1)$ is a solution of the original problem if and only if $\Phi(\underset{\sim}{F})$ is a solution of the derived problem (whether the problem is fixed or variable terminal time).

Define the mapping Φ as follows. If $\underset{\sim}{F}(\cdot) \in \mathscr{F}(t_1)$, let $\underset{\sim}{u}(\cdot) = \Phi(\underset{\sim}{F}(\cdot))$ be the absolutely continuous function from $[0,t_1]$ to E_3 that is given by

$$\underset{\sim}{u}(t) = \int_0^t \frac{\underset{\sim}{F}(s)}{M(s;\underset{\sim}{F})}\, ds, \qquad 0 \le t \le t_1. \tag{2.4}$$

We shall show that Φ is one-to-one from $\mathscr{F}(t_1)$ onto $\mathscr{U}(t_1)$, and that $\Phi^{-1} = \Psi$, where $\underset{\sim}{F}(\cdot) = \Psi(\underset{\sim}{u}(\cdot))$ for $\underset{\sim}{u} \in \mathscr{U}(t_1)$ is defined by

$$\underset{\sim}{F}(t) = \exp\{\mu(t;\underset{\sim}{u})\} \frac{d\underset{\sim}{u}(t)}{dt}, \qquad 0 \le t \le t_1, \tag{2.5}$$

$$\mu(t;\underset{\sim}{u}) = -\frac{1}{A}\int_0^t \left\|\frac{d\underset{\sim}{u}(s)}{ds}\right\| ds + \ell n\, M_0, \qquad 0 \le t \le t_1. \tag{2.6}$$

Note that $\underset{\sim}{F}(t)$ is defined by (2.5) for almost all $t \in [0,t_1]$, since an absolutely continuous function has a derivative almost everywhere. At points where $d\underset{\sim}{u}/dt$ does not exist, $\underset{\sim}{F}(t)$ may be defined arbitrarily. Since $d\underset{\sim}{u}/dt$ is summable, the integral in (2.6) is finite.

Consider eqs. (2.3) with $\underset{\sim}{u} = \Phi(\underset{\sim}{F})$, where $\underset{\sim}{F} \in \mathscr{F}(t_1)$. It is clear that $\dot{\underset{\sim}{\rho}}(\cdot;\underset{\sim}{u})$ is absolutely continuous in $[0,t_1]$, and that, a.e. in $[0,t_1]$,

$$\ddot{\underset{\sim}{\rho}}(t;\underset{\sim}{u}) = \underset{\sim}{G}(\underset{\sim}{\rho}(t;\underset{\sim}{u}),t) + \frac{\underset{\sim}{F}(t)}{M(t;\underset{\sim}{F})}. \tag{2.7}$$

Since

$$\rho(0;u) = r_0, \quad \dot{\rho}(0;u) = z(0;u) + u(0) = v_0, \tag{2.8}$$

it follows that (replacing u by $\Phi(F)$), for all $t\in[0,t_1]$,

$$\rho(t;\Phi(F)) = r(t;F), \quad z(t;\Phi(F)) + \Phi(F)(t) = \dot{r}(t;F). \tag{2.9}$$

Hence, $\Phi(F)\in\mathscr{U}(t_1)$ by definition of $\mathscr{F}(t_1)$ and $\mathscr{U}(t_1)$, or $\Phi(\mathscr{F}(t_1))\subset\mathscr{U}(t_1)$. Also (by (2.2), (2.4) and (2.6)), for all $t\in[0,t_1]$, we have that

$$M(t;F) = \exp\{\mu(t;\Phi(F))\}. \tag{2.10}$$

Let us show that $\Psi(\mathscr{U}(t_1))\subset\mathscr{F}(t_1)$. Thus, let $F = \Psi(u)$, where $u\in\mathscr{U}(t_1)$. It follows from (2.5) and (2.6) that $\exp\{\mu(\cdot;u)\}$ is absolutely continuous in $[0,t_1]$, that $\exp\{\mu(0;u)\} = M_0$ and that, a.e. in $[0,t_1]$,

$$\frac{d}{dt}\,[\exp\{\mu(t;u)\}] = -A^{-1}\|F(t)\|, \tag{2.11}$$

so that $\int_0^{t_1}\|F(t)\|dt < AM_0$, and (see (2.2)), for all $t\in[0,t_1]$,

$$\exp\{\mu(t;u)\} = M(t;\Psi(u)). \tag{2.12}$$

It is clear from (2.3) that $\dot{\rho}(t;u)$ is absolutely continuous in $[0,t_1]$. Also (see (2.3), (2.5) and (2.12)), $\rho(t;u)$ satisfies eq. (2.7) a.e. in $[0,t_1]$. By definition of $\mathscr{U}(t_1)$, $u(0) = 0$, so that relations (2.8) are satisfied. Hence, for all $t\in[0,t_1]$,

$$\rho(t;u) = r(t;\Psi(u)), \quad z(t;u) + u(t) = \dot{r}(t;\Psi(u)), \tag{2.13}$$

from which it follows that $F\in\mathscr{F}(t_1)$. Now the relation $\Phi^{-1} = \Psi$ is a consequence of (2.2), (2.4)-(2.6), and (2.12), and it only remains to show that Φ maps all of the solutions of the original variational problem onto all of the solutions of the derived problem. But it is a consequence of (2.6) and (2.12) that if $u_i \in \mathscr{U}(t_i)$, i = 1,2, then $M(t_1;\Psi(u_1)) > M(t_2;\Psi(u_2))$ if and only if $\int_0^{t_1} \|\dot{u}_1(s)\|ds < \int_0^{t_2} \|\dot{u}_2(s)\|ds$, and this immediately implies the desired result.

Note that eqs. (2.9), (2.12), and (2.13) describe the correspondence between solutions of eqs. (2.1) and (2.2) and solutions of eqs. (2.3) when u and $T = F$ correspond under the mappings Φ and Ψ.

If $u(\cdot)$ is an absolutely continuous function from $[0,t_1]$ to E_3, then

$$\int_0^{t_1} \left\|\frac{du(t)}{dt}\right\| dt \quad = \quad \text{STV } u, \tag{2.14}$$

where STV u, the <u>strong total variation</u> of u, is defined (see [5, p.59]) as follows:

$$\text{STV } u \quad = \quad \sup \sum_{i=1}^{m} \|u(\tau_i) - u(\tau_{i-1})\|,$$

with the supremum taken over all finite partitions $0 = \tau_0 < \tau_1 < \ldots < \tau_m = t_1$ of $[0,t_1]$. For scalar-valued functions (where STV reduces to the total variation), relation (2.14) is well-known. The proof of (2.14) (see, e.g., [6, p.209]) carries over from the scalar-valued to the vector-valued case with only minor modification.

Thus, the original fixed terminal time problem is equivalent to the problem of finding, for a given $t_1 > 0$, an element $\hat{u} \in \mathscr{U}(t_1)$ such that

$$\mathrm{STV}\,\hat{u} = \inf_{u \in \mathscr{U}(t_1)} \mathrm{STV}\,u; \tag{2.15}$$

and the variable terminal time problem is equivalent to that of finding a number $t_1 > 0$ and an element $\hat{u} \in \mathscr{U}(t_1)$ such that

$$\mathrm{STV}_{[0,t_1]}\hat{u} = \inf_{\substack{u \in \mathscr{U}(t) \\ t>0}} \mathrm{STV}_{[0,t]}u, \tag{2.16}$$

where $\mathrm{STV}_{[0,t]}$ denotes the strong total variation over the interval $[0,t]$.

Unfortunately, there is, in general, no element $\hat{u} \in \mathscr{U}(t_1)$ that achieves the infimum in the right-hand side of (2.15) or of (2.16). To circumvent this difficulty, we shall embed the sets $\mathscr{U}(t)$ in larger sets $\mathscr{H}(t)$ possessing the following two properties: 1. If $u(\cdot)$ is any element of $\mathscr{H}(t)$, then there exist functions $u_n(\cdot) \in \mathscr{U}(t)$, $n = 1,2,\ldots$, such that $u_n(s) \to u(s)$ as $n \to \infty$ for all $s \in [0,t]$, and $\mathrm{STV}\,u_n \to \mathrm{STV}\,u$ as $n \to \infty$ (see theorem 4 in section 8). 2. There is an element $\hat{u} \in \mathscr{H}(t)$ such that $\mathrm{STV}_{[0,t]}\hat{u} = \inf_{u \in \mathscr{H}(t)} \mathrm{STV}_{[0,t]}u$ (see theorem 3 in section 8). Consequently, $\inf_{u \in \mathscr{U}(t)} \mathrm{STV}_{[0,t]}u = \inf_{u \in \mathscr{H}(t)} \mathrm{STV}_{[0,t]}u$.

For each $\bar{t}$, $0 < \bar{t} < \infty$, we define $\mathscr{H}(\bar{t})$ as follows. Let $\mathscr{B}(\bar{t}) = \{u(\cdot): u$ from $[0,\bar{t}]$ to E_3 and continuous from the right in $(0,\bar{t})$, $u(0) = 0$, $\mathrm{STV}_{[0,\bar{t}]}u < \infty\}$. For every $w \in \mathscr{B}(\bar{t})$, eqs. (2.3) with $u = w$ have a unique solution[2] in $[0,\bar{t}]$ that satisfies the initial conditions $z(0) = v_0$, $\rho(0) = r_0$. We shall also denote this solution by $z(t;w)$, $\rho(t;w)$. Then, for each $\bar{t} > 0$, let

$$\mathscr{H}(\bar{t}) = \{w(\cdot): w \in \mathscr{B}(\bar{t}), h_i(\rho(\bar{t};w), z(\bar{t};w) + w(\bar{t}), \bar{t}) = 0 \text{ for } i = 1,\ldots,\nu\}. \tag{2.17}$$

It is obvious that $\mathscr{U}(\bar{t}) \subset \mathscr{H}(\bar{t}) \subset \mathscr{B}(\bar{t})$.

We shall denote by $\mathcal{B}(\infty)$ the set of all functions from $[0,\infty)$ to E_3 whose restrictions on $[0,\bar{t}]$, for every $\bar{t} > 0$, belong to $\mathcal{B}(\bar{t})$.

We shall henceforth be concerned with the extended variational problems defined as follows.

<u>The extended variable terminal time problem consists in finding a number</u> t_1, $0 < t_1 < \infty$, <u>and an element</u> $\hat{\underset{\sim}{u}} \in \mathcal{H}(t_1)$ <u>such that</u>

$$STV_{[0,t_1]}\hat{\underset{\sim}{u}} = \inf_{\substack{\underset{\sim}{u}\in\mathcal{H}(t) \\ t>0}} STV_{[0,t]}\underset{\sim}{u}.$$

The extended fixed terminal time problem is analogously defined.

Sections 3-6 are devoted to the derivation of necessary conditions that solutions of the extended variational problems must satisfy. In section 5, we consider the variable terminal time problem, and in section 6, the fixed terminal time problem.

3. <u>Variational equations:</u> In this section, t_1 is an arbitrary fixed positive number and $\hat{\underset{\sim}{u}}(\cdot)$ is an arbitrary fixed element of $\mathcal{B}(t_1)$. Denote $\underset{\sim}{\rho}(t;\hat{\underset{\sim}{u}})$ and $\underset{\sim}{z}(t;\hat{\underset{\sim}{u}})$, for $0 \leq t \leq t_1$, by $\hat{\underset{\sim}{\rho}}(t)$ and $\hat{\underset{\sim}{z}}(t)$, respectively. Let $\Lambda(\cdot)$ denote the continuous matrix-valued function on $[0,t_1]$ whose i,j-th element Λ_{ij} is given by

$$\Lambda_{ij}(t) = \frac{\partial G_i(\hat{\underset{\sim}{\rho}}(t),t)}{\partial r_j}; \quad i,j = 1,2,3; \quad 0 \leq t \leq t_1.$$

We shall also use the notation

$$\Lambda(t) = \frac{\partial \underset{\sim}{G}(\hat{\underset{\sim}{\rho}}(t),t)}{\partial \underset{\sim}{r}}, \quad 0 \leq t \leq t_1. \tag{3.1}$$

For every function $\underset{\sim}{u}(\cdot) \in \mathcal{B}(t_1)$, let $\delta \underset{\sim}{z}(\cdot;\underset{\sim}{u})$ and $\delta \underset{\sim}{x}(\cdot;\underset{\sim}{u})$ denote the absolutely continuous functions from $[0,t_1]$ to E_3 that satisfy the equations

$$\begin{aligned} \frac{d}{dt}[\delta \underset{\sim}{z}(t;\underset{\sim}{u})] &= \Lambda(t)\delta \underset{\sim}{\rho}(t;\underset{\sim}{u}), \\ \frac{d}{dt}[\delta \underset{\sim}{\rho}(t;\underset{\sim}{u})] &= \delta \underset{\sim}{z}(t;\underset{\sim}{u}) + \underset{\sim}{u}(t) - \hat{\underset{\sim}{u}}(t) \end{aligned} \tag{3.2}$$

almost everywhere in $[0,t_1]$, and assume the initial values

$$\delta \underset{\sim}{z}(0;\underset{\sim}{u}) = \delta \underset{\sim}{\rho}(0;\underset{\sim}{u}) = 0. \tag{3.3}$$

We shall refer to eqs. (3.2) as the variational equations associated with $\hat{\underset{\sim}{u}}(t)$. Since these equations are linear, their solution is given by the well-known variations of parameters formula, which here takes the form

$$\begin{aligned} \delta \underset{\sim}{z}(t;\underset{\sim}{u}) &= -\int_0^t [\dot{X}_1(t)\dot{Y}_1(s)+\dot{X}_2(t)\dot{Y}_2(s)][\underset{\sim}{u}(s)-\hat{\underset{\sim}{u}}(s)]ds, \\ \delta \underset{\sim}{\rho}(t;\underset{\sim}{u}) &= -\int_0^t [X_1(t)\dot{Y}_1(s)+X_2(t)\dot{Y}_2(s)][\underset{\sim}{u}(s)-\hat{\underset{\sim}{u}}(s)]ds, \end{aligned} \tag{3.4}$$

where the 3×3 matrices $X_i(t)$ and $Y_i(t)$ $(i = 1 \text{ or } 2)$ satisfy the differential equations

$$\ddot{X}_i(t) = \Lambda(t)X_i(t), \quad \ddot{Y}_i(t) = Y_i(t)\Lambda(t), \quad 0 \le t \le t_1, \quad i = 1 \text{ or } 2, \tag{3.5}$$

and the initial conditions

$$X_1(0) = \dot{X}_2(0) = -\dot{Y}_1(0) = Y_2(0) = I, \quad \dot{X}_1(0) = X_2(0) = Y_1(0) = \dot{Y}_2(0) = 0, \tag{3.6}$$

I being the identity matrix. The matrices X_i, Y_i also satisfy the following identity:

$$\begin{pmatrix} X_1(t) & X_2(t) \\ \dot{X}_1(t) & \dot{X}_2(t) \end{pmatrix} \begin{pmatrix} -\dot{Y}_1(t) & Y_1(t) \\ -\dot{Y}_2(t) & Y_2(t) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad 0 \leq t \leq t_1. \tag{3.7}$$

In conventional physical models of gravitational fields, the function $\underset{\sim}{G}(\cdot,t)$, for every fixed t, is the gradient of a twice continuously differentiable scalar-valued function on E_3. Under this hypothesis, $\Lambda(t)$ is symmetric for every t, $0 \leq t \leq t_1$, in which case (3.5) and (3.6) imply that $Y_1(t) = -X_2^T(t)$ and $Y_2(t) = X_1^T(t)$ for $0 \leq t \leq t_1$. This computationally useful result, which is known as Schmidt's theorem, but is apparently originally due to Siegel [7, page 14], was brought to my attention by O. K. Smith.

Integrating eqs. (3.4) by parts, using (3.7) and the fact that $\underset{\sim}{u}(0) = \underset{\sim}{\hat{u}}(0) = 0$, we obtain

$$\begin{aligned} \delta\underset{\sim}{z}(t;\underset{\sim}{u}) &= \int_0^t [\dot{X}_1(t)Y_1(s)+\dot{X}_2(t)Y_2(s)]d[\underset{\sim}{u}(s)-\underset{\sim}{\hat{u}}(s)] - \underset{\sim}{u}(t) + \underset{\sim}{\hat{u}}(t), \\ \delta\underset{\sim}{\rho}(t;\underset{\sim}{u}) &= \int_0^t [X_1(t)Y_1(s)+X_2(t)Y_2(s)]d[\underset{\sim}{u}(s)-\underset{\sim}{\hat{u}}(s)], \end{aligned} \tag{3.8}$$

the integrals in (3.8) being in the sense of Stieljes.

For every $\underset{\sim}{u}(\cdot)\in\mathcal{B}(t_1)$ and real number α, let $\delta\underset{\sim}{x}(\underset{\sim}{u},\alpha)$ be the element in $E_3 \times E_3 \times E_1$ given by

$$\underset{\sim}{\delta x}(\underset{\sim}{u},\alpha) = (\delta\underset{\sim}{\rho}(t_1;\underset{\sim}{u}),\delta\underset{\sim}{z}(t_1;\underset{\sim}{u})+\underset{\sim}{u}(t_1)-\hat{\underset{\sim}{u}}(t_1),STV_{[0,t_1]}\underset{\sim}{u}+\alpha), \tag{3.9}$$

and let

$$\underset{\sim}{\omega}_0 = \underset{\sim}{\delta x}(\hat{\underset{\sim}{u}},0) = (0,0,STV\,\hat{\underset{\sim}{u}}). \tag{3.10}$$

Now define the set W in $E_3 \times E_3 \times E_1$ as follows:

$$W = \{\underset{\sim}{\delta x}(\underset{\sim}{u},\alpha):\underset{\sim}{u}\in\mathcal{B}(t_1),\alpha \geq 0\}. \tag{3.11}$$

Clearly, $\underset{\sim}{\omega}_0 \in W$.

Since, for every $\underset{\sim}{u}$ and $\underset{\sim}{w}$ in $\mathcal{B}(t_1)$ and real number β, we have

$$STV(\underset{\sim}{u}+\underset{\sim}{w}) \leq STV\,\underset{\sim}{u} + STV\,\underset{\sim}{w}, \quad STV(\beta\underset{\sim}{u}) = |\beta|STV\,\underset{\sim}{u}, \tag{3.12}$$

it follows at once that W is convex.

The set W is analogous to the cone of attainability described in [8, Chapter 2], and is also patterned closely after the convex set of variations introduced by Warga in [9, section III].

Let $\underset{\sim}{\eta}$ be an arbitrary nonzero row vector in E_7. If $\underset{\sim}{\eta} = (\underset{\sim}{\eta}_1,\underset{\sim}{\eta}_2,\eta_7)$, where $\underset{\sim}{\eta}_1\in E_3,\underset{\sim}{\eta}_2\in E_3$, and $\eta_7\in E_1$, let $\underset{\sim}{p}(t;\underset{\sim}{\eta})$, for $0 \leq t \leq t_1$, be the row vector defined by

$$\underset{\sim}{p}(t;\underset{\sim}{\eta}) = \sum_{i=1}^{2}\underset{\sim}{\zeta}_i(\underset{\sim}{\eta})Y_i(t), \quad \underset{\sim}{\zeta}_i(\underset{\sim}{\eta}) = \underset{\sim}{\eta}_1 X_i(t_1) + \underset{\sim}{\eta}_2\dot{X}_i(t_1), \quad i = 1 \text{ or } 2. \tag{3.13}$$

It follows at once from (3.7) and (3.13) that

$$\underset{\sim}{p}(t_1;\underset{\sim}{\eta}) = \underset{\sim}{\eta}_2, \quad \dot{\underset{\sim}{p}}(t_1;\underset{\sim}{\eta}) = -\underset{\sim}{\eta}_1\,. \tag{3.14}$$

If we consider $p(\cdot;\eta)$ to be a function from $[0,t_1]$ to E_3 (for η fixed), we conclude, by virtue of (3.5) and (3.13), that $p(\cdot;\eta)$ is twice continuously differentiable and that

$$\ddot{p}(t;\eta) = p(t;\eta)\Lambda(t), \quad 0 \le t \le t_1. \tag{3.15}$$

Lemma 1. <u>If there is a nonzero vector</u> $\bar{\eta} = (\bar{\eta}_1,\ldots,\bar{\eta}_7)\in E_7$ <u>such that</u> $\bar{\eta}\cdot\delta x \le \bar{\eta}\cdot\omega_0$ <u>for all</u> $\delta x\in W$, <u>then</u> $\bar{\eta}_7 < 0$.

Proof. The hypothesis of the lemma, together with the definitions of W, ω_0, and $p(t;\bar{\eta})$ (see (3.8)-(3.11) and (3.13)) imply that, for every $u\in\mathcal{B}(t_1)$,

$$\int_0^{t_1} p(t;\bar{\eta})du(t) + \bar{\eta}_7 \text{ STV } u \le \int_0^{t_1} p(t;\bar{\eta})d\hat{u}(t) + \bar{\eta}_7 \text{ STV } \hat{u}. \tag{3.16}$$

We first show that $\bar{\eta}_7 \neq 0$. Suppose the contrary. Then, (3.16) takes the form

$$\int_0^{t_1} p(t;\bar{\eta})du \le \int_0^{t_1} p(t;\bar{\eta})d\hat{u}. \tag{3.17}$$

Since (3.17) must hold for every $u\in\mathcal{B}(t_1)$, it follows that $p(t;\bar{\eta}) \equiv 0$ in $[0,t_1]$. Hence, $\dot{p}(t;\bar{\eta}) \equiv 0$. In particular, $p(0;\bar{\eta}) = \dot{p}(0;\bar{\eta}) = 0$, i.e., (see (3.6) and (3.13)) $\zeta_1(\bar{\eta}) = \zeta_2(\bar{\eta}) = 0$. But this implies that (see (3.7) and (3.13)) $(\bar{\eta}_1,\ldots,\bar{\eta}_6) = 0$, i.e., $\bar{\eta} = 0$, and this contradiction shows that $\bar{\eta}_7 \neq 0$.

By hypothesis, $\bar{\eta}\cdot\delta x(\hat{u},1) \le \bar{\eta}\cdot\omega_0$, so that (see (3.9)-(3.11))$\bar{\eta}_7 \le 0$. Since $\bar{\eta}_7 \neq 0$, $\bar{\eta}_7 < 0$.

Lemma 2. <u>Suppose that $\hat{u}(\cdot) \not\equiv 0$. If there is a vector $\bar{\eta} = (\bar{\eta}_1, \ldots, \bar{\eta}_7) \in E_7$ with $\bar{\eta}_7 = -1$ such that $\bar{\eta} \cdot \delta x \leq \bar{\eta} \cdot \omega_0$ for all $\delta x \in W$, then</u>

$$\max_{0 \leq t \leq t_1} \|p(t;\bar{\eta})\| = 1, \tag{3.18}$$

<u>and</u>

$$\int_0^{t_1} p(t;\bar{\eta}) d\hat{u}(t) = \mathrm{STV}_{[0,t_1]} \hat{u}. \tag{3.19}$$

Proof. Let $\Omega = \max_{0 \leq t \leq t_1} \|p(t;\bar{\eta})\|$, and let $\tau \in [0,t_1]$ be such that $\|p(\tau;\bar{\eta})\| = \Omega$. Define $\tilde{w}(\cdot) \in \mathcal{B}(t_1)$ as follows:

$$\tilde{w}(t) = \begin{cases} 0 \text{ for } 0 \leq t < \tau \\ K\,[p(\tau;\bar{\eta})]^T \text{ for } \tau \leq t \leq t_1 \end{cases}$$

(an obvious modification must be made if $\tau = 0$), where $K > 0$ is arbitrary. Then

$$\int_0^{t_1} p(t;\bar{\eta}) d\tilde{w}(t) = K\Omega^2. \tag{3.20}$$

As in the proof of lemma 1, we can show that (3.16) is satisfied for every $u \in \mathcal{B}(t_1)$ and in particular, for $u = \tilde{w}$. Since STV $\tilde{w} = K\Omega$, we conclude that

$$\int_0^{t_1} p(t;\bar{\eta}) d\tilde{w} - \mathrm{STV}\,\tilde{w} = K\Omega(\Omega - 1) \leq \int_0^{t_1} p(t;\bar{\eta}) d\hat{u} - \mathrm{STV}\,\hat{u}. \tag{3.21}$$

But (3.21) must hold for every $K > 0$, which implies that $\Omega \leq 1$.

It is easily verified that

$$\int_0^{t_1} \underset{\sim}{p}(t;\underset{\sim}{\overline{\eta}})d\underset{\sim}{\hat{u}}(t) \leq \left[\max_{0\leq t\leq t_1} \|\underset{\sim}{p}(t;\underset{\sim}{\overline{\eta}})\|\right] \text{STV } \underset{\sim}{\hat{u}} = \Omega' \text{ STV } \underset{\sim}{\hat{u}},$$

so that

$$\int_0^{t_1} \underset{\sim}{p}(t;\underset{\sim}{\overline{\eta}})d\underset{\sim}{\hat{u}} - \text{STV } \underset{\sim}{\hat{u}} \leq (\Omega-1)\text{STV } \underset{\sim}{\hat{u}} \leq 0. \tag{3.22}$$

But (3.16), with $\underset{\sim}{u} \equiv 0$, gives rise to the inequality

$$\int_0^{t_1} \underset{\sim}{p}(t;\underset{\sim}{\overline{\eta}})d\underset{\sim}{\hat{u}} - \text{STV } \underset{\sim}{\hat{u}} \geq 0. \tag{3.23}$$

Combining (3.22) and (3.23), we obtain (3.19).

If $\Omega < 1$, it follows from (3.19) and (3.22) that STV $\underset{\sim}{\hat{u}} = 0$, i.e., $\underset{\sim}{\hat{u}} \equiv 0$. This contradiction shows that $\Omega = 1$, i.e., (3.18) holds. This completes the proof of lemma 2.

Lemma 3. <u>The interior of</u> W <u>is not empty</u>.

Proof. Since W is convex, it is sufficient to show that W does not belong to any flat in E_7 of dimension less than seven. Suppose the contrary. Then there is a nonzero vector $\underset{\sim}{\overline{\eta}} = (\overline{\eta}_1,\ldots,\overline{\eta}_7)$ in E_7 such that (because $\underset{\sim}{\omega}_0 \in W$) $\underset{\sim}{\overline{\eta}}\cdot\underset{\sim}{\omega}_0 = \underset{\sim}{\overline{\eta}}\cdot\delta\underset{\sim}{x}$ for every $\delta\underset{\sim}{x} \in W$. In particular, $\underset{\sim}{\overline{\eta}}\cdot\underset{\sim}{\omega}_0 = \underset{\sim}{\overline{\eta}}\cdot\delta\underset{\sim}{x}(\underset{\sim}{\hat{u}},1)$, so that (see (3.9)-(3.11)) $\overline{\eta}_7 = 0$. But this contradicts lemma 1, and thereby proves lemma 3.

If $\underset{\sim}{u}_1(\cdot),\ldots,\underset{\sim}{u}_7(\cdot)$ are arbitrary fixed functions in $\mathcal{U}(t_1)$, we define the function $\underset{\sim}{u}(\cdot,\cdot)$ from $[0,\infty) \times E_7$ to E_3 as follows:

$$\underset{\sim}{u}(t,\delta_1,\ldots,\delta_7) = \underset{\sim}{u}(t,\underset{\sim}{\delta}) = \begin{cases} \left(1 - \sum_{j=1}^{7} \delta_j\right)\hat{\underset{\sim}{u}}(t) + \sum_{j=1}^{7} \delta_j \underset{\sim}{u}_j(t) & \text{for } 0 \le t < t_1, \\ \left(1 - \sum_{j=1}^{7} \delta_j\right)\hat{\underset{\sim}{u}}(t_1^-) + \sum_{j=1}^{7} \delta_j \underset{\sim}{u}_j(t_1^-) & \text{for } t_1 \le t < \infty. \end{cases} \tag{3.24}$$

Note that, for every fixed $\underset{\sim}{\delta} \in E_7$, $\underset{\sim}{u}(\cdot,\underset{\sim}{\delta}) \in \mathcal{B}(\infty)$. For ease of notation, let

$$\underset{\sim}{\rho}(t,\underset{\sim}{\delta}) = \underset{\sim}{\rho}(t;\underset{\sim}{u}(\cdot,\underset{\sim}{\delta})), \quad \underset{\sim}{z}(t,\underset{\sim}{\delta}) = \underset{\sim}{z}(t;\underset{\sim}{u}(\cdot,\underset{\sim}{\delta})), \quad 0 \le t < \infty. \tag{3.25}$$

We shall also consider $\underset{\sim}{\rho}(\cdot,\cdot)$ and $\underset{\sim}{z}(\cdot,\cdot)$ to be functions from $[0,\infty) \times E_7$ to E_3. It is easily verified that, for $0 \le t \le t_1$,

$$\underset{\sim}{\rho}(t,0) = \hat{\underset{\sim}{\rho}}(t), \quad \underset{\sim}{z}(t,0) = \hat{\underset{\sim}{z}}(t). \tag{3.26}$$

It follows from well-known theorems on the dependence of solutions of differential equations on parameters that $\partial\underset{\sim}{z}(t,\underset{\sim}{\delta})/\partial\delta_i$ and $\partial\underset{\sim}{\rho}(t,\underset{\sim}{\delta})/\partial\delta_i$ exist and are continuous functions of t and $\underset{\sim}{\delta}$ in $[0,\infty) \times E_7$. In addition, for fixed $\underset{\sim}{\delta}$, these derivatives are absolutely continuous functions of t which, for almost all $t \in [0,t_1]$, satisfy the equations

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\underset{\sim}{z}(t,\underset{\sim}{\delta})}{\partial\delta_i}\right) &= \frac{\partial\underset{\sim}{G}(\underset{\sim}{\rho}(t,\underset{\sim}{\delta}),t)}{\partial\underset{\sim}{r}} \frac{\partial\underset{\sim}{\rho}(t,\underset{\sim}{\delta})}{\partial\delta_i} \\ \frac{d}{dt}\left(\frac{\partial\underset{\sim}{\rho}(t,\underset{\sim}{\delta})}{\partial\delta_i}\right) &= \frac{\partial\underset{\sim}{z}(t,\underset{\sim}{\delta})}{\partial\delta_i} + \underset{\sim}{u}_i(t) - \hat{\underset{\sim}{u}}(t), \end{aligned} \tag{3.27}$$

together with the initial conditions $\partial z(0,\delta)/\partial\delta_i = \partial\rho(0,\delta)/\partial\delta_i = 0$. In particular, it follows from (3.26) and (3.1)-(3.3) that, for $0 \leq t \leq t_1$,

$$\left(\frac{\partial z(t,\delta)}{\partial\delta_i}\right)_{\delta = 0} = \delta z(t;u_i), \quad \left(\frac{\partial\rho(t,\delta)}{\partial\delta_i}\right)_{\delta = 0} = \delta\rho(t;u_i), \quad i = 1,\ldots,7. \tag{3.28}$$

4. A fundamental lemma: In this and the next section we shall suppose that $\hat{u}(\cdot)$ is a solution of the extended variable terminal time problem, and that $t_1 > 0$ is the corresponding terminal time. We shall keep the notation introduced in section 3.

Let

$$\hat{\rho}(t_1) = \bar{r}, \quad \hat{z}(t_1) + \hat{u}(t_1) = \bar{v}. \tag{4.1}$$

By hypothesis, $(\bar{r},\bar{v})\in H(t_1)$, or $h_i(\bar{r},\bar{v},t_1) = 0$ for $i = 1,\ldots,\nu$, and

$$STV_{[0,t_1]}\hat{u} \leq STV_{[0,t]}u \text{ for every } u\in\mathcal{H}(t) \text{ and every } t > 0. \tag{4.2}$$

For (x,y,t) in a neighborhood of $(\bar{r},\bar{v},t_1)$, there is a parametric representation of the manifolds $H(t) = \{(x,y): x\in E_3, y\in E_3, h_i(x,y,t) = 0 \text{ for } i = 1,\ldots,\nu\}$ of the following form:

$$\begin{aligned} x &= \lambda_1(\sigma,t), \\ y &= \lambda_2(\sigma,t), \end{aligned} \tag{4.3}$$

where $\lambda_1(\cdot,\cdot)$ and $\lambda_2(\cdot,\cdot)$ are functions from a neighborhood of $(0,t_1)$ in $E_{6-\nu} \times E_1$ (let this neighborhood be of the form $N_1 \times N_2$, where $N_1\subset E_{6-\nu}$ and $N_2\subset E_1$) to E_3, possessing continuous first partial derivatives with respect to t and the coordinates of σ, and

$$\begin{aligned} \bar{r} &= \lambda_1(0,t_1), \\ \bar{v} &= \lambda_2(0,t_1). \end{aligned} \tag{4.4}$$

If we have a fixed endpoint problem, so that $\nu = 6$, λ_1 and λ_2 are continuously differentiable functions from a neighborhood to t_1 to E_3 with

$$\lambda_1(t_1) = \bar{r}, \quad \lambda_2(t_1) = \bar{v}. \tag{4.5}$$

Let $\nabla h_i(x,y,t)$ denote the vector in E_6 defined by

$$\nabla h_i = \left(\frac{\partial h_i}{\partial x_1}, \frac{\partial h_i}{\partial x_2}, \frac{\partial h_i}{\partial x_3}, \frac{\partial h_i}{\partial y_1}, \frac{\partial h_i}{\partial y_2}, \frac{\partial h_i}{\partial y_3}\right), \quad i = 1,\ldots,\nu.$$

By definition of a smooth manifold, the vectors $\nabla h_i(x,y,t)$, $i = 1,\ldots,\nu$, are linearly independent whenever $(x,y)\in H(t)$. Let T denote the hyperplane of dimension $6-\nu$ that is tangent to $H(t_1)$ at $(\bar{r},\bar{v})$, i.e.,

$$T = \{(x,y): x\in E_3, y\in E_3, [\nabla h_i(\bar{r},\bar{v},t_1)]\cdot(x-\bar{r},y-\bar{v}) = 0, \quad i = 1,\ldots,\nu\}. \tag{4.6}$$

If $\nu = 6$, T consists of the single point $(\bar{r},\bar{v})$. Now define the set Q in E_7 as follows:

$$Q = \{(x-\bar{r},y-\bar{v},\mathrm{STV}\,\hat{u}+\alpha):(x,y)\in T,\alpha \leq 0\}. \tag{4.7}$$

It is easily seen that Q is convex. If $\nu = 6$, Q is the ray $L = \{\delta x(\hat{u},\alpha):\alpha \leq 0\}$.

Let

$$\hat{\zeta} = \left(\hat{z}(t_1)+\hat{u}(\bar{t_1}) - \frac{\partial\lambda_1(0,t_1)}{\partial t}, G(\bar{r},t_1) - \frac{\partial\lambda_2(0,t_1)}{\partial t}, 0\right); \quad \hat{\zeta}\in E_7; \tag{4.8}$$

and, for every $\underset{\sim}{u}(\cdot)\in\mathcal{B}(t_1)$, $\alpha\in E_1$ and $\Delta\in E_1$, let

$$\delta\underset{\sim}{\xi}(\underset{\sim}{u},\alpha,\Delta) = \delta\underset{\sim}{x}(\underset{\sim}{u},\alpha) + \Delta\hat{\underset{\sim}{\zeta}}, \tag{4.9}$$

and let

$$V = \{\delta\underset{\sim}{\xi}(\underset{\sim}{u},\alpha,\Delta):\underset{\sim}{u}\in\mathcal{B}(t_1),\alpha \geq 0,-\infty < \Delta < \infty\}. \tag{4.10}$$

It is clear that $W\subset V$, that V is convex, and that $\underset{\sim}{\omega}_0\in V\cap Q$ (see (3.10), (4.6), (4.7), (4.9), and (4.10)).

We now prove a fundamental lemma.

Lemma 4. <u>If there is a number</u> $\lambda > 0$ <u>such that</u> $\hat{\underset{\sim}{u}}(\cdot)$ <u>is continuous in</u> $(t_1-\lambda,t_1)$, <u>then</u> Q <u>does not meet the interior of</u> V.

This lemma is similar to lemma 11 in [8, page 112]. The proof we shall give below is based on the proof given by Warga of an analogous lemma [9, lemma 3.1].

Proof. Let us assume that the lemma is false, so that there is a point $\underset{\sim}{q}\in Q$ that belongs to the interior of V. Let

$$\underset{\sim}{q} = (\underset{\sim}{\overline{x}}-\underset{\sim}{\overline{r}},\underset{\sim}{\overline{y}}-\underset{\sim}{\overline{v}},\text{STV }\hat{\underset{\sim}{u}}+\overline{\alpha}), \tag{4.11}$$

where $(\underset{\sim}{\overline{x}},\underset{\sim}{\overline{y}})\in T$ and $\overline{\alpha} \leq 0$. If $\overline{\alpha} = 0$, we replace $\overline{\alpha}$ by $-\epsilon$, where $\epsilon > 0$ is sufficiently small, and thereby obtain a point that also belongs to both Q and the interior of V. Thus, without loss of generality, we shall suppose that $\overline{\alpha} < 0$ in (4.11).

Since $\underset{\sim}{q}$ is an interior point of V, there are seven points $\underset{\sim}{\chi}_1,\ldots,\underset{\sim}{\chi}_7$, in V, such that $\underset{\sim}{\omega}_0$, $\underset{\sim}{\chi}_1,\ldots,\underset{\sim}{\chi}_7$ are vertices of a 7-simplex which contains $\underset{\sim}{q}$ in its interior. Let $\chi_j = \delta\xi(w_j,\alpha_j,\Delta_j)$, where $w_j(\cdot)\in\mathcal{B}(t_1)$ and $\alpha_j \geq 0$. For every ϵ, $0 < \epsilon < t_1$, and $j = 1,\ldots,7$, define $u_j(\cdot;\epsilon)\in\mathcal{B}(t_1)$ as follows:

$$u_j(t;\epsilon) = \begin{cases} w_j(t) & \text{if } 0 \leq t < t_1-\epsilon \text{ or } t = t_1 \\ w_j(t_1^-) & \text{if } t_1-\epsilon \leq t < t_1. \end{cases}$$

It is easily seen that $\delta\xi(u_j(\cdot;\epsilon),\alpha_j,\Delta_j) \to \chi_j$ as $\epsilon \to 0$ for each j. Hence, for any fixed $\epsilon > 0$ sufficiently small, the points ω_0 and $\delta\xi(u_j(\cdot;\epsilon),\alpha_j,\Delta_j)$ for $j = 1,\ldots,7$ are vertices of a 7-simplex which contains q in its interior. Let ϵ_0 be one such ϵ, denote $u_j(\cdot;\epsilon_0)$ simply by $u_j(\cdot)$, and let

$$\omega_j = \delta\xi(u_j(\cdot),\alpha_j,\Delta_j), \quad j = 1,\ldots,7. \tag{4.12}$$

Then the vectors $(\omega_j-\omega_0)$, $j = 1,\ldots,7$, are linearly independent, and there are positive numbers $\gamma_0,\gamma_1,\ldots,\gamma_7$ such that

$$q = \sum_{j=0}^{7}\gamma_j\omega_j, \quad \sum_{j=0}^{7}\gamma_j = 1. \tag{4.13}$$

Note that the functions $u_j(\cdot)$, $j = 1,\ldots,7$, are constant in $[t_1-\epsilon_0,t_1)$.

Let $\tau(\cdot)$ be the function from E_7 to E_1 defined by

$$\tau(\delta_1,\ldots,\delta_7) = \tau(\delta) = t_1 + \sum_{j=1}^{7}\delta_j\Delta_j, \tag{4.14}$$

let $\lambda_0 = \min\{\lambda,\epsilon_0\}$, and let N be a neighborhood of 0 in E_7 such that $\tau(\delta)\in N_2$ and $\tau(\delta) > t_1-\lambda_0$ whenever $\delta\in N$. Let $u(\cdot,\delta)$, $\rho(\cdot,\delta)$, and $z(\cdot,\delta)$ be defined as in section 3 (see (3.24) and (3.25)). Note that $u(\cdot,\cdot)$ is continuous in t and

$\underset{\sim}{\delta}$ for $t \in (t_1 - \lambda_0, \infty)$ and all $\underset{\sim}{\delta} \in E_7$.

For each $\underset{\sim}{\delta} \in N$, let $\bar{\underset{\sim}{u}}(\cdot;\underset{\sim}{\delta}) \in \mathcal{B}(t_1)$ and $\tilde{\underset{\sim}{u}}(\cdot;\underset{\sim}{\delta}) \in \mathcal{B}(\tau(\underset{\sim}{\delta}))$ be defined as follows:

$$\bar{\underset{\sim}{u}}(t;\underset{\sim}{\delta}) = \hat{\underset{\sim}{u}}(t) + \sum_{j=1}^{7} \delta_j[\underset{\sim}{u}_j(t) - \hat{\underset{\sim}{u}}(t)], \quad 0 \leq t \leq t_1, \tag{4.15}$$

$$\tilde{\underset{\sim}{u}}(t;\underset{\sim}{\delta}) = \underset{\sim}{u}(t,\underset{\sim}{\delta}) \text{ for } 0 \leq t < \tau(\underset{\sim}{\delta}), \quad \tilde{\underset{\sim}{u}}(\tau(\underset{\sim}{\delta});\underset{\sim}{\delta}) = \bar{\underset{\sim}{u}}(t_1;\underset{\sim}{\delta}). \tag{4.16}$$

It is clear that $\tilde{\underset{\sim}{u}}(t;0) = \hat{\underset{\sim}{u}}(t)$ for $0 \leq t \leq t_1$. If $\tau(\underset{\sim}{\delta}) = t_1$, then $\tilde{\underset{\sim}{u}}(t;\underset{\sim}{\delta}) = \bar{\underset{\sim}{u}}(t;\underset{\sim}{\delta})$ for $0 \leq t \leq t_1$, and obviously $STV_{[0,\tau(\underset{\sim}{\delta})]}\tilde{\underset{\sim}{u}}(\cdot;\underset{\sim}{\delta}) = STV_{[0,t_1]}\bar{\underset{\sim}{u}}(\cdot;\underset{\sim}{\delta})$. It easily follows that the same equality holds if $\tau(\underset{\sim}{\delta}) > t_1$, and it is not difficult to show that, if $\tau(\underset{\sim}{\delta}) < t_1$, then

$$STV_{[0,\tau(\underset{\sim}{\delta})]}\tilde{\underset{\sim}{u}}(\cdot;\underset{\sim}{\delta}) \leq STV_{[0,t_1]}\bar{\underset{\sim}{u}}(\cdot;\underset{\sim}{\delta}), \tag{4.17}$$

i.e., (4.17) is satisfied for all $\underset{\sim}{\delta} \in N$.

Note that (see (4.16) and (3.25)), for $0 \leq t \leq \tau(\underset{\sim}{\delta})$,

$$\underset{\sim}{\rho}(t;\tilde{\underset{\sim}{u}}(\cdot;\underset{\sim}{\delta})) = \underset{\sim}{\rho}(t,\underset{\sim}{\delta}), \quad \underset{\sim}{z}(t;\tilde{\underset{\sim}{u}}(\cdot;\underset{\sim}{\delta})) = \underset{\sim}{z}(t,\underset{\sim}{\delta}). \tag{4.18}$$

Since $(\bar{\underset{\sim}{x}},\bar{\underset{\sim}{y}})$ and $(\bar{\underset{\sim}{r}},\bar{\underset{\sim}{v}})$ both belong to T, the entire line segment joining the two points (assuming that they are distinct) belongs to T. Denote this segment by $\hat{\ell}$; $\hat{\ell}$ is tangent to $H(t_1)$ at $(\bar{\underset{\sim}{r}},\bar{\underset{\sim}{v}})$. Hence, $\hat{\ell}$ is tangent at $(\bar{\underset{\sim}{r}},\bar{\underset{\sim}{v}})$ to a smooth curve Γ on $H(t_1)$. Let Γ be represented parametrically as $\{(\underset{\sim}{\lambda}_1(\underset{\sim}{\sigma}(s),t_1), \underset{\sim}{\lambda}_2(\underset{\sim}{\sigma}(s),t_1)): -1 < s < 1\}$, where $\underset{\sim}{\sigma}(\cdot)$ is a continuously differentiable function from $(-1,1)$ to N_1, $\underset{\sim}{\sigma}(0) = 0$, and

$$\left(\sum_{j=1}^{6-\nu} \frac{\partial \lambda_1(0,t_1)}{\partial \sigma_j} \frac{d\sigma_j(0)}{ds}, \sum_{j=1}^{6-\nu} \frac{\partial \lambda_2(0,t_1)}{\partial \sigma_j} \frac{d\sigma_j(0)}{ds} \right) = (\bar{x}-\bar{r},\bar{y}-\bar{v}). \quad (4.19)$$

If $(\bar{x},\bar{y}) = (\bar{r},\bar{v})$, and this must be true if $\nu = 6$, we let Γ consist of the single point $(\bar{r},\bar{v})$, or, equivalently, let $\sigma(s) \equiv 0$, in which case (4.19) remains valid.

Let $\Theta(\cdot,\cdot)$ be the function from $N \times (-1,1)$ to E_7 defined as follows:

$$\Theta(\delta_1,\ldots,\delta_7,s) = \Theta(\delta,s) =$$

$$\left(f_1(\delta)-g_1(\delta,s), f_2(\delta)+f_3(\delta)-g_2(\delta,s), \sum_{j=1}^{7} \delta_j[\mathrm{STV}\ u_j-\mathrm{STV}\ \hat{u}+\alpha_j]-s\bar{\alpha} \right), \quad (4.20)$$

where the $f_i(\cdot)$ are functions from N to E_3 defined as follows:

$$f_1(\delta) = \rho(\tau(\delta),\delta); \quad f_2(\delta) = z(\tau(\delta),\delta); \quad f_3(\delta) = \tilde{u}(\tau(\delta);\delta), \quad (4.21)$$

and the $g_i(\cdot,\cdot)$ are functions from $N \times (-1,1)$ defined by

$$g_k(\delta,s) = \lambda_k(\sigma(s),\tau(\delta)), \quad k = 1 \text{ or } 2. \quad (4.22)$$

We shall show that $\Theta(\cdot,\cdot)$ is continuously differentiable in $N \times (-1,1)$.

It follows from (4.15), (4,16) and (4.21) that $f_3(\cdot)$ is differentiable and that

$$\frac{\partial f_3(\delta)}{\partial \delta_i} = u_i(t_1) - \hat{u}(t_1). \quad (4.23)$$

It was shown in section 3 that the partial derivatives $\partial\rho(t,\delta)/\partial\delta_i$ and $\partial z(t,\delta)/\partial\delta_i$ exist and are continuous functions of t and δ in $[0,\infty) \times E_7$.

Further, $\partial z(\cdot,\cdot)/\partial t$ exists and is continuous in $[0,\infty) \times E_7$, and inasmuch as $u(\cdot,\cdot)$ is continuous in $(t_1-\lambda_0,\infty) \times E_7$, $\partial \rho(\cdot,\cdot)/\partial t$ exists and is continuous in $(t_1-\lambda_0,\infty) \times E_7$ (see (3.25), (2.3) and footnote 2). It now follows from (4.21), (4.14), and (2.3) that $f_1(\cdot)$ and $f_2(\cdot)$ have continuous derivatives in N, and that, for $\delta \in N$,

$$\frac{\partial f_1(\delta)}{\partial \delta_i} = \left(\frac{\partial \rho(t,\delta)}{\partial \delta_i}\right)_{t = \tau(\delta)} + [z(\tau(\delta),\delta) + u(\tau(\delta),\delta)]\Delta_i$$

$$\frac{\partial f_2(\delta)}{\partial \delta_i} = \left(\frac{\partial z(t,\delta)}{\partial \delta_i}\right)_{t = \tau(\delta)} + G(\rho(\tau(\delta),\delta),\tau(\delta))\Delta_i . \tag{4.24}$$

Because of the differentiability properties that we have assumed for the functions $\sigma(\cdot)$, $\lambda_1(\cdot,\cdot)$, and $\lambda_2(\cdot,\cdot)$, it follows from (4.22) that $\lambda_1(\cdot,\cdot)$ and $\lambda_2(\cdot,\cdot)$ have continuous first derivatives in $N \times (-1,1)$, and

$$\frac{\partial g_k(\delta,s)}{\partial s} = \sum_{j=1}^{6-\nu} \frac{\partial \lambda_k(\sigma(s),\tau(\delta))}{\partial \sigma_j} \frac{d\sigma_j(s)}{ds},$$

$$\frac{\partial g_k(\delta,s)}{\partial \delta_i} = \frac{\partial \lambda_k(\sigma(s),\tau(\delta))}{\partial t} \Delta_i, \quad k = 1 \text{ or } 2. \tag{4.25}$$

It follows from (4.20)-(4.25) that $\Theta(\cdot,\cdot)$ has continuous first partial derivatives in $N \times (-1,1)$. In addition, by virtue of (4.14), (3.28), (3.26), (3.24), (4.1), (4.8), (3.9), (3.10), (4.9) and (4.12),

$$\left(\frac{\partial \Theta(\delta,s)}{\partial \delta_i}\right)_{\substack{\delta = 0 \\ s = 0}} = \omega_i - \omega_0 . \tag{4.26}$$

Also, it follows from (4.20), (4.25), (4.14), (4.19), (4.11), (3.10), and (4.13) that

$$\left(\frac{\partial \underset{\sim}{\Theta}(\underset{\sim}{\delta}, s)}{\partial s} \right)_{\substack{\underset{\sim}{\delta} = 0 \\ s = 0}} = - \sum_{j=1}^{7} \gamma_j (\underset{\sim}{\omega}_j - \underset{\sim}{\omega}_o). \tag{4.27}$$

If $\nu = 6$, the $\underset{\sim}{\lambda}_i$ are functions only of t, and obvious notation changes must be made in (4.8), (4.22), and (4.25), after which eqs. (4.26) and (4.27) follow in the same way.

Now consider the vector equation

$$\underset{\sim}{\Theta}(\underset{\sim}{\delta}, s) = 0 \tag{4.28}$$

for the unknown $\underset{\sim}{\delta}$ as a function of s. For s = 0, it is easily seen (see (4.20)-(4.22), (4.14)-(4.16), (3.26), (4.1) and (4.4)) that (4.28) has the solution $\underset{\sim}{\delta} = 0$. Because of (4.26), the Jacobian of eq. (4.28) at $\underset{\sim}{\delta} = 0$, s = 0 is the determinant of the matrix whose columns are the vectors $(\underset{\sim}{\omega}_i - \underset{\sim}{\omega}_o)$. This Jacobian does not vanish inasmuch as these vectors are, by hypothesis, linearly independent. Since $\underset{\sim}{\Theta}(\cdot,\cdot)$ has continuous first partial derivatives with respect to all of its arguments in a neighborhood of $(0,0) \in E_7 \times E_1$, we can appeal to the implicit function theorem and solve eq. (4.28) for $\underset{\sim}{\delta}$ as a continuously differentiable function of s in a neighborhood of s = 0. Say $\underset{\sim}{\delta} = \underset{\sim}{\phi}(s) = (\phi_1(s), \ldots, \phi_7(s))$. Then $\underset{\sim}{\phi}(0) = 0$ and $d\underset{\sim}{\phi}/ds$ can be obtained by differentiating (4.28) implicitly:

$$\sum_{j=1}^{7} \frac{\partial \Theta(\phi(s),s)}{\partial \delta_j} \frac{d\phi_j(s)}{ds} + \frac{\partial \Theta(\phi(s),s)}{\partial s} = 0.$$

In particular, for s = 0, we obtain, by virtue of (4.26) and (4.27),

$$\sum_{j=1}^{7} (\omega_j - \omega_0) \frac{d\phi_j(0)}{ds} = \sum_{j=1}^{7} \gamma_j(\omega_j - \omega_0). \qquad (4.29)$$

Since the vectors $(\omega_j - \omega_0)$, $j = 1,\ldots,7$, are linearly independent, (4.29) implies that $d\phi_j(0)/ds = \gamma_j$ for each j. Recalling that $\gamma_j > 0$ and $\phi_j(0) = 0$ for each j, we conclude that $\phi_j(s) > 0$ if s is positive and sufficiently small. Also, $\phi(s) \to 0$ as $s \to 0$.

Thus, let $\bar{s}$, $0 < \bar{s} < 1$, be sufficiently small that $\phi(\bar{s}) \in N$, $\phi_j(\bar{s}) > 0$ for each j, and $\sum_{j=1}^{7} \phi_j(\bar{s}) < 1$. Denote $\phi(\bar{s})$ and $\phi_j(\bar{s})$ by $\bar{\delta}$ and $\bar{\delta}_j$, respectively, so that $\Theta(\bar{\delta},\bar{s}) = 0$. Let $\bar{\tau} = \tau(\bar{\delta})$. It follows from (4.20)-(4.22) that

$$\rho(\bar{\tau},\bar{\delta}) = \lambda_1(\sigma(\bar{s}),\bar{\tau}), \qquad (4.30)$$

$$z(\bar{\tau},\bar{\delta}) + \tilde{u}(\bar{\tau};\bar{\delta}) = \lambda_2(\sigma(\bar{s}),\bar{\tau}), \qquad (4.31)$$

$$\sum_{j=1}^{7} \bar{\delta}_j[\text{STV } u_j - \text{STV } \hat{u} + \alpha_j] = \bar{s}\bar{\alpha}. \qquad (4.32)$$

But (4.30), (4.31), (4.18), and the representation (4.3) of H(t) imply that

$$h_i(\rho(\bar{\tau};\tilde{u}(\cdot;\bar{\delta})),z(\bar{\tau};\tilde{u}(\cdot;\bar{\delta})) + \tilde{u}(\bar{\tau};\bar{\delta}),\bar{\tau}) = 0, \quad i = 1,\ldots,\nu, \qquad (4.33)$$

i.e. (see (2.17)), $\tilde{\mathbf{u}}(\cdot;\bar{\boldsymbol{\delta}}) \in \mathcal{U}(\bar{\tau})$. But it follows from (4.17), (4.15), (3.12), (4.32), the non-negatively of the α_j and $\bar{\delta}_j$, and the relations $\bar{s} > 0$, $\bar{\alpha} < 0$, that

$$STV_{[0,\bar{\tau}]}\tilde{\mathbf{u}}(\cdot;\bar{\boldsymbol{\delta}}) \leq STV_{[0,t_1]}\hat{\mathbf{u}}(\cdot) + \bar{s}\bar{\alpha} < STV_{[0,t_1]}\hat{\mathbf{u}}(\cdot),$$

contradicting (4.2), and thereby proving lemma 4.

5. *The necessary conditions*: We now prove the following theorem which provides necessary conditions for the extended variable terminal time problem.

Theorem 1. *Let* $\hat{\mathbf{u}}(\cdot)$ *be a nontrivial* (i.e., $\hat{\mathbf{u}} \not\equiv 0$) *solution of the extended variable terminal time problem, let* t_1 *be the corresponding terminal time, and let* $\hat{\boldsymbol{\rho}}(t)$ *be the corresponding trajectory. Suppose that the points of discontinuity of* $\hat{\mathbf{u}}(\cdot)$ *do not cluster at* t_1. *Then there exists a twice differentiable (column) vector-valued function* $\boldsymbol{\psi}(\cdot)$ *from* $[0,t_1]$ *to* E_3 *such that*

$$\ddot{\boldsymbol{\psi}}(t) = \left(\frac{\partial \mathbf{G}(\hat{\boldsymbol{\rho}}(t),t)}{\partial \mathbf{r}}\right)^T \boldsymbol{\psi}(t) \tag{5.1}$$

for all $t \in [0,t_1]$,

$$\max_{0 \leq t \leq t_1} \|\boldsymbol{\psi}(t)\| = 1, \tag{5.2}$$

and

$$\int_0^{t_1} [\boldsymbol{\psi}(t)]^T \, d\hat{\mathbf{u}}(t) = STV_{[0,t_1]}\hat{\mathbf{u}}. \tag{5.3}$$

Let (4.3) *and* (4.4) *be a parametric representation of* $H(t)$ *in a neighborhood of* $(\hat{\rho}(t_1), z(t_1;\hat{u})+\hat{u}(t_1), t_1) = (\bar{r}, \bar{v}, t_1)$, *and let* T *be the hyperplane tangent to* $H(t_1)$ *at* $(\bar{r}, \bar{v})$. *Then* $\psi(\cdot)$ *satisfies the following transversality conditions*:

$$\dot{\psi}(t_1)\cdot(x-\bar{r}) = \psi(t_1)\cdot(y-\bar{v}) \text{ for all } (x,y)\in T, \tag{5.4}$$

or, equivalently,

$$(\dot{\psi}(t_1), -\psi(t_1)) = \sum_{i=1}^{\nu} \mu_i \nabla h_i(\bar{r},\bar{v},t_1) \text{ for some real constants } \mu_1,\ldots,\mu_\nu, \tag{5.4'}$$

and

$$-\psi(t_1)\cdot\left[G(\bar{r},t_1) - \frac{\partial\lambda_2(0,t_1)}{\partial t}\right] + \dot{\psi}(t_1)\cdot\left[\bar{v} - \frac{\partial\lambda_1(0,t_1)}{\partial t}\right] = \dot{\psi}(t_1)\cdot[\hat{u}(t_1)-\hat{u}(t_1^-)]\geq 0. \tag{5.5}$$

If $\|\psi(t)\| < 1$ *in some interval* $[t',t'']\subset[0,t_1]$, *then* $\hat{u}(\cdot)$ *is constant for* $t' \leq t \leq t''$. *If* $\hat{u}(\tau) \neq \hat{u}(\tau^-)$ *for some* $\tau\in(0,t_1]$, *then* $\|\psi(\tau)\| = 1$, *and there is a number* κ_τ *such that*

$$\hat{u}(\tau) - \hat{u}(\tau^-) = \kappa_\tau \psi(\tau), \quad \kappa_\tau > 0. \tag{5.6}$$

If $0 = \hat{u}(0) \neq \hat{u}(0^+)$, *then* $\|\psi(0)\| = 1$, *and* $\hat{u}(0^+) = \kappa_0 \psi(0)$, *where* $\kappa_0 > 0$. *In particular, if* $D = \{t: \|\psi(t)\| = 1, 0 \leq t \leq t_1\}$ *is a finite set, then* $\hat{u}(\cdot)$ *is a step function whose points of discontinuity all belong to* D, *and whose jumps are given by* (5.6) *or the modification thereof if* $\tau = 0$.

Note that eq. (5.4) is satisfied trivially if $\nu = 6$, since, in this case, T consists of the single point $(\bar{r},\bar{v})$. Also, if $\nu = 6$, $\partial\lambda_i(0,t_1)/\partial t$ (i = 1 and 2) in (5.5) should be replaced by $d\lambda_i(t_1)/dt$.

Proof. By lemma 4, Q does not meet the interior of V. Since $\omega_0 \in V \cap Q$, ω_0 is a boundary point of V. Now both V and Q are convex, and $V \supset W$, so that, by virtue of lemma 3, the interior of V is not empty. Hence, there is a supporting hyperplane to V at ω_0 that separates V from Q. Let $\bar{\eta} = (\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_7) \neq 0$, where $\bar{\eta}_1 \in E_3$, $\bar{\eta}_2 \in E_3$, and $\bar{\eta}_7 \in E_1$, be a normal to this hyperplane directed so that

$$\bar{\eta}\cdot\delta\xi \leq \bar{\eta}\cdot\omega_0 \leq \bar{\eta}\cdot\theta \text{ for all } \delta\xi \in V, \quad \theta \in Q. \tag{5.7}$$

By virtue of lemma 1, relations (5.7) allow us to conclude that $\bar{\eta}_7 < 0$. Without loss of generality we shall assume that $\bar{\eta}_7 = -1$. If we let $\psi(t) = [p(t;\bar{\eta})]^T$, eqs (5.2) and (5.3) follow from lemma 2 (see (3.18) and (3.19)), and eq. (5.1) follows from (3.15) and (3.1).

Let (x,y) be an arbitrary point of T. Then (see (4.6) and (4.7)), $(x-\bar{r}, y-\bar{v}, STV\,\hat{u}) \in Q$, and $(-x+\bar{r}, -y+\bar{v}, STV\,\hat{u}) \in Q$. Consequently, by virtue of (5.7) and (3.10),

$$\bar{\eta}_1\cdot(x-\bar{r}) + \bar{\eta}_2\cdot(y-\bar{v}) = 0.$$

Taking (3.14) into account, we obtain (5.4). The equivalence of (5.4) and (5.4') follows at once from the definition of T (see (4.6)).

Consider the points $\delta\xi(\hat{u},0,\pm 1) = \omega_0 \pm \hat{\zeta}$ of V (see (4.9), (4.10), and (3.10)). It follows from (5.7) that $\bar{\eta}\cdot(\pm\hat{\zeta}) \leq 0$, i.e., $\bar{\eta}\cdot\hat{\zeta} = 0$, and the equality in (5.5) follows from (4.8), (4.1), and (3.14).

The final conclusions of the theorem are direct consequences of (5.2) and (5.3) (see [3, theorem 3]).

It only remains to prove the inequality in (5.5). If $\hat{u}(t_1) = \hat{u}(t_1^-)$, the inequality is obvious. If $\hat{u}(t_1) \neq \hat{u}(t_1^-)$, then $\|\psi(t_1)\| = 1$, and, since $\|\psi(t)\| \leq 1$ for $0 \leq t < t_1$,

$$0 \leq \left(\frac{d\|\psi(t)\|^2}{dt}\right)_{t = t_1} = 2\psi(t_1)\cdot\dot{\psi}(t_1). \tag{5.8}$$

Relations (5.8) and (5.6) imply the inequality in (5.5), completing the proof of theorem 1.

The vector-valued function $(\psi(\cdot),-\dot{\psi}(\cdot))$ is analogous to the adjoint variable in the formulation of the Pontryagin maximum principle, or to the Lagrange multipliers of the classical calculus of variations. Relation (5.3) corresponds to the maximum principle itself, or to the Weierstrass E-condition.

If the set D is finite—say $D = \{\tau_1,\ldots,\tau_\ell\}$—then $\hat{u}(\cdot)$ is determined by $6+\ell$ scalar parameters: six initial values for eq. (5.1), and the ℓ constants κ_{τ_i} in (5.6). Indeed, given the values of these parameters and the initial values $z(0)$, $\rho(0)$, it is possible to "simultaneously solve" eqs. (2.3) with $u = \hat{u}$ and eq. (5.1), and determine $\hat{u}$ through (5.6).

6. <u>The fixed terminal time problem:</u> In this section we shall derive the necessary conditions for the extended fixed terminal time problem. Thus, let $t_1 > 0$ be fixed, and let $\hat{u}(\cdot)$ be a solution of the corresponding extended fixed terminal time problem. We shall keep the notation introduced in sec-

tion 3. Let $(\bar{r},\bar{v})$ be defined by (4.1), so that, by hypothesis, $h_i(\bar{r},\bar{v},t_1) = 0$ for $i = 1,\dots,\nu$, and

$$STV_{[0,t_1]}\hat{u} \le STV_{[0,t_1]}u \text{ for every } u \in \mathcal{H}(t_1). \qquad (6.1)$$

We define T and Q as in section 4 (see (4.6) and (4.7)). Corresponding to lemma 4, we have the following proposition.

Lemma 5. The set Q does not meet the interior of W.

Note that here it is not necessary to assume that $\hat{u}(\cdot)$ is continuous in $(t_1-\lambda,t_1)$ for some $\lambda > 0$.

Proof. The derivation is almost identical to that of lemma 4, with certain simplifications, and we shall only outline the necessary arguments. Assuming the contrary, we show that there are points $\omega_j = \delta x(u_j,\alpha_j)$, $j = 1,\dots,7$, such that each $\omega_j \in W$ and $\omega_0,\omega_1,\dots,\omega_7$ are the vertices of a 7-simplex that contains a point q of the form (4.11), with $\bar{\alpha} < 0$, in its interior. We let $\tau(\delta) = t_1$ for all $\delta \in E_7$, and consider solutions of eq. (4.28) near $s = 0$, where $\Theta(\cdot,\cdot)$ is again given by (4.20)-(4.22) and (4.16), and the function $\sigma(\cdot)$ has all of the properties described in section 4. Relations (4.26) and (4.27) can now be derived as in section 4, except that (4.26) can be obtained without having to show that $\partial\rho(t,\delta)/\partial t$ exists in a neighborhood of t_1 (since $\tau(\cdot) \equiv t_1$). The continuity of $\hat{u}$ in $(t_1-\lambda,t_1)$ was used only in showing the existence of this derivate, and consequently the extra continuity hypothesis can here be dispensed with. It then follows as before that there is a vector $\bar{\delta}$ possessing the same properties as in section 4 such that eqs. (4.32) and (4.33), with $\bar{\tau} = t_1$, are satisfied.

But these equations are inconsistent with relation (6.1), and we have a contradiction. This completes the proof of lemma 5.

We now have the following theorem.

Theorem 2. Let $\hat{u}(\cdot)$ be a nontrivial (i.e., $\hat{u} \not\equiv 0$) solution of the extended fixed terminal time problem, t_1 being the terminal time, and let $\hat{\rho}(\cdot)$ be the corresponding trajectory. Then there exists a function $\psi(\cdot)$ from $[0,t_1]$ to E_3 such that $\psi(\cdot)$ and $\hat{u}(\cdot)$ satisfy all of the conditions stated in theorem 1 with the possible exception of (5.5). If the points of discontinuity of $\hat{u}(\cdot)$ cluster at t_1, then, in addition,

$$\left(\frac{d\|\psi(t)\|}{dt}\right)_{t = t_1} = 0 , \quad \|\psi(t_1)\| = 1 . \tag{6.2}$$

Proof. Just as the existence of a vector $\bar{\eta} = (\bar{\eta}_1,\bar{\eta}_2,\bar{\eta}_7)$ which satisfies (5.7) followed from lemma 4, it is here a consequence of lemma 5 that there is a vector $\bar{\eta} = (\bar{\eta}_1,\bar{\eta}_2,\bar{\eta}_7)$ such that

$$\bar{\eta}\cdot\delta x \leq \bar{\eta}\cdot\omega_0 \leq \bar{\eta}\cdot\theta \text{ for all } \delta x \in W \text{ and } \theta \in Q . \tag{6.3}$$

With the exception of (5.5), the conclusions of theorem 1 now follow from (6.3) as in section 5. Since $\|\psi(t)\| = 1$ at all points of discontinuity of $\hat{u}$, and $\|\psi(t)\|$ is differentiable (and certainly continuous) at $t = t_1$, the last sentence of theorem 2 follows at once.

Note that if $\hat{u}(\cdot)$ is a solution of the extended variable terminal time problem, it is a fortiori a solution of a fixed terminal time problem. Therefore, theorem 2 also provides necessary conditions for the case excluded in theorem 1; i.e., if the points of discontinuity of $\hat{u}(\cdot)$ do cluster at t_1, the conclusions of theorem 1 remain valid, with the exception that (5.5) must be replaced by (6.2).

7. <u>Examples:</u> Let us apply theorems 1 and 2 to some problems that are of contemporary interest.

First consider the variable terminal time, fixed endpoint problem (sometimes referred to as the "rendezvous" problem) in which $\lambda_1(t)$ and $\lambda_2(t)$ (the functions that describe H(t)) represent the position and velocity, respectively, of an actual or fictitious target at the time t. The equations of motion of such a target can be written in the form

$$\frac{d\lambda_1(t)}{dt} = \lambda_2(t) , \quad \frac{d\lambda_2(t)}{dt} = G(\lambda_1(t),t) + a(t) , \tag{7.1}$$

where $a(t)$ is the non-gravitational acceleration experienced by the target. Since (7.1) must hold, in particular, when $t = t_1$, relations (5.5) in this case take the form (see (4.5))

$$\psi(t_1)\cdot a(t_1) = \dot{\psi}(t_1)\cdot[\hat{u}(t_1)-\hat{u}(t_1^-)] \geq 0.$$

If $a(t) \equiv 0$, i.e., if the target is in a "free-fall" trajectory, we obtain

$$\dot{\psi}(t_1)\cdot[\hat{u}(t_1)-\hat{u}(t_1^-)] = 0 , \tag{7.2}$$

which implies that either $\hat{u}(t_1) = \hat{u}(t_1^-)$, or that $\|\psi(t_1)\| = 1$ and (see (5.6)) $\psi(t_1)\cdot\dot{\psi}(t_1) = 0$, i.e., relations (6.2) are satisfied.

Also consider the following three variable endpoint problems.

The first, the so-called "intercept" problem, is the problem in which the vehicle terminal position is specified (but may depend on the terminal time), and the terminal velocity is arbitrary. In this case, eqs. (4.3) can be put in the form $\underset{\sim}{x} = \underset{\sim}{\lambda}_1(t)$, $\underset{\sim}{y} = \underset{\sim}{\sigma}$, and

$$H(t_1) = T = \{(\underset{\sim}{x},\underset{\sim}{y}): \underset{\sim}{x} = \underset{\sim}{\lambda}_1(t_1)\} .$$

The transversality condition (5.4') in this case implies that $\underset{\sim}{\psi}(t_1) = 0$. Consequently, $\|\underset{\sim}{\psi}(t)\| < 1$ for $t' \le t \le t_1$ and some $t' < t_1$, so that, by theorem 1, $\hat{\underset{\sim}{u}}(t)$ is constant for $t' \le t \le t_1$. This conclusion is valid whether the problem is with fixed or variable terminal time. For the variable terminal time problem, relations (5.5) take the form (since $\hat{\underset{\sim}{u}}(t_1) = \hat{\underset{\sim}{u}}(t_1^-)$ and $\underset{\sim}{\psi}(t_1) = 0$)

$$\dot{\underset{\sim}{\psi}}(t_1)\cdot\left[\underset{\sim}{\bar{v}} - \frac{d\underset{\sim}{\lambda}_1(t_1)}{dt}\right] = 0 .$$

As a second example, consider the case where the terminal velocity is specified (but may depend on the time) and the terminal position is arbitrary. Then, eqs. (4.3) can be put in the form $\underset{\sim}{x} = \underset{\sim}{\sigma}$, $\underset{\sim}{y} = \underset{\sim}{\lambda}_2(t)$, and the transversality condition (5.4') implies that $\dot{\underset{\sim}{\psi}}(t_1) = 0$ whether the problem is for fixed or variable terminal time. For the variable terminal time problem, (5.5) takes the form

$$\underset{\sim}{\psi}(t_1)\cdot\left[\underset{\sim}{G}(\underset{\sim}{\bar{r}},t_1) - \frac{d\underset{\sim}{\lambda}_2(t_1)}{dt}\right] = 0 .$$

In the third example, the "transfer to a specified orbit" problem, we shall assume that $\underset{\sim}{G}(\underset{\sim}{r},t)$ is independent of t. Here, the vehicle terminal position and velocity are to be the same as the position and velocity at any point on a specified solution curve (i.e., orbit) of eq. (2.1) with $\underset{\sim}{T} \equiv 0$. Thus, for every $t > 0$,

$$H(t) = \left\{(\underset{\sim}{x}(s),\underset{\sim}{y}(s)): -\infty<s<\infty,\ \frac{d\underset{\sim}{x}(s)}{ds} = \underset{\sim}{y},\ \frac{d\underset{\sim}{y}(s)}{ds} = \underset{\sim}{G}(\underset{\sim}{x}(s)),\ \underset{\sim}{x}(0) = \underset{\sim}{a},\ \underset{\sim}{y}(0) = \underset{\sim}{b}\right\},$$

where $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are given vectors in E_3 that specify the orbit. Then, in the notation of section 4,

$$T = \{(\bar{\underset{\sim}{r}}+\sigma\bar{\underset{\sim}{v}},\ \bar{\underset{\sim}{v}}+\sigma \underset{\sim}{G}(\bar{\underset{\sim}{r}})): -\infty < \sigma < \infty\} ,$$

and the transversality condition (5.4) takes the form

$$\bar{\underset{\sim}{v}}\cdot\dot{\underset{\sim}{\psi}}(t_1) = \underset{\sim}{G}(\bar{\underset{\sim}{r}})\cdot\underset{\sim}{\psi}(t_1) . \tag{7.3}$$

Since H is independent of t, the functions $\underset{\sim}{\lambda}_1$ and $\underset{\sim}{\lambda}_2$ can be chosen to be independent of t, so that, for the variable terminal time problem, we have (by virtue of (5.5) and (7.3)) that (7.2) holds, i.e., either $\hat{\underset{\sim}{u}}(t_1) = \hat{\underset{\sim}{u}}(t_1^-)$ or eqs. (6.2) hold.

8. <u>Existence and approximation theorems:</u> In this section we shall prove that the sets $\mathcal{H}(t)$ possess the two properties described in section 2. We first prove an almost self-evident, but nevertheless interesting, lemma.

We shall say that a function $\underset{\sim}{h}(\cdot)$ from [0,1] to E_3 is <u>regular</u> if $\underset{\sim}{h}(\cdot)$ is continuous and piecewise smooth, and if $d\underset{\sim}{h}/dt$ is of strong bounded variation in [0,1].

Lemma 6. <u>Let $\underset{\sim}{h}(\cdot)$ be a regular function from [0,1] to E_3, $\bar{t}$ an arbitrary positive number, and $\underset{\sim}{z}_0$ and $\underset{\sim}{z}_1$ arbitrary vectors in E_3. Then there exists a</u>

function $\bar{\mathbf{u}}(\cdot) \in \mathcal{B}(\bar{t})$ *such that the solution* $\mathbf{z}(\cdot)$, $\boldsymbol{\rho}(\cdot)$ *of eqs.* (2.3), *with* $\mathbf{u} = \bar{\mathbf{u}}$ *and initial values* $\boldsymbol{\rho}(0) = \mathbf{h}(0)$ *and* $\mathbf{z}(0) = \mathbf{z}_0$, *satisfies the equation* $\boldsymbol{\rho}(t) = \mathbf{h}(t/\bar{t})$ *for all* $t \in [0,\bar{t}]$ *as well as the boundary condition* $\mathbf{z}(\bar{t}) + \bar{\mathbf{u}}(\bar{t}) = \mathbf{z}_1$.

Proof. For each $t \in [0,\bar{t}]$, let

$$\boldsymbol{\zeta}(t) = \int_0^t \mathbf{G}(\mathbf{h}(s/\bar{t}),s)ds + \mathbf{z}_0 , \tag{8.1}$$

and let the function $\bar{\mathbf{w}}(\cdot) \in \mathcal{B}(\bar{t})$ be defined as follows:

$$\bar{\mathbf{w}}(t) = \begin{cases} \left(\dfrac{d\mathbf{h}(s)}{ds}\right)_{s = (t/\bar{t})^+} & \text{for } 0 < t < \bar{t} \\ 0 & \text{for } t = 0 \\ \bar{t}\mathbf{z}_1 & \text{for } t = \bar{t} \;. \end{cases}$$

Since the function $\mathbf{G}(\cdot,\cdot)$ is bounded, it follows that the function $\boldsymbol{\zeta}(\cdot)$ defined by (8.1) is of strong bounded variation in $[0,\bar{t}]$. Let $\bar{\mathbf{u}}(\cdot) = (1/\bar{t})\bar{\mathbf{w}}(\cdot) - \boldsymbol{\zeta}(\cdot)$. Then $\bar{\mathbf{u}} \in \mathcal{B}(\bar{t})$. If we set $\mathbf{z}(t) = \boldsymbol{\zeta}(t)$ and $\boldsymbol{\rho}(t) = \mathbf{h}(t/\bar{t})$ for $t \in [0,\bar{t}]$, it can be verified directly that the functions $\mathbf{z}(\cdot)$ and $\boldsymbol{\rho}(\cdot)$ are a solution of eqs. (2.3) with $\mathbf{u} = \bar{\mathbf{u}}$ that satisfy the initial and boundary conditions prescribed in the statement of the lemma.

Let $K(\cdot)$ be a function whose domain is $[0,\infty)$ and whose range is the class of subsets of E_3. Preserving the notation of section 2, we shall in addition denote, for every $\bar{t} > 0$, by $\tilde{\mathcal{H}}(\bar{t};K(\cdot))$ the following subset of $\mathcal{H}(\bar{t})$:

$$\tilde{\mathcal{H}}(\bar{t};K(\cdot)) = \{\mathbf{w}(\cdot) : \mathbf{w}(\cdot) \in \mathcal{H}(\bar{t}),\ \boldsymbol{\rho}(t;\mathbf{w}) \in K(t) \text{ for every } t \in [0,\bar{t}]\} .$$

We now prove the following existence theorem.

Theorem 3. *Let* t_1 *be a given positive number and let* $K(\cdot)$ *be a function from* $[0,\infty)$ *into the class of all closed subsets of* E_3 *with the property that there is at least one regular function* $h(\cdot)$ *from* $[0,1]$ *to* E_3 *such that* $h(t/t_1)\in K(t)$ *for every* $t\in[0,t_1]$, $h(0) = r_0$, *and* $(h(1),y)\in H(t_1)$ *for some* $y\in E_3$. *Then* $\tilde{\mathcal{H}}(t_1;K(\cdot))$ *is not empty, and there is an element* $\hat{u}(\cdot)\in\tilde{\mathcal{H}}(t_1;K(\cdot))$ *such that*

$$STV_{[0,t_1]}\hat{u}(\cdot) = \inf_{u\in\tilde{\mathcal{H}}(t_1;K(\cdot))} STV_{[0,t_1]}u(\cdot) . \qquad (8.2)$$

Proof. The fact that $\tilde{\mathcal{H}}(t_1;K(\cdot))$ is not empty follows immediately from lemma 6. If $\tilde{\mathcal{H}}(t_1;K(\cdot))$ is a finite set, the theorem is trivial. Thus, let $u_n(\cdot)$, $n = 1,2,\ldots$, be a sequence of elements of $\tilde{\mathcal{H}}(t_1;K(\cdot))$ such that

$$\lim_{n\to\infty} STV_{[0,t_1]}u_n = \inf_{u\in\tilde{\mathcal{H}}(t_1;K(\cdot))} STV_{[0,t_1]}u . \qquad (8.3)$$

Denote the functions $z(\cdot;u_n)$ and $\rho(\cdot;u_n)$ by $z_n(\cdot)$ and $\rho_n(\cdot)$, respectively. Because the function $G(\cdot,\cdot)$ is bounded, the derivatives $\dot{z}_n(\cdot)$ as well as the functions $z_n(\cdot)$ themselves are uniformly bounded, and the $z_n(\cdot)$ are equicontinuous on $[0,t_1]$. Since $u_n(0) = 0$ for each n, and the numbers $STV_{[0,t_1]}u_n$, $n = 1,2,\ldots$, are bounded, it follows that the functions $u_n(\cdot)$ are uniformly bounded on $[0,t_1]$. Consequently, the derivatives $\dot{\rho}_n(\cdot)$, which exist almost everywhere in $[0,t_1]$, are uniformly bounded and the functions $\rho_n(\cdot)$ are themselves uniformly bounded and equicontinuous on $[0,t_1]$. Appealing to Arzela's theorem [6, p. 122] and the Helly selection theorem [11, p. 222], we conclude that there is a subsequence of the $\underset{\sim}{u}_n(\cdot)$ (which we shall continue to denote by $\underset{\sim}{u}_n$, without loss of generality), and functions $\underset{\sim}{u}_\infty(\cdot)$, $\underset{\sim}{z}_\infty(\cdot)$, and $\underset{\sim}{\rho}_\infty(\cdot)$ from $[0,t_1]$ to E_3, where $\underset{\sim}{u}_\infty$ is of strong bounded variation and $\underset{\sim}{z}_\infty$ and $\underset{\sim}{\rho}_\infty$ are continuous, such that, for every $t\in[0,t_1]$,

$$\lim_{n\to\infty} \underset{\sim}{u}_n(t) = \underset{\sim}{u}_\infty(t), \quad \lim_{n\to\infty} \underset{\sim}{\rho}_n(t) = \underset{\sim}{\rho}_\infty(t), \quad \lim_{n\to\infty} \underset{\sim}{z}_n(t) = \underset{\sim}{z}_\infty(t). \tag{8.4}$$

Also, it is easily seen that $\lim_{n\to\infty} \text{STV}\, \underset{\sim}{u}_n \geq \text{STV}[\lim_{n\to\infty} \underset{\sim}{u}_n]$, i.e. (see (8.3) and (8.4)),

$$\text{STV}\, \underset{\sim}{u}_\infty \leq \inf_{\underset{\sim}{u}\in\tilde{\mathcal{H}}(t_1;K(\cdot))} \text{STV}\, \underset{\sim}{u}\ . \tag{8.5}$$

Now define the function $\hat{\underset{\sim}{u}}(\cdot)$ from $[0,t_1]$ to E_3 as follows:

$$\hat{\underset{\sim}{u}}(t) = \begin{cases} \underset{\sim}{u}_\infty(t^+) & \text{for } 0 < t < t_1 \\ \underset{\sim}{u}_\infty(t) & \text{for } t = 0 \text{ or } t = t_1\ . \end{cases} \tag{8.6}$$

Since $\underset{\sim}{u}_\infty(\cdot)$ is of strong bounded variation, $\underset{\sim}{u}_\infty(t^+)$ exists for each $t\in(0,t_1)$, and since there are at most a denumerable number of points at which a function of bounded variation is discontinuous, $\hat{\underset{\sim}{u}}(t) = \underset{\sim}{u}_\infty(t)$ for almost all t in $[0,t_1]$. It is easily seen that $\text{STV}\,\hat{\underset{\sim}{u}} \leq \text{STV}\,\underset{\sim}{u}_\infty$, so that, by virtue of (8.5),

$$\text{STV}\,\hat{\underset{\sim}{u}} \leq \inf_{\underset{\sim}{u}\in\tilde{\mathcal{H}}(t_1;K(\cdot))} \text{STV}\, \underset{\sim}{u}\ . \tag{8.7}$$

It follows from (8.4) and (8.6) that $\hat{\underset{\sim}{u}}(0) = 0$, so that $\hat{\underset{\sim}{u}}(\cdot)\in\mathcal{B}(t_1)$. In addition, because of (8.4), (8.6), and the continuity of the functions h_i we have that $\underset{\sim}{\rho}_\infty(0) = \underset{\sim}{r}_o$, $\underset{\sim}{z}_\infty(0) = \underset{\sim}{v}_o$, and $h_i(\underset{\sim}{\rho}_\infty(t_1),\underset{\sim}{z}_\infty(t_1)+\hat{\underset{\sim}{u}}(t_1),t_1) = 0$

for $i = 1,\ldots,\nu$. Since the sets $K(t)$ are closed by hypothesis, it follows from (8.4) that $\rho_\infty(t)\in K(t)$ for all $t\in[0,t_1]$. Consequently, if we can show that $\rho_\infty(\cdot) = \rho(\cdot;\hat{u})$ and that $z_\infty(\cdot) = z(\cdot;\hat{u})$, we can conclude that $\hat{u}\in\tilde{\mathcal{H}}(t_1;K(\cdot))$, and (8.2) is then an immediate consequence of (8.7).

By virtue of (2.3),

$$z_n(t) = v_0 + \int_0^t G(\rho_n(s),s)ds ,$$

$$\rho_n(t) = r_0 + \int_0^t [z_n(s)+u_n(s)]ds , \quad 0 \leq t \leq t_1 .$$

Since the functions G, z_n, and u_n are uniformly bounded, we can appeal to the Lebesgue dominated convergence theorem, and conclude that

$$z_\infty(t) = v_0 + \int_0^t G(\rho_\infty(s),s)ds ,$$

$$\rho_\infty(t) = r_0 + \int_0^t [z_\infty(s)+\hat{u}(s)]ds , \quad 0 \leq t \leq t_1 ,$$

where we have also used the fact that $u_\infty(t) = \hat{u}(t)$ for almost all $t\in[0,t_1]$. It now follows immediately (see footnote 2) that $z_\infty(\cdot) = z(\cdot;\hat{u})$ and $\rho_\infty(\cdot) = \rho(\cdot;\hat{u})$, completing the proof of theorem 3.

If $K(t) = E_3$ for every $t \geq 0$, then $\tilde{\mathcal{H}}(t_1;K(\cdot)) = \mathcal{H}(t_1)$, and it is evident that there exists a regular function $h(\cdot)$ with the required properties. The existence theorem promised in section 2 then follows at once from theorem 3.

We now prove the following theorem.

Theorem 4. <u>Let</u> $\hat{u}(\cdot)\in\mathcal{H}(t_1)$ <u>for</u> $t_1 > 0$. <u>Then there exist functions</u> $u_n(\cdot)\in\mathcal{A}(t_1)$, $n = 1,2,\ldots$, <u>such that</u>: 1. <u>the derivatives</u> du_n/dt <u>are essentially bounded</u>, 2. $u_n(t) \to \hat{u}(t)$ <u>for each</u> $t\in[0,t_1]$ <u>as</u> $n \to \infty$, <u>and</u> 3. $\lim_{n\to\infty} \mathrm{STV}\ u_n = \mathrm{STV}\ \hat{u}$.

Proof. We first prove two lemmas.

Lemma 7. <u>Let</u> $w_1(\cdot)$, $w_2(\cdot)$,... <u>denote a sequence of uniformly bounded functions in</u> $\mathcal{B}(t_1)$ <u>such that</u> $w_n(t) \to \hat{u}(t)$ <u>as</u> $n\to\infty$ <u>for each</u> $t\in[0,t_1]$, <u>where</u> $\hat{u}(\cdot)\in\mathcal{B}(t_1)$. <u>Then</u> $\rho(t;w_n) \to \rho(t;\hat{u})$ <u>and</u> $z(t;w_n) \to z(t;\hat{u})$ <u>as</u> $n\to\infty$ <u>uniformly in</u> $[0,t_1]$.

Proof. Let $\theta_n(\cdot)$ be the scalar-valued function on $[0,t_1]$ defined by $\theta_n(t) = \|\rho(t;w_n)-\rho(t;\hat{u})\| + \|z(t;w_n)-z(t;\hat{u})\|$. Then it follows from (2.3) and the boundedness of the partial derivatives $\partial G_i/\partial r_j$ that $\dot{\theta}_n(t) \leq R\theta_n(t) + \|\hat{u}(t)-w_n(t)\|$ for some positive constant R and all $t\in[0,t_1]$. Now $\theta_n(0) = 0$, so that, for $0 \leq t \leq t_1$,

$$\theta_n(t) \leq R\int_0^t \theta_n(\tau)d\tau + \int_0^{t_1} \|\hat{u}(\tau)-w_n(\tau)\|d\tau .$$

It follows from the Lebesgue dominated convergence theorem that $\int_0^{t_1} \|\hat{u}(\tau) - w_n(\tau)\|d\tau \to 0$ and $n \to \infty$, and the lemma is now an immediate consequence of Gronwall's inequality.

Lemma 8. <u>For every</u> $\epsilon > 0$ <u>there exists a</u> $\delta > 0$, <u>depending only on</u> ϵ, <u>such that the following proposition holds</u>: <u>If</u> $w(\cdot)$ <u>is any absolutely continuous function in</u> $\mathcal{B}(t_1)$ <u>whose derivative is essentially bounded</u>, <u>and</u> $\hat{\rho}$ <u>and</u> $\hat{z}$ <u>are any vectors in</u> E_3 <u>that satisfy the inequality</u> $\|\hat{\rho}\| + \|\hat{z}\| < \delta$, <u>then</u>

there exists an absolutely continuous function $u(\cdot) \in \mathcal{B}(t_1)$ with essentially bounded derivative such that $STV(u-w) < \epsilon$ and

$$\rho(t_1;u) = \rho(t_1;w) + \hat{\rho}, \quad z(t_1;u) + u(t_1) = z(t_1;w) + w(t_1) + \hat{z}. \tag{8.8}$$

Proof. Fix $\epsilon > 0$. Let $\tilde{G} = \sup_{r,t} \|G(r,t)\|$, $\tilde{\epsilon} = \min \{\epsilon/(7+2\tilde{G}), 1, t_1\}$, and $\delta = (\tilde{\epsilon})^2$. Let $w(\cdot)$ be an arbitrary, fixed, absolutely continuous function in $\mathcal{B}(t_1)$, and let $\hat{\rho}$ and $\hat{z}$ be arbitrary fixed vectors in E_3 such that $\|\hat{\rho}\| + \|\hat{z}\| < \delta$. Define the element $\hat{w}(\cdot)$ of $\mathcal{B}(t_1)$ as follows:

$$\hat{w}(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq t_1-\tilde{\epsilon} \\ (t-t_1+\tilde{\epsilon})m_1 & \text{for } t_1-\tilde{\epsilon} \leq t \leq t_1-\tilde{\epsilon}/2 \\ (t-t_1)m_2+\hat{z} & \text{for } t_1-\tilde{\epsilon}/2 \leq t \leq t_1 , \end{cases}$$

where $m_1 = (4\hat{\rho}/\tilde{\epsilon}^2)-\hat{z}/\tilde{\epsilon}$ and $m_2 = (3\hat{z}/\tilde{\epsilon})-4\hat{\rho}/\tilde{\epsilon}^2$. It can be immediately verified that $\hat{w}(\cdot)$ is absolutely continuous, that its derivative is essentially bounded, and that

$$\hat{w}(t_1) = \hat{z}, \quad \int_0^{t_1} \hat{w}(t)dt = \hat{\rho}. \tag{8.9}$$

Also,

$$STV\, \hat{w}(\cdot) = \frac{\tilde{\epsilon}}{2} [\|m_1\| + \|m_2\|] \leq \frac{4\|\hat{\rho}\|}{\tilde{\epsilon}} + 2\|\hat{z}\| \leq \frac{4\delta}{\tilde{\epsilon}} + 2\delta = 4\tilde{\epsilon} + 2\tilde{\epsilon}^2 < 7\tilde{\epsilon}. \tag{8.10}$$

Define the absolutely continuous functions $\bar{\rho}(\cdot)$, $\bar{z}(\cdot)$, $\zeta(\cdot)$, and $u(\cdot)$ from $[0,t_1]$ to E_3 as follows:

$$\bar{\rho}(t) = r_0 + \int_0^t [z(\tau;w)+w(\tau)+\hat{w}(\tau)]d\tau \ ,$$

$$\bar{z}(t) = v_0 + \int_0^t G(\bar{\rho}(\tau),\tau)d\tau \ , \tag{8.11}$$

$$\zeta(t) = z(t;w) - \bar{z}(t) \ ,$$

$$u(t) = \hat{w}(t) + w(t) + \zeta(t) \ .$$

Since the function $G(\cdot,\cdot)$ is bounded, it follows that the derivatives of $z(\cdot;w)$ and of $\bar{z}(\cdot)$, and consequently of $\zeta(\cdot)$ and of $u(\cdot)$, are essentially bounded. Now (8.11) and (2.3) imply that, for $0 \le t \le t_1$,

$$\bar{\rho}(t) = \rho(t;u) \ , \quad \bar{z}(t) = z(t;u) \ ,$$

and eqs. (8.8) therefore follow from (8.9), (2.3) and (8.11).

Further, $u(\cdot)-w(\cdot) = \hat{w}(\cdot)+\zeta(\cdot)$, so that (see (8.10))

$$\text{STV}(u-w) \le \text{STV}\,\hat{w} + \text{STV}\,\zeta < 7\tilde{\epsilon} + \text{STV}\,\zeta \ . \tag{8.12}$$

Since $\hat{w}(t) = 0$ for $0 \le t \le t_1-\tilde{\epsilon}$, it is a consequence of (8.11) and (2.3) that, in this interval, $\rho(t;w) = \bar{\rho}(t)$ and $z(t;w) = \bar{z}(t)$; i.e., $\zeta(t) = 0$ for $0 \le t \le t_1-\tilde{\epsilon}$. Because $\zeta(\cdot)$ is absolutely continuous,

$$\text{STV}\,\zeta = \int_0^{t_1} \|\dot{\zeta}(\tau)\|d\tau = \int_{t_1-\tilde{\epsilon}}^{t_1} \|\dot{\zeta}(\tau)\|d\tau \ .$$

But $\|\dot{\zeta}\| \le 2\tilde{G}$ (see (8.11) and (2.3)), so that STV $\zeta \le 2\tilde{G}\tilde{\epsilon}$; i.e. (see (8.12)),

$STV(\underset{\sim}{u}-\underset{\sim}{w}) < (7+2\tilde{G})\tilde{\epsilon} \leq \epsilon$. This completes the proof of lemma 8.

We now turn to the proof of theorem 4. Since $\hat{\underset{\sim}{u}}(\cdot) \in \mathcal{H}(t_1)$, $\hat{\underset{\sim}{u}}$ is of bounded variation and continuous from the right in $(0,t_1)$. Consequently, $\hat{\underset{\sim}{u}}(\cdot)$ has the representation $\hat{\underset{\sim}{u}}(\cdot) = \hat{\underset{\sim}{u}}_1(\cdot) + \hat{\underset{\sim}{u}}_2(\cdot)$, where $\hat{\underset{\sim}{u}}_1(\cdot)$ and $\hat{\underset{\sim}{u}}_2(\cdot)$ are in $\mathcal{B}(t_1)$, $\hat{\underset{\sim}{u}}_1(\cdot)$ is continuous, and $\hat{\underset{\sim}{u}}_2(\cdot)$ is the jump function of $\hat{\underset{\sim}{u}}(\cdot)$. Let $\tau_1, \tau_2, \ldots$ denote the points of discontinuity of $\hat{\underset{\sim}{u}}(\cdot)$ (or, equivalently, of $\hat{\underset{\sim}{u}}_2(\cdot)$). Without loss of generality we shall assume that they are infinite in number. Let $\bar{\underset{\sim}{v}}_i = \hat{\underset{\sim}{u}}_2(\tau_i) - \hat{\underset{\sim}{u}}_2(\tau_i^-)$ if $\tau_i \neq 0$; $\bar{\underset{\sim}{v}}_i = \hat{\underset{\sim}{u}}_2(0^+) - \hat{\underset{\sim}{u}}_2(0)$ if $\tau_i = 0$. Then

$$STV\,\hat{\underset{\sim}{u}} = STV\,\hat{\underset{\sim}{u}}_1 + STV\,\hat{\underset{\sim}{u}}_2\,, \qquad STV\,\hat{\underset{\sim}{u}}_2 = \sum_{i=1}^{\infty} \|\bar{\underset{\sim}{v}}_i\| < \infty\,.$$

Let $s(\cdot;\tau)$ denote the unit step function at $t = \tau$:

$$s(t;\tau) = \begin{cases} 0 \text{ for } t < \tau \\ 1 \text{ for } t > \tau, \end{cases} \qquad s(\tau;\tau) = \begin{cases} 1 \text{ if } \tau \neq 0 \\ 0 \text{ if } \tau = 0\,. \end{cases}$$

For each positive integer n, define the absolutely continuous functions $\underset{\sim}{w}_n'(\cdot)$, $\underset{\sim}{w}_n''(\cdot) \in \mathcal{B}(t_1)$ as follows:

$$\underset{\sim}{w}_n'(t) = \hat{\underset{\sim}{u}}_1(it_1/n) + [(nt/t_1)-i][\hat{\underset{\sim}{u}}_1((i+1)t_1/n) - \hat{\underset{\sim}{u}}_1(it_1/n)]$$

$$\text{for } \frac{it_1}{n} \leq t \leq (i+1)\frac{t_1}{n}\,, \quad i = 0,\ldots,n-1\,;$$

$$\underset{\sim}{w}_n''(\cdot) = \sum_{i=1}^{n} \bar{\underset{\sim}{v}}_i s(\cdot;\tau_i)\,.$$

Let $\hat{w}_n(\cdot) = w'_n(\cdot)+w''_n(\cdot)$. It is clear that $\hat{w}_n(t) \to \hat{u}(t)$ uniformly in $[0,t_1]$ as $n \to \infty$, that each $\hat{w}_n(\cdot)\in\mathcal{B}(t_1)$, and that

$$\mathrm{STV}\,\hat{w}_n = \mathrm{STV}\,w'_n + \sum_{i=1}^{n} \|\bar{v}_i\| \leq \mathrm{STV}\,\hat{u}_1 + \mathrm{STV}\,\hat{u}_2 = \mathrm{STV}\,\hat{u}. \qquad (8.13)$$

For every $\epsilon > 0$ and positive integer j, define open intervals $I_{j,\epsilon}$ as follows:

$$I_{j,\epsilon} = (0,\epsilon) \text{ if } \tau_j = 0 ; \quad I_{j,\epsilon} = (\tau_j-\epsilon,\tau_j) \text{ if } \tau_j > 0,$$

and let $r(\cdot;\tau_j,\epsilon)$ denote the absolutely continuous, real-valued functions defined as follows:

$$r(t;\tau_j,\epsilon) = \begin{cases} 0 & \text{for } 0 \leq t \leq \tau_j-\epsilon \\ \frac{1}{\epsilon}(t-\tau_j+\epsilon) & \text{for } \tau_j - \epsilon \leq t \leq \tau_j \\ 1 & \text{for } \tau_j \leq t \leq t_1 \end{cases} \quad \text{if } \tau_j > 0 ,$$

$$r(t;\tau_j,\epsilon) = \begin{cases} \frac{t}{\epsilon} & \text{for } 0 \leq t \leq \epsilon \\ 1 & \text{for } \epsilon \leq t \leq t_1 \end{cases} \quad \text{if } \tau_j = 0 .$$

Let $\epsilon_1,\epsilon_2,\ldots$ be a strictly decreasing sequence of positive numbers, with $\epsilon_n < 1/n$ for each n, such that, for every n, the intervals $I_{1,\epsilon_n},\ldots,I_{n,\epsilon_n}$ are mutually disjoint and all contained in $[0,t_1]$. Let the absolutely continuous function $w_n(\cdot)\in\mathcal{B}(t_1)$ be defined as follows:

$$w_n(\cdot) = w_n'(\cdot) + \sum_{i=1}^{n} r(\cdot;\tau_i,\epsilon_n)\overline{v}_i \ .$$

It is easily seen that the derivative of $w_n(\cdot)$ is essentially bounded, that

$$w_n(t) = \hat{w}_n(t) \text{ for } t \notin \bigcup_{i=1}^{n} I_{i,\epsilon_n}; \ \|w_n(t)-\hat{w}_n(t)\| < \|\overline{v}_i\| \text{ for } t \in I_{i,\epsilon_n}; \ i = 1,\ldots,n; \tag{8.14}$$

and that (see (8.13))

$$\text{STV } w_n \leq \text{STV } w_n' + \sum_{i=1}^{n} \|\overline{v}_i\| \leq \text{STV } \hat{u} \ . \tag{8.15}$$

Now, $\|\overline{v}_i\| \to 0$ as $i \to \infty$; i.e., for any fixed $\overline{\epsilon} > 0$, there is an $i' > 0$ such that $\|\overline{v}_i\| < \overline{\epsilon}$ for all $i > i'$. Consequently, it follows from (8.14) that $\|w_n(t)-\hat{w}_n(t)\| < \overline{\epsilon}$ for all $t \notin \bigcup_{i=1}^{i'} I_{i,\epsilon_n}$. Since, for each i, $I_{i,\epsilon_n} \supset I_{i,\epsilon_{n+1}}$ and $\bigcap_{n=1}^{\infty} I_{i,\epsilon_n} = \emptyset$, we conclude that $(w_n(t)-\hat{w}_n(t)) \to 0$ as $n \to \infty$ for every $t \in [0,t_1]$.

Recalling that $\hat{w}_n(t) \to \hat{u}(t)$, we obtain that

$$\lim_{n\to\infty} w_n(t) = \hat{u}(t) \text{ for all } t \in [0,t_1] \ . \tag{8.16}$$

Further, since $w_n(0) = 0$ for each n, it follows from (8.15) that the $w_n(\cdot)$ are uniformly bounded.

Appealing to lemmas 7 and 8, we can assert that there exist absolutely continuous functions $u_n(\cdot) \in \mathcal{B}(t_1)$ with essentially bounded derivatives such that

$$\rho(t_1;u_n) = \rho(t_1;\hat{u}) ,$$

$$z(t_1;u_n) + u_n(t_1) = z(t_1;\hat{u}) + \hat{u}(t_1) ,$$

and

$$\text{STV } (u_n - w_n) \to 0 \text{ as } n \to \infty .$$

Thus, each $u_n \in \mathscr{A}(t_1)$, and $(w_n(t) - u_n(t)) \to 0$ as $n \to \infty$ for all $t \in [0,t_1]$, which, by virtue of (8.16), implies that

$$\lim_{n\to\infty} u_n(t) = \hat{u}(t) \text{ for all } t \in [0,t_1] . \qquad (8.17)$$

Now STV $u_n \leq$ STV w_n+STV$(u_n - w_n)$. Therefore (see (8.15)), lim $\sup_{n\to\infty}$STV $u_n \leq$ STV $\hat{u}$. But it follows from (8.17) that lim $\inf_{n\to\infty}$STV $u_n \geq$ STV $\hat{u}$. Consequently, $\lim_{n\to\infty}$STV u_n exists and is equal to STV $\hat{u}$. This completes the proof of theorem 4.

According to theorems 1 and 2, there is good reason to expect that a solution $\hat{u}(\cdot)$ of the extended variational problem is a step function. The result of theorem 4, as well as the method for constructing the approximating functions in the proof thereof, together with eq. (2.5) indicate that, loosely speaking, <u>a minimum-fuel thrust program generally consists of a finite number of impulses</u>.

9. <u>Bounded thrust problem and limit theorem:</u> In this section we shall consider the original fixed terminal time variational problem described in section 2 in the presence of the additional constraint that $\|F(t)\| \leq \mu$ for all t, where $\mu < \infty$ is a given positive constant. This constraint is a mathematical representation of the physical fact that the magnitude of the thrust vector of a rocket engine is always limited.

The problem may be precisely formulated as follows. Preserving the notation of section 2, for given positive numbers t_1 and μ, let $\hat{\mathcal{F}}(t_1,\mu) =$ $= \{F(\cdot): F \in \mathcal{F}(t_1), \|F(t)\| \leq \mu$ for all $t \in [0,t_1]\}$. Then we shall consider the problem of finding (for given t_1 and μ) an element $\hat{F}(\cdot) \in \hat{\mathcal{F}}(t_1,\mu)$ such that $M(t_1;\hat{F}) \geq M(t_1;F)$ for every $F \in \hat{\mathcal{F}}(t_1,\mu)$. We shall refer to this problem as the μ-bounded problem. (It is to be understood in this section that all problems are fixed terminal time.)

We shall show that the μ-bounded problem always has a solution if μ is sufficiently large; that, in a certain sense, the solutions of the μ-bounded problem tend to a solution of the extended fixed terminal time problem when $\mu \to \infty$; and that the necessary conditions for the latter problem can, in a sense, be obtained by passing to the limit in the necessary conditions for the μ-bounded problem.

We shall first prove the existence theorem.

Theorem 5. If μ is sufficiently large, then $\hat{\mathcal{F}}(t_1,\mu)$ is not empty and there exists an element $\hat{F}(\cdot) \in \hat{\mathcal{F}}(t_1,\mu)$ such that $M(t_1;\hat{F}) \geq M(t_1;F)$ for every $F \in \hat{\mathcal{F}}(t_1,\mu)$; i.e., the μ-bounded problem admits a solution.

Proof. Let us show that $\hat{\mathcal{F}}(t_1,\mu)$ is not empty for μ sufficiently large.

According to theorems 3 and 4, there exists a function $u(\cdot) \in \mathcal{U}(t_1)$ with essentially bounded derivative. Let STV $u = \alpha$ and let $\|du(t)/dt\| < \beta < \infty$ for almost all $t \in [0,t_1]$. If $F(\cdot)$ is now defined by means of relations (2.5) and (2.6), it follows that $F(\cdot) \in \mathcal{F}(t_1)$ and $\|F(t)\| \leq (M_0\beta)$ for almost all $t \in [0,t_1]$. Changing the values of $F(t)$ for t in a set of measure zero if necessary, we conclude that $F(\cdot) \in \hat{\mathcal{F}}(t_1,\mu)$ whenever $\mu \geq M_0\beta$.

Denoting r_i by x_{i+3} and $\dot{r}_i$ by x_i (where i = 1,2, or 3), and (-M) by x_7, eqs. (2.1) and (2.2) may be rewritten as follows:

$$\begin{aligned} \dot{x}_i(t) &= x_{i-3}(t), \quad i = 4,5,6 , \\ \dot{x}_i(t) &= G_i(x_4,x_5,x_6,t) - \frac{T_i(t)}{x_7(t)} , \quad i = 1,2,3 , \\ \dot{x}_7(t) &= \frac{\|\underset{\sim}{T}(t)\|}{A} , \quad \underset{\sim}{T} = (T_1,T_2,T_3) . \end{aligned} \tag{9.1}$$

In order to show the existence of an element $\hat{\underset{\sim}{F}}$ in $\mathcal{F}(t_1,\mu)$ that maximizes $M(t_1;\underset{\sim}{F})$, we shall first consider a system which, in a sense, is more general than (9.1), and which, following Warga [12], we shall refer to as the relaxed system. Namely, we consider the following system of equations for the scalar variables $y_1,\ldots,y_7$:

$$\begin{aligned} \dot{y}_i(t) &= y_{i-3}(t), \quad i = 4,5,6 , \\ \dot{y}_i(t) &= G_i(y_4,y_5,y_6,t) - \frac{T_i(t)}{y_7(t)} , \quad i = 1,2,3 , \\ \dot{y}_7(t) &= \frac{\|\underset{\sim}{T}(t)\|+\tilde{T}(t)}{A} , \quad \underset{\sim}{T} = (T_1,T_2,T_3) . \end{aligned} \tag{9.2}$$

If $\underset{\sim}{F}(\cdot)$ is a measurable function from $[0,\infty)$ to E_3 and $\tilde{F}(\cdot)$ is a summable function from $[0,\infty)$ to E_1 such that the inequality $\int_0^\infty [\|\underset{\sim}{F}(t)\|+\tilde{F}(t)]dt < AM_0$ holds, we shall denote by $(y_1(t;\underset{\sim}{F},\tilde{F}),\ldots,y_7(t;\underset{\sim}{F},\tilde{F})) = \underset{\sim}{y}(t;\underset{\sim}{F},\tilde{F})$ the solution for $t \geq 0$ of eqs. (9.2), with $\underset{\sim}{T}(t) = \underset{\sim}{F}(t)$ and $\tilde{T}(t) = \tilde{F}(t)$, that assumes the initial values $(y_1(0;\underset{\sim}{F},\tilde{F}),\ldots,y_3(0;\underset{\sim}{F},\tilde{F})) = \underset{\sim}{v}_0$, $(y_4(0;\underset{\sim}{F},\tilde{F}),\ldots,y_6(0;\underset{\sim}{F},\tilde{F})) = \underset{\sim}{r}_0$, $y_7(0;\underset{\sim}{F},\tilde{F}) = -M_0$. Let $\mathcal{F}_R(t_1,\mu)$ denote the class of all pairs $(\underset{\sim}{F}(\cdot), \tilde{F}(\cdot))$

such that 1. $\underset{\sim}{F}(\cdot)$ is a measurable function from $[0,t_1]$ to E_3, 2. $\tilde{F}(\cdot)$ is a summable function from $[0,t_1]$ into $[0,\infty)$, 3. $\int_0^{t_1}[\|\underset{\sim}{F}(t)\|+\tilde{F}(t)]dt < AM_0$, 4. $\|\underset{\sim}{F}(t)\|+\tilde{F}(t) \leq \mu$ for all $t\in[0,t_1]$, and 5. $h_i(y_4(t_1;\underset{\sim}{F},\tilde{F}),\ldots,y_6(t_1;\underset{\sim}{F},\tilde{F}), y_1(t_1;\underset{\sim}{F},\tilde{F}),\ldots,y_3(t_1;\underset{\sim}{F},\tilde{F}),t_1) = 0$ for $i = 1,\ldots,\nu$. Then we shall consider the <u>relaxed</u> μ-<u>bounded problem</u> which consists in finding an element $(\hat{\underset{\sim}{F}}(\cdot), \bar{F}(\cdot))\in\mathcal{F}_R(t_1,\mu)$ such that $y_7(t_1;\hat{\underset{\sim}{F}},\bar{F}) \leq y_7(t_1;\underset{\sim}{F},\tilde{F})$ for every $(\underset{\sim}{F}(\cdot), \tilde{F}(\cdot))\in\mathcal{F}_R(t_1,\mu)$. According to a theorem of Warga [12, Theorem 3.3], such a minimizing element always exists so long as $\mathcal{F}_R(t_1,\mu)$ is not empty, and we shall show below that this set is not empty whenever μ is sufficiently large.

Indeed, it follows at once from (9.1) and (9.2) that, if $\tilde{F}_0(\cdot)$ denotes the function which vanishes identically, then, for every measurable function $\underset{\sim}{F}(\cdot)$ from $[0,t_1]$ to E_3 with $\int_0^{t_1}\|\underset{\sim}{F}(t)\|dt < AM_0$,

$$(y_1(t;\underset{\sim}{F},\tilde{F}_0),\ldots,y_3(t;\underset{\sim}{F},\tilde{F}_0)) = \dot{\underset{\sim}{r}}(t;\underset{\sim}{F}),$$

$$(y_4(t;\underset{\sim}{F},\tilde{F}_0),\ldots,y_6(t;\underset{\sim}{F},\tilde{F}_0)) = \underset{\sim}{r}(t;\underset{\sim}{F}),$$

$$y_7(t;\underset{\sim}{F},\tilde{F}_0) = -M(t;\underset{\sim}{F})$$

for every $t\in[0,t_1]$. It then follows at once that $\underset{\sim}{F}(\cdot)\in\hat{\mathcal{F}}(t_1,\mu)$ if and only if $(\underset{\sim}{F}(\cdot), \tilde{F}_0(\cdot))\in\mathcal{F}_R(t_1,\mu)$. Inasmuch as we have shown that $\hat{\mathcal{F}}(t_1,\mu)$ is not empty for μ sufficiently large, we conclude that $\mathcal{F}_R(t_1,\mu)$ is not empty for μ large enough.

We shall prove below that if $(\hat{\underset{\sim}{F}}(\cdot), \bar{F}(\cdot))$ is a solution of the relaxed μ-bounded problem (and we have shown that such a solution exists if μ is sufficiently large), then $\bar{F}(t) = \tilde{F}_0(t) = 0$ for almost all $t\in[0,t_1]$. By what was said above, this will imply that $\hat{\underset{\sim}{F}}(\cdot)\in\hat{\mathcal{F}}(t_1,\mu)$ and that $M(t_1;\hat{\underset{\sim}{F}}) =$

$= -y_7(t_1;\hat{F},\tilde{F}_o) \geq -y_7(t_1;F,\tilde{F}_o) = M(t_1;F)$ for every $F(\cdot) \in \mathcal{F}(t_1,\mu)$; i.e., $\hat{F}(\cdot)$ is a solution of the μ-bounded problem.

It easily follows from the Pontryagin maximum principle [8, Chapter 1] that if $(\hat{F}(\cdot), \bar{F}(\cdot))$ is a solution of the relaxed μ-bounded problem, then there exists a twice differentiable function $\psi(\cdot)$ from $[0,t_1]$ to E_3, and an absolutely continuous function $\psi(\cdot)$ from $[0,t_1]$ to E_1, with $\|\psi(\cdot)\|+\psi(\cdot)$ not vanishing identically, such that

$$\frac{d^2\psi(t)}{dt^2} = \left(\frac{\partial G(y_4(t;\hat{F},\bar{F}),\ldots,y_6(t;\hat{F},\bar{F}),t)}{\partial r}\right)^T \psi(t) , \quad 0 \leq t \leq t_1 ; \quad (9.3)$$

$$\frac{d\psi(t)}{dt} = -[y_7(t;\hat{F},\bar{F})]^{-2}[\psi(t)\cdot\hat{F}(t)] \text{ for almost all } t , \quad 0 \leq t \leq t_1 ; \quad (9.4)$$

$$\psi(t_1) \leq 0 ; \quad (9.5)$$

$$\psi(t)\,\frac{\|\hat{F}(t)\|+\bar{F}(t)}{A} - \frac{\psi(t)\cdot\hat{F}(t)}{y_7(t;\hat{F},\bar{F})} = \max_{\substack{V\in E_3 \\ V\geq 0 \\ \|V\|+V\leq\mu}} \left[\psi(t)\,\frac{\|V\|+V}{A} - \frac{\psi(t)\cdot V}{y_7(t;\hat{F},\bar{F})}\right],$$

for almost all t, $0 \leq t \leq t_1$. (9.6)

It follows from (9.6) that, for almost all $t\in[0,t_1]$, either $\bar{F}(t) = 0$ or $\|\psi(t)\| = 0$. Hence, if the zeros of $\|\psi(\cdot)\|$ are isolated, then $\bar{F}(t) = 0$ for almost all t. If the zeros of $\|\psi(\cdot)\|$ in $[0,t_1]$ are not isolated, then $\psi(t) \equiv 0$. For, if t_2 is an accumulation point of zeros of $\|\psi(\cdot)\|$, then $\psi(t_2) = \dot{\psi}(t_2) = 0$, which, because of the uniqueness of solutions of eqs. (9.3), implies that $\psi(t) \equiv 0$. If $\psi(t) \equiv 0$, it follows from (9.4) and (9.5) and the fact that ψ and ψ cannot both vanish identically, that $\psi(t) = \text{const.}<0$, and (9.6) then implies that $\|\hat{F}(t)\| = \bar{F}(t) = 0$ almost everywhere in $[0,t_1]$. Q.E.D.

According to the Pontryagin maximum principle as applied to eqs. (9.1), if $\hat{F}(\cdot)$ is any solution of the μ-bounded problem, there exists a twice differentiable function $\psi(\cdot)$ from $[0,t_1]$ to E_3 and an absolutely continuous function $\psi(\cdot)$ from $[0,t_1]$ to E_1, with $\|\psi(\cdot)\|+\psi(\cdot)$ not vanishing identically, such that

$$\ddot{\psi}(t) = \left(\frac{\partial G(r(t;\hat{F}),t)}{\partial r}\right)^T \psi(t) , \quad 0 \leq t \leq t_1 ; \qquad (9.7)$$

$$\dot{\psi}(t) = -[M(t;\hat{F})]^{-2}[\psi(t)\cdot\hat{F}(t)] \text{ for almost all } t , \quad 0 \leq t \leq t_1 ; \qquad (9.8)$$

$$\psi(t_1) \leq 0 ; \qquad (9.9)$$

$$\frac{\psi(t)\|\hat{F}(t)\|}{A} + \frac{\psi(t)\cdot\hat{F}(t)}{M(t;\hat{F})} = \max_{\substack{V\in E_3\\ \|V\|\leq\mu}} \left[\frac{\psi(t)\|V\|}{A} + \frac{\psi(t)\cdot V}{M(t;\hat{F})}\right], \text{ for almost all } t,\ 0\leq t\leq t_1; \qquad (9.10)$$

$$\dot{\psi}(t_1)\cdot[x-r(t_1;\hat{F})] = \psi(t_1)\cdot[y-\dot{r}(t_1;\hat{F})] \text{ for all } (x,y)\in T , \qquad (9.11)$$

where T is the hyperplane tangent to $H(t_1)$ at $(r(t_1;\hat{F}), \dot{r}(t_1;\hat{F}))$. We shall say that such a pair $(\psi(\cdot), \psi(\cdot))$ is an adjoint function which corresponds to $\hat{F}(\cdot)$.

It follows from (9.10) that, for almost all $t\in[0,t_1]$,

$$\hat{F}(t) = \begin{cases} 0 & \text{if } \|\psi(t)\| < \tilde{\psi}(t) \\ \alpha\psi(t), \text{ where } 0 \le \alpha \le \mu/\|\psi(t)\|, & \text{if } \|\psi(t)\| = \tilde{\psi}(t) \ne 0 \\ \mu\psi(t)/\|\psi(t)\| & \text{if } 0 \ne \|\psi(t)\| > \tilde{\psi}(t) , \end{cases} \tag{9.12}$$

$$\|\hat{F}(t)\| = \mu \qquad \text{if } 0 = \|\psi(t)\| > \tilde{\psi}(t) ,$$

where

$$\tilde{\psi}(t) = - M(t;\hat{F})\psi(t)/A . \tag{9.13}$$

It is easily verified that $\tilde{\psi}(\cdot)$ is differentiable on $[0,t_1]$ and that (see (9.8) and (2.2))

$$\frac{d\tilde{\psi}(t)}{dt} = \begin{cases} 0 & \text{if } \|\psi(t)\|-\tilde{\psi}(t) \le 0 \\ \frac{\mu}{AM(t;\hat{F})} [\|\psi(t)\|-\tilde{\psi}(t)] & \text{if } \|\psi(t)\|-\tilde{\psi}(t) \ge 0 . \end{cases} \tag{9.14}$$

Hence, $d\tilde{\psi}(t)/dt \ge 0$ for all t, $0 \le t \le t_1$.

Thus, let $\hat{F}(\cdot)$ be a solution of the μ-bounded problem, let $(\psi(\cdot),\psi(\cdot))$ be a corresponding adjoint function, and let $\tilde{\psi}(\cdot)$ be defined by (9.13). It follows as before that if $\psi(t) \not\equiv 0$ on $[0,t_1]$, then the zeros of $\|\psi(\cdot)\|$ are isolated. Further, if $\psi(t) \equiv 0$ on $[0,t_1]$, then (see (9.8), (9.9), (9.13) and (9.12)) $\psi(t) = \text{const.} < 0$, $\tilde{\psi}(t) > 0$ for all $t \in [0,t_1]$, and $\hat{F}(t) = 0$ almost everywhere in $[0,t_1]$. Note that $(\alpha\psi(\cdot), \alpha\psi(\cdot))$ is an adjoint function corresponding to $\hat{F}(\cdot)$ for every $\alpha > 0$. Thus, if $\{t: \hat{F}(t) \ne 0, 0 \le t \le t_1\}$ is of positive measure, then $\psi(t) \ne 0$ for some $t \in [0,t_1]$, and, multiplying $(\psi(\cdot),\psi(\cdot))$ by a suitable positive constant if necessary, we may assume that

$$\max_{0 \le t \le t_1} \|\psi(t)\| = 1 . \tag{9.15}$$

Adjoint functions $(\psi(\cdot),\psi(\cdot))$ that satisfy (9.15) will be said to be <u>normalized</u>.

Let $F_0(\cdot)$ denote the function from $[0,t_1]$ to E_3 defined by $F_0(t) = 0$ for all t, $0 \le t \le t_1$.

We now prove the following limit theorem.

Theorem 6. <u>Let</u> $\tilde{\mu}_1, \tilde{\mu}_2, \ldots$ <u>be a sequence of positive numbers such that</u> $\tilde{\mu}_i \to \infty$ <u>as</u> $i \to \infty$. <u>Then, if</u> $F_0(\cdot) \notin \mathcal{F}(t_1)$, <u>there exists a subsequence</u> $\mu_1, \mu_2, \ldots$ <u>of the</u> $\tilde{\mu}_i$, <u>solutions</u> $F_i(\cdot)$ <u>of the</u> μ_i<u>-bounded problems, normalized adjoint functions</u> $(\overline{\psi}_i(\cdot),\overline{\Psi}_i(\cdot))$ <u>corresponding to the</u> F_i, <u>and functions</u> $\hat{u}(\cdot)$ <u>and</u> $\psi(\cdot)$ <u>from</u> $[0,t_1]$ <u>to</u> E_3 <u>such that</u> $\psi(\cdot)$ <u>is twice differentiable and</u>

1. $\hat{u}(\cdot)$ <u>is a solution of the extended fixed terminal time problem,</u>

2. $$\overline{\psi}_i(t) \underset{i\to\infty}{\to} \psi(t), \tag{9.16}$$

3. $$-M(t;F_i)\overline{\Psi}_i(t)/A \underset{i\to\infty}{\to} 1, \tag{9.17}$$

4. $$\int_0^t \frac{F_i(s)}{M(s;F_i)} ds \underset{i\to\infty}{\to} \hat{u}(t), \tag{9.18}$$

5. $$r(t;F_i) \underset{i\to\infty}{\to} \rho(t;\hat{u}), \tag{9.19}$$

6. $$\dot{r}(t;F_i) \underset{i\to\infty}{\to} z(t;\hat{u})+\hat{u}(t), \tag{9.20}$$

7. $$M(t_1;F_i) \underset{i\to\infty}{\to} M_0 \exp[-A^{-1}STV\,\hat{u}], \tag{9.21}$$

where 2, 3, and 5 hold for every $t \in [0,t_1]$ and the convergence is uniform with respect to $t \in [0,t_1]$; and 4 and 6 hold for almost all $t \in [0,t_1]$ including 0, t_1 and the points of continuity of $\hat{u}(\cdot)$. Also, $\hat{u}(\cdot)$ and $\psi(\cdot)$ satisfy eqs. (5.1)-(5.4), where $\hat{\rho}(t) = \rho(t;\hat{u})$, and $\bar{r}$, $\bar{v}$, and T are defined as in the statement of theorem 1.

Proof. According to theorem 5, solutions of the μ-bounded problem exist when $\mu \geq \mu^*$, where $\mu^* < \infty$ is a sufficiently large positive constant. For every $\mu \geq \mu^*$, let $\hat{F}_\mu(\cdot) \in \mathcal{F}(t_1)$ be a solution of the μ-bounded problem. Since $F_0(\cdot) \notin \mathcal{F}(t_1)$, $\{t\colon \hat{F}_\mu(t) \neq 0,\ 0 \leq t \leq t_1\}$ is of positive measure for every $\mu \geq \mu^*$, and, by virtue of the immediately preceding discussion, there exists a normalized adjoint function, which we shall denote by $(\psi_\mu(\cdot),\psi_\mu(\cdot))$, corresponding to each $\hat{F}_\mu$ with $\mu \geq \mu^*$.

Let $u_\mu(\cdot) = \Phi(\hat{F}_\mu(\cdot))$; i.e.,

$$u_\mu(t) = \int_0^t \frac{\hat{F}_\mu(s)}{M(s;\hat{F}_\mu)}\, ds\ , \quad 0 \leq t \leq t_1\ . \tag{9.22}$$

It follows from the discussion in section 2 that, for each $\mu \geq \mu^*$, $u_\mu(\cdot) \in \mathcal{U}(t_1)$ and (see (2.6), (2.10), and (2.14))

$$\text{STV}\ u_\mu(\cdot) = A \ln\left[\frac{M_0}{M(t_1;\hat{F}_\mu)}\right]. \tag{9.23}$$

Since $\hat{F}_{\mu^*}(\cdot) \in \mathcal{F}(t_1,\mu)$ when $\mu \geq \mu^*$,

$$M(t_1;\hat{F}_\mu) \geq M(t_1;\hat{F}_{\mu^*}) \text{ for } \mu \geq \mu^*\ . \tag{9.24}$$

By virtue of (9.23) and (9.24), we can conclude that $\text{STV}\ u_\mu \leq \text{STV}\ u_{\mu^*}$ for all

$\mu \geq \mu^*$.

The functions $\psi_\mu(\cdot)$ are uniformly bounded by definition, and satisfy eq. (9.7) with $\hat{F}$ replaced by $\hat{F}_\mu$. Since $G(\cdot,\cdot)$ has bounded first partial derivatives the functions $\ddot{\psi}_\mu(\cdot)$ are also uniformly bounded. But if the ψ_μ and the $\ddot{\psi}_\mu$ are uniformly bounded, then the functions $\dot{\psi}_\mu(\cdot)$ must also possess this property. Consequently, the functions ψ_μ and $\dot{\psi}_\mu$ for $\mu \geq \mu^*$ are equicontinuous as well as uniformly bounded.

Just as was done in the proof of theorem 3, we can now show with the aid of relation (2.9) that there exist functions $\hat{u}(\cdot)$, $\psi(\cdot)$ and a subsequence $\mu_1, \mu_2, \ldots$ of the $\tilde{\mu}_i$ such that $\hat{u} \in \mathcal{H}(t_1)$, and, denoting ψ_{μ_i} by $\overline{\psi}_i$ and $\hat{F}_{\mu_i}$ by F_i, such that: (a) (9.16) and (9.19) are satisfied uniformly in $[0,t_1]$, (b) (9.18) and (9.20) hold for almost all $t \in [0,t_1]$ including 0, t_1 and the points of continuity of $\hat{u}(\cdot)$, (c) (5.1), (5.2), and (5.4) are satisfied, and (d) $\lim_{i\to\infty} \text{STV}\, u_{\mu_i}$ exists and

$$\text{STV}\, \hat{u} \leq \lim_{i\to\infty} \text{STV}\, u_{\mu_i} \quad . \tag{9.25}$$

We now shall verify (9.21). Let $\tilde{u}(\cdot) \in \mathcal{H}(t_1)$ be a solution of the extended fixed terminal time problem. Then (see theorem 4) there exist functions $\tilde{u}_i(\cdot) \in \mathcal{H}(t_1)$, $i = 1,2,\ldots,$and positive constants $M_1, M_2, \ldots$ such that $\lim_{n\to\infty} \text{STV}\, \tilde{u}_n = \text{STV}\, \tilde{u}$, and $\|d\tilde{u}_i(t)/dt\| < M_i < \infty$ for almost all $t \in [0,t_1]$ and every $i = 1,2,\ldots$. Setting $\tilde{F}_n = \Psi(\tilde{u}_n)$, we conclude on the basis of (2.5) and (2.6) that $\|\tilde{F}_n(t)\| \leq M_o M_n$ almost everywhere in $[0,t_1]$, i.e., (modifying the values of $\tilde{F}_n(t)$ for t in a set of measure zero, if necessary) $\tilde{F}_n \in \hat{\mathcal{F}}(t_1,\mu)$ for every $\mu \geq M_o M_n$. Consequently, for each n = 1,2,..., there is an integer I, depending on n, such that $M(t_1;F_i) \geq M(t_1;\tilde{F}_n)$ for every $i \geq I(n)$, or (see (9.23)), STV $u_{\mu_i} \leq$ STV $\tilde{u}_n$ for every $i \geq I(n)$. Therefore (see (9.25)),

$$\text{STV}\ \hat{u} \leq \lim_{i\to\infty} \text{STV}\ u_{\mu_i} \leq \lim_{i\to\infty} \text{STV}\ \tilde{u}_i = \text{STV}\ \tilde{u}\ .$$

But $\hat{u} \in \mathcal{H}(t_1)$ and $\tilde{u}$ is a solution of the extended fixed terminal time problem, so that STV $\tilde{u} \leq$ STV $\hat{u}$. Therefore, STV $\hat{u}$ = STV $\tilde{u}$, and $\hat{u}$ is a solution of this problem also. Consequently, (9.25) is actually an equality, and (9.21) now follows at once from (9.23).

We now verify (9.17). Denote $\psi_{\mu_i}(\cdot)$ by $\bar{\Psi}_i(\cdot)$, and let

$$\tilde{\Psi}_i(t) = - M(t;F_i)\ \bar{\Psi}_i(t)/A, \quad 0 \leq t \leq t_1, \quad i = 1,2,\ldots\ . \tag{9.26}$$

We shall show that $\tilde{\Psi}_i(0) \to 1$ as $i \to \infty$, and that $\tilde{\Psi}_i(t) \leq 1$ for every i = 1,2,... and $t\in[0,t_1]$. Since each $\tilde{\Psi}_i$ is differentiable, and $d\tilde{\Psi}_i(t)/dt \geq 0$ for each i and t, this will imply that $\tilde{\Psi}_i(t) \to 1$ uniformly in $[0,t_1]$ as $i \to \infty$, i.e., that (9.17) holds uniformly in $[0,t_1]$.

Let $E_i = \{t\colon \|\bar{\Psi}_i(t)\| > \tilde{\Psi}_i(t),\ 0 \leq t \leq t_1\}$, and let $|E_i|$ denote the Lebesgue measure of E_i. According to (9.12), $\|F_i(t)\| = \mu_i$ when $t\in E_i$, so that (see (2.2)) $0 \leq M(t_1;F_i) \leq M_o - \mu_i|E_i|/A$. Consequently,

$$\mu_i|E_i| \leq AM_o\ , \quad i = 1,2,\ldots, \tag{9.27}$$

and, since $\lim_{i\to\infty}\mu_i = \infty$, $|E_i| \to 0$ as $i \to \infty$.

Now suppose that $\tilde{\psi}_i(\bar{t}) > 1$ for some $\bar{t}\in[0,t_1]$ and some $i = 1,2,\ldots$. Since $\tilde{\psi}_i(\cdot)$ satisfies eq. (9.14) with $\underset{\sim}{F}$ replaced by $\underset{\sim}{F}_i$, μ by μ_i, and $\underset{\sim}{\psi}$ by $\bar{\underset{\sim}{\psi}}_i$, and $\|\bar{\underset{\sim}{\psi}}_i(t)\| \leq 1$ for every $t\in[0,t_1]$, it follows that $\tilde{\psi}_i(t) = \tilde{\psi}_i(\bar{t}) > 1$ for for every $t\in[0,t_1]$, and, by virtue of (9.12), that $\underset{\sim}{F}_i(t) = 0$ for almost all $t\in[0,t_1]$. This implies that $\underset{\sim}{F}_0\in\mathcal{F}(t_1)$, which is a contradiction, so that $\tilde{\psi}_i(t) \leq 1$ for every i and t.

Because $M(t;\underset{\sim}{F}_i) \geq M(t_1;\underset{\sim}{F}_i) \geq M(t_1;\hat{\underset{\sim}{F}}_{\mu^*}) = M^*$ (see (9.24)), and $\|\bar{\underset{\sim}{\psi}}_i(t)\| \leq 1$ for each i and $t\in[0,t_1]$, it follows from (9.14) that, for every i and t,

$$\frac{d\tilde{\psi}_i(t)}{dt} \leq \frac{\mu_i}{AM^*}\,[1-\tilde{\psi}_i(t)]c_{E_i}(t) \;,$$

where $c_{E_i}(t)$ is the characteristic function of E_i. Thus,

$$\frac{d\tilde{\psi}_i(t)}{dt} = \frac{\mu_i}{AM^*}\,[1-\tilde{\psi}_i(t)]c_{E_i}(t) - \tilde{\phi}_i(t) \;, \quad 0 \leq t \leq t_1 \;, \qquad (9.28)$$

where $\tilde{\phi}_i(t) \geq 0$ for all $t\in[0,t_1]$. But the solution of (9.28) is given by

$$\tilde{\psi}_i(t) = 1-\left\{\exp\left[-\int_0^t \frac{\mu_i}{AM^*}\,c_{E_i}(s)ds\right]\right\}\left\{1-\tilde{\psi}_i(0)+\int_0^t \tilde{\phi}_i(s)\exp\left[\int_0^s \frac{\mu_i}{AM^*}\,c_{E_i}(\tau)d\tau\right]ds\right\},$$

so that

$$\tilde{\psi}_i(t) \leq 1 - [1-\tilde{\psi}_i(0)]\exp\left[-\frac{\mu_i}{AM^*}\,|E_i|\right] \leq 1 - [1-\tilde{\psi}_i(0)]A^* \;, \quad 0 \leq t \leq t_1 \;, \qquad (9.29)$$

where $A^* = \exp\,[-M_0/M^*] > 0$ (see (9.27)). We shall show that (9.29) implies that $\tilde{\psi}_i(0) \to 1$ as $i \to \infty$.

For each i, let τ_i be any value in $[0,t_1]$ such that $\|\bar{\psi}_i(\tau_i)\| = 1$. (Such values exist by definition of the $\bar{\psi}_i = \psi_{\mu_i}$). If $\tilde{\psi}_i(\tau_i) = 1$, it follows from (9.29) and the fact that $\tilde{\psi}_i(0) \leq 1$ that $\tilde{\psi}_i(0) = 1$. Now suppose that $\tilde{\psi}_i(\tau_i) < 1$. Then $\tau_i \in E_i$, and let (τ_i', τ_i'') be the largest open interval contained in E_i which contains τ_i in its closure. We shall suppose that i is sufficiently large that $(\tau_i', \tau_i'') \neq (0,t_1)$. Then, $\|\bar{\psi}_i(\tau_i')\| = \tilde{\psi}_i(\tau_i')$ and/or $\|\bar{\psi}_i(\tau_i'')\| = \tilde{\psi}_i(\tau_i'')$. Without loss of generality, we shall assume that $\|\bar{\psi}_i(\tau_i')\| = \tilde{\psi}_i(\tau_i')$. Since $0 \leq \tau_i - \tau_i' \leq |E_i|$, and there is a constant $K > 0$ such that $\|d\bar{\psi}_i(t)/dt\| < K$ for each i = 1,2,... and $t \in [0,t_1]$, we have that

$$0 \leq 1 - \|\bar{\psi}_i(\tau_i')\| = \|\bar{\psi}_i(\tau_i)\| - \|\bar{\psi}_i(\tau_i')\| = \int_{\tau_i'}^{\tau_i} \frac{d}{dt} \|\bar{\psi}_i(t)\| dt \leq \int_{\tau_i'}^{\tau_i} \|\frac{d}{dt} \bar{\psi}_i(t)\| dt < K|E_i| .$$

But $|E_i| \to 0$ as $i \to \infty$, so that $1 - \|\bar{\psi}_i(\tau_i')\| = 1 - \tilde{\psi}_i(\tau_i')$ will be non-negative and arbitrarily small if i is sufficiently large. Consequently, we can conclude, by virtue of (9.29), that $\lim_{i\to\infty}[1-\tilde{\psi}_i(0)]A^* = 0$, i.e., that $\lim_{i\to\infty}\tilde{\psi}_i(0) = 1$. This completes the verification of (9.17).

It only remains to prove that (5.3) holds.

Let $G_i = \{t: \|\bar{\psi}_i(t)\| \geq \tilde{\psi}_i(t),\ 0 \leq t \leq t_1\}$. It follows from (9.22) and (9.12) that $du_{\mu_i}(s)/dt = 0$ when $s \notin G_i$, $0 \leq s \leq 1$, and that, for almost all $s \in G_i$, either $du_{\mu_i}(s)/dt$ is a non-negative scalar multiple of $\bar{\psi}_i(s)$ or $\bar{\psi}_i(s) = 0$. Hence,

$$\int_0^{t_1} [\bar{\psi}_i(t)]^T \frac{du_{\mu_i}(t)}{dt}\, dt = \int_{G_i} \|\bar{\psi}_i(t)\| \left\|\frac{du_{\mu_i}(t)}{dt}\right\| dt . \qquad (9.30)$$

If $t \in G_i$, $1 \geq \|\overline{\Psi}_i(t)\| \geq \tilde{\Psi}_i(t) \geq \tilde{\Psi}_i(0)$, so that, by virtue of (9.30),

$$\tilde{\Psi}_i(0) \int_{G_i} \left\| \frac{du_{\mu_i}(t)}{dt} \right\| dt \leq \int_0^{t_1} [\overline{\Psi}_i(t)]^T \frac{du_{\mu_i}(t)}{dt} dt \leq \int_{G_i} \left\| \frac{du_{\mu_i}(t)}{dt} \right\| dt. \quad (9.31)$$

Since (9.25) has been shown to be an equality, we have, by virtue of (2.14), that

$$\int_{G_i} \left\| \frac{du_{\mu_i}(t)}{dt} \right\| dt = \int_0^{t_1} \left\| \frac{du_{\mu_i}(t)}{dt} \right\| dt = \text{STV } u_{\mu_i} \underset{i\to\infty}{\to} \text{STV } \hat{u} . \quad (9.32)$$

Also, $\tilde{\Psi}_i(0) \to 1$ as $i \to \infty$, so that (9.31) and (9.32) yield that

$$\int_0^{t_1} [\overline{\Psi}_i(t)]^T \frac{du_{\mu_i}(t)}{dt} dt \underset{i\to\infty}{\to} \text{STV } \hat{u} . \quad (9.33)$$

Finally, it follows from (9.16), (9.18), (9.22) and the Helly-Bray theorem [6, p. 288] that

$$\int_0^{t_1} [\overline{\Psi}_i(t)]^T \frac{du_{\mu_i}(t)}{dt} dt = \int_0^{t_1} [\overline{\Psi}_i(t)]^T du_{\mu_i}(t) \underset{i\to\infty}{\to} \int_0^{t_1} [\psi(t)]^T d\hat{u}(t), \quad (9.34)$$

and (5.3) is now an immediate consequence of (9.33) and (9.34).

Corollary 1. Let $\hat{u}$ be a solution of the extended fixed terminal time problem, and suppose that $\hat{u} \neq F_0$. Then, if $\hat{F}_\mu$ is any solution of the μ-bounded problem (for every μ sufficiently large), we have that

1. $M(t_1;\hat{F}_\mu) \underset{\mu\to\infty}{\to} M_0 \exp[-A^{-1}\text{STV } \hat{u}]$.

Further, if $\hat{\mathbf{u}}(\cdot)$ *is the unique solution of the extended fixed terminal time problem, and* $\mathbf{u}_\mu$ *is given by* (9.22), *then*

2. $\mathbf{u}_\mu(t) \underset{\mu\to\infty}{\to} \hat{\mathbf{u}}(t)$,
3. $\dot{\mathbf{r}}(t;\hat{\mathbf{F}}_\mu) \underset{\mu\to\infty}{\to} \mathbf{z}(t;\hat{\mathbf{u}}) + \hat{\mathbf{u}}(t)$,
4. $\mathbf{r}(t;\hat{\mathbf{F}}_\mu) \underset{\mu\to\infty}{\to} \boldsymbol{\rho}(t;\hat{\mathbf{u}})$,

where 4 *holds uniformly in* $[0,t_1]$, *and* 2 *and* 3 *hold for almost all* $t\in[0,t_1]$ *including* $0,t_1$, *and the points of continuity of* $\hat{\mathbf{u}}(\cdot)$. *If, in addition, the function* $\boldsymbol{\psi}(\cdot)$ *that satisfies relations* (5.1)-(5.4) *is unique, and* $(\boldsymbol{\psi}_\mu,\psi_\mu)$ *is any normalized adjoint function corresponding to* $\hat{\mathbf{F}}_\mu$, *then the following limits exist uniformly in* $[0,t_1]$:

5. $\boldsymbol{\psi}_\mu(t) \underset{\mu\to\infty}{\to} \boldsymbol{\psi}(t)$,
6. $-M(t;\hat{\mathbf{F}}_\mu)\,\psi_\mu(t)/A \underset{\mu\to\infty}{\to} 1$.

The proof of the corollary is straightforward and is therefore omitted.

REFERENCES

1. D. F. Lawden, Optimal Trajectories for Space Navigation, Butterworths, London, 1963.

2. G. M. Ewing, A fundamental problem of navigation in free space, Quart. Appl. Math., 18 (1961), pp. 355-362.

3. L. W. Neustadt, Optimization, a moment problem, and nonlinear programming, J. SIAM, Ser. A.: Control, 2 (1964), pp. 33-53.

4. R. W. Rishel, An extended Pontryagin principle for control systems whose control laws contain measures, Mathematical Analysis Note No. 48 (1964), Aero-Space Division, Boeing Co., Seattle, Washington.

5. E. Hille and R. S. Phillips, Functional Analysis and Semi-Groups, Amer. Math. Soc., Providence, R.I., 1957.

6. L. M. Graves, The Theory of Functions of Real Variables, 2nd edition, McGraw-Hill, New York, N.Y., 1956.

7. C. L. Siegel, Vorlesungen über Himmelsmechanik, Springer-verlag, Berlin, 1956.

8. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience Division of John Wiley, New York, N.Y., 1962.

9. J. Warga, Necessary conditions for minimum in relaxed variational problems, J. Math. Anal. Appl., 4 (1962), pp. 129-145.

10. A. N. Kolmogorov and S. V. Fomin, Elements of the Theory of Functions and Functional Analysis, Vol. 1, Graylock, Rochester, N.Y., 1957.

REFERENCES (Concluded)

11. I. P. Natanson, Theory of Functions of a Real Variable, Vol. 1, Ungar, New York, N.Y., 1955.

12. J. Warga, Relaxed variational problems, J. Math. Anal. Appl., 4 (1962), pp. 111-128.

FOOTNOTES

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1. By a solution of eq. (2.2) we here mean an absolutely continuous function M(·) that satisfies (2.2) for almost all $t > 0$. The inequality $\int_0^\infty \|F\| dt < AM_0$ implies that $M(t) > \overline{M}_0$ for some positive constant $\overline{M}_0$ and all $t > 0$. Physically, the first inequality signifies that the rocket can not provide thrust once the fuel has been consumed. By a solution of eq. (2.1) we mean a function $r(\cdot)$, whose time derivative $\dot{r}(\cdot)$ exists (for all $t > 0$) with $\dot{r}(\cdot)$ absolutely continuous, that satisfies eq. (2.1) for almost all $t > 0$.

2. If $w(\cdot) \in \mathcal{B}(\bar{t})$, w has at most a denumerable number of points of discontinuity, and the discontinuities of w are of the first kind. By a solution of eqs. (2.3), with $u(t) = w(t)$, we here mean a continuously differentiable function $z(\cdot)$ that satisfies the first equation everywhere, and an absolutely continuous function $\rho(\cdot)$ that satisfies the second equation at all points of continuity of $w(\cdot)$.