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ACCELERATION OF A HYPERSONIC BOUNDARY LAYER APPROACHING A CORNER

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FOREWORD

This report contains the work submitted by G. R. Olsson to the Graduate School of The University of Michigan in August 1967 in partial fulfillment of the requirements for the degree of Doctor of Philosophy. The committee for the dissertation consisted of professors A. F. Messiter (chairman), T. C. Adamson, Jr., J. D. Murray, M. Sichel, and W. W. Willmarth.

ABSTRACT

An asymptotic description of the acceleration of a laminar hypersonic boundary layer approaching a sharp corner is obtained. The description assumes small interaction with the outer inviscid flow. Viscous forces are neglected except in a thin sublayer. The initial part of the expansion takes place over a distance $O(M_e\bar{\delta})$, where M_e is the external Mach number, and $\bar{\delta}$ is the boundary-layer thickness. Here the transverse pressure gradient is small, and a solution can be obtained analytically. Within a distance $O(\bar{\delta})$ from the corner, the effect of streamline curvature is essential, and a numerical solution is obtained by the method of integral relations for a single strip. The solution for surface pressure is compared with experimental results for a particular case, and an approximate velocity profile at the corner is calculated. Possibilities for improving the accuracy of the calculation, both by refining the numerical procedure and by including higher order effects, are considered.

CONTENTS

SYMBOLS

A	Parameter defined in equation 5.35
A_i	i = 0, 1. Constants defined in equation II-11
a	Speed of sound
a	(Sec. 4) $1 - \phi^2$
$a_m(\tilde{x})$	Coefficients in representation for $\tilde{\rho}\tilde{u}$ in equation 5.18
$a_m(\xi)$	Coefficients in representation for Θ in equation 6.13
B	Function in equation 5.194
B_i	i = 0, 1. Functions in equation II-7
B_{ij}	i, j = 1, 2. Functions defined in equations 5.138 through 5.141
b	(Sec. 4) ϕ^2
$b_m(\tilde{x})$	Coefficients in representation for $\tilde{\rho}\tilde{u}\tilde{v}$ in equation 5.19
$b_m(\xi)$	Coefficients in representation for Θ^{-1} in equation 6.13
C	Function in equation 5.197
C_i	i = 1, 2. Functions defined in equations 5.142 and 5.143
c	(Sec. 4) $1 - \phi^2(1 - \alpha^2\beta^2)$
$c_n(\tilde{x})$	Coefficients in representation for $\tilde{\rho}\tilde{u}$, equation 5.34
D	Function defined in equation 5.146
D_{ij}	i, j = 1, . . . , 4. Constants appearing in equations 5.171, 5.174, 5.177, 5.182, etc.
$d_n(\tilde{x})$	Coefficients in representation for $\tilde{\rho}\tilde{u}\tilde{v}$, equation 5.48
E	Entropy function in the isentropic relation
F_i	i = 0, 1, 2, 3. Functions defined in equations 5.110, 5.120, 5.121, and 5.155
$f(M_e)$	Scale factor in intermediate limit, equation 3.54
f_j, f_k	Weighting functions in the method of integral relations
G	Function defined in equation 5.151
g	Transformed stream function defined by equations I-27 and I-28
H	Function defined in equation 5.157
$H_i^{(j)}$	Nonhomogeneous terms in equations 5.177, 5.182, and 5.189 through 5.192

h	Enthalpy
h	(Sec. 5) Function defined in equation 5.45
$I(\beta)$	Function defined following equation 4.20
I_i	$i = 1, 2$. Integrals defined in equations 4.22
i	Index
$J_i^{(j)}$	Constants defined in equations 5.163 through 5.166
j	Index
K_i	$i = 1, 2$. Integrals defined in equations 4.20
k	Heat conductivity
k	0, 1 for a wedge or a cone, respectively
k	Index
L	Length of the body measured along the surface
L_i	$i = 1, \ldots, n$. Arbitrary functions introduced in equation 5.1
M	Mach number, q/a
M	Number of strips in the calculation by the method of integral relations for the initially subsonic part of boundary layer
m	Index
N	Number of strips in the calculation by the method of integral relations for the initially supersonic part of boundary layer
n	Index
P_i	$i = 1, \ldots, n$. Arbitrary functions introduced in equation 5.1
Pr	Prandtl number
p	Pressure
Q_i	$i = 1, \ldots, n$. Arbitrary functions introduced in equation 5.2
q	Magnitude of the velocity
R	Reynolds number, $\bar{\rho}\bar{u}\bar{L}/\bar{\mu}$
r	Radial coordinate
r	Body radius
s	Coordinate (equal to $\hat{x}$) in the direction of the free stream
T	Temperature
U	Nondimensional velocity at the outer edge of the sublayer
u	Velocity component parallel to the wall
V_1	v_1/q_1

v	Velocity component normal to the wall
w	Transformed normal velocity component defined in equation 6.7
X	Transformed x coordinate in equation 5.80
x	Coordinate measured parallel to the wall
y	Coordinate measured normal to the wall
y_i	Dependent variables defined in equations 4.43 and I-38
Z_i	$i = 1, \ldots, 4$. Functions defined in equation 4.46
α	$\dot{g}(0) = 0.4696$
α	y_1/δ_c in integral-relations analysis
α_k	y_k/δ_c
β	Similarity variable defined in equation I-25
β_n	Function defined in equation 5.38
γ	Ratio of specific heats
$\Delta(x)$	Layer thickness introduced in material following equation 5.1
δ	Boundary-layer thickness
δ^*	Boundary-layer displacement thickness
ϵ	$1 - \phi$
ζ	Transformed coordinate defined in equation 6.21
ζ	(App. I) $\beta - 1.21678$
η	Transformed normal coordinate defined in equation 6.1
Θ	$(\partial u^{\dagger}/\partial\eta)^{-1}$
θ	Flow deflection angle
λ	Constant in equations II-11 and II-12
μ	Viscosity coefficient
ξ	Transformed x coordinate defined in equation 6.1
ρ	Density
σ	Shock wave inclination angle
τ	Typical flow deflection
ϕ	Transformed s coordinate defined in equation 4.9
Ψ	Transformed stream function defined in equation I-23
ψ	Stream function
ω	Exponent in power-law viscosity-temperature relation, equation I-6

Subscripts

b	Condition in the base region
c	Critical condition
e	Condition at the outer edge of the undisturbed boundary layer
i	(App. III and fig. 4) Value at the surface which would be predicted by inviscid-flow theory
i, j, k, m, n	Indices
max	Condition at infinite Mach number
M	Condition on the strip boundary $\tilde{y} = y_M$
M + N	Condition on the strip boundary $\tilde{y} = \delta_c$
o	Condition as $\tilde{y} \to 0$
SL	Sublayer condition
t	Stagnation condition
w	Wall condition
δ	Condition at the outer edge of the disturbed layer
∞	Condition in the undisturbed free stream
1	Condition in the undisturbed boundary layer for $\bar{x}/L = 1$
0, 1, 2	Indices which indicate conditions on the strip boundaries $\tilde{y} = y_k$, $k = 0, 1, 2$
—	(bar over letter) Dimensional quantity
$\wedge$	Condition in the first outer limit
~	Condition in the second outer limit
†	Condition in the sublayer limit
•	(dot over letter) $\partial/\partial\beta$ or $\partial/\partial\xi$
'	$d/d\tilde{x}$

ACCELERATION OF A HYPERSONIC BOUNDARY LAYER APPROACHING A CORNER

1
INTRODUCTION

The expansion of an inviscid supersonic flow at a sharp corner takes place through a centered Prandtl-Meyer expansion fan. Of course, this description neglects any effect of the viscous boundary layer at the solid surface. For nonzero viscosity, the expansion actually begins somewhat upstream from the corner and is completed somewhat downstream from the corner. At any given Mach number, the acceleration of the boundary layer, here assumed to be laminar, takes place over a distance which decreases as the Reynolds number increases. The details of the boundary-layer expansion are of interest, for example, in relation to the calculation of base flows. The present investigation was motivated by the need for proper initial conditions for the study of the hypersonic near wake. This work is concerned specifically with the portion of the expansion which occurs just upstream from the corner, for the case of hypersonic flow over a slender wedge or cone.

The inviscid hypersonic flow past a slender wedge or cone is described by hypersonic small-disturbance theory [1]. Although the shock wave is rather close to the surface, there is a significant range of Mach numbers and Reynolds numbers for which the boundary-layer thickness is small compared with the shock-layer thickness, while the boundary layer remains laminar. For this range, the approximate boundary-layer velocity and temperature profiles can be obtained by neglecting interaction with the outer flow.

At the base of the wedge or cone, the boundary layer expands rather rapidly and in a complicated manner. Since the acceleration and the pressure gradient are quite large in a relatively small region near the corner, the flow is approximately an inviscid rotational flow. Since, in this approximation, the no-slip condition is violated for a short distance upstream from the corner, a viscous sublayer, with a thickness that is small compared to the boundary-layer thickness, must also exist. As in conventional boundary-layer theory, and for the same reasons, the sublayer is considered to have only a small effect on the expansion of the outer part of the boundary layer.

It has been experimentally determined that the base pressure, for the range of parameters of present interest, is considerably smaller than the pressure in the region between the shock wave and the body surface. Except for the sublayer, the entire boundary layer, therefore, accelerates to supersonic speed. Downstream from the corner, when the complicated interaction has been completed, the inviscid-flow approximation predicts a highly rotational outer shear layer and a velocity discontinuity corresponding to the sublayer described above. Below the sublayer is a region of recirculating flow at lower velocity and more nearly constant pressure.

Existing solutions for the near-wake region have been obtained without complete knowledge of the initial shear-layer profiles. For example, the integral method used by Reeves and Lees [2] to analyze the near wake of a circular cylinder is extended by Golik, Webb, and Lees [3] in a study of the wake behind a wedge. In each of these papers, the importance of the boundary-layer expansion at a corner is pointed out, but the flow near a sharp corner is not studied in detail.

Some recent attempts have been made to study the portion of the expansion just downstream from a corner. The model proposed by Weinbaum [4] assumes a highly rotational outer shear layer with velocity discontinuity across a sublayer located essentially at the zero streamline. In this model, flow near the zero streamline expands to the base pressure, but an overexpansion occurs elsewhere in the boundary layer. A curved, centered expansion fan from the corner leads to reflected waves which are predominantly expansions; these, in turn, are reflected as compressions from the constant-pressure zero streamline and ultimately coalesce to form the lip-shock wave. Weiss [5] incorporates this model in a calculation of the near wake of a wedge. He uses the full Navier-Stokes equations in the recirculation region, boundary-layer equations in the sublayer, and a rotational-characteristics calculation in the remainder of the shear layer; an iteration procedure is used in obtaining a solution. However, detailed pressure measurements by Hama [6] suggest that even the flow close to the surface overexpands, and, in fact, completely turns the corner. The lip shock occurs as a separation shock, originating very close to the base at a point just below the corner. A more detailed theoretical investigation of the lip-shock wave formation would require accurate information concerning profiles of the flow properties at the corner.

It is evident from Hama's [6] data for laminar flow that a significant portion of the pressure drop does indeed occur upstream from the corner. In fact, since the base pressure is sufficiently low, the air very close to the surface (i.e., just outside the thin sublayer) is expected to accelerate at least to sonic speed, and details downstream from the corner will not influence the upstream flow. Therefore, the flow upstream can be studied without further knowledge of the lip-shock wave.

The upstream influence of the corner is associated with the propagation of disturbances through the subsonic part of the boundary layer. A basic assumption in this study is that the extent of upstream influence is small compared with the length of the body. In the present approximation, it is found that this requirement is satisfied provided that the boundary-layer thickness is small compared with the shock-layer thickness. Because the changes take place in a small region, the problem is approximately two-dimensional for a cone as well as for a wedge.

The idea of describing abrupt changes in a boundary layer by the inviscid-flow equations has appeared in the literature in several other contexts. For example, Morkovin [7] has observed experimentally the effect of an expansion wave impinging upon a boundary layer on the wall of a supersonic wind tunnel and has successfully predicted the post-interaction velocity profile by an inviscid-flow calculation. That is, given the initial conditions in the boundary layer, he calculated the final velocity profile from the Bernoulli equation, the entropy equation, and the measured value of the final pressure. Except for the effect of a viscous sublayer, his prediction agrees well with the experimentally-determined velocity profile. Lighthill [8] uses a similar concept in analyzing the interaction of a supersonic boundary layer with a disturbance sufficiently weak that separation does not occur. He introduces small perturbations on a parallel shear flow and neglects viscous shear forces except in a sublayer. Zakkay and Tani [9] consider a problem of boundary-layer acceleration at a sharp corner without considering separation. Their interest is primarily in the boundary-layer development downstream from the corner, and they assume that the changes close to the corner are described by inviscid-flow equations. The concept of a sublayer again appears. For the same case, a calculation describing changes close to the corner is given by Hunt and Sibulkin [10]. They use a momentum integral and assume that pressure is constant along radial lines.

After the present study had been completed it was brought to the authors' attention that closely related work has been carried out by Neiland and Sychev [11] and by Matveeva and Neiland [12]. Neiland and Sychev consider compressible boundary-layer flow at a rounded corner having radius of curvature $O(\bar{\delta})$. For a distance $O(\bar{\delta})$ in the stream direction, they obtain inviscid-flow equations, except in a viscous sublayer of thickness $O\left(R_w^{-1/4}\bar{\delta}\right)$, where the boundary-layer equations are required. Matveeva and Neiland use a similar approximation to formulate a description of a supersonic boundary layer approaching a sharp corner. They carry out a one-strip calculation by the method of integral relations. The numerical integration is started by use of an asymptotic solution valid upstream where the perturbation in pressure is characterized by a small nondimensional parameter, Δ. In this region, nonlinear inviscid-flow equations are required at a distance $O(\bar{\delta}\Delta^{1/2})$ from the wall, and the disturbances extend over a distance $O(\bar{\delta}\Delta^{-1/2})$ in the stream direction.

In the present study, the case of zero wall heat transfer and unity Prandtl number is considered, primarily because of simplifications in the equations. The initial boundary-layer profile is then obtained quite easily (app. 1). As the boundary layer approaches the corner, the pressure drop causes the boundary layer to become thinner because the changes in stream-tube area for the subsonic portion of the layer are dominant. Flow deflections at the outer edge of the boundary layer remain small, even if relative pressure changes are of the order one, because the flow is hypersonic. In the early stages of the expansion, the flow deflection is also small throughout the boundary layer. Since the profile eventually becomes entirely supersonic, this behavior cannot persist all the way to the corner; eventually the spreading of streamlines in the supersonic region must dominate. In a second region close to the corner, the streamline deflection remains small at the outer edge but can become quite large inside the layer because the fluid is free to turn inward when it reaches the corner. The boundary layer in these two regions might be called subcritical and supercritical. This distinction is discussed by Lees and Reeves [13].

A numerical solution of the problem to be discussed here has been obtained by Baum [14], who used a finite-difference method to solve the boundary-layer equations. For the initially supersonic part of the flow, an acceleration term was retained in the transverse momentum equation. Consequently, the normal pressure gradient was nonzero. Weiss and Nelson [15] have obtained an approximate solution by using a stream-tube calculation (zero normal pressure gradient) for the fluid which is initially at subsonic speed and a Prandtl-Meyer expansion for the initially supersonic part. In the present investigation, approximate equations are derived which are expected to be correct in an asymptotic sense for the case of a sufficiently thin hypersonic boundary layer, a simple method is shown for obtaining approximate numerical results, and the procedures for studying the largest neglected terms are considered.

A more detailed physical description of the flow is given in section 2. In section 3 the asymptotic nature of the approximation is discussed, order estimates are given for the two regions of inviscid flow and for the sublayer, approximate differential equations are obtained for each of these regions, and the appropriate matching conditions are given. An analytical solution is derived in section 4 for the upstream region in which the normal pressure gradient is negligible. In section 5, the full inviscid-flow equations for the region closer to the corner are studied by using the method of integral relations. Numerical results are obtained for a one-strip calculation, and a procedure for carrying out a two-strip analysis is described. An attempt at a generalization to an arbitrary number of strips is also discussed. A composite expansion of the solutions for the wall-pressure ratio is obtained and compared with an experiment [6] for a particular case, and an approximate velocity profile at the corner is calculated. In section 6, an approximate formulation of the sublayer problem is derived by using the method of integral relations. Results and conclusions are summarized in section 7.

2
PHYSICAL DESCRIPTION OF THE FLOW

The inviscid flow over a slender pointed body at high Mach number (see fig. 1) can be described by using the approximations of hypersonic small-disturbance theory. The parameter $(M_\infty \tau)^{-1}$ is taken to be of order one, where τ is some typical value of the flow deflection, for example, the body thickness ratio. Order estimates for the flow variables can be obtained from the shock-wave relations [1]. The shock is inclined at a small angle of order τ, and, therefore, the velocity component in the direction of the free stream remains approximately unchanged. The velocity $\bar{u}_e$ just outside the boundary layer is approximately equal to the freestream velocity $\bar{u}_\infty$ (and, in fact, to the maximum velocity $\bar{q}_{max}$). Relative changes in pressure and temperature at the shock wave are of order one or larger. At the outer edge of the boundary layer, the pressure $\bar{p}_e$ is of order $\bar{\rho}_\infty \bar{u}_\infty^2 \tau^2$, the temperature $\bar{T}_e$ is of order $M_\infty^2 \tau^2 \bar{T}_\infty$, and the density $\bar{\rho}_e$ is of order $\bar{\rho}_\infty$. It follows that the Mach number M_e is of order τ^{-1}.

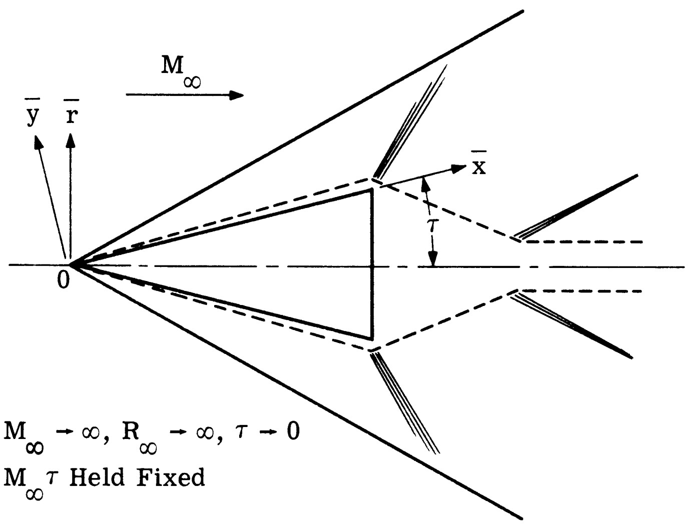

FIGURE 1. SLENDER BODY IN A HYPERSONIC FLOW

If the boundary-layer thickness is small compared with the distance from the body surface to the shock wave, then the inviscid-flow equations remain approximately correct in the region between the boundary layer and the shock wave. For a boundary layer with zero wall heat transfer, the thickness is proportional to $R_w^{-1/2}$, where R_w is a Reynolds number based on $\bar{u}_e$ and on thermodynamic properties evaluated at the surface. Then the assumption of a thin boundary layer requires that a viscous interaction parameter $M_e R_w^{-1/2}$ be small. If the body is a wedge or cone, the pressure is constant in the boundary layer. The temperature in the boundary layer is large, of order $M_e^2 \overline{T}_e$, and the density is therefore small. Since the mass flow is small, the boundary-layer thickness $\overline{\delta}$ can be taken equal to the displacement thickness $\overline{\delta}^*$ [16].

It is known from experiment that for a wide variety of body shapes in high-speed flow, the pressure $\overline{p}_b$ in a neighborhood of the base of the body is considerably smaller than $\overline{p}_e$. Thus, the nondimensional pressure drop $(\overline{p}_e - \overline{p}_b)/\overline{p}_e$ at the base of the body is of order unity, and the boundary layer, in separating from the body, will undergo a significant acceleration. Outside the boundary layer, this acceleration influences the flow through a change in the boundary-layer displacement thickness. The pressure varies with the flow deflection angle θ_δ at the outer edge of the boundary layer according to the Prandtl-Meyer formula for a simple-wave expansion. Inside the boundary layer, the large pressure drop at the corner causes the pressure and velocity gradients, $\overline{p}_{\overline{x}}$ and $\overline{u}_{\overline{x}}$, to increase greatly over their values in the upstream, undisturbed boundary layer, while the viscous shear stress remains of the same order as farther upstream.

These remarks suggest that the accelerating flow in the boundary layer (see fig. 2) can be described approximately by inviscid-flow equations. Propagation of disturbances upstream through the subsonic portion of the boundary layer will cause a significant portion of the acceleration to occur near the surface of the body upstream from the corner. However, the no-slip condition at the body surface cannot be satisfied by a solution to inviscid-flow equations. Therefore, a viscous sublayer must exist in which the viscous shear stress is of the same order as the streamwise pressure gradient $\overline{p}_{\overline{x}}$ and the inertia term $\overline{\rho}\overline{u}\overline{u}_{\overline{x}}$. The balance of viscous and inertia terms in the streamwise momentum equation provides the estimate of the order of magnitude of the thickness $\overline{\delta}_{SL}$ of the sublayer, which is found to be considerably

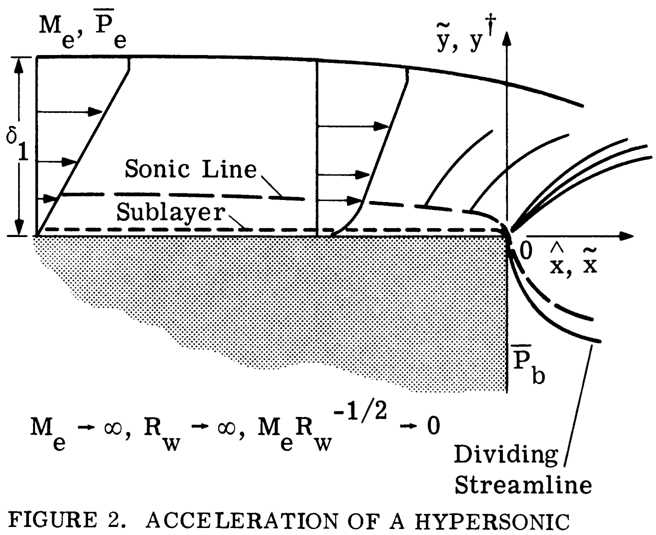

FIGURE 2. ACCELERATION OF A HYPERSONIC BOUNDARY LAYER APPROACHING A CORNER

smaller than the boundary-layer thickness, provided that the interaction parameter $M_e R_w^{-1/2}$ is small. The sublayer flow is described by the boundary-layer equations with a pressure gradient.

In a first approximation (as in conventional boundary-layer theory), the sublayer is ignored and an inviscid-flow calculation is made with the normal velocity component $\bar{v}$ set equal to zero at the wall. Then, from a knowledge of the pressure and velocity distributions along the wall, calculated from the inviscid equations, the sublayer equations can be solved, giving the variation in sublayer displacement thickness along the wall. In a second approximation (not carried out in this work), the normal velocity component $\bar{v}$ at the wall would be related to the rate of change in the sublayer displacement thickness $d\bar{\delta}_{SL}{*}/d\bar{x}$ while, otherwise, the inviscid-flow equations would still apply. Thus, for most of the boundary layer, the primary effect of viscosity is the variation in sublayer displacement thickness.

If the base pressure is sufficiently low, a streamline at the surface (i.e., just outside the viscous sublayer) will accelerate to the sonic condition at the corner. The sonic line is not expected to intersect the surface upstream from the corner because streamlines near the surface would have to bend away from the surface as the pressure continues to decrease. If the pressure is at least as low as the base pressure immediately downstream from the corner, as expected from experiment [6], the flow at the surface must have reached supersonic speed at the corner. A similar situation occurs for the flow in a convergent nozzle exhausting to a low pressure. The sonic condition must occur at the nozzle exit to permit the flow to adjust to the ambient pressure. Thus, in the present problem, the sonic line will intersect the corner in a first approximation, and the portion of the acceleration of the boundary layer which takes place upstream from the corner can be analyzed, independent of a knowledge of the base pressure.

The flow deflection angle θ will be equal to zero at the wall and comparable in magnitude with M_e^{-1} at the outer edge of the accelerating boundary layer. Two possibilities arise for the order of magnitude of θ in the layer. To match θ to the value $\theta_\delta = O\left(M_e^{-1}\right)$ at the outer edge, one might anticipate a region where the flow deflection angle is comparable in magnitude with M_e^{-1} throughout the boundary layer. In such a region, the streamline curvature is small. Therefore, the normal pressure gradient can be neglected, and the flow is described by inviscid-boundary-layer equations. (For another application of these equations, see reference 17.) A second region might occur in which θ is of order unity throughout the layer. The boundary condition at the outer edge, in the first approximation, would then require that θ be equal to zero. In this region, the normal and streamwise pressure gradients would be comparable in magnitude because of significant streamline curvature. A considerable divergence of streamlines will actually occur because, in the immediate neighborhood of the corner, a streamline just ouside the viscous sublayer turns rather sharply around the corner.

The two possible choices for the order estimate of θ correspond to two rather different physical effects. A decrease in the pressure will cause the subsonic portion of the layer to contract and the supersonic portion to widen. Initially, the subsonic portion of the layer is dominant, and $d\bar{\delta}^*/d\bar{p} > 0$. This is designated as the subcritical condition [13] and is analogous to subsonic flow in a convergent nozzle. The pressure change generated is communicated smoothly through the subsonic portion of the layer a considerable distance upstream. Lighthill [8] finds the inverse logarithmic decrement of upstream influence for small disturbances in a subcritical shear layer to be

$$M_e^2\left(1 - M_e^2\right)^{-1/2}\int_0^{\bar{\delta}} M^{-2}(\bar{y})[1 - M^2(\bar{y})]\,d\bar{y} \qquad (2.1)$$

where M(0) is the Mach number at a point just outside of the viscous sublayer. Clearly, in an accelerating boundary layer, a point may be reached at which the integral in equation 2.1 vanishes. At such a "critical point," some average Mach number in the layer is sonic, and small disturbances can propagate upstream only through a distance of the same order as the boundary-layer thickness. When the expression in equation 2.1 is set equal to zero, the condition

$$\bar{\delta}^{-1}\int_0^{\bar{\delta}} M^{-2}(\bar{y})\,d\bar{y} \equiv \langle M^{-2}\rangle = 1 \qquad (2.2)$$

results. Therefore, $\langle M^{-2}\rangle$ is the appropriate function for determining whether or not the layer is subcritical. In a subcritical flow, $\langle M^{-2}\rangle > 1$, while, in a supercritical flow, we have $\langle M^{-2}\rangle < 1$. In the supercritical region, we would expect $d\bar{\delta}^*/d\bar{p} < 0$ because the widening of streamlines in the supersonic region is dominant as the pressure continues to decrease. Since the flow deflection at the outer edge is not expected to change sign, the boundary layer in the present problem can become supercritical only in a region where the stream tubes near the wall can be displaced inward. It is shown in the next section that this effect can occur within a distance of order $\bar{\delta}$ upstream from the corner.

3
ASYMPTOTIC REPRESENTATIONS

3.1. LIMIT PROCESSES

The equations of hypersonic small-disturbance theory for flow past slender bodies are obtained from the full inviscid-flow equations by taking the limit

$$M_\infty \rightarrow \infty, \ \tau \rightarrow 0$$

$$\bar{x}/L, \qquad \bar{r}/\tau L, \qquad M_\infty \tau \text{ held fixed} \qquad (3.1)$$

Here $\bar{x}$ and $\bar{r}$ are dimensional coordinates measured from the front of the body (see fig. 1), M_∞ is the free-stream Mach number, τ is the body thickness ratio, and L is the length of the body. For a sufficiently high Reynolds number, the shape of the shock wave for a cone or a wedge is given in the form

$$\bar{r}/\tau\bar{x} = \text{constant} \tag{3.2}$$

In this approximation, the Mach number behind the shock wave is of order τ^{-1} (see app. I).

The approximate equations describing a hypersonic laminar boundary layer on a slender body are obtained in the limit

$$M_e \to \infty, \qquad R_w \to \infty, \qquad M_e R_w^{-1/2} \to 0$$
$$\bar{x}/L, \qquad \bar{y}/R_w^{-1/2}L \text{ held fixed} \tag{3.3}$$

where M_e is the Mach number just outside the boundary layer, and R_w is the Reynolds number based upon $\bar{u}_e$, L, and the thermodynamic properties evaluated at the wall. The condition $M_e R_w^{-1/2} \to 0$ arises from the requirement that the ratio of boundary-layer thickness to shock-layer thickness vanish in the limit 3.3 (see app. I).

In the present study, the nondimensional parameters, in the case of a chemically inert, laminar, continuum flow, may be chosen as

$$M_e, R_w, \gamma, \text{Pr} \tag{3.4}$$

where γ is the ratio of specific heats, and Pr is the Prandtl number. We will consider a limit

$$M_e \to \infty, \qquad R_w \to \infty, \qquad M_e R_w^{-1/2} \to 0$$
$$\gamma, \text{Pr held fixed} \tag{3.5}$$

Since we will be concerned with a small region near the corner at the base of the body, both the $\bar{x}$ and $\bar{y}$ coordinates will be stretched in some manner.

The equations of motion for two-dimensional planar, nonreacting, laminar, continuum flow are

$$(\bar{\rho}\bar{u})_{\bar{x}} + (\bar{\rho}\bar{v})_{\bar{y}} = 0 \tag{3.6}$$

$$\bar{\rho}(\bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}}) = -\bar{p}_{\bar{x}} + (\bar{\mu}\bar{u}_{\bar{y}})_{\bar{y}} + \ldots \tag{3.7}$$

$$\bar{\rho}(\bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}}) = -\bar{p}_{\bar{y}} + \ldots \tag{3.8}$$

$$\bar{\rho}(\bar{u}\bar{h}_{\bar{x}} + \bar{v}\bar{h}_{\bar{y}}) - (\bar{u}\bar{p}_{\bar{x}} + \bar{v}\bar{p}_{\bar{y}}) = (\bar{k}\bar{T}_{\bar{y}})_{\bar{y}} + \bar{\mu}\bar{u}_{\bar{y}}^2 + \ldots \tag{3.9}$$

For simplicity, the terms shown for viscous and heat conduction effects are only those important in the boundary-layer equations. In the case of axisymmetric flow past a cone, it will be shown that the effect of body curvature on the acceleration of the boundary layer at the corner of the base of the body is negligible in a first approximation.

We now consider the boundary conditions to be imposed at the wall and at the outer edge of the accelerating boundary layer. At $\bar{y} = 0$ we have

$$\bar{u}(\bar{x}, 0) = \bar{v}(\bar{x}, 0) = 0 \tag{3.10}$$

As the boundary layer undergoes rapid acceleration, it will initially become thinner if it is in the subcritical condition in which $d\bar{\delta}^*/d\bar{p} > 0$, where $\bar{\delta}^*$ is the displacement thickness of the boundary layer (see sec. 2). Then the pressure at the outer edge of the layer will be related to the flow deflection by the simple-wave relation in hypersonic small-disturbance theory [1].

$$\frac{\bar{p}(\bar{x}, \bar{\delta})}{\bar{\rho}_e \bar{u}_e^2} = \frac{1}{\gamma M_e^2}\left[1 + \frac{\gamma - 1}{2} M_e \theta(\bar{x}, \bar{\delta})\right]^{2\gamma/(\gamma - 1)} \tag{3.11}$$

Notice that the boundary-layer thickness $\bar{\delta}$ in the limit 3.5 is equal to the displacement thickness $\bar{\delta}^*$ (see eqs. I-34, I-35, and ref. 16), so that

$$\theta(\bar{x}, \bar{\delta}) = d\bar{\delta}^*/d\bar{x}\left[1 + O\left(M_e^{-2}\right)\right] \tag{3.12}$$

with

$$\bar{v}/\bar{u} = \tan \theta$$
$$\sim \theta$$

when θ is small. From the limit 3.1 and when $\Delta\bar{p}/\bar{p} = O(1)$, we have

$$\theta(\bar{x}, \bar{\delta}) = O\left(M_e^{-1}\right) \tag{3.13}$$

If in addition to the limit 3.5, $\bar{y}/R_w^{-1/2}L$ and some stretched $\bar{x}$ coordinate are held fixed in a manner such that

$$(\bar{x}/L) - 1 \to 0 \tag{3.14}$$

the equations obtained in a first approximation will not contain terms representing viscous effects. Regardless of the limit chosen, the order of magnitude of the flow deflection should be such that a first approximation to the continuity equation shows a balance between streamline divergence and change of mass flux. If $\Delta\bar{p}/\bar{p} = O(1)$, it follows that

$$\theta = O(\Delta\bar{y}/\Delta\bar{x}) \tag{3.15}$$

where $\Delta\bar{x}$ and $\Delta\bar{y}$ are taken to be of the same order as the relevant lengths, respectively, in the $\bar{x}$ and $\bar{y}$ directions.

If $\tilde{y} \equiv \bar{y}/R_w^{-1/2}L$ is held fixed, any of four possible sets of equations might be obtained, consistent with the limits 3.5 and 3.14. For $\hat{x} \equiv [(\bar{x}/L) - 1]/M_e R_w^{-1/2}$ held fixed, equation 3.15 gives

$$\theta = O\left(M_e^{-1}\right) \tag{3.16}$$

throughout the layer; the $\bar{y}$-momentum equation simplifies to $\bar{p}_{\bar{y}} \approx 0$, and no further approximation is made in the boundary condition (3.11). For $\tilde{x} \equiv [(\bar{x}/L) - 1)]/R_w^{-1/2}$ held fixed, equation 3.15 gives

$$\theta = O(1) \tag{3.17}$$

in the boundary layer; terms for both pressure and inertia appear in the first approximation to the $\bar{y}$-momentum equation. Since $\theta = O\left(M_e^{-1}\right)$ at the outer edge, the leading term in θ is required to approach zero as $\bar{y} \to \bar{\delta}$, and equation 3.11 is replaced by the requirement that $\bar{p}$ remain bounded as $\bar{y} \to \bar{\delta}$. For the class of limits $\hat{x} \to 0$ but $\tilde{x} \to -\infty$, we obtain the approximate form both for the $\bar{y}$-momentum equation and for the boundary condition at $\bar{y} = \bar{\delta}$. This case turns out to be important for obtaining higher approximations. Finally, for the class of limits $\hat{x} \to -\infty$ with $(\bar{x}/L) - 1 \to 0$, we find

$$\theta = o\left(M_e^{-1}\right) \tag{3.18}$$

throughout the boundary layer. Here the flow properties are only slightly perturbed from their undisturbed values, a situation related to that studied by Lighthill [8].

Following the physical arguments of the previous section, we will assume that limits for $\hat{x}$ or $\tilde{x}$ held fixed mustboth be considered, with $\hat{x}$ fixed to correspond to the subcritical condition and $\tilde{x}$ fixed to correspond to the supercritical condition. The choices of $\hat{x}$ and $\tilde{x}$ show the same dependence on Reynolds number as those given in references 11 and 12. Since in each case the no-slip condition (3.10) is lost, we will refer to those as the first and second outer limits, respectively, and we will later introduce a sublayer limit in which $\tilde{y} \to 0$ at a prescribed rate. Specification of the sublayer limit is determined, as in ordinary boundary-layer theory, by the requirement that terms for viscosity and inertia in the $\bar{x}$-momentum equation (3.7) be of the same order of magnitude.

The notation to be introduced for the boundary-layer limit, the first outer limit, the second outer limit, and the sublayer limit are summarized in table I.

3.2. FIRST OUTER LIMIT

The first outer limit is described by the conditions

$$\begin{gathered} M_e \to \infty,\ R_w \to \infty,\ M_e R_w^{-1/2} \to 0 \\ \hat{x},\ \tilde{y},\ \gamma,\ \mathrm{Pr}\ \text{held fixed} \end{gathered} \tag{3.19}$$

TABLE I. COORDINATE NOTATION AND LEADING TERMS IN ASYMPTOTIC EXPANSIONS

Original Variables	Boundary Layer	First Outer Limit	Second Outer Limit	Sublayer
$\bar{x}$	$x = \frac{\bar{x}}{L}$	$\hat{x} = \frac{\bar{x}/L - 1}{M_e R_w^{-1/2}}$	$\tilde{x} = \frac{\bar{x}/L - 1}{R_w^{-1/2}}$	$\tilde{x} = \frac{\bar{x}/L - 1}{R_w^{-1/2}}$
$\bar{y}$	$\tilde{y} = \frac{\bar{y}/L}{R_w^{-1/2}}$	$\tilde{y} = \frac{\bar{y}/L}{R_w^{-1/2}}$	$\tilde{y} = \frac{\bar{y}/L}{R_w^{-1/2}}$	$y^\dagger = \frac{\bar{y}/L}{R_w^{-3/4}}$
$\bar{u}/\bar{u}_e$	$u(x, \tilde{y})$	$\hat{u}(\hat{x}, \tilde{y})$	$\tilde{u}(\tilde{x}, \tilde{y})$	$U(\tilde{x})u^\dagger(\tilde{x}, y^\dagger)$
$\bar{v}/\bar{u}_e$	$R_w^{-1/2}v(x, \tilde{y})$	$M_e^{-1}\hat{v}(\hat{x}, \tilde{y})$	$\tilde{v}(\tilde{x}, \tilde{y})$	$R_w^{-1/4}U(\tilde{x})v^\dagger(\tilde{x}, y^\dagger)$
$\bar{p}/\bar{\rho}_w\bar{u}_e^{2}$	$p_e = \frac{\gamma - 1}{2\gamma}$	$\hat{p}(\hat{x})$	$\tilde{p}(\tilde{x}, \tilde{y})$	$p^\dagger(\tilde{x})$
$\bar{\rho}/\bar{\rho}_w$	$\rho(x, \tilde{y})$	$\hat{\rho}(\hat{x}, \tilde{y})$	$\tilde{\rho}(\tilde{x}, \tilde{y})$	$\rho^\dagger(\tilde{x}, y^\dagger)$

where $\hat{x} \equiv [(\bar{x}/L) - 1]\big/M_e R_w^{-1/2}$ and $\tilde{y} \equiv \bar{y}\big/R_w^{-1/2}L$. We wish to obtain the appropriate asymptotic representations for $\bar{u}$, $\bar{v}$, $\bar{p}$, and $\bar{\rho}$ and the system of equations and boundary or initial conditions applicable in a first approximation. We note that in a boundary layer in the limit 3.3

$$\bar{u}/\bar{u}_e = O(1) \tag{3.20}$$

$$\bar{\rho}/\bar{\rho}_e = O\left(M_e^{-2}\right) \tag{3.21}$$

These relations are also applicable in the limit 3.19. Thus from equations 3.16 and 3.20 we have

$$\bar{v}/\bar{u}_e = O\left(M_e^{-1}\right) \tag{3.22}$$

and if $\Delta\bar{p}/\bar{p}$ is assumed to be of order one, equation 3.21 gives

$$\Delta\bar{p}\Big/\left(\bar{\rho}_w\bar{u}_e^{2}\right) = O(1) \tag{3.23}$$

From the $\bar{x}$-momentum equation (3.7), we see that

$$\begin{aligned}\Delta\bar{u}/\bar{u}_e &= O\left[\Delta\bar{p}\Big/\left(\bar{\rho}_w\bar{u}_e^{2}\right)\right] \\ &= O(1)\end{aligned} \tag{3.24}$$

In the $\bar{y}$-momentum equation (3.8), we have

$$\frac{\overline{\rho}\,\overline{u}\,\overline{v}_{\bar{x}}}{\overline{\rho}_w\overline{u}_e^2} = O\left(M_e^{-2}R_w^{1/2}L^{-1}\right) \tag{3.25}$$

Since $\Delta\bar{y}$ is of the order $R_w^{-1/2}L$, in a first approximation, $\bar{p}_{\bar{y}} \approx 0$, and $\bar{p}$ is only a function of $\hat{x}$.

From equations 3.20 through 3.25, we see that the asymptotic representations for $\bar{u}$, $\bar{v}$, $\bar{p}$, and $\bar{\rho}$ in the limit 3.19 are given by

$$\begin{aligned}
\bar{u}/\bar{u}_e &\sim \hat{u}(\hat{x}, \tilde{y}) + \dots \\
\bar{v}/\bar{u}_e &\sim M_e^{-1}\hat{v}(\hat{x}, \tilde{y}) + \dots \\
\bar{p}/\bar{\rho}_w\bar{u}_e^2 &\sim \hat{p}(\hat{x}) + \dots \\
\bar{\rho}/\bar{\rho}_w &\sim \hat{\rho}(\hat{x}, \tilde{y}) + \dots
\end{aligned} \tag{3.26}$$

Substitution of equations 3.26 into equations 3.6 through 3.9 provides the relations for the first approximation in the limit 3.19

$$(\hat{\rho}\hat{u})_{\hat{x}} + (\hat{\rho}\hat{v})_{\tilde{y}} + o(1) = 0 \tag{3.27}$$

$$\hat{\rho}\left(\hat{u}\hat{u}_{\hat{x}} + \hat{v}\hat{u}_{\tilde{y}}\right) = -\hat{p}_{\hat{x}} + O\left(M_eR_w^{-1/2}\right) \tag{3.28}$$

$$0 = -\hat{p}_{\tilde{y}} + O\left(M_e^{-2}\right) \tag{3.29}$$

$$\hat{\rho}\left(\hat{u}\hat{h}_{\hat{x}} + \hat{v}\hat{h}_{\tilde{y}}\right) = \hat{u}\hat{p}_{\hat{x}} + O\left(M_e^{-2}\right) + O\left(M_eR_w^{-1/2}\right) \tag{3.30}$$

where $\hat{h} \equiv \bar{h}/\bar{u}_e^2$. Since the equations in the first outer limit are inviscid-flow equations, the no-slip condition $\bar{u}(\bar{x}, 0) = 0$ must be dropped, and only the requirement

$$\hat{v}(\hat{x}, 0) = 0 \tag{3.31}$$

is retained. For convenience we introduce a nondimensional, stretched boundary-layer thickness

$$\delta = \bar{\delta}/R_w^{-1/2}L \tag{3.32}$$

which remains finite in the limit 3.19. Then the pressure is related to the flow deflection at the outer edge of the layer from equations 3.11 and 3.13 by

$$\hat{p}(\hat{x}) = \frac{\gamma - 1}{2\gamma}\left[1 + \frac{\gamma - 1}{2}\hat{v}(\hat{x}, \delta)\right]^{2\gamma/(\gamma-1)} \tag{3.33}$$

Upstream we have the initial condition

$$\hat{u}(-\infty, \tilde{y}) = u_1(\tilde{y}) \tag{3.34}$$

where $u_1(\tilde{y})$ is the velocity profile found from the solution to the boundary-layer equations with $\bar{x}/L = 1$ (see app. I).

For a large and negative $\hat{x}$, it follows from equations 3.11 and 3.18 that the pressure disturbances are small, and the flow problem becomes similar to the one studied by Lighthill [8]. Lighthill found that the region of disturbed flow extends a distance of order $R_w^{-3/8}L$ in the stream direction. The variation in the displacement thickness of a viscous sublayer has a first-order effect on the perturbations in the outer, inviscid portion of the layer and is of order $R_w^{-5/8}L$ in magnitude:

$$\left.\begin{aligned} \hat{x} &= O\left(R_w^{1/8}\right) \\ \bar{\delta}^*_{SL}/\bar{\delta} &= O\left(R_w^{1/8}\right) \end{aligned}\right. \tag{3.35}$$

Depending upon the order of magnitude of the pressure disturbance, there are two possible order estimates for the change $\Delta\bar{u}$ in the velocity of any fluid element in the sublayer, namely

$$\Delta\bar{u} = \begin{cases} O(\bar{u}) \\ o(\bar{u}) \end{cases} \tag{3.36}$$

In the first case, the sublayer equations are nonlinear, and a balance of the orders of magnitude of $\bar{\rho}\bar{u}\bar{u}_{\bar{x}}$, $\bar{p}_{\bar{x}}$, and $(\bar{\mu}\bar{u}_{\bar{y}})_{\bar{y}}$ in equation 3.7 with

$$\Delta\bar{x} = O\left(R_w^{-3/8}L\right)$$

$$\Delta\bar{y} = O\left(R_w^{-5/8}L\right)$$

provides the order estimates

$$\bar{u}/\bar{u}_e = O\left(R_w^{-1/8}\right) \tag{3.37}$$

$$\Delta\bar{p}/\left(\bar{\rho}_w\bar{u}_e^2\right) = O\left(R_w^{-1/4}\right) \tag{3.38}$$

In the outer, inviscid region, the orders of magnitude of $\bar{\rho}\bar{u}\bar{u}_{\bar{x}}$ and $\bar{p}_{\bar{x}}$ in equation 3.7 are equated, giving, along a streamline,

$$\Delta\bar{u}\Big/\bar{u}_e = O\left(R_w^{-1/4}\right) \tag{3.39}$$

In the second case in equation 3.36, the sublayer equations may be linearized, and $\Delta\bar{p}\Big/\left(\bar{\rho}_w\bar{u}_e^2\right) = o\left(R_w^{-1/4}\right)$. (Note that Lighthill's order estimates may be reconstructed by a systematic consideration of orders of magnitude in the limit $R_w \rightarrow \infty$.)

The solution for $\hat{u}(\hat{x}, \tilde{y})$ is obtained in section 4. It is shown in equation 4.29 that as $\hat{x} \rightarrow -\infty$, $\hat{u}(\hat{x}, \tilde{y})$ has the form

$$\hat{u}(\hat{x}, \tilde{y}) = \begin{cases} u_1(\tilde{y}) + O(\hat{x}^{-2}), \ \tilde{y} > 0 \\ \\ O(\hat{x}^{-1}), \ \tilde{y} = 0 \end{cases} \tag{3.40}$$

Thus, if $\hat{x} = O\left(R_w^{1/8}\right)$, the order of magnitude of $\hat{u}(\hat{x}, \tilde{y})$, as found from equation 3.40, agrees with the order estimates given in equations 3.37 and 3.39. That is, the velocity profile obtained in the first outer limit has the possibility of matching upstream with a solution to the problem described by Lighthill.

3.3. SECOND OUTER LIMIT

The second outer limit is described by the conditions

$$\left.\begin{array}{c} M_e \rightarrow \infty, \ R_w \rightarrow \infty, \ M_e R_w^{-1/2} \rightarrow 0 \\ \tilde{x}, \ \tilde{y}, \ \gamma, \ \mathrm{Pr} \text{ held fixed} \end{array}\right\} \tag{3.41}$$

where $\tilde{x} \equiv [(\bar{x}/L) - 1]\Big/R_w^{-1/2}$, and $\tilde{y} \equiv \bar{y}\Big/\left(R_w^{-1/2}L\right)$. We wish to obtain the appropriate asymptotic representations for $\bar{u}$, $\bar{v}$, $\bar{p}$, and $\bar{\rho}$, and the system of equations and boundary conditions applicable in a first approximation. The order estimates 3.20 and 3.21 are also valid in the limit 3.41, and, thus, from equations 3.15 and 3.20, and 3.41, we have

$$\bar{v}/\bar{u}_e = O(1) \tag{3.42}$$

In order for the inertia and pressure gradient terms to balance in the $\bar{x}$- and $\bar{y}$-momentum equations 3.7 and 3.8, we then must have

$$\Delta\bar{p}\Big/\left(\bar{\rho}_w\bar{u}_e^2\right) = O(1) \tag{3.43}$$

$$\Delta\bar{u}\Big/\bar{u}_e = O(1) \tag{3.44}$$

From equations 3.20, 3.21, and 3.42 through 3.44, we see that the asymptotic representations for $\bar{u}$, $\bar{v}$, $\bar{p}$, and $\bar{\rho}$ in the limit 3.41 are

$$
\begin{aligned}
\bar{u}/\bar{u}_e &\sim \tilde{u}(\tilde{x}, \tilde{y}) + \ldots \\
\bar{v}/\bar{u}_e &\sim \tilde{v}(\tilde{x}, \tilde{y}) + \ldots \\
\bar{p}\Big/\left(\bar{\rho}_w \bar{u}_e^2\right) &\sim \tilde{p}(\tilde{x}, \tilde{y}) + \ldots \\
\bar{\rho}/\bar{\rho}_w &\sim \tilde{\rho}(\tilde{x}, \tilde{y}) + \ldots
\end{aligned}
\tag{3.45}
$$

Substitution of equations 3.45 into equations 3.6 through 3.9 provides the relations for the first approximation in the limit 3.41.

$$(\tilde{\rho}\tilde{u})_{\tilde{x}} + (\tilde{\rho}\tilde{v})_{\tilde{y}} + o(1) = 0 \tag{3.46}$$

$$\tilde{\rho}(\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}}) = -\tilde{p}_{\tilde{x}} + O\left(R_w^{-1/2}\right) \tag{3.47}$$

$$\tilde{\rho}(\tilde{u}\tilde{v}_{\tilde{x}} + \tilde{v}\tilde{v}_{\tilde{y}}) = -\tilde{p}_{\tilde{y}} + O\left(R_w^{-1/2}\right) \tag{3.48}$$

$$\tilde{\rho}(\tilde{u}\tilde{h}_{\tilde{x}} + \tilde{v}\tilde{h}_{\tilde{y}}) = \tilde{u}\tilde{p}_{\tilde{x}} + \tilde{v}\tilde{p}_{\tilde{y}} + O\left(R_w^{-1/2}\right) \tag{3.49}$$

where $\tilde{h} \equiv \bar{h}\big/\bar{u}_e^2$. Because these relations for the second outer limit are the inviscid-flow equations, the no-slip condition $\bar{u}(\bar{x}, 0) = 0$ must be abandoned, and only the requirement

$$\tilde{v}(\tilde{x}, 0) = 0 \tag{3.50}$$

is retained at the wall. At the outer edge of the layer, $\bar{v}/\bar{u}$ is of the order M_e^{-1} so that the boundary condition on $\tilde{v}$ at $\tilde{y} = \delta$ becomes

$$\tilde{v}(\tilde{x}, \delta) = 0 \tag{3.51}$$

while $\tilde{p}(\tilde{x}, \delta)$ must remain bounded. The equations 3.46 through 3.49 are elliptic when the local Mach number

$$M = \left[(\tilde{\rho}\tilde{u}^2\big/\gamma\tilde{p})\right]^{1/2} \tag{3.52}$$

is less than one. Thus, it is appropriate in this limit to specify a downstream condition. As discussed in section 2, for a sufficiently low base pressure, the sonic line will intersect the corner, $\tilde{x} = \tilde{y} = 0$, in a first approximation where the effect of the viscous sublayer is neglected. Therefore, we have the condition

$$M(0, 0) = 1 \tag{3.53}$$

To obtain initial conditions upstream, we assume that for any given flow quantity the solutions in the first and second outer limits can be matched. That is, we assume that both solutions are valid for some class of intermediate limits such that

$$\begin{gathered} \tilde{x} \to -\infty, \ \tilde{x}/M_e \to 0 \\ \tilde{x}/f(M_e) \text{ fixed} \end{gathered} \tag{3.54}$$

where $1 \ll f(M_e) \ll M_e$ as $M_e \to \infty$. In a first approximation the matching appears straightforward, and it is expected that the matching conditions become simply

$$\tilde{u}(-\infty, \tilde{y}) = \hat{u}(0, \tilde{y}) \tag{3.55}$$

$$\tilde{p}(-\infty, \tilde{y}) = \hat{p}(0) \tag{3.56}$$

$$\tilde{v}(-\infty, \tilde{y}) = 0 \tag{3.57}$$

See reference 18 for a further discussion of matching.

For higher approximations, the situation is quite different. As $\tilde{x} \to -\infty$, the asymptotic expansion of the normal velocity component $\bar{v}$ for the second outer limit (3.41) gives

$$\bar{v}/\bar{u}_e \sim (\text{constant})\ \tilde{x}^{-3}$$

as $\tilde{x} \to -\infty$. See the representation for $\bar{v}$ described by equations 5.48 and 5.93, for example. In the first outer limit, the asymptotic expansion of $\bar{v}$, for $\hat{x} = \tilde{x}/M_e \to 0$, is found from the solutions given by equations 4.17, 4.18, and 4.33 to be

$$\begin{aligned} \bar{v}/\bar{u}_e &\sim (\text{constant})\ M_e^{-1} |\hat{x}|^{-1/2} \\ &\sim (\text{constant})\ M_e^{-1} \left|\tilde{x}/M_e\right|^{-1/2} \end{aligned}$$

as $\tilde{x}/M_e \to 0$. Thus, these two expansions for $\bar{v}$ do not have the same functional form, and the higher order matching cannot be carried out. Since these expressions are of the same order of magnitude when $\tilde{x} = O\left(M_e^{1/5}\right)$, it is expected that approximate differential equations must also be derived for $f(M_e) = O\left(M_e^{1/5}\right)$ in equation 3.54, i.e., for the distinguished limit in which $\tilde{x}/M_e^{1/5}$ is held fixed. It can be shown that in this limit the order estimates for the perturbations in the dependent variables along a streamline are

$$\Delta\bar{u}/\bar{u}_e = O\left(M_e^{-2/5}\right)$$

$$\bar{v}/\bar{u}_e = O\left(M_e^{-3/5}\right)$$
$$\Delta\bar{p}/\left(\bar{\rho}_w\bar{u}_e^2\right) = O\left(M_e^{-2/5}\right)$$
$$\Delta\bar{\rho}/\bar{\rho}_w = O\left(M_e^{-2/5}\right)$$

In the governing equations in this limit, both the approximation $\bar{p}_{\bar{y}} \approx 0$ employed in the first outer limit (3.19) and the approximation $\bar{v}(\bar{x}, \bar{\delta}) \approx 0$ applied in the second outer limit are valid. A solution to these equations would be required as part of a higher order approximation.

In equations 3.6 through 3.9, the effect of body curvature has been neglected. The extent of upstream influence of the corner in which the disturbances are still a first-order effect is found to be

$$(\bar{x}/L) - 1 = O\left(M_e R_w^{-1/2}\right)$$

Since the radius of a slender cone is equal to $\tau\bar{x}$, the relative variation in the radial coordinate $\bar{r}$ when $\hat{x} \equiv [(\bar{x}/L) - 1]/M_e R_w^{-1/2}$ is held fixed is

$$\Delta\bar{r}/\bar{r}_1 = O\left(M_e R_w^{-1/2}\right)$$

where $\bar{r}_1 = \tau L$ is the radius of the base of the cone. Thus, the variation in $\bar{r}$ is relatively small, and may be classed as a second-order effect.

3.4. COMPOSITE EXPANSIONS OF SOLUTIONS IN THE FIRST AND SECOND OUTER LIMITS

As long as the asymptotic expansions obtained in various limits in a solution to a singular perturbation problem have a common region of validity, a single uniformly valid expansion can be constructed from them. One method of constructing composite expansions is by additive composition. The sum of the expansions is corrected by subtracting the part they have in common so that it is not counted twice [18].

In the present problem, we will construct a composite expansion of the solutions in the first and second outer limits to provide the necessary boundary values in the sublayer problem. Following the rule for additive composition we have, to the first order,

$$\frac{\bar{u}\left(\tilde{x}, \tilde{y}; M_e, R_w\right)}{\bar{u}_e} \sim \hat{u}\left(\frac{\tilde{x}}{M_e}, \tilde{y}\right) + \tilde{u}(\tilde{x}, \tilde{y}) - \hat{u}(0, \tilde{y}) + \ldots \tag{3.58}$$

where $\tilde{x}/M_e$ has been substituted for $\hat{x}$ and where the common part, to the first order, is found from equation 3.55. Since only M_e appears in the expansion, it is necessary, in this approach, to specify M_e but not R_w in carrying out the solution to the viscous sublayer equations. When the sublayer solution has been obtained, all three expansions may be combined to give, in a first approximation,

$$\frac{\bar{u}\left(\tilde{x}, \tilde{y}; M_e, R_w\right)}{\bar{u}_e} \sim \hat{u}\left(\frac{\tilde{x}}{M_e}, \tilde{y}\right) + \tilde{u}(\tilde{x}, \tilde{y}) - \hat{u}(0, \tilde{y}) + U(\tilde{x})\, u^{\dagger}\left(\tilde{x}, \frac{\tilde{y}}{R_w^{-1/4}}\right)$$

$$- \hat{u}\left(\frac{\tilde{x}}{M_e}, 0\right) - \tilde{u}(\tilde{x}, 0) + \hat{u}(0, 0) + \ldots \tag{3.59}$$

where $\bar{u}/\bar{u}_e \sim U(\tilde{x})u^{\dagger}\left(\tilde{x}, \tilde{y}/R_w^{-1/4}\right) + \ldots$ in the sublayer (eq. 3.64). Composite expansions for the other flow properties are found in a similar manner. In particular, the first-order, uniformly valid representation for $\bar{v}$ is simply

$$\frac{\bar{v}\left(\tilde{x}, \tilde{y}; M_e, R_w\right)}{\bar{u}_e} \sim \tilde{v}(\tilde{x}, \tilde{y}) + \ldots$$

3.5. SUBLAYER LIMIT

The conditions applicable in the sublayer limit are found when an approximate stretching factor for the $\bar{y}$-coordinate in the sublayer is determined. Again, the order estimates for $\bar{u}$ and $\bar{\rho}$ given in equations 3.20 and 3.21 are applicable in the viscous sublayer also. Because the pressure gradient in the sublayer is equal to the pressure gradient given by the outer inviscid solution for $\tilde{y} = 0$, equation 3.23 also provides the order estimate for $\Delta\bar{p}$ in the sublayer. Thus, for the $\bar{\rho}\bar{u}\bar{u}_{\bar{x}}$ and $\bar{p}_{\bar{x}}$ terms in equation 3.7 to balance in the sublayer,

$$\Delta\bar{u}/\bar{u}_e = O(1) \tag{3.60}$$

If we choose to consider a limit in which $M_e \to \infty$, it is again necessary to introduce the two $\bar{x}$ coordinates, $\hat{x}$ and $\tilde{x}$. It seems, however, that to take the hypersonic limit in studying the sublayer does not lead to simplification, but rather to increased complication. A different approach is first to obtain the composite solutions for the first and second outer limits (e.g., equation 3.58) and then to use these results evaluated at $\tilde{y} = 0$ as the boundary conditions at the outer edge of the sublayer. To accomplish this, of course, it is necessary to specify the value of M_e. Then $\tilde{x} = [(\bar{x}/L) - 1]/R_w^{-1/2}$ is an appropriate $\bar{x}$ coordinate in the sublayer, and

$$\frac{\bar{\rho}\bar{u}\bar{u}_{\bar{x}}}{\bar{\rho}_w\bar{u}_e^2} = O\left(R_w^{1/2}L^{-1}\right)$$
$$\frac{\left(\bar{\mu}\bar{u}_{\bar{y}}\right)_{\bar{y}}}{\bar{\rho}_w\bar{u}_e^2} = O\left(R_w^{-1}\bar{\delta}_{SL}^{-2}L\right) \tag{3.61}$$

Thus, $\bar{\delta}_{SL} = O\left(R_w^{-3/4}L\right)$, and the conditions in the sublayer limit are

$$R_w \to \infty$$
$$\tilde{x},\ y^{\dagger},\ M_e,\ \gamma,\ Pr \text{ held fixed} \tag{3.62}$$

where $\tilde{x} \equiv [(\bar{x}/L) - 1]\Big/R_w^{-1/2}$ and $y^{\dagger} \equiv \bar{y}\Big/R_w^{-3/4}L$. This choice is in agreement with the result given in references 11 and 12. The order of magnitude of the flow deflection in the sublayer is found from equations 3.15 and 3.62 to be

$$\bar{v}/\bar{u}_e = O\left(R_w^{-1/4}\right) \tag{3.63}$$

From equations 3.20, 3.21, 3.23, 3.60, and 3.63, we see that the asymptotic representations for $\bar{u}$, $\bar{v}$, $\bar{p}$, and $\bar{\rho}$ in the limit 3.62 are

$$\bar{u}\Big/\bar{u}_e \sim U(\tilde{x})u^{\dagger}(\tilde{x}, y^{\dagger}) + \ldots$$
$$\bar{v}\Big/\bar{u}_e \sim R_w^{-1/4}U(\tilde{x})v^{\dagger}(\tilde{x}, y^{\dagger}) + \ldots$$
$$\bar{p}\Big/\left(\bar{\rho}_w\bar{u}_e^2\right) \sim p^{\dagger}(\tilde{x}) + \ldots \tag{3.64}$$
$$\bar{\rho}\Big/\bar{\rho}_w \sim \rho^{\dagger}(\tilde{x}, y^{\dagger}) + \ldots$$

where $U(\tilde{x})$ is the nondimensional $\bar{x}$ component of velocity at the outer edge of the sublayer and is found from equation 3.58 with $\tilde{y} = 0$. The introduction of $U(\tilde{x})$ into equations 3.64 simplifies the boundary conditions on $u^{\dagger}$. Substitution of equations 3.64 into equations 3.6 through 3.9 provides the relations for the first approximation in the limit 3.62.

$$(\rho^{\dagger}Uu^{\dagger})_{\tilde{x}} + (\rho^{\dagger}Uv^{\dagger})_{y^{\dagger}} + o(1) = 0 \tag{3.65}$$

$$\rho^{\dagger}U\left[u^{\dagger}(Uu^{\dagger})_{\tilde{x}} + v^{\dagger}Uu^{\dagger}_{y^{\dagger}}\right] = -p^{\dagger}_{\tilde{x}} + U\left(\mu^{\dagger}u^{\dagger}_{y^{\dagger}}\right)_{y^{\dagger}} + O\left(R_w^{-1/2}\right) \tag{3.66}$$

$$0 = -p^{\dagger}_{y^{\dagger}} + O\left(R_w^{-1/2}\right) \tag{3.67}$$

$$\rho^{\dagger}U\left(u^{\dagger}h^{\dagger}_{\tilde{x}} + v^{\dagger}h^{\dagger}_{y^{\dagger}}\right) - Uu^{\dagger}p^{\dagger}_{\tilde{x}} = \left(k^{\dagger}T^{\dagger}_{y^{\dagger}}\right)_{y^{\dagger}} + \mu^{\dagger}U^{2}\left(u^{\dagger}_{y^{\dagger}}\right)^{2} + O\left(R_{w}^{-1/2}\right) \tag{3.68}$$

where $\mu^{\dagger} \equiv \bar{\mu}/\bar{\mu}_w$, $k^{\dagger} \equiv \bar{k}/\bar{k}_w$, $h^{\dagger} \equiv \bar{h}/\bar{u}_e^2$. The boundary conditions (3.10) become

$$u^{\dagger}(\tilde{x}, 0) = v^{\dagger}(\tilde{x}, 0) = 0 \tag{3.69}$$

while, at the outer edge of the sublayer, $y^{\dagger} \to \infty$, and

$$u^{\dagger}(\tilde{x}, \infty) = 1 \tag{3.70}$$

4
SOLUTION IN THE FIRST OUTER LIMIT: $(\bar{x}/L) - 1 = O(M_e R_w^{-1/2})$

4.1. TRANSFORMATION OF THE EQUATIONS

The approximate equations 3.27 through 3.30, obtained by taking a limit of the exact equations for $\hat{x}$ fixed, can be integrated directly by the following procedure. First, von Mises variables s, $\hat{\psi}$ are introduced.

$$\begin{aligned} s &= \hat{x} \\ \hat{\psi}_{\hat{x}} &= -\hat{\rho}\hat{v} \\ \hat{\psi}_{\tilde{y}} &= \hat{\rho}\hat{u} \end{aligned} \tag{4.1}$$

Then in the case of adiabatic flow of a thermally and calorically perfect gas over an insulated body, the set of equations (4.1) becomes

$$(\hat{v}/\hat{u})_{\hat{\psi}} = (1/\hat{\rho}\hat{u})_s \tag{4.2}$$

$$\hat{u}^2 + \frac{2\gamma}{\gamma - 1}\frac{\hat{p}}{\hat{\rho}} = 1 \tag{4.3}$$

$$\hat{p} = E(\hat{\psi})\hat{\rho}^{\gamma} \tag{4.4}$$

where $\hat{u}$, $\hat{v}$, $\hat{p}$, and $\hat{\rho}$ are now functions of s and $\hat{\psi}$. The boundary and initial conditions (eqs. 3.31, 3.33, and 3.34) are

$$\hat{u}(-\infty, \hat{\psi}) = \dot{g}(\beta) \tag{4.5}$$

$$\hat{v}(s, 0) = 0 \tag{4.6}$$

$$\hat{p}(s) = \frac{\gamma - 1}{2\gamma}\left[1 + \frac{\gamma - 1}{2}\hat{v}(s, \infty)\right]^{\frac{2\gamma}{\gamma-1}} \tag{4.7}$$

where $\dot{g}(\beta) = u_1$, the Blasius boundary-layer solution $u(x, \tilde{y})$ evaluated at $x \equiv \bar{x}/L = 1$ (see app. I). Notice that any streamline can be identified by specifying either the appropriate value of $\hat{\psi}$ or the corresponding value of β. From equations 4.3, 4.4, and 4.5,

$$E(\hat{\psi}) = \frac{\gamma - 1}{2\gamma}(1 - \dot{g}^2)^\gamma \tag{4.8}$$

Now let us introduce a new coordinate

$$\phi = 1 + \frac{\gamma - 1}{2}\hat{v}(s, \infty) \tag{4.9}$$

Thus, from equations 4.3, 4.4, 4.7, 4.8, and 4.9 we obtain

$$\hat{u} = [1 - \phi^2(1 - \dot{g}^2)]^{1/2} \tag{4.10}$$

$$\hat{p} = \frac{\gamma - 1}{2\gamma}\phi^{2\gamma/(\gamma-1)} \tag{4.11}$$

$$\hat{\rho} = [1 - \dot{g}^2]^{-1}\phi^{2/(\gamma-1)} \tag{4.12}$$

The normal velocity component $\hat{v}$ and the transformation $s = s(\phi)$ are found from integration of equation 4.2.

$$\hat{v} = \hat{u}(ds/d\phi)^{-1}\int_0^{\hat{\psi}} (1/\hat{\rho}\hat{u})_\phi \, d\hat{\psi} \tag{4.13}$$

Since $\hat{v} \to -\frac{2}{\gamma - 1}(1 - \phi)$ as $\hat{\psi} \to \infty$, letting $\hat{\psi} \to \infty$ in equation 4.13 yields

$$ds/d\phi = -\frac{\gamma - 1}{2}(1 - \phi)^{-1}\int_0^{\infty} (1/\hat{\rho}\hat{u})_\phi \, d\hat{\psi} \tag{4.14}$$

The normal coordinate $\tilde{y}$ is found by integrating the third of equations 4.1.

$$\tilde{y} = \int_0^{\hat{\psi}} (1/\hat{\rho}\hat{u}) \, d\hat{\psi} \tag{4.15}$$

The integrals in equations 4.13 through 4.15 can be evaluated by substituting for $\hat{\rho}$, $\hat{u}$ from equations 4.10 and 4.12 and carrying out the differentiation with respect to ϕ. Then the variable of integration is changed from $\hat{\psi}$ to β. Since $x_1 = r_1 = 1$, equations I-26 and I-31 yield

$$d\hat{\psi} = \left(\frac{2}{2k+1}\right)^{1/2} \dot{g}\, d\beta \tag{4.16}$$

The resulting integrals are

$$ds/d\phi = \left(\frac{2}{2k+1}\right)^{1/2} \phi^{-\frac{\gamma+1}{\gamma-1}} (1-\phi)^{-1} \int_0^\infty K_1(\phi, \beta)\, d\beta \tag{4.17}$$

$$\hat{v} = -\frac{2}{\gamma-1}[1 - \phi^2(1 - \dot{g}^2)]^{1/2}(1-\phi) \frac{\int_0^\beta K_1(\phi, \beta)\, d\beta}{\int_0^\infty K_1(\phi, \beta)\, d\beta} \tag{4.18}$$

$$\tilde{y} = \left(\frac{2}{2k+1}\right)^{1/2} \phi^{-\frac{2}{\gamma-1}} \int_0^\beta K_2(\phi, \beta)\, d\beta \tag{4.19}$$

where K_1 and K_2 are given by

$$\begin{aligned} K_1(\phi, \beta) &= \dot{g}(1 - \dot{g}^2)\left[1 - \frac{\gamma+1}{2}\phi^2(1 - \dot{g}^2)\right][1 - \phi^2(1 - \dot{g}^2)]^{-3/2} \\ K_2(\phi, \beta) &= \dot{g}(1 - \dot{g}^2)[1 - \phi^2(1 - \dot{g}^2)]^{-1/2} \end{aligned} \tag{4.20}$$

The function $I(\beta)$ will be defined by

$$I(\beta) = \left(\frac{2}{2k+1}\right)^{1/2} \phi^{-\frac{\gamma+1}{\gamma-1}} (1-\phi)^{-1} \int_0^\beta K_1(\phi, \beta)\, d\beta$$

and will be used in the calculation of $ds/d\phi$. Equation 4.17 now could be written as $ds/d\phi = I(\infty)$.

The kernel K_1 can be written in terms of the Mach number $M = (\hat{\rho}\hat{u}^2/\gamma\hat{p})^{\frac{1}{2}}$ with

$$M^{-2}(1 - M^2) = -\left[1 - \frac{\gamma+1}{2}\phi^2(1 - \dot{g}^2)\right][1 - \phi^2(1 - \dot{g}^2)]^{-1}$$

Substituting $d\hat{\psi} = \hat{\rho}\hat{u}\, d\tilde{y}$, we get

$$ds/d\phi = -\phi^{-1}(1 - \phi)^{-1}\int_0^\delta \frac{1 - M^2}{M^2} d\tilde{y} \tag{4.21}$$

Thus $ds/d\phi$ is proportional to Lighthill's result, equation 2.1, for the inverse logarithmic decrement of upstream influence for small disturbances. The derivative $ds/d\phi$ clearly vanishes at some point where the Mach number just outside the sublayer is still subsonic. This "critical point" represents the downstream limit of validity of the solutions obtained in terms of $\hat{x}$ and $\tilde{y}$. As the pressure continues to drop, the spreading of streamlines in the supersonic portion of the layer will be dominant. This is expected to occur only when the flow becomes free to turn inward within a distance $\Delta\bar{x} = O\left(R_w^{-1/2}L\right)$ upstream from the corner. Since $\hat{x} = O(1)$ corresponds to a distance $\Delta\bar{x} = O\left(M_e R_w^{-1/2}L\right)$ upstream from the corner, the critical point is located at $\hat{x} = 0$.

4.2. ASYMPTOTIC EXPANSION FOR $s \to -\infty$

To obtain asymptotic expansions for the dependent variables as $s \to -\infty$, we introduce $\epsilon = 1 - \phi$ so that $\epsilon \to 0$ when $\phi \to 1$. The integral in equation 4.17 can be split into two parts by introducing

$$\begin{aligned} I_1 &= \int_0^{\beta_0} K_1(\phi, \beta)\, d\beta \\ I_2 &= \int_{\beta_0}^{\infty} K_1(\phi, \beta)\, d\beta \end{aligned} \tag{4.22}$$

where β_0 is defined to satisfy $\epsilon \ll \beta_0^2 \ll 1$. Then in I_1 the approximation (see eq. I-41)

$$\dot{g}(\beta) \sim \alpha\beta$$

as $\beta \to 0$ can be used, while in I_2 the approximation

$$\epsilon \ll \dot{g}^2(\beta) \tag{4.23}$$

is valid. Thus, we obtain

$$I_1 = -\frac{\gamma - 1}{4\alpha}\int_0^{\beta_0} (2\epsilon + \alpha^2\beta^2)^{-3/2} d(\alpha^2\beta^2) + O(1) = \frac{\gamma - 1}{2\alpha}\left[-\frac{1}{(2\epsilon)^{1/2}} + \frac{1}{\alpha\beta_0}\right] + O(1) \tag{4.24}$$

$$I_2 \sim -\frac{\gamma - 1}{2}\int_{\beta_0}^{\infty}\left(\frac{1}{\dot{g}^2} - \frac{2\gamma}{\gamma - 1} + \frac{\gamma + 1}{\gamma - 1}\dot{g}^2\right)d\beta \tag{4.25}$$

Because the integral is not bounded for $\beta_0 \to 0$, we add and subtract a term to remove the singularity.

$$I_2 \sim -\frac{\gamma - 1}{2}\int_{\beta_0}^{\infty}\left[\left(\frac{1}{\dot{g}^2} - \frac{1}{(\alpha\beta)^2}\right) - \frac{2\gamma}{\gamma - 1} + \frac{\gamma + 1}{\gamma - 1}\dot{g}^2\right]d\beta - \frac{\gamma - 1}{2\alpha^2\beta_0} \tag{4.26}$$

When we recombine the integrals I_1 and I_2, the terms in β_0^{-1} cancel out so that

$$ds/d\phi \sim -\left(\frac{1}{2k + 1}\right)^{1/2}\frac{\gamma - 1}{2\alpha}\frac{1}{(1 - \phi)^{3/2}}$$

$$s \sim -(\gamma - 1)\alpha^{-1}(2k + 1)^{-1/2}(1 - \phi)^{-1/2} \tag{4.27}$$

or

$$\phi \sim 1 - (\gamma - 1)^2\alpha^{-2}(2k + 1)^{-1}s^{-2} \tag{4.28}$$

From equations 4.10 through 4.12 and 4.28, as $s \to -\infty$,

$$\hat{u} \sim \begin{cases} \frac{\gamma - 1}{\alpha}\left(\frac{2}{2k + 1}\right)^{1/2}s^{-1}, & \beta = 0 \\ \dot{g} + O(s^{-2}), & \beta > 0 \end{cases} \tag{4.29}$$

$$\hat{p} = \frac{\gamma - 1}{2\gamma} + O(s^{-2}) \tag{4.30}$$

$$\hat{\rho} = (1 - \dot{g}^2)^{-1} + O(s^{-2}) \tag{4.31}$$

while from equations 4.18 and 4.28 we have

$$\hat{v} \sim (\text{constant})s^{-2} \text{ for } \beta > 0 \tag{4.32}$$

as $s \to -\infty$. Thus, in the upstream limit $s \to -\infty$, the flow properties decay algebraically as anticipated by the discussion of equations 3.40.

4.3. EXPANSION AT THE CRITICAL POINT

Although $ds/d\phi$ vanishes at the critical point, $d^2s/d\phi^2$ there is nonzero, so that a Taylor series expansion of the integral of equation 4.17 yields

$$-s \propto (\phi - \phi_c)^2 \tag{4.33}$$

as $\phi \to \phi_c$ where ϕ_c is found from

$$\int_0^\infty K_1(\phi_c, \beta)\, d\beta = 0 \tag{4.34}$$

Since

$$d\delta/ds = \hat{v}(\phi, \infty) = -\frac{2}{\gamma - 1}(1 - \phi) \tag{4.35}$$

we have that

$$d^2\delta/ds^2 = \frac{2}{\gamma - 1}(ds/d\phi)^{-1} \tag{4.36}$$

Thus, in a neighborhood of the critical point,

$$\begin{aligned} d\delta/ds &\propto \text{constant} + O[(-s)^{1/2}] \\ d^2\delta/ds^2 &\propto (-s)^{-1/2} \end{aligned} \tag{4.37}$$

as $s \to 0$. Thus, although the streamline slope at the outer edge of the boundary layer remains bounded, the curvature, in terms of the stretched coordinate s, becomes large at the critical point.

4.4. NUMERICAL RESULTS

To obtain $\hat{u}$, $\hat{v}$, $\hat{p}$, and $\hat{\rho}$ as functions of $\hat{x}$ and $\tilde{y}$, it is necessary to integrate equations 4.17 and 4.20 numerically. These integrations have been carried out using the IBM 7090 digital computer at The University of Michigan Computing Center. The algorithms employed are programmed in the MAD language [19]. The numerical technique employed is first to integrate equations 4.20 and I-39 on β, using the Runge-Kutta fourth-order method, which is a standard computer library subroutine [20, 21]. Thus, the integration on β consists of solving the following set of first-order, ordinary differential equations:

$$dy_1/d\beta = y_2 \tag{4.38}$$

$$dy_2/d\beta = y_3 \tag{4.39}$$

$$dy_3/d\beta = -y_1 y_3 \tag{4.40}$$

$$dy_4/d\beta = \left(\frac{2}{2k + 1}\right)^{1/2} K_2 \phi^{-\frac{2}{\gamma - 1}} \tag{4.41}$$

$$dy_5/d\beta = \left(\frac{2}{2k + 1}\right)^{1/2} \phi^{-\frac{\gamma+1}{\gamma-1}} (1 - \phi)^{-1} K_1 \tag{4.42}$$

where

$$
\begin{aligned}
y_1 &= g(\beta) \\
y_2 &= \dot{g}(\beta) \\
y_3 &= \ddot{g}(\beta) \\
y_4 &= \tilde{y} \\
y_5 &= \left(\frac{2}{2k+1}\right)^{1/2} \phi^{-\frac{\gamma+1}{\gamma-1}} (1-\phi)^{-1} \int_0^\beta K_1 \, d\beta
\end{aligned}
\tag{4.43}
$$

Equations 4.38 through 4.40 are introduced to generate the function $\dot{g}(\beta)$, which appears in the expressions for K_1 and K_2. To start the numerical integrations it is necessary to utilize asymptotic expansions for $y_1, \ldots, y_5$ as $\beta \to 0$. The expansions for y_1, y_2, and y_3 are given by equations I-41, while results for expansions of $\tilde{y}$ and y_5 are

$$
\tilde{y} \sim \left(\frac{2}{2k+1}\right)^{1/2} \phi^{-\frac{2}{\gamma-1}} \frac{1}{2\alpha}(Z_3 - Z_4) + \ldots \tag{4.44}
$$

$$
y_5 \sim \left(\frac{2}{2k+1}\right)^{1/2} \phi^{-\frac{\gamma+1}{\gamma-1}} (1-\phi)^{-1} \frac{1}{2\alpha}\left\{\left(1 - \frac{\gamma+1}{2}\phi^2\right) Z_1 + [(\gamma+1)\phi^2 - 1] Z_2\right\} + \ldots \tag{4.45}
$$

as $\beta \to 0$, where $\alpha = 0.4696$ and where

$$
\begin{aligned}
Z_1 &= -2b^{-1}(c^{-1/2} - a^{-1/2}) \\
Z_2 &= 2b^{-2}(c^{1/2} - 2a^{1/2} + ac^{-1/2}) \\
Z_3 &= 2b^{-1}(c^{1/2} - a^{1/2}) \\
Z_4 &= 2b^{-2}\left(\frac{1}{3}c^{3/2} - ac^{1/2} + \frac{2}{3}a^{3/2}\right)
\end{aligned}
\tag{4.46}
$$

Here $a \equiv 1 - \phi^2$, $b \equiv \phi^2$, and $c \equiv 1 - \phi^2(1 - \alpha^2\beta^2)$. Since we find $1 - \phi^2$ to be a numerically small quantity over its possible range of values, β^2 has not been neglected in comparison with $1 - \phi^2$ as $\beta \to 0$, in equations 4.44 and 4.45. With the initial values of the dependent variables given, the numerical integration proceeds, step by step, up to a value of β sufficiently large for the asymptotic expansion for $\dot{g}$ as $\beta \to \infty$, given in equation I-30, to be applicable. In terms of the values calculated in the last step in the numerical integration, the values for $\beta \to \infty$ are

$$
\delta \sim y_4(\phi, \beta) + 0.662\left(\frac{2}{2k+1}\right)^{1/2} \phi^{-\frac{2}{\gamma-1}} \zeta^{-2} \exp\left(-\frac{1}{2}\zeta^2\right) \tag{4.47}
$$

$$ds/d\phi \sim y_5(\phi, \beta) + 0.662\left(\frac{2}{2k+1}\right)^{1/2} \phi^{-\frac{\gamma+1}{\gamma-1}}(1-\phi)^{-1}\zeta^{-2} \exp\left(-\frac{1}{2}\zeta^2\right) \tag{4.48}$$

where $\zeta = \beta - 1.21678$. Now $ds/d\phi$ can be calculated for any given value of ϕ from equation 4.48. The critical point is located when $ds/d\phi = 0$. The critical values are found by a simple iteration scheme to be

$$\begin{aligned} M_{o,c} &= 0.4482 \\ \hat{p}_c/p_e &= 0.8712 \\ \phi_c &= 0.980495 \end{aligned} \tag{4.49}$$

when $\gamma = 1.4$. We retain six significant figures in ϕ_c because of the factor $1 - \phi$ which appears in the equations. Although the value of ϕ_c is quite close to one, the drop in pressure is more pronounced because $\hat{p} \propto \phi^7$ when $\gamma = 1.4$. Then $s = 0$ when $\phi = \phi_c$, and the second step in the numerical solution is to compute the integral

$$s = \int_{\phi_c}^{\phi} (ds/d\phi)\, d\phi \tag{4.50}$$

where $ds/d\phi$ is found in equation 4.48. This is carried out by Gaussian quadratures [20], another standard computer library subroutine.

It should be noted that when the results are considered as functions of $\hat{x}/\delta_1$ and $\tilde{y}/\delta_1$, where δ_1 is given in equation I-37, they are applicable to both wedge-shaped and conical configurations. This is because not only δ_1 but also s and $\tilde{y}$ contain the factor $[2/(2k + 1)]^{1/2}$ (see eqs. 4.17 and 4.19).

The velocity $\bar{u}/\bar{u}_e$ and the function $I(\beta)$ are tabulated against $\bar{y}/\bar{\delta}$ and β for several values of ϕ in table II. In table III, the quantities ϕ, M_o, $\bar{p}/\bar{p}_e$, $\bar{u}_o/\bar{u}_e$, $d\phi/ds$, and $\bar{\delta}/\bar{\delta}_1$ are tabulated against s/δ_1. Figure 3 shows M_o, $\bar{u}_o/\bar{u}_e$, $\bar{p}/\bar{p}_e$, $\bar{\delta}/\bar{\delta}_1$, and ϕ plotted as functions of $\hat{x}/\delta_1$. The flow properties are seen to change rather rapidly as $\hat{x} \to 0$, a result expected from the expansions in equations 4.28 through 4.32 for $s \to 0$.

The results in the solution for the first outer limit will be utilized further in constructing composite expansions. This topic is discussed in section 5 after the solution in the second outer limit has been obtained.

TABLE II. SOLUTION IN FIRST OUTER LIMIT, PART ONE

ϕ = 1.000000

β	$\bar{y}/\bar{\delta}_1$	$\bar{u}/\bar{u}_e$	I(β)
0.000	0.0000	0.0000	0.000E + 00
0.010	0.0059	0.0047	0.000E + 00
0.500	0.2910	0.2342	-0.000E + 00
1.000	0.5502	0.4606	-0.000E + 00
1.500	0.7514	0.6614	-0.000E + 00
2.000	0.8834	0.8166	-0.000E + 00
2.500	0.9545	0.9167	-0.000E + 00
3.000	0.9854	0.9688	-0.000E + 00
3.500	0.9962	0.9904	-0.000E + 00
4.000	0.9992	0.9975	-0.000E + 00
4.500	0.9999	0.9994	-0.000E + 00
0.000	1.0009	1.0000	0.000E + 00

ϕ = 0.999000

β	$\bar{y}/\bar{\delta}_1$	$\bar{u}/\bar{u}_e$	I(β)
0.000	0.0000	0.0447	0.000E + 00
0.010	0.0003	0.0450	-0.423E + 02
0.500	0.2305	0.2382	-0.361E + 04
1.000	0.4888	0.4623	-0.385E + 04
1.500	0.6905	0.6623	-0.370E + 04
2.000	0.8231	0.8170	-0.355E + 04
2.500	0.8946	0.9168	-0.345E + 04
3.000	0.9256	0.9689	-0.341E + 04
3.500	0.9364	0.9904	-0.340E + 04
4.000	0.9394	0.9975	-0.339E + 04
4.500	0.9402	0.9994	-0.339E + 04
0.000	0.9412	1.0000	-0.339E + 04

ϕ = 0.995000

β	$\bar{y}/\bar{\delta}_1$	$\bar{u}/\bar{u}_e$	I(β)
0.000	0.0000	0.0999	0.000E + 00
0.010	0.0001	0.1000	-0.745E + 00
0.500	0.1966	0.2536	-0.331E + 03
1.000	0.4519	0.4691	-0.367E + 03
1.500	0.6559	0.6656	-0.335E + 03
2.000	0.7906	0.8186	-0.304E + 03
2.500	0.8635	0.9175	-0.286E + 03
3.000	0.8952	0.9691	-0.277E + 03
3.500	0.9062	0.9905	-0.274E + 03
4.000	0.9092	0.9975	-0.273E + 03
4.500	0.9100	0.9994	-0.273E + 03
0.000	0.9111	1.0000	-0.273E + 03

ϕ = 0.990000

β	$\bar{y}/\bar{\delta}_1$	$\bar{u}/\bar{u}_e$	I(β)
0.000	0.0000	0.1411	0.000E + 00
0.010	0.0001	0.1411	-0.128E + 00
0.500	0.1738	0.2714	-0.960E + 02
1.000	0.4264	0.4773	-0.109E + 03
1.500	0.6332	0.6698	-0.923E + 02
2.000	0.7708	0.8206	-0.764E + 02
2.500	0.8454	0.9184	-0.667E + 02
3.000	0.8779	0.9695	-0.623E + 02
3.500	0.8892	0.9906	-0.607E + 02
4.000	0.8923	0.9975	-0.603E + 02
4.500	0.8931	0.9994	-0.602E + 02
0.000	0.8942	1.0000	-0.600E + 02

ϕ = 0.985000

β	$\bar{y}/\bar{\delta}_1$	$\bar{u}/\bar{u}_e$	I(β)
0.000	0.0000	0.1726	0.000E + 00
0.010	0.0001	0.1726	-0.447E - 01
0.500	0.1596	0.2881	-0.424E + 02
1.000	0.4103	0.4854	-0.481E + 02
1.500	0.6202	0.6740	-0.367E + 02
2.000	0.7607	0.8226	-0.257E + 02
2.500	0.8371	0.9193	-0.190E + 02
3.000	0.8704	0.9698	-0.160E + 02
3.500	0.8819	0.9907	-0.149E + 02
4.000	0.8852	0.9976	-0.146E + 02
4.500	0.8859	0.9994	-0.146E + 02
0.000	0.8871	1.0000	-0.145E + 02

ϕ = 0.980495

β	$\bar{y}/\bar{\delta}_1$	$\bar{u}/\bar{u}_e$	I(β)
0.000	0.0000	0.1965	0.000E + 00
0.010	0.0001	0.1966	-0.224E - 01
0.500	0.1507	0.3023	-0.239E + 02
1.000	0.4002	0.4925	-0.267E + 02
1.500	0.6130	0.6776	-0.175E + 02
2.000	0.7563	0.8244	-0.887E + 01
2.500	0.8343	0.9200	-0.361E + 01
3.000	0.8683	0.9701	-0.122E + 01
3.500	0.8801	0.9908	-0.378E + 00
4.000	0.8835	0.9976	-0.140E + 00
4.500	0.8842	0.9994	-0.833E - 01
0.000	0.8854	1.0000	0.459E - 03

TABLE III. SOLUTION IN THE FIRST OUTER LIMIT, PART TWO

ϕ	M_o	$\bar{p}/\bar{p}_e$	$\bar{u}_o/\bar{u}_e$	$d\phi/ds$	$\bar{\delta}/\bar{\delta}_1$	s/δ_1
1.000000	0.0000	1.0000	1.0000	0.000E + 00	1.0000	-0.0000
0.999000	0.1001	0.9930	0.0447	-0.339E + 04	0.9412	-3.3437
0.998000	0.1416	0.9861	0.0632	-0.122E + 04	0.9309	-1.8937
0.997000	0.1736	0.9792	0.0774	-0.655E + 03	0.9229	-1.2456
0.996000	0.2006	0.9723	0.0894	-0.406E + 03	0.9165	-0.8711
0.995000	0.2244	0.9655	0.0999	-0.273E + 03	0.9111	-0.6291
0.994000	0.2461	0.9587	0.1094	-0.192E + 03	0.9065	-0.4625
0.993000	0.2660	0.9520	0.1181	-0.140E + 03	0.9026	-0.3431
0.992000	0.2846	0.9453	0.1262	-0.104E + 03	0.8993	-0.2552
0.991000	0.3020	0.9387	0.1339	-0.788E + 02	0.8965	-0.1893
0.990000	0.3186	0.9321	0.1411	-0.600E + 02	0.8942	-0.1392
0.989000	0.3344	0.9255	0.1479	-0.459E + 02	0.8922	-0.1010
0.988000	0.3496	0.9190	0.1545	-0.351E + 02	0.8905	-0.0717
0.987000	0.3641	0.9125	0.1607	-0.266E + 02	0.8891	-0.0495
0.986000	0.3781	0.9060	0.1667	-0.199E + 02	0.8880	-0.0327
0.985000	0.3917	0.8996	0.1726	-0.145E + 02	0.8871	-0.0203
0.984000	0.4049	0.8932	0.1782	-0.101E + 02	0.8864	-0.0114
0.983000	0.4177	0.8869	0.1836	-0.651E + 01	0.8859	-0.0055
0.982000	0.4301	0.8806	0.1889	-0.355E + 01	0.8856	-0.0018
0.981000	0.4422	0.8743	0.1940	-0.108E + 01	0.8854	-0.0002
0.980495	0.4482	0.8712	0.1965	0.459E - 03	0.8854	0.0000

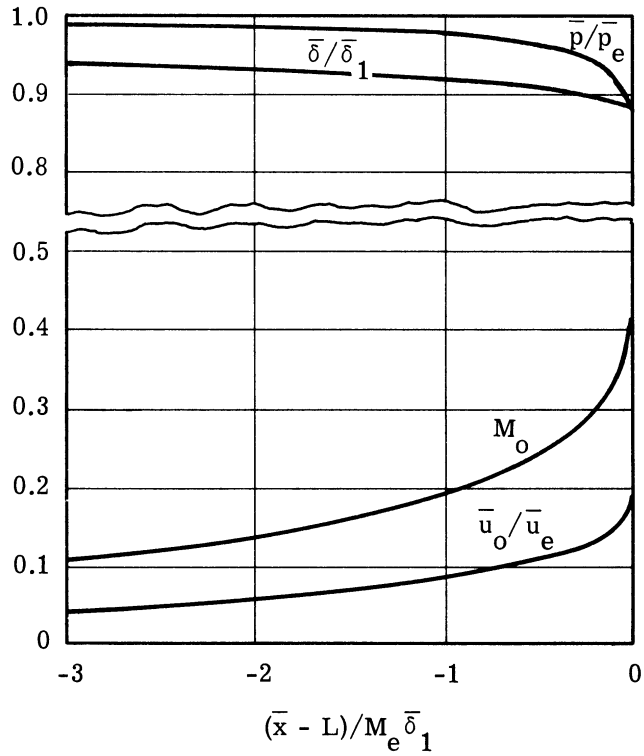

FIGURE 3. SOLUTION FOR M_o, $\bar{u}_o/\bar{u}_e$, $\bar{p}/\bar{p}_e$, AND $\bar{\delta}/\bar{\delta}_1$ IN THE FIRST OUTER LIMIT

5
SOLUTION IN THE SECOND OUTER LIMIT: $(\bar{x}/L)-1 = O(R_w^{-1/2})$

5.1. FORMULATION BY THE METHOD OF INTEGRAL RELATIONS

The equations 3.46 through 3.49 in the variables $\tilde{x} = [(\bar{x}/L) - 1]R_w^{-1/2}$ and $\tilde{y} = \bar{y}/R_w^{-1/2}L$, describing the flow in the second outer limit for the first approximation, are the full inviscid-rotational-flow relations, and the flow contains both subsonic and supersonic regions. We formulate a numerical procedure for obtaining the solution to this problem by an application of Dorodnitsyn's method of integral relations [22]. In this method, the differential equations are integrated across horizontal strips bounded by the lines $y_j = y_j(x)$. A system of first-order, ordinary differential equations is obtained in which the dependent variables are the flow properties on the strip boundaries. These equations, then, are in a form well suited to numerical integration using high-speed electronic digital computing machinery.

In the present case, numerical results are obtained for a one-strip calculation, and a procedure is described for carrying out a two-strip analysis. An attempt to generalize the calculation to an arbitrary number of strips is also discussed.

In an application of the method of integral relations, there may be n equations of the form

$$\partial P_i/\partial x + \partial Q_i/\partial y = L_i \tag{5.1}$$

where i = 1, . . . , n with $a \leqq x \leqq b$ and $0 \leqq y \leqq \Delta(x)$ and where P_i, Q_i, and L_i are known functions of the independent and dependent variables. We integrate equation 5.1 on y from the lower boundary y_{j-1} to the upper boundary y_j of each of N strips, where

$$y_j = (j/N)\Delta x \tag{5.2}$$

and obtain n first-order, quasi-linear, ordinary differential equations of the form

$$\frac{d}{dx}\int_{y_{j-1}}^{y_j} P_i\,dy = \int_{y_{j-1}}^{y_j} L_i\,dy + P_i(x, y_j)y_j' - P_i(x, y_{j-1})y_{j-1}' + Q_i(x, y_j) - Q_i(x, y_{j-1}) \tag{5.3}$$

where j = 1, 2, . . . , N. In other applications of the method, equation 5.1 is first multiplied by weighting functions $f_j(y)$ and then integrated from $y = 0$ to $y = \Delta(x)$. (See sec. 6 for an application to boundary layers.) Equation 5.3 would be obtained using weighting functions

$$f_j(y) = \begin{cases} 0, & y < y_{j-1} \\ 1, & y_{j-1} \leqq y \leqq y_j \\ 0, & y > y_j \end{cases}$$

If a total of n boundary conditions are given at $y = 0$ and $y = \Delta(x)$, we have $n(N + 1)$ equations in terms of x. Presumably, n suitable initial and/or boundary conditions are given at $x = a$ and $x = b$. We represent P_i, Q_i, and L_i by interpolation formulas containing simple functions of y with unknown functions of x as coefficients. Then the integrations over y may be carried out analytically. The new functions in x can be expressed in terms of the dependent variables evaluated at the edges of the strips. Thus, we obtain a system of nN first-order, ordinary, quasi-linear differential equations plus n boundary conditions at $y = 0$ and Δ in $n(N + 1)$ unknown functions.

In reference 22 there are comparisons of the results of calculations using the method of integral relations for various numbers of strips with exact solutions and experimental results. It is found that a two-strip calculation provides a reasonable degree of accuracy in a number of different applications. In appendix II, it is shown that a two-strip calculation by the method of integral relations for a boundary layer on a flat plate is comparable in accuracy with a Pohlhausen calculation.

In our present problem, equations 3.46 through 3.49 in the case of an adiabatic flow of a thermally and calorically perfect gas over an insulated body can be written in the form

$$(\widetilde{\rho}\widetilde{u})_{\widetilde{x}} + (\widetilde{\rho}\widetilde{v})_{\widetilde{y}} = 0 \tag{5.4}$$

$$(\widetilde{\rho}\widetilde{u}\widetilde{v})_{\widetilde{x}} + (\widetilde{p} + \widetilde{\rho}\widetilde{v}^2)_{\widetilde{y}} = 0 \tag{5.5}$$

$$\widetilde{u}^2 + \widetilde{v}^2 + [2\gamma/(\gamma - 1)]\widetilde{p}/\widetilde{\rho} = 1 \tag{5.6}$$

$$\widetilde{p} = E\widetilde{\rho}^{\gamma} \tag{5.7}$$

where $E = [(\gamma - 1)/2\gamma](1 - \dot{g}^2)^{\gamma}$ and where $\dot{g} = u(1, \widetilde{y})$ (see app. II). Thus, in this problem there are two differential equations (5.4 and 5.5) and two integrated expressions (5.6 and 5.7). The latter two relations are valid along streamlines, whose locations are found from

$$d\widetilde{y}/d\widetilde{x} = \widetilde{v}/\widetilde{u} \tag{5.8}$$

The initial conditions are prescribed by the first-order matching with the solution in the first outer limit, equations 3.55 and 3.56. As noted in the discussion following equation 4.21, the critical point in our approximation is expected to occur at $\hat{x} \equiv (\bar{x} - L)/R_w^{-1/2}L = 0$. Further decrease in pressure requires a significant turning inward of the streamlines, which can take place only within a distance $\Delta\bar{x} = O\left(R_w^{-1/2}L\right)$ upstream from the corner. For convenience, we introduce the notation

$$u_c(\tilde{y}) \equiv \hat{u}(0, \tilde{y})$$

$$p_c \equiv \hat{p}(0)$$

Then equations 3.55 and 3.56 become

$$\begin{aligned} \tilde{u}(-\infty, \tilde{y}) &= u_c(\tilde{y}) \\ \tilde{p}(-\infty, \tilde{y}) &= p_c \end{aligned} \tag{5.9}$$

In a first approximation, the boundary condition at $\tilde{y} = 0$ is

$$\tilde{v}(\tilde{x}, 0) = 0 \tag{5.10}$$

(In a second approximation, $\theta(\tilde{x}, 0)$ would be set equal to the flow deflection angle at the outer edge of the viscous sublayer.)

Before the hypersonic limit is taken, the boundary condition at the outer edge of the layer is a Prandtl-Meyer relation between $\tilde{p}$ and $\tilde{v}$. In the hypersonic limit, equations 3.12 and 3.51 give

$$d\delta/d\tilde{x} = 0$$

and, therefore

$$\begin{aligned} \delta &= \delta_c \\ \tilde{v}(\tilde{x}, \delta_c) &= 0 \end{aligned} \tag{5.11}$$

In the subsonic region, equations 5.4 and 5.5 are elliptic so that in this region it is appropriate to impose downstream conditions. In applications of the method of integral relations, the downstream boundary condition takes the form of a requirement that the integral curves must pass through saddle-point singularities of the differential equations derived from the integral relations [22]. The corner, $\tilde{x} = \tilde{y} = 0$, is located by the requirement that it is a sonic point. Thus, the special downstream condition on the line $\tilde{y} = 0$ is

$$M(0, 0) = 1 \tag{5.12}$$

and the corner is a singular point in the flow.

In the present application of the method of integral relations, the use of equations 5.6 and 5.7 is simplified if we choose the strip boundaries to be streamlines rather than lines y_j =

$(j/N)\delta$ as in equation 5.2. We let y_j be the value of $\tilde{y}$ on the upper boundary of the j^{th} strip and note that the functions $y_j(\tilde{x})$ are to be found at each point from integration of

$$dy_j/d\tilde{x} = v_j/u_j \tag{5.13}$$

where $u_j \equiv \tilde{u}(\tilde{x}, y_j)$ and $v_j \equiv \tilde{v}(\tilde{x}, y_j)$.

Because of the different nature of the boundary conditions in the supersonic region as compared with the region that is initially subsonic, we treat these regions separately in constructing the integral relations. In the region that is initially subsonic, we introduce M strips bounded by the streamlines $y_j = y_j(\tilde{x})$, $j = 0, \ldots, M$, where y_M is the normal coordinate of the streamline that is initially sonic. In the supersonic region, we introduce N strips bounded by the streamlines $y_k = y_k(\tilde{x})$, $k = M, \ldots, M + N$, where $y_{M+N} = \delta_c$; there is a total of M + N strips.

We now consider the application of the method of integral relations in the region that is initially subsonic. It is convenient to write the integral relations, derived from integrations on $\tilde{y}$ of equations 5.4 and 5.5, in the form

$$\int_{y_{j-1}}^{y_j} (\tilde{\rho}\tilde{u})_{\tilde{x}}\, d\tilde{y} = -(\rho_j v_j - \rho_{j-1} v_{j-1}) \tag{5.14}$$

$$\int_{y_{j-1}}^{y_j} (\tilde{\rho}\tilde{u}\tilde{v})_{\tilde{x}}\, d\tilde{y} = -\left(p_j - p_{j-1} + \rho_j v_j^2 - \rho_{j-1} v_{j-1}^2\right) \tag{5.15}$$

where $j = 1, \ldots, M$. Here $\rho_j \equiv \tilde{\rho}(\tilde{x}, y_j)$, etc. Equations 5.6 and 5.7, written for the strip boundaries $\tilde{y} = y_j(\tilde{x})$, become

$$u_j^2 + v_j^2 + [2\gamma/(\gamma - 1)]p_j/\rho_j = 1 \tag{5.16}$$

$$p_j = E_j \rho_j^{\gamma} \tag{5.17}$$

where the terms $E_j = [(\gamma - 1)/2\gamma]\left(1 - \dot{g}_j^2\right)^{\gamma}$ are constants and where $j = 0, \ldots, M$.

Although $\tilde{\rho} \to \infty$ as $\tilde{y} \to \delta$ in the hypersonic limit, $\tilde{\rho}\tilde{u}$ is finite and bounded everywhere in the region which is initially subsonic. Thus, an appropriate interpolation formula for $\tilde{\rho}\tilde{u}$ to be used in carrying out the integration in equation 5.14 is

$$\widetilde{\rho}\widetilde{u} = \sum_{m=0}^{M} a_m(\tilde{x})(\tilde{y}/\delta_c)^m \tag{5.18}$$

Similarly, a representation for $\widetilde{\rho}\widetilde{u}\widetilde{v}$ to be substituted into equation 5.15 is

$$\widetilde{\rho}\widetilde{u}\widetilde{v} = \sum_{m=0}^{M} b_m(\tilde{x})(\tilde{y}/\delta_c)^m \tag{5.19}$$

Thus substitution of equations 5.18 and 5.19 into equations 5.14 and 5.15, respectively, transforms these relations into the ordinary differential equations

$$\sum_{m=0}^{M} (m + 1)^{-1}\left(\alpha_j^{m+1} - \alpha_{j-1}^{m+1}\right)a'_m(\tilde{x}) = -\delta_c^{-1}(\rho_j v_j - \rho_{j-1}v_{j-1}) \tag{5.20}$$

$$\sum_{m=0}^{M} (m + 1)^{-1}\left(\alpha_j^{m+1} - \alpha_{j-1}^{m+1}\right)b'_m(x) = -\delta_c^{-1}\left(p_j - p_{j-1} + \rho_j v_j^2 - \rho_{j-1}v_{j-1}^2\right) \tag{5.21}$$

where $\alpha_j = y_j/\delta_c$ and $j = 1, \ldots, M$. From equation 5.13, we obtain the equation for the streamline slope,

$$\alpha'_j = \delta_c^{-1}v_j/u_j \tag{5.22}$$

Equations 5.18 and 5.19, when evaluated on the strip boundaries, are a system of linear algebraic equations in the a_m and b_m in terms of the values of $\widetilde{\rho}\widetilde{u}$ and $\widetilde{\rho}\widetilde{u}\widetilde{v}$ on the strip boundaries.

$$\sum_{m=0}^{M} \alpha_j^m a_m = \rho_j u_j \tag{5.23}$$

$$\sum_{m=0}^{M} \alpha_j^m b_m = \rho_j u_j v_j \tag{5.24}$$

where $j = 0, \ldots, M$ in these relations.

Thus, in the application of the method of integral relations to the portion of the flow that is initially subsonic, we have 7M + 5 equations in 7M + 6 unknown functions of $\tilde{x}$: u_j, v_j, p_j, ρ_j, a_m, b_m (where j, m = 0, . . . , M) and α_j (where j = 1, . . . , M). The 7M + 5 equations are: the boundary condition (5.10), which gives $v_o = 0$; equations 5.16, 5.17, 5.23, and 5.24 in which

$j = 0, \ldots, M$; and equations 5.20, 5.21, and 5.22 in which $j = 1, \ldots, M$. The extra unknown function appears because the boundary condition at $\tilde{y} = \delta_c$ has not yet been employed.

Now we consider the application of the method of integral relations in the region that is initially supersonic. We introduce in this region N strips bounded by the streamlines $y_k = y_k(\tilde{x})$, $k = M, \ldots, M + N$. The integral relations may be written in the form

$$\int_{y_{k-1}}^{y_k} (\tilde{\rho}\tilde{u})_{\tilde{x}}\, d\tilde{y} = -(\rho_k v_k - \rho_{k-1} v_{k-1}) \tag{5.25}$$

$$\int_{y_{k-1}}^{y_k} (\tilde{\rho}\tilde{u}\tilde{v})_{\tilde{x}}\, d\tilde{y} = -\left(p_k - p_{k-1} + \rho_k v_k^2 - \rho_{k-1} v_{k-1}^2\right) \tag{5.26}$$

where $k = M + 1, \ldots, M + N$. Equations 5.6 and 5.7 become

$$u_k^2 + v_k^2 + [2\gamma/(\gamma - 1)]p_k/\rho_k = 1 \tag{5.27}$$

$$p_k = E_k \rho_k^{\gamma} \tag{5.28}$$

where the terms $E_k = [(\gamma - 1)/2\gamma]\left(1 - \dot{g}_k^2\right)^{\gamma}$ are constants and where $k = M, \ldots, M + N - 1$.

Let us now consider the representation for $\tilde{\rho}\tilde{u}$ to be substituted into equation 5.25. Using equation 5.6 and the relation $\tilde{v}(\tilde{x}, \tilde{\delta}) = 0$ we have, in the hypersonic limit,

$$\tilde{\rho} = O[(1 - \tilde{u})^{-1}] \tag{5.29}$$

as $\tilde{y} \to \delta$. From equation 4.10 it can be seen that $u_c(\tilde{y})$ has the same form of asymptotic behavior for $\tilde{y} \to \delta_c$ as $u_1(\tilde{y})$ in the upstream boundary layer. It follows from equation I-30 that

$$\beta = O\{[-\log (1 - \tilde{u})]^{1/2}\}$$

$$\partial u/\partial\beta = O\{(1 - \tilde{u})[-\log (1 - \tilde{u})]^{1/2}\} \tag{5.30}$$

as $\tilde{y} \to \delta$. In the boundary layer at $x = 1$, we see from equations I-19, I-25, and I-31 that

$$d\beta = [2/(2k + 1)]^{1/2} \rho\, d\tilde{y} \tag{5.31}$$

Then equations 5.29-5.31 lead us to the results

$$\partial\tilde{y}/\partial\tilde{u} = O\{[-\log(1 - \tilde{u})]^{-1/2}\}$$

$$1 - \tilde{y}/\delta = O\{(1 - \tilde{u})[-\log(1 - \tilde{u})]^{1/2}\} \tag{5.32}$$

$$\tilde{\rho} = O\{(1 - \tilde{y}/\delta)^{-1}[-\log(1 - \tilde{u})]^{-1/2}\} \tag{5.33}$$

as $\tilde{y} \to \delta$. Consistent with other applications of this method to boundary-layer problems [22], we approximate the integral in equation 5.25 by employing an interpolation formula for $\tilde{\rho}\tilde{u}$ that omits the logarithmic factor

$$\tilde{\rho}\tilde{u} = (1 - \tilde{y}/\delta_c)^{-1}\sum_{n=0}^{N} c_n(\tilde{x})(\tilde{y}/\delta_c)^n \tag{5.34}$$

We introduce

$$A = \sum_{n=0}^{N} c_n(\tilde{x}) \tag{5.35}$$

where A is constant, as will be shown in equation 5.43. A nondimensional stream function $\tilde{\psi}$ is given by

$$\tilde{\psi} - \tilde{\psi}_M = \int_{y_M}^{\tilde{y}} \tilde{\rho}\tilde{u}\, d\tilde{y} \tag{5.36}$$

which, from equation 5.34, becomes

$$\tilde{\psi} - \tilde{\psi}_M = \delta_c \sum_{n=0}^{N} c_n(\tilde{x}) \int_{\alpha_M}^{\alpha} (1 - \xi)^{-1}\xi^n d\xi = -\delta_c\left[A \log\left(\frac{1 - \alpha}{1 - \alpha_M}\right) + \sum_{n=0}^{N} c_n(\tilde{x})\beta_n(\alpha, \alpha_M)\right] \tag{5.37}$$

where $\alpha \equiv \tilde{y}/\delta_c$ and where $\beta_n(\alpha, \alpha_M)$ is given by

$$\beta_n(\alpha, \alpha_M) \equiv -\int_{\alpha_M}^{\alpha} (1 - \xi)^{-1}(\xi^M - 1)\, d\xi \tag{5.38}$$

For instance, $\beta_0 = 0$, $\beta_1 = \alpha - \alpha_M$, $\beta_2 = \alpha - \alpha_M + (1/2)\left(\alpha^2 - \alpha_M^2\right)$, etc. Thus, we have, from equation 5.37,

$$1 - \tilde{y}/\delta_c = (1 - \alpha_M) \exp \left[-(\delta_c A)^{-1}(\tilde{\psi} - \tilde{\psi}_M) - A^{-1} \sum_{n=0}^{N} c_n(\tilde{x})\beta_n(\alpha, \alpha_M) \right] \tag{5.39}$$

We can write the isentropic relation 5.7 in the form

$$\tilde{p}/p_c = (\tilde{\rho}/\rho_c)^{\gamma} \tag{5.40}$$

Then we substitute for $\tilde{\rho}$ and ρ_c from equation 5.34 and get

$$\tilde{p}/p_c = \left[\left(\frac{1 - \tilde{y}_c/\delta_c}{1 - \tilde{y}/\delta_c} \right) \frac{u_c \sum c_n(\tilde{y}/\delta_c)^n}{\tilde{u} \sum c_{n,c}(\tilde{y}_c/\delta_c)^n} \right]^{\gamma} \tag{5.41}$$

where $\tilde{y}_c$ is the value of $\tilde{y}$ on the same streamline at the critical point. Now we substitute for the values of $1 - \tilde{y}/\delta_c$ and $1 - \tilde{y}_c/\delta_c$ from equation 5.39.

$$\frac{\tilde{p}}{p_c} = \left[\left(\frac{1 - \alpha_{M,c}}{1 - \alpha_M} \right) \frac{u_c \sum c_n(\tilde{y}/\delta_c)^n}{\tilde{u} \sum c_{n,c}(\tilde{y}_c/\delta_c)^n} \right]^{\gamma} \exp \left[-\frac{\gamma}{\delta_c} \left(\frac{A - A_c}{AA_c} \right) (\tilde{\psi} - \tilde{\psi}_M) - \gamma \sum \left(\frac{c_{n,c}\beta_{n,c}}{A_c} - \frac{c_n\beta_n}{A} \right) \right] \tag{5.42}$$

In equation 5.42, $\beta_n = \beta_n(\alpha, \alpha_M)$, $\beta_{M,c} = \beta_M(\alpha_c, \alpha_{M,c})$, and $c_{n,c} = c_n(-\infty)$. For the pressure p_{M+N} at the outer edge of the layer to remain bounded and nonzero,

$$A = A_c = \text{constant} \tag{5.43}$$

Then p_{M+N} is given from equations 5.41 and 5.42 as

$$p_{M+N} = p_c[(1 - \alpha_{M,c})/(1 - \alpha_M)] \exp(-\gamma h) \tag{5.44}$$

where

$$h \equiv A^{-1} \sum_{M=0}^{N} [\beta_{n,c}(1, \alpha_{M,c})c_{n,c} - \beta_n(1, \alpha_M)c_n(\tilde{x})]$$

$$p_c = [(\gamma - 1)/2\gamma]\phi_c^{2\gamma/(\gamma-1)} \tag{5.45}$$

We see from equations 5.39 and 5.43 that ignoring the logarithmic factor in equation 5.32 in constructing a representation for $\widetilde{\rho u}$ is equivalent to assuming that

$$\tilde{u} \sim 1 - f(\tilde{x}) \exp\left[-\tilde{\psi}/(\delta_c A_c)\right] \tag{5.46}$$

as $\tilde{\psi} \to \infty$.

With the interpolation formula 5.34 for $\widetilde{\rho u}$ substituted in the integral, equation 5.25 becomes

$$\sum_{n=0}^{N} \beta_n(\alpha_k, \alpha_{k-1}) c_n'(\tilde{x}) = \delta_c^{-1}(\rho_k v_k - \rho_{k-1} v_{k-1}) \tag{5.47}$$

where $k = M + 1, \ldots, M + N$.

From equation 5.47 we see that $\widetilde{\rho v}$ is bounded and nonzero in the limit $\tilde{y} \to \delta_c$. Thus, an appropriate interpolation formula for $\widetilde{\rho u v}$ in this case is

$$\widetilde{\rho u v} = \sum_{n=0}^{N} d_n(\tilde{x})(\tilde{y}/\delta_c)^n \tag{5.48}$$

and substitution of this relation into equation 5.26 gives us

$$\sum_{n=0}^{N} (n + 1)^{-1}\left(\alpha_k^{n+1} - \alpha_{k-1}^{n+1}\right) d_n'(\tilde{x}) = -\delta_c^{-1}\left(p_k - p_{k-1} + \rho_k v_k^2 - \rho_{k-1} v_{k-1}^2\right) \tag{5.49}$$

where $k = M + 1, \ldots, M + N$.

Equations 5.34 and 5.48, evaluated on each strip in the supersonic region, give

$$\sum_{n=0}^{N} \alpha_k^n c_n = (1 - \alpha_k)\rho_k u_k \tag{5.50}$$

$$\sum_{n=0}^{N} \alpha_k^n d_n = \rho_k u_k v_k \tag{5.51}$$

where $k = M, \ldots, M + N - 1$. To utilize equation 5.47 for $k = M + N$, it is necessary to substitute for $\rho_{M+N} v_{M+N}$ from equation 5.51. We note that $u_{M+N} = 1$, and obtain

$$\lim_{\tilde{y} \to \delta_c} (\widetilde{\rho v}) = \sum_{n=0}^{N} d_n \tag{5.52}$$

From equation 5.8 we get

$$\alpha_k'(\tilde{x}) = \delta_c^{-1} v_k/u_k \tag{5.53}$$

where $k = M, \ldots, M + N - 1$.

This application of the method of integral relations for the region that is initially supersonic also can be applied across the entire layer in the case of a one-strip calculation.

In the application of the method of integral relations to the supersonic portion of the flow, we end up with $7N + 3$ equations in $7N + 4$ unknown functions. The $7N + 4$ functions are u_k, v_k, p_k, ρ_k, α_k (where $k = M, \ldots, M + N - 1$); c_n, d_n (where $n = 0, \ldots, N$); p_{M+N}; and v_{M+N}. The $7N + 3$ equations are the condition 5.11, which gives $v_{M+N} = 0$; equations 5.27, 5.28, 5.50, 5.51, and 5.53 for $k = M, \ldots, M + N - 1$; equations 5.47 and 5.49 for $k = M + 1, \ldots, M + N$; and equations 5.35 and 5.44. Here the extra unknown function appears because the boundary condition at $\tilde{y} = 0$ has not been employed.

When we compare the equations obtained for the supersonic region with the equations obtained for the region that is initially subsonic, we see that the five dependent variables u_M, v_M, p_M, ρ_M, and α_M have been counted twice. Thus, there are $7(M + N) + 5$ equations in $7(M + N) + 5$ unknown functions.

In the preceding discussion we did not obtain a relation for evaluating the constant A, defined in equation 5.35. When we expand p_o and $\tilde{p}_\delta$ for $M_o \to M_{o,c}$ and form the difference $\tilde{p}_\delta - p_o$, we see that there are two possibilities for the selection of A.

$$\tilde{p}_\delta - p_o = \begin{cases} O(M_o - M_{o,c}) \\ o(M_o - M_{o,c}) \end{cases} \tag{5.54}$$

In the first case, the perturbations in u_k, v_k, p_k, etc. are all of the same order of magnitude, and the perturbations will decay exponentially in $\tilde{x}$ as $\tilde{x} \to -\infty$. The second case is obtained when A is chosen so that the terms in $\tilde{p}_\delta - p_o$ of the order $M_o - M_{o,c}$ vanish identically. Thus, in this instance a relation for evaluating A is obtained, while in the first case no relation determining A is found. In the present problem the boundary layer on the body is subcritical (see sec. 2), and, after an acceleration described approximately in the first outer limit, the boundary-layer profile reaches the critical state, in that $\langle M^{-2} \rangle = 1$. Therefore, the initial condition for the equations in the second outer limit is special in the sense that the initial profile is critical, and we can expect A to have a particular value, corresponding to the application of the second part of equation 5.54.

In support of the above argument, we will show that the second part of equation 5.54 is a necessary condition for

$$(d\bar{\delta}^*/d\bar{p}_\delta)_c = 0 \tag{5.55}$$

to be satisfied. Here $\bar{\delta}^*$ is the boundary layer displacement thickness, and $\bar{p}_\delta$ is the pressure at $\tilde{y} = \delta$. From equation I-35 and the definition of the stream function we have, in the hypersonic limit,

$$\delta^* = \int_0^\infty (1/\tilde{\rho}\tilde{u})\, d\tilde{\psi} \tag{5.56}$$

where $\delta^* = \bar{\delta}^*/R_w^{-1/2}L$ and $\tilde{\psi}_{\tilde{y}} = \tilde{\rho}\tilde{u}$. We introduce a small perturbation in the pressure in a neighborhood of the critical point and obtain

$$\Delta\delta^* \sim -\int_0^{\delta_c} (\Delta\tilde{p}/\gamma p_c)\left(1 - M_c^{-2}\right) d\tilde{y} \sim -(\Delta\tilde{p}_\delta/\gamma p_c)\int_0^{\delta_c} (\Delta\tilde{p}/\Delta\tilde{p}_\delta)\left(1 - M_c^{-2}\right) d\tilde{y} \tag{5.57}$$

as $\Delta\tilde{p} \to 0$. From equation 5.55 we see that

$$\Delta\delta^* = o(\Delta\tilde{p}_\delta) \tag{5.58}$$

as $\Delta\tilde{p}_\delta \to 0$ near the critical point. Also, from equations 4.17, 4.21, and 4.34 we have

$$\int_0^{\delta_c} \left(1 - M_c^{-2}\right) d\tilde{y} = 0 \tag{5.59}$$

Thus, in view of the results expressed in equations 5.58 and 5.59 and since $\Delta\tilde{p}/\Delta\tilde{p}_\delta$ is monotonic in $\tilde{y}$, a necessary condition for the orders of magnitude in equation 5.57 to match is

$$\Delta\tilde{p} - \Delta\tilde{p}_\delta = o(\Delta\tilde{p}_\delta) \tag{5.60}$$

as $\Delta\tilde{p}_\delta \to 0$. This condition is equivalent to the second part of equation 5.54.

Thus, A is to be selected so that the second part of equation 5.54 is satisfied. The perturbations in the dependent variables are not all of the same order of magnitude; they will decay algebraically in $\tilde{x}$ as $\tilde{x} \to -\infty$. We find, in this case, that along a streamline

$$\tilde{v} = O[(\Delta M)^{3/2}]$$

as $\tilde{x} \to -\infty$, which is consistent with transonic small-disturbance theory.

In the use of the Bernoulli equation and the entropy equation in the application of the method of integral relations to this problem, it is convenient to introduce the Mach number as a new dependent variable. Then we may replace the Bernoulli and entropy relations by

$$p_i = \frac{\gamma - 1}{2\gamma}\left[\left(1 + \frac{\gamma - 1}{2} M_i^2\right)\left(1 - \dot{g}_i^2\right)\right]^{-\gamma/(\gamma-1)} \tag{5.61}$$

$$\rho_i = \left(1 + \frac{\gamma - 1}{2} M_i^2\right)^{-1/(\gamma-1)}\left(1 - \dot{g}_i^2\right)^{-\gamma/(\gamma-1)} \tag{5.62}$$

$$p_i'/p_i = -\gamma M_i\left(1 + \frac{\gamma - 1}{2} M_i^2\right)^{-1} M_i' \tag{5.63}$$

$$\rho_i'/\rho_i = -M_i\left(1 + \frac{\gamma - 1}{2} M_i^2\right)^{-1} M_i' \tag{5.64}$$

Also, for adiabatic flow,

$$q_i^2 \equiv u_i^2 + v_i^2 \tag{5.65}$$

$$= \frac{\gamma - 1}{2} M_i^2\left(1 + \frac{\gamma - 1}{2} M_i^2\right)^{-1} \tag{5.66}$$

$$q_i'/q_i = M_i^{-1}\left(1 + \frac{\gamma - 1}{2} M_i^2\right)^{-1} M_i' \tag{5.67}$$

In the above equations, $i = 0, \ldots, M + N - 1$.

To proceed further it is necessary to specify the values of M and N.

5.2. SOLUTION BY THE METHOD OF INTEGRAL RELATIONS FOR ONE STRIP

The simplest possible calculation to carry out is a single-strip application with $M = 0$, $N = 1$. Then equations 5.27, 5.28, 5.50, 5.51, with $k = 0$; equations 5.47 and 5.49, with $k = 1$; and equations 5.35 and 5.44 become, with $v_o = \alpha_o = v_1 = 0$ and $\alpha_1 = 1$:

$$u_o^2 + [2\gamma/(\gamma - 1)]p_o/\rho_o = 1 \tag{5.68}$$

$$p_o = [(\gamma - 1)/2\gamma]\rho_o^{\gamma} \tag{5.69}$$

$$c_o + c_1 = A \tag{5.70}$$

$$p_1 = p_c \exp(-\gamma h) \tag{5.71}$$

where $h = A^{-1}(c_{1,c} - c_1)$ and $p_c = [(\gamma - 1)/2\gamma]\phi_c^{2\gamma/(\gamma-1)}$.

$$c_1' = \delta_c^{-1}(d_o + d_1) \tag{5.72}$$

$$\frac{1}{2}d_1' = -\delta_c^{-1}(p_1 - p_o) \tag{5.73}$$

$$c_o = \rho_o u_o \tag{5.74}$$

$$d_o = 0 \tag{5.75}$$

From equations 5.70 and 5.72 through 5.75, we obtain

$$(\rho_o u_o)' = -\delta_c^{-1} d_1 \tag{5.76}$$

$$d_1' = -2\delta_c^{-1}(p_1 - p_o) \tag{5.77}$$

from which we get the second-order, ordinary differential equation

$$(\rho_o u_o)'' = 2\delta_c^{-2}(p_1 - p_o) \tag{5.78}$$

A form of the equations more convenient for carrying out the numerical solution is obtained by introducing the Mach number M_o as the independent variable. From equations 5.64 and 5.67, with $i = 0$, we have

$$(\rho_o u_o)' = \rho_o u_o M_o^{-1}\left(1 - M_o^2\right)\left(1 + \frac{\gamma - 1}{2} M_o^2\right)^{-1} M_o' \tag{5.79}$$

Since $\tilde{x}$ does not appear explicitly in the equations, the initial value of $\tilde{x}$ is arbitrary, and we can introduce a new $\tilde{x}$ coordinate, X, depending upon M_o such that

$$\tilde{x} = X(M_o) - X(1) \tag{5.80}$$

Then the boundary condition $M(0, 0) = 1$ will automatically be satisfied. The differential equations in the new variables are

$$\frac{dX}{dM_o} = \frac{\delta_c \rho_o u_o}{\sqrt{d_1^2} M_o}\left(\frac{1 - M_o^2}{1 + \frac{\gamma - 1}{2} M_o^2}\right) \tag{5.81}$$

$$\frac{d\left(d_1^2\right)}{dM_o} = \frac{4\rho_o u_o (p_1 - p_o)\left(1 - M_o^2\right)}{M_o\left(1 + \frac{\gamma - 1}{2} M_o^2\right)} \tag{5.82}$$

The integration is to be carried out over the interval $M_{o,c} \leqq M_o \leqq 1$, and the values of $\tilde{x}$ are recovered from equation 5.80. In equation 5.81 we have chosen

$$d_1 = -\sqrt{d_1^2} \tag{5.83}$$

so that $dX/dM_o > 0$, which corresponds to a decreasing pressure. If $d_1 = +\sqrt{d_1^2}$ is chosen, we would obtain $\Delta M_o < 0$ and $\Delta p_o > 0$ for $\Delta X > 0$.

We now obtain the asymptotic expansions for d_1 and X as $M_o \to M_{o,c}$. From equation 5.61 with i = 0, we get

$$p_o \sim p_c\left[1 - \gamma M_{o,c}\left(1 + \frac{\gamma - 1}{2} M_{o,c}^2\right)^{-1} \Delta M_o + \ldots\right] \tag{5.84}$$

as $\Delta M_o \to 0$, where $\Delta M_o \equiv M_o - M_{o,c}$. For i = 0, equations 5.62 and 5.67 yield

$$\rho_o u_o \sim \rho_{o,c} u_{o,c}\left[1 + M_{o,c}^{-1}\left(1 - M_{o,c}^2\right)\left(1 + \frac{\gamma - 1}{2} M_{o,c}^2\right)^{-1} \Delta M_o + \ldots\right] \tag{5.85}$$

so that, with equations 5.70, 5.71, 5.74, and 5.85, the expansion for p_1 is

$$p_1 \sim p_c\left[1 - \gamma A^{-1} \rho_{o,c} u_{o,c} M_{o,c}^{-1}\left(1 - M_{o,c}^2\right)\left(1 + \frac{\gamma - 1}{2} M_{o,c}^2\right)^{-1} \times \Delta M_o + \ldots\right] \tag{5.86}$$

as $\Delta M_o \to 0$. Therefore, the expansion for $p_1 - p_o$ as $\Delta M_o \to 0$ is

$$p_1 - p_o \sim \gamma p_c M_{o,c}\left(1 + \frac{\gamma - 1}{2} M_{o,c}^2\right)^{-1}\left[1 - A^{-1} \rho_{o,c} u_{o,c} M_{o,c}^{-2} \times \left(1 - M_{o,c}^2\right)\right] \Delta M_o + \ldots \tag{5.87}$$

The condition expressed in the second part of equation 5.54 implies, in this case, that $p_1 - p_o = o(\Delta M_o)$. This condition is satisfied when

$$A = \rho_{o,c} u_{o,c} M_{o,c}^{-2}\left(1 - M_{o,c}^2\right) \tag{5.88}$$

The numerical value is A = 0.708. Thus, $p_1 - p_o = O(\Delta M_o)^2$, and from equation 5.86 we get

$$d\left(d_1^2\right)\Big/ dM_o \sim (\text{constant})(\Delta M_o)^2 \tag{5.89}$$

Since $d_1 \to 0$ as $\Delta M_o \to 0$ because $d_1 = O(\theta)$, we have

$$d_1 \sim (\text{constant})(\Delta M_o)^{3/2} \tag{5.90}$$

as $\Delta M_o \to 0$. Then $\theta = O[(\Delta M_o)^{3/2}$ as $\tilde{x} \to -\infty$. We substitute equation 5.90 into the expansion of equation 5.81 and get

$$dX/dM_o \sim (\text{constant})(\Delta M_o)^{-3/2}$$

$$X \sim (\text{constant})(\Delta M_o)^{-1/2} \tag{5.91}$$

The expansions for M_o and d_1 in terms of X are

$$M_o \sim M_{o,c} + (\text{constant})X^{-2} \tag{5.92}$$

$$d_1 \sim (\text{constant})X^{-3} \tag{5.93}$$

as $X \to -\infty$. Note that, in order to obtain the numerical constants in the expansions 5.90 and 5.91 or 5.92 and 5.93, it is necessary to carry out the expansion of $p_1 - p_o$ to the order $(\Delta M_o)^2$, as $\Delta M_o \to 0$.

When $\tilde{x} \to 0$, it can be seen from equations 5.81 and 5.82 that

$$M_o \sim 1 + (\text{constant})|\tilde{x}|^{1/2}$$

$$d_1 \sim d_1(0) + (\text{constant})\tilde{x}$$

In general, it is necessary to employ the asymptotic expansions for the dependent variables in order to get the numerical integration of the differential equations started. In the special case of one strip, however, the numerical integration of the differential equations 5.81 and 5.82 can be carried out without using the expansion for $\Delta M_o \to 0$. The reason for this is that $d\left(d_1^2\right)/dM_o$ is bounded at $M_o = M_{o,c}$, and is also only a function of the independent variable M_o. Once a value of d_1 has been found at a point $M_o > M_{o,c}$, the integration of equation 5.81 may be started and carried out simultaneously with the integration of equation 5.82. The numerical results will be discussed at the end of this section.

5.3. SOLUTION BY THE METHOD OF INTEGRAL RELATIONS FOR TWO STRIPS

The next step is to consider a two-strip calculation with M = N = 1. In the region that is initially subsonic, the governing equations are 5.16, 5.17, 5.23, and 5.24 for j = 0, 1; 5.20, 5.21, and 5.22 for j = 1; and $v_o = \alpha_o = 0$. We also define $\alpha \equiv \alpha_1$. Thus, we have the equations

$$u_o^2 + [2\gamma/(\gamma - 1)]p_o/\rho_o = 1 \tag{5.94}$$

$$u_1^2 + v_1^2 + [2\gamma/(\gamma - 1)]p_1/\rho_1 = 1 \qquad (5.95)$$

$$p_o = E_o\rho_o^\gamma \qquad (5.96)$$

$$p_1 = E_1\rho_1^\gamma \qquad (5.97)$$

where $E_o = (\gamma - 1)/2\gamma$ and $E_1 = [(\gamma - 1)/2\gamma]\left(1 - \dot{g}_1^2\right)^\gamma$,

$$\alpha a_o' + \frac{1}{2}\alpha^2 a_1' = -\delta_c^{-1}\rho_1 v_1 \qquad (5.98)$$

$$\alpha b_o' + \frac{1}{2}\alpha^2 b_1' = -\delta_c^{-1}\left(p_1 - p_o + \rho_1 v_1^2\right) \qquad (5.99)$$

$$\alpha' = \delta_c^{-1} v_1/u_1 \qquad (5.100)$$

$$a_o = \rho_o u_o \qquad (5.101)$$

$$a_o + \alpha a_1 = \rho_1 u_1 \qquad (5.102)$$

$$b_o = 0 \qquad (5.103)$$

$$\alpha b_1 = \rho_1 u_1 v_1 \qquad (5.104)$$

We solve for a_1, b_1, a_1', and b_1' from equations 5.100 through 5.104 and obtain

$$a_1 = \alpha^{-1}(\rho_1 u_1 - \rho_o u_o) \qquad (5.105)$$

$$b_1 = \alpha^{-1}\rho_1 u_1 v_1 \qquad (5.106)$$

$$a_1' = \alpha^{-1}\left[(\rho_1 u_1)' - (\rho_o u_o)' - \delta_c^{-1} a_1 v_1/u_1\right] \qquad (5.107)$$

$$b_1' = \alpha^{-1}(\rho_1 u_1 v_1)' - \delta_c^{-1}\alpha^{-2}\rho_1 v_1^2 \qquad (5.108)$$

Then the differential equations 5.98 and 5.99 become

$$(\rho_o u_o)' + (\rho_1 u_1)' = \delta_c^{-1}\alpha^{-1}\rho_1 v_1[(\alpha a_1/\rho_1 u_1) - 2] \qquad (5.109)$$

$$(\rho_1 u_1 v_1)' = -\delta_c^{-1}\alpha^{-1}\left[\rho_1 v_1^2 + 2\alpha(p_1 - p_o)\right] \equiv F_2 \qquad (5.110)$$

The relations derived for the supersonic region (discounting those already noted and with $v_2 = 0$, $\alpha_2 = 1$) are, in this case,

$$c_o + c_1 = A \tag{5.111}$$

$$p_2 = p_c[(1 - \alpha_c)/(1 - \alpha)] \exp(-\gamma h) \tag{5.112}$$

where $h = A^{-1}[(1 - \alpha_c)c_{1,c} - (1 - \alpha)c_1]$,

$$(1 - \alpha)c'_1 = \delta_c^{-1}(d_o + d_1 - \rho_1 v_1) \tag{5.113}$$

$$(1 - \alpha)d'_o + \frac{1}{2}(1 - \alpha^2)d'_1 = -\delta_c^{-1}\left(p_2 - p_1 - \rho_1 v_1^2\right) \tag{5.114}$$

$$c_o + \alpha c_1 = (1 - \alpha)\rho_1 u_1 \tag{5.115}$$

$$d_o + \alpha d_1 = \rho_1 u_1 v_1 \tag{5.116}$$

We find c_o, c_1, and c'_1 from equations 5.100, 5.111, and 5.115 to be

$$c_o = \rho_1 u_1 - \alpha(1 - \alpha)^{-1}A \tag{5.117}$$

$$c_1 = -\rho_1 u_1 + (1 - \alpha)^{-1}A \tag{5.118}$$

$$c'_1 = -(\rho_1 u_1)' + \delta_c^{-1}A(1 - \alpha)^{-2}v_1/u_1 \tag{5.119}$$

Then equation 5.113 becomes

$$(\rho_1 u_1)' = \delta_c^{-1}(1 - \alpha)^{-1}\left[(1 - \alpha)^{-1}Av_1/u_1 - (d_o + d_1) + \rho_1 v_1\right] \equiv F_1 \tag{5.120}$$

When we substitute the result in equation 5.120 into equation 5.109, we get

$$(\rho_o u_o)' = \delta_c^{-1}\alpha^{-1}(1 - \alpha)^{-2}\left\{\alpha\left[(1 - \alpha)^2 a_1 - A\right]v_1/u_1 + \alpha(1 - \alpha)(d_o + d_1) - (1 - \alpha)(2 - \alpha)\rho_1 v_1\right\} \equiv F_o \tag{5.121}$$

From equations 5.61, 5.62, 5.65, 5.66, and 5.67, with i = 0, 1 (these equations are equivalent to equations 5.94 through 5.97) we have

$$p_o = \frac{\gamma - 1}{2\gamma}\left(1 + \frac{\gamma - 1}{2}M_o^2\right)^{-\gamma/(\gamma - 1)} \tag{5.122}$$

$$\rho_o = \left(1 + \frac{\gamma - 1}{2} M_o^2\right)^{-1/(\gamma-1)} \tag{5.123}$$

$$u_o^2 = \frac{\gamma - 1}{2} M_o^2 \left(1 + \frac{\gamma - 1}{2} M_o^2\right)^{-1} \tag{5.124}$$

$$p_1 = \frac{\gamma - 1}{2\gamma} \left[\left(1 + \frac{\gamma - 1}{2} M_1^2\right)\left(1 - \dot{g}_1^2\right)\right]^{-\gamma/(\gamma-1)} \tag{5.125}$$

$$\rho_1 = \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{-1/(\gamma-1)} \left(1 - \dot{g}_1^2\right)^{-\gamma/(\gamma-1)} \tag{5.126}$$

$$\rho_1^{-1} \rho_1' = -M_1 \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{-1} M_1' \tag{5.127}$$

$$q_1^2 = u_1^2 + v_1^2 \tag{5.128}$$

$$= \frac{\gamma - 1}{2} M_1^2 \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{-1} \tag{5.129}$$

$$q_1^{-1} q_1' = M_1^{-1} \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{-1} M_1' \tag{5.130}$$

Thus, we wish to express $(\rho_o u_o)'$, $(\rho_1 u_1)'$, and $(\rho_1 u_1 v_1)'$ in terms of M_o', M_1', and V_1'. The expression for $(\rho_o u_o)'$ in terms of M_o' is given in equation 5.79. We note that

$$(\rho_1 u_1)' = \rho_1 u_1 \left(\rho_1^{-1} \rho_1' + u_1^{-1} u_1'\right) = F_1 \tag{5.131}$$

$$(\rho_1 u_1 v_1)' = \rho_1 u_1 v_1 \left[(\rho_1 u_1)^{-1} F_1 + v_1^{-1} v_1'\right] = F_2 \tag{5.132}$$

where F_1 and F_2 are defined in equations 5.120 and 5.110, respectively. We also have the relations

$$u_1^{-1} u_1' = q_1^{-1} q_1' - V_1 \left(1 - V_1^2\right)^{-1} V_1' \tag{5.133}$$

$$v_1^{-1} v_1' = q_1^{-1} q_1' + V_1^{-1} V_1' \tag{5.134}$$

Then from equations 5.79, 5.121, 5.127, and 5.131 through 5.134, the differential equations for M_o, M_1, and V_1 are

$$M_o' = \left[\frac{M_o\left(1 + \frac{\gamma - 1}{2} M_o^2\right)}{\rho_o u_o \left(1 - M_o^2\right)}\right] F_o \tag{5.135}$$

$$B_{11}M_1' + B_{12}V_1' = C_1 \tag{5.136}$$

$$B_{21}M_1' + B_{22}V_1' = C_2 \tag{5.137}$$

where the B_{ij} and C_j terms are given by

$$B_{11} = \rho_1 u_1 M_1^{-1}\left(1 - M_1^2\right)\left(1 + \frac{\gamma - 1}{2}M_1^2\right)^{-1} \tag{5.138}$$

$$B_{12} = -\rho_1 u_1 V_1\left(1 - V_1^2\right)^{-1} \tag{5.139}$$

$$B_{21} = \rho_1 u_1 q_1 V_1 M_1^{-1}\left(1 + \frac{\gamma - 1}{2}M_1^2\right)^{-1} \tag{5.140}$$

$$B_{22} = \rho_1 u_1 q_1 \tag{5.141}$$

$$C_1 = F_1 \tag{5.142}$$

$$C_2 = F_2 - q_1 V_1 F_1 \tag{5.143}$$

We solve equations 5.136 and 5.137 for M_1' and V_1' and obtain

$$M_1' = D^{-1}(C_1B_{22} - C_2B_{12}) \tag{5.144}$$

$$V_1' = D^{-1}(C_2B_{11} - C_1B_{21}) \tag{5.145}$$

where D is given from

$$D = B_{11}B_{22} - B_{12}B_{21} \tag{5.146}$$

Substitution of equations 5.138 through 5.141 into equation 5.146 gives

$$D = \rho_1^2 q_1^3 M_1^{-1}\left(1 + \frac{\gamma - 1}{2}M_1^2\right)^{-1}\left[1 - \left(u_1^2/a_1^2\right)\right] \tag{5.147}$$

Here a_1 is the local speed of sound on the strip $\tilde{y} = \alpha\delta_c$. Since this line is initially sonic ($q_{1,c} = a_{1,c}$) and since the initial value of v_1 is zero, then $q_{1,c} = u_{1,c}$, and

$$D_c = 0 \tag{5.148}$$

The implication of equation 5.148 will be discussed further when the asymptotic expansions for $\tilde{x} \to -\infty$ are considered.

There is a fourth differential equation, deriving from equation 5.114. If we pick F_o, defined in equation 5.121, as the fourth dependent variable, a relation equivalent to equation 5.82 will be obtained when M_o is chosen to be the independent variable. From equations 5.116 and 5.121 we get

$$d_o = (1 - \alpha)^{-1}\rho_1 u_1 v_1 - \alpha G \tag{5.149}$$

$$d_1 = -(1 - \alpha)^{-1}\rho_1 u_1 v_1 + G \tag{5.150}$$

where

$$G = \delta_c F_o + \rho_1 q_1 V_1\left\{-(\rho_1 u_1)^{-1}\left[a_1 - (1 - \alpha)^{-2}A\right] + \alpha^{-1}(1 - \alpha)^{-1}(2 - \alpha)\right\} \tag{5.151}$$

By differentiating equations 5.149 and 5.150, we obtain

$$d_o' = -\alpha G' + (1 - \alpha)^{-1}F_2 + \left[(1 - \alpha)^{-2}\rho_1 u_1 v_1 - G\right]\delta_c^{-1}v_1/u_1 \tag{5.152}$$

$$d_1' = G' - (1 - \alpha)^{-1}F_2 - (1 - \alpha)^{-2}\delta_c^{-1}\rho_1 v_1^2 \tag{5.153}$$

so that equation 5.114 becomes

$$G' = -(1 - \alpha)^{-1}F_2 + 2(1 - \alpha)^{-2}\delta_c^{-1}\left[(1 - \alpha)Gv_1/u_1 - (p_2 - p_1) + \frac{1}{2}\rho_1 v_1^2\right] \tag{5.154}$$

We obtain the differential equation for F_o by differentiating equation 5.151 and solving for F_o':

$$\begin{aligned} F_o' = \delta_c^{-1}G' - \delta_c^{-1}\rho_1 q_1 V_1\Big\{&(\rho_1 q_1)^{-1}(\rho_1 q_1)' + V_1^{-1}V_1' \\ &+ (\rho_1 u_1)^{-2}F_1\left[a_1 - (1 - \alpha)^{-2}A\right] - (\rho_1 u_1)^{-1} \\ &\times\left[\alpha^{-1}\left(F_1 - F_o - \delta_c^{-1}a_1 v_1/u_1\right) - 2(1 - \alpha)^{-3}\delta_c^{-1}Av_1/u_1\right] \\ &- \alpha^{-2}(1 - \alpha)^{-2}(\alpha^2 - 4\alpha + 2)\delta_c^{-1}v_1/u_1\Big\} \equiv F_3 \end{aligned} \tag{5.155}$$

In equation 5.155, V_1' and G' are given by equations 5.145 and 5.154, respectively. Also, from equations 5.132 and 5.135 we have

$$(\rho_1 q_1)^{-1}(\rho_1 q_1)' = M_1^{-1}\left(1 - M_1^2\right)\left(1 + \frac{\gamma - 1}{2}M_1^2\right)^{-1}M_1' \tag{5.156}$$

Thus, with M_o as the independent variable, the differential equations, obtained from equations 5.135, 5.144, 5.145, and 5.155, are

$$\frac{dX}{dM_o} = \frac{\rho_o u_o\left(1 - M_o^2\right)}{M_o\left(1 + \frac{\gamma - 1}{2}M_o^2\right)\sqrt{F_o^2}} \equiv H \tag{5.157}$$

$$dM_1/dM_o = HD^{-1}(C_1B_{22} - C_2B_{12}) \tag{5.158}$$

$$dV_1/dM_o = HD^{-1}(C_2B_{11} - C_1B_{21}) \tag{5.159}$$

$$\frac{dF_o^2}{dM_o} = \frac{2\rho_o u_o\left(1 - M_o^2\right)F_3}{M_o\left(1 + \frac{\gamma - 1}{2}M_o^2\right)} \tag{5.160}$$

$$d\alpha/dM_o = \delta_c^{-1}HV_1\left(1 - V_1^2\right)^{-1/2} \tag{5.161}$$

where X is defined in equation 5.80. In this case it is necessary to choose

$$F_o = +\sqrt{F_o^2} \tag{5.162}$$

in order to guarantee that dX/dM_o is positive.

To begin the numerical integration of equations 5.157 through 5.161, it is necessary to utilize the asymptotic expansions for $M_o \to M_{o,c}$. This is because dF_o^2/dM_o, for example, is a function of M_1 and V_1 as well as M_o. Also, since $p_1 - p_o = O\left[(\Delta M_o)^2\right]$ as $\Delta M_o \to 0$ and since dV_1/dM_o and dF_o^2/dM_o are, in part, dependent on $p_1 - p_o$, it is necessary to carry out the expansions to second order in ΔM_o as $M_o \to M_{o,c}$.

From the expansions 5.90 and 5.91 obtained in a single-strip calculation, we expect the expansions to be of the form

$$M_1 \sim M_{1,c} + J_1^{(1)}\Delta M_o + J_1^{(2)}(\Delta M_o)^2 + \ldots \tag{5.163}$$

$$V_1 \sim J_2^{(1)}(\Delta M_o)^{3/2} + J_2^{(2)}(\Delta M_o)^{5/2} + \ldots \tag{5.164}$$

$$F_o \sim J_3^{(1)}(\Delta M_o)^{3/2} + J_3^{(2)}(\Delta M_o)^{5/2} + \ldots \tag{5.165}$$

$$\alpha \sim \alpha_c + J_4^{(1)}\Delta M_o + J_4^{(2)}(\Delta M_o)^2 + \ldots \tag{5.166}$$

Because the initial value of X is arbitrary, it is not required to obtain an expansion for $X(M_o)$.

Since $M_{1,c} = 1$, we have $D_c = 0$ (see eqs. 5.147 and 5.148), and

$$D = O(\Delta M_o) \tag{5.167}$$

as $\Delta M_o \to 0$. Then, to the order ΔM_o, the expansions of equations 5.158 through 5.161 are

$$F_1 = O\left[(\Delta M_o)^2\right] \tag{5.168}$$

$$p_1 - p_o = O\left[(\Delta M_o)^2\right] \tag{5.169}$$

$$p_2 - p_o = O\left[(\Delta M_o)^2\right] \tag{5.170}$$

$$D_{42}J_2^{(1)} + D_{43}J_3^{(1)} = 0 \tag{5.171}$$

where

$$D_{42} = \rho_o u_o\left(1 - M_o^2\right)M_o^{-1}\left(1 + \frac{\gamma - 1}{2}M_{o,c}^2\right)^{-1} \tag{5.172}$$

$$D_{43} = -\delta_c J_4^{(1)} \tag{5.173}$$

We get from equation 5.168 the result

$$D_{12}J_2^{(1)} + D_{13}J_3^{(1)} = 0 \tag{5.174}$$

where

$$D_{12} = \alpha_c^{-1}\left(\alpha_c^2 + 2\alpha_c - 2\right)\rho_{1,c}u_{1,c} + (1 - \alpha_c)a_{1,c} \tag{5.175}$$

$$D_{13} = -(1 - \alpha_c)\delta_c \tag{5.176}$$

From equation 5.169, to order ΔM_o, we obtain

$$D_{21}J_1^{(1)} = H_2^{(1)} \tag{5.177}$$

where

$$D_{21} = M_{1,c}\left(1 + \frac{\gamma - 1}{2}M_{1,c}^2\right)^{-1} \tag{5.178}$$

$$H_2^{(1)} = M_{o,c}\left(1 + \frac{\gamma - 1}{2}M_{o,c}^2\right)^{-1} \tag{5.179}$$

Now we require the expansion for p_2. From equation 5.112, we find

$$p_2 \sim p_c\left\{1 + \left[(1 - \alpha_c)^{-1} + \gamma A^{-1}\rho_{1,c}u_{1,c}\right]J_4^{(1)}\Delta M_o - \gamma A^{-1}(1 - \alpha_c)\Delta(\rho_1 u_1)\right\} \tag{5.180}$$

while the expansion for $\rho_1 u_1$ as $\Delta M_o \to 0$ gives

$$\Delta(\rho_1 u_1) = O\left[(\Delta M_o)^2\right] \tag{5.181}$$

so that equation 5.170 becomes

$$D_{34}J_4^{(1)} = H_3^{(1)} \tag{5.182}$$

where

$$H_3^{(1)} = H_2^{(1)} \tag{5.183}$$

$$D_{34} = \gamma^{-1}(1 - \alpha_c)^{-1} + A^{-1}\rho_{1,c}u_{1,c} \tag{5.184}$$

When $J_4^{(1)}$ is calculated from equation 5.182, equations 5.171 and 5.174 become a system of two linear homogeneous equations in $J_2^{(1)}$ and $J_3^{(1)}$. For a nontrivial solution of these equations to exist, the determinant of the coefficients must vanish.

$$D_{12}D_{43} - D_{13}D_{42} = 0 \tag{5.185}$$

Now D_{43}, through equations 5.173 and 5.184, contains A. Thus, equation 5.185 provides a relation for determining A, and we get

$$D_{43} = D_{12}^{-1}D_{13}D_{42} \tag{5.186}$$

$$D_{34} = -\delta_c D_{43}^{-1}H_2^{(1)} \tag{5.187}$$

$$A = \rho_{1,c}u_{1,c}\left[D_{34} - \gamma^{-1}(1 - \alpha_c)^{-1}\right]^{-1} \tag{5.188}$$

However, now equations 5.171 and 5.174 are linearly dependent, and either $J_2^{(1)}$ or $J_3^{(1)}$ will remain undetermined. An additional relationship involving the first-order perturbations is found from the second-order relations. In the second-order expansions of equations 5.158 through 5.161, we obtain

$$D_{12}J_2^{(2)} + D_{13}J_3^{(2)} \qquad = H_1^{(2)} \tag{5.189}$$

$$D_{21}J_1^{(2)} \qquad = H_2^{(2)} \tag{5.190}$$

$$D_{34}J_4^{(2)} = H_3^{(2)} \tag{5.191}$$

$$D_{42}J_2^{(2)} + D_{43}J_3^{(2)} \qquad = H_4^{(2)} \tag{5.192}$$

The coefficient matrix D_{ij} is the same for all orders of magnitude. The nonhomogeneous terms, $H_i^{(2)}$, are functions of the first-order perturbation constants, $J_k^{(1)}$. But A has been chosen so that the determinant of the coefficient matrix in equations 5.189 and 5.192 vanishes. Thus, we must have

$$\begin{vmatrix} H_1^{(2)} & D_{13} \\ H_4^{(2)} & D_{43} \end{vmatrix} = 0 \tag{5.193}$$

and it appears that the last of the first-order perturbations is found from this equation.

Equations 5.189 and 5.192 now will contain only one independent relationship, and it appears that one second-order coefficient, $J_2^{(2)}$, for instance, will be determined by a third-order relationship equivalent to equation 5.198.

In utilizing the asymptotic expansions 5.163 through 5.166 to start the numerical integration of equations 5.157 through 5.161, it is necessary for dM_1/dM_o, dV_1/dM_o, etc. to be correct only to the second order. Hence it would be permissible to choose the value of $J_2^{(2)}$, for instance, arbitrarily, as this only incurs a relative error of the order ΔM_o or smaller in the initial estimates of the derivatives.

5.4. GENERALIZATION TO AN ARBITRARY NUMBER OF STRIPS

If an arbitrary number of strips were utilized in the application of the integral relations 5.20, 5.21, 5.47, and 5.49, we would expect to obtain differential equations of the form

$$dX/dM_o = \left(1 - M_o^2\right)B/F \tag{5.194}$$

$$dM_i/dM_o = \left(1 - M_o^2\right)\Delta_{1,i}/\Delta F \tag{5.195}$$

$$dV_i/dM_o = \left(1 - M_o^2\right)\Delta_{2,i}/\Delta F \tag{5.196}$$

$$dF^2/dM_o = \left(1 - M_o^2\right)C \tag{5.197}$$

$$d\alpha_i/dM_o = \left(1 - M_o^2\right)Bv_i/u_iF \tag{5.198}$$

where $F = \pm\sqrt{F^2}$ and where $i = 1, \ldots, M + N - 1$. In general, Δ has a simple zero at each of the points where $1 - u_i/a_i = 0$. When $M + N = 2$, $\Delta \equiv D$ is given in equations 5.146 and 5.147.

In the case of strip boundaries that are initially subsonic, downstream boundary conditions are obtained from the requirement that the solution be regular at the points where $\Delta = 0$ [22]. This condition takes the form

$$\Delta_{1,i} = 0 \tag{5.199}$$

at the points where $u_i = a_i$. In the asymptotic expansions of M_i, V_i, and F, one perturbation should be undetermined on each strip boundary. The solution is found by guessing values of the initial perturbations in the V_i, for example. Trial integrations are carried out until equation 5.199 is satisfied, and then the integral curves will pass through the saddle-point singularities at the points where $u_i = a_i$.

As an example, we will consider a three-strip calculation (M = 2, N = 1) where $M_{1,c} < 1$, and $M_{2,c} = 1$. The asymptotic expansions of the dependent variables, to first order, are of the form

$$\begin{aligned}
\Delta M_1 &\sim J_1^{(1)} \Delta M_o \\
\Delta M_2 &\sim J_2^{(1)} \Delta M_o \\
V_1 &\sim J_3^{(1)} (\Delta M_o)^{3/2} \\
V_2 &\sim J_4^{(1)} (\Delta M_o)^{3/2} \\
F &\sim J_5^{(1)} (\Delta M_o)^{3/2} \\
\Delta \alpha_1 &\sim J_6^{(1)} \Delta M_o \\
\Delta \alpha_2 &\sim J_7^{(1)} \Delta M_o
\end{aligned} \tag{5.200}$$

Then the $J_1^{(1)}$ terms are related by the equations

$$D_{11}J_5^{(1)}J_1^{(1)} + D_{13}J_3^{(1)} + D_{14}J_4^{(1)} + D_{15}J_5^{(1)} = 0 \qquad (5.201)$$

$$D_{23}J_3^{(1)} + D_{24}J_4^{(1)} + D_{25}J_5^{(1)} = 0 \qquad (5.202)$$

$$D_{31}J_1^{(1)} = H_3^{(1)} \qquad (5.203)$$

$$D_{42}J_2^{(1)} = H_4^{(1)} \qquad (5.204)$$

$$D_{57}J_7^{(1)} = H_5^{(1)} \qquad (5.205)$$

$$D_{63}J_3^{(1)} + D_{65}J_6^{(1)}J_5^{(1)} = 0 \qquad (5.206)$$

$$D_{74}J_4^{(1)} + D_{75}J_7^{(1)}J_5^{(1)} = 0 \qquad (5.207)$$

where the D_{ij} and the $H_i^{(1)}$ terms are known constants, except for D_{57}, which will depend upon A. Since there is one strip boundary that is initially subsonic, we expect one of the $J_i^{(1)}$ terms to remain arbitrary in the solution of equations 5.201 through 5.207. When we substitute for $J_1^{(1)}$ and $J_7^{(1)}$ from equations 5.203 and 5.205, equations 5.201, 5.202, and 5.207 become three linear homogeneous equations in $J_3^{(1)}$, $J_4^{(1)}$, and $J_5^{(1)}$. The determinant of the coefficients in these equations must vanish, providing a relation for A equivalent to equation 5.185, and one of the coefficients, $J_3^{(1)}$, for instance, remains undetermined. In the second-order equations, the same coefficient matrix appears since the terms in $J_1^{(2)}$ and $J_7^{(2)}$ may be grouped with the nonhomogeneous terms. Thus there appears to be a second-order equation in $J_3^{(1)}$, equivalent to equation 5.193, and we do not seem to have the proper initial conditions in the subsonic region.

Now let us consider a three-strip calculation (M = 1, N = 2) where $M_{1,c} = 1$ and $M_{2,c} > 1$. In this case, the relations for the $J_i^{(1)}$ are the same as in equations 5.201 through 5.207 when the subscripts 1 and 2, and 6 and 7 are interchanged except in that D_{22} now has changed sign. Thus, further investigation is required to distinguish between the initial conditions on strip boundaries that are initially subsonic and the initial conditions on strip boundaries that are initially supersonic. The difficulty may be related to the fact that the approximation $\tilde{p}_{\tilde{y}} \approx 0$ as $\tilde{x} \to -\infty$ causes the characteristics to degenerate to the single family of lines $\tilde{x}$ = constant.

A possible alternative approach for obtaining numerical solutions to this problem would be to apply the method of integral relations only to the portion of the flow that is initially subsonic. Then one function remains undetermined, the normal velocity component on the streamline that is initially sonic. A numerical method of characteristics or a finite difference technique such as that employed by Baum [14] then would be applied in the supersonic region. Since the subsonic portion of the layer is relatively thin, comprising less than 20 percent of the initial thickness of the layer, a single-strip calculation by the method of integral relations for the subsonic portion of the layer together with another numerical technique applied in the supersonic region is likely to provide a relatively high degree of accuracy. Also, it would not be necessary to carry out trial integrations of the equations in this instance. However, further investigation to ascertain the proper treatment of the initial conditions in the subsonic region would be required in this case also.

5.5. EVALUATION OF RESULTS AND COMPARISON WITH EXPERIMENTAL DATA

The numerical integration of equations 5.81 and 5.82 applicable in a single-strip calculation (M = 0, N = 1) has been carried out by using the IBM 7090 digital computer at The University of Michigan Computing Center. The algorithms employed are programmed in the MAD language [19]. The numerical technique applied is the Runge-Kutta fourth-order method, which is a standard computer library subroutine [20, 21]. First, $d\left(d_1^2\right)/dM_o$, which is only a function of the independent variable M_o, is integrated from the initial point, $M_o = M_{o,c}$ (where $d_1 = 0$) to a point where $\Delta M_o = M_o - M_{o,c}$ is small. Now the integration of dX/dM_o in equation 5.81 can be started since a finite value of d_1 has been calculated. An asymptotic expansion for $X(M_o)$ as $M_o \to M_{o,c}$ is not required since the initial value of X is arbitrary. Then, integration of equations 5.81 and 5.82 proceeds step by step up to the corner where $M_o = 1$. X(1) is determined, and the value of $\tilde{x}$ at each point is recovered from equation 5.80. The numerical results for $\gamma = 1.4$ are presented in table IV.

When the results of our calculations for the first approximation are plotted as a function of $\tilde{x}/\delta_1 \equiv (\bar{x} - L)/\bar{\delta}_1$, they are independent of the Reynolds number R_w (see, for example, eq. 3.58, the composite solution for $\bar{u}/\bar{u}_e$ in a first approximation). Also, the results in a first approximation are the same for a wedge as for a cone. The only conical effect is a factor of $1/\sqrt{3}$ that appears in the formula for the boundary-layer thickness $\bar{\delta}_1$ (see eq. I-37).

These premises can be tested by replotting Hama's [6] data as a function of $(\bar{x} - L)/\bar{\delta}_1$. In figure 4, some of Hama's wall-pressure-ratio data for laminar flow (which appears as fig. 4 of ref. 14) are presented. The ratio $\bar{p}_o/\bar{p}_{e,i}$ is plotted against $\bar{x}$ for three different Mach numbers, where $\bar{p}_o$ is the measured surface pressure and $\bar{p}_{e,i}$ is the surface pressure predicted by inviscid-flow theory. These data are replotted in figure 5, with abscissa $\tilde{x}/\delta_1 \equiv (\bar{x} - L)/\bar{\delta}_1$. The necessary calculations for the reduction of Hama's data are presented in appendix III. The

TABLE IV. SOLUTION IN THE SECOND OUTER LIMIT FOR A SINGLE-STRIP CALCULATION

$\tilde{x}/\delta_1$	M_o	$\bar{u}_o/\bar{u}_e$	$\bar{p}_o/\bar{p}_e$	$\bar{p}_1/\bar{p}_e$	d_1	$\tilde{x}/\delta_1$	M_o	$\bar{u}_o/\bar{u}_e$	$\bar{p}_o/\bar{p}_e$	$\bar{p}_1/\bar{p}_e$	d_1
0.0000	0.4482	0.1965	0.8712	0.8712	0.0000	-0.4353	0.7390	0.3138	0.6957	0.7676	-0.0273
-171.4514	0.4490	0.1969	0.8708	0.8708	-0.0000	-0.3935	0.7490	0.3176	0.6892	0.7656	-0.0283
-16.7019	0.4590	0.2011	0.8655	0.8656	-0.0003	-0.3550	0.7590	0.3214	0.6827	0.7637	-0.0293
-10.7671	0.4690	0.2053	0.8601	0.8606	-0.0007	-0.3193	0.7690	0.3252	0.6762	0.7619	-0.0302
-8.0810	0.4790	0.2095	0.8547	0.8556	-0.0013	-0.2865	0.7790	0.3290	0.6697	0.7602	-0.0312
-6.4751	0.4890	0.2136	0.8492	0.8508	-0.0019	-0.2561	0.7890	0.3327	0.6632	0.7586	-0.0321
-5.3831	0.4990	0.2178	0.8436	0.8461	-0.0027	-0.2281	0.7990	0.3365	0.6567	0.7570	-0.0330
-4.5823	0.5090	0.2220	0.8379	0.8416	-0.0034	-0.2024	0.8090	0.3402	0.6502	0.7556	-0.0338
-3.9652	0.5190	0.2261	0.8322	0.8371	-0.0043	-0.1786	0.8190	0.3439	0.6437	0.7542	-0.0346
-3.4725	0.5290	0.2302	0.8265	0.8328	-0.0052	-0.1568	0.8290	0.3476	0.6371	0.7529	-0.0354
-3.0689	0.5390	0.2343	0.8206	0.8286	-0.0061	-0.1369	0.8390	0.3513	0.6306	0.7517	-0.0362
-2.7313	0.5490	0.2384	0.8148	0.8245	-0.0071	-0.1186	0.8490	0.3550	0.6242	0.7506	-0.0369
-2.4444	0.5590	0.2425	0.8088	0.8205	-0.0080	-0.1019	0.8590	0.3586	0.6177	0.7496	-0.0376
-2.1974	0.5690	0.2466	0.8028	0.8167	-0.0091	-0.0868	0.8690	0.3622	0.6112	0.7486	-0.0383
-1.9825	0.5790	0.2507	0.7968	0.8129	-0.0101	-0.0730	0.8790	0.3658	0.6048	0.7477	-0.0389
-1.7937	0.5890	0.2547	0.7907	0.8093	-0.0112	-0.0607	0.8890	0.3694	0.5983	0.7469	-0.0395
-1.6267	0.5990	0.2588	0.7846	0.8058	-0.0122	-0.0496	0.8990	0.3730	0.5919	0.7462	-0.0400
-1.4779	0.6090	0.2628	0.7785	0.8024	-0.0133	-0.0398	0.9090	0.3766	0.5855	0.7455	-0.0405
-1.3446	0.6190	0.2668	0.7723	0.7991	-0.0144	-0.0312	0.9190	0.3801	0.5791	0.7450	-0.0410
-1.2247	0.6290	0.2708	0.7660	0.7959	-0.0155	-0.0237	0.9290	0.3837	0.5727	0.7444	-0.0414
-1.1163	0.6390	0.2748	0.7598	0.7928	-0.0166	-0.0174	0.9390	0.3872	0.5664	0.7440	-0.0418
-1.0180	0.6490	0.2787	0.7535	0.7899	-0.0177	-0.0120	0.9490	0.3907	0.5601	0.7436	-0.0421
-0.9285	0.6590	0.2827	0.7471	0.7870	-0.0188	-0.0077	0.9590	0.3942	0.5538	0.7433	-0.0423
-0.8468	0.6690	0.2866	0.7408	0.7842	-0.0199	-0.0044	0.9690	0.3976	0.5475	0.7431	-0.0425
-0.7721	0.6790	0.2906	0.7344	0.7815	-0.0210	-0.0020	0.9790	0.4011	0.5413	0.7429	-0.0427
-0.7036	0.6890	0.2945	0.7280	0.7790	-0.0221	-0.0005	0.9890	0.4045	0.5351	0.7428	-0.0428
-0.6408	0.6990	0.2984	0.7216	0.7765	-0.0231	-0.0000	0.9990	0.4079	0.5289	0.7428	-0.0428
-0.5829	0.7090	0.3022	0.7151	0.7741	-0.0242	0.0000	1.0000	0.4082	0.5283	0.7428	-0.0428
-0.5297	0.7190	0.3061	0.7087	0.7719	-0.0252						
-0.4806	0.7290	0.3100	0.7022	0.7697	-0.0263						

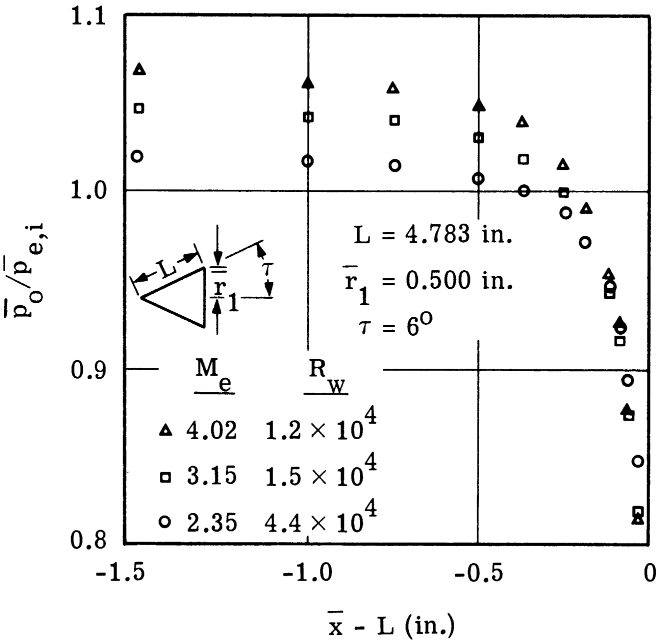

FIGURE 4. HAMA'S DATA ON THE WALL-PRESSURE RATIO FOR LAMINAR FLOW [14, fig. 4]

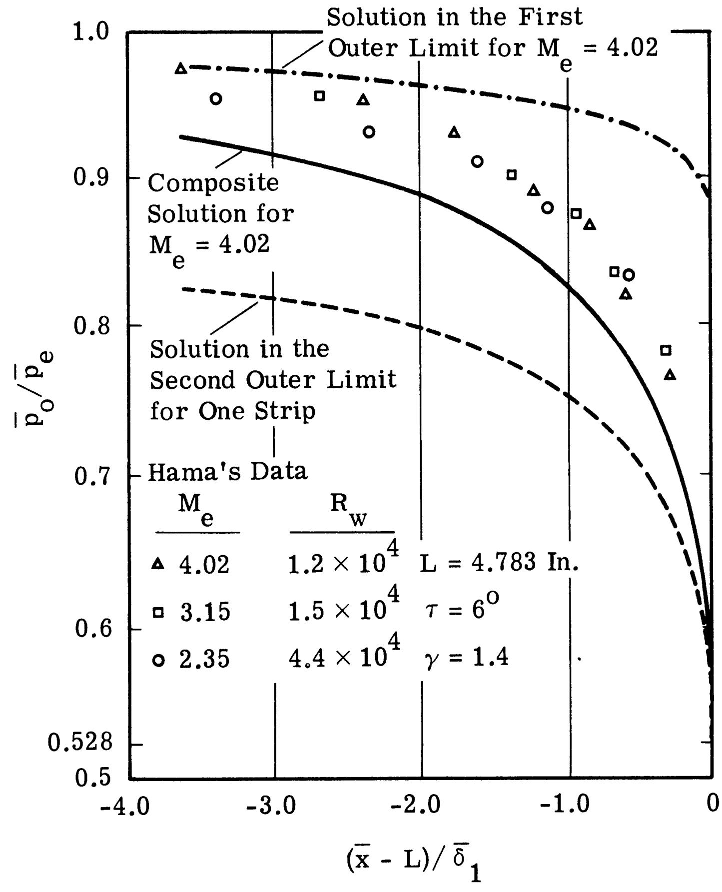

FIGURE 5. COMPARISON OF THE COMPOSITE SOLUTION FOR THE WALL-PRESSURE RATIO AND HAMA'S DATA

reference values M_e and p_e are chosen to correspond to the measurements at the static-pressure orifice farthest upstream from the corner. This orifice is located at $(\bar{x} - L) = -1.5$ in., where $L = 4.783$ in. The largest experimental Mach number presented here corresponds to $M_e \approx 4.02$, $R_w \approx 1.2 \times 10^4$, and $\bar{\delta}_1 \approx 0.10$ in. Although even this value of M_e might be rather low for use of a hypersonic theory, the other sources of error to be considered seem to be at least as important.

We see in figure 5 that the wall-pressure-ratio measurements are independent of the Reynolds number R_w within a discrepancy of $\Delta\bar{p}_o/\bar{p}_o \approx 0.02$ over the range $1.2 \times 10^4 \leqq R_w \leqq 4.4 \times 10^4$. Thus, the theoretical prediction of Reynolds number independence appears to be confirmed by the experimental data.

Also presented in figure 5 is the wall-pressure-ratio solution in the first outer limit for $M_e = 4.02$, the solution in the second outer limit (which is independent of both Mach and Reynolds numbers) for the one-strip calculation by the method of integral relations, and the composite solution for $M_e = 4.02$. The discrepancy between the calculated solution and the experimental data is observed to be $|\Delta\bar{p}_o/\bar{p}_o| \approx 0.06$ or less. This is a fairly good result, as compared with the sort of accuracy usually obtained in a one-strip calculation by the method of integral relations (see ref. 22 and app. II). This may be because the calculation by the method of integral relations comprises only part of the solution; the equations in the first outer limit have an exact solution.

The changes in velocity profile for the accelerating boundary layer are shown in figure 6 by plots of $\bar{u}/\bar{u}_e$ vs. $\tilde{\psi}$. The two solid curves represent the initial velocity profile $\bar{u}/\bar{u}_e = \hat{u}(-\infty, \tilde{y}) = \dot{g}(\beta)$ and the profile $\bar{u}/\bar{u}_e = \hat{u}(0, \tilde{y})$ given by the upstream solution 4.10 at $\tilde{x} = 0$. It is evident that the initial acceleration of fluid particles along streamlines is significant primarily near the wall, but plots of $\bar{u}/\bar{u}_e$ vs. $\tilde{y}$ would show that the resulting displacement of streamlines is significant all across the layer. For the solution in terms of $\tilde{x}$ and $\tilde{y}$, it is probably consistent with a one-strip calculation by the method of integral relations to choose only a linear variation of $\tilde{u}$ with $\tilde{y}$.

$$u = u_o + (1 - u_o)\tilde{y}/\delta_c \tag{5.208}$$

If equation 5.37 is specialized for a one-strip calculation, we have

$$\tilde{\psi} = -\delta_c[A \log (1 - \tilde{y}/\delta_c) + c_1(\tilde{x})\tilde{y}/\delta_c] \tag{5.209}$$

These two equations can be combined to give plots of $\bar{u}/\bar{u}_e$ vs. $\tilde{\psi}$ for $\tilde{x} \to -\infty$ and $\tilde{x} = 0$. The plots are shown as dotted curves in figure 6. Rather good agreement is obtained between the approximate form for $\tilde{u}$ as $\tilde{x} \to -\infty$ and the solution 4.10 for $\hat{u}$ at $\hat{x} = 0$.

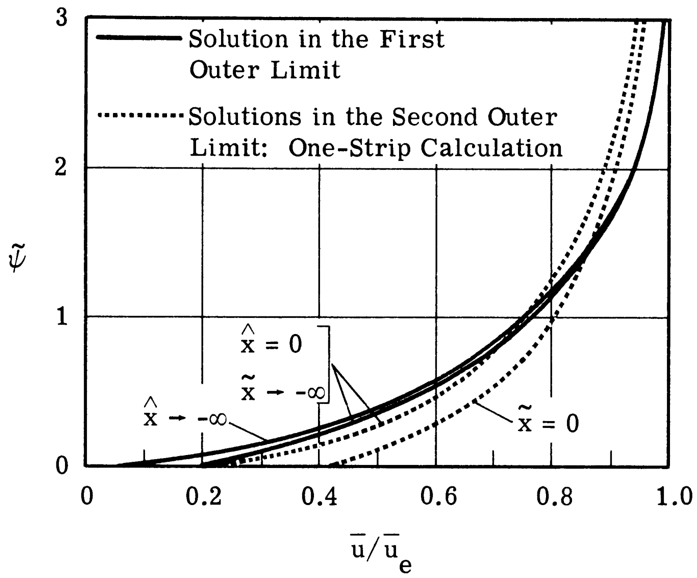

FIGURE 6. VELOCITY PROFILE AT THREE STATIONS: $\hat{x} \to -\infty$, $\hat{x} = 0(\tilde{x} \to -\infty)$, AND $\tilde{x} = 0$

We shall now consider the implications of some higher order effects. The chief correction required because of finite Reynolds number probably involves the sublayer displacement thickness effect. The relative sublayer displacement thickness, $\bar{\delta}_{SL}/\bar{\delta}_1 = O\left(R_w^{-1/4}\right)$ (see eq. 6.35), also causes a pressure disturbance, $\Delta\bar{p}/\bar{p}_e = O\left(R_w^{-1/4}\right)$. In Hama's experiment, $R_w^{-1/4} \approx 0.10$ for $M_e = 4.02$. Since $\bar{p}_{\bar{x}} < 0$, the sublayer thickness $\bar{\delta}_{SL}$ will decrease monotonically, and, since $|\bar{p}_{\bar{x}}|$ increases monotonically as $(\bar{x} - L)/\bar{\delta}_1 \to 0$, the sharpest decrease in $\bar{\delta}_{SL}$ will occur near the corner. This will tend to cause a further increase in $|\bar{p}_{\bar{x}}|$ near the corner. The overall drop in the pressure, $[\bar{p}(0, 0) - \bar{p}_e]/\bar{p}_e$, is fixed because of the sonic condition at the corner. Thus, an increase in $|\bar{p}_{\bar{x}}|$ near the corner will be balanced by a decrease in $|\bar{p}_{\bar{x}}|$ farther upstream, and a calculated wall-pressure-ratio distribution that includes the correction for the sublayer effect will lie above the solution for the first-order theory shown in figure 5. Thus, the sublayer effect may account for a substantial part of the discrepancy between the predicted and measured values in figure 5.

Disturbances in the pressure of the order $R_w^{-1/4}$ also arise when $(\bar{x}/L) - 1 = O\left(R_w^{-3/8}\right)$ (see eq. 3.38). It would be consistent to treat this effect in conjunction with an analysis of the sublayer.

Another Reynolds number effect appears through the displacement effect of the boundary layer on the outer inviscid flow. The order of magnitude of this effect is characterized by the interaction parameter, $M_e R_w^{-1/2}$. In Hama's experiment, $M_e R_w^{-1/2} \approx 0.04$, and it is pointed out in reference 14 that viscous interaction has a noticeable effect is raising the initial pressure

$\bar{p}_e$. The parameter $M_e R_w^{-1/2}$ also is a measure of the three-dimensional effect in flow over a cone (see sec. 3).

The most significant Mach number effect is a result of the solution in the first outer limit. An increase in M_e should result in a greater upstream influence and thus a lower value of $\bar{p}_o/\bar{p}_e$ for a given value of $(\bar{x} - L)/\bar{\delta}_1$ (see fig. 5). However, the experimental results in figure 5 do not seem to bear this out. This predicted Mach number effect may be obscured by the sublayer influence previously discussed. Since in the experiment, the test conditions for larger values of M_e also have smaller values of R_w, the two effects tend to offset each other.

Another Mach number effect appears if we attempt to calculate higher-order terms. It is then also necessary to consider a limit 3.54 in which $\tilde{x}/M_e^{1/5}$ is held fixed, and, in this limit, we find $\Delta\bar{p}/\bar{p} = O\left(M_e^{-2/5}\right)$.

In principle, it would be possible to generalize the present theory by the following procedures:

(1) Include the viscous sublayer effect by carrying out the calculations suggested in section 4.
(2) Obtain a solution for the limit 3.54 with $\tilde{x}/M_e^{1/5}$ fixed
(3) Generalize the equations to include nonadiabatic and real gas effects

We shall now examine certain aspects of the application of the method of integral relations to this problem. The system of nonlinear partial differential equations is reduced to a system of quasi-linear, first-order, ordinary differential equations. The character of the downstream boundary condition in the subsonic region where the equations are elliptic is simplified to consist of one downstream condition on each strip boundary that is initially subsonic (see eq. 5.199). Also, the procedure for obtaining the numerical solution of the equations is well adapted to the use of high-speed electronic digital computers. Increased accuracy in the numerical solution is achieved, in principle, by increasing the number of strips used in the application, rather than by increasing the number of iterations in a relaxation technique, for example.

However, in the application of any integral technique, certain properties of the full equations may be only approximated or obscured. Although accurate representations for the flow properties along the strip boundaries might be attained with the method of integral relations, profiles normal to the strips are found only by interpolation of the values on the strip boundaries. In the present study, the property profiles at $\bar{x} = L$ are of particular interest since they would provide the initial conditions for a calculation of the near wake of the body.

In the present application of the method, we have found that

$$M_o \sim 1 + (\text{constant})|\tilde{x}|^{1/2} \tag{5.210}$$

as $\tilde{x} \to 0$. This behavior for $\tilde{x} \to 0$ is also found by Gold and Holt [23] in a one-strip integral-relations calculation of supersonic flow past a flat-faced cylinder. However, results obtained by Vaglio-Laurin [24] and by Fal'kovich and Chernov [25], for example would seem to suggest that the correct behavior is

$$M_o \sim 1 + (\text{constant})|\tilde{x}|^{2/5} \tag{5.211}$$

as $\tilde{x} \to 0$ instead of equation 5.210. A possible explanation of this discrepancy is that the integral relations resemble a description of a generalized one-dimensional flow as discussed by Shapiro [26] where, for instance,

$$dM^2/dx = G(x)(1 - M^2)^{-1} \tag{5.212}$$

(see eq. 8.71a in ref. 26). When $G(x) \neq 0$ at the point where $M = 1$, the expansion for M near the sonic point is similar to that obtained in equation 5.210. In general, there is a requirement that $G(x) = 0$ when $M = 1$, and equation 5.212 has properties similar to those of equations 5.195 and 5.196. Belotserkovskii, Sedova, and Shugaev [27] avoid this difficulty for the related problem of inviscid supersonic flow over a blunt axisymmetric body with a corner. They obtain a solution by the method of integral relations, with strip boundaries equally spaced between the axis of symmetry and the limiting characteristic, and join their result with Vaglio-Laurin's solution near the corner.

6
SOLUTION IN THE SUBLAYER

6.1. DORODNITSYN TRANSFORMATION

The sublayer equations describe a compressible boundary layer which extends upstream to infinity. Problems of this type are discussed by Neiland [28], and a numerical solution for the boundary layer approaching a corner is given by Matveeva and Neiland. In the present work, we employ Dorodnitsyn's method [22] to derive the integral relations for N = 1 and 2. The appropriate form of the Dorodnitsyn transformation in this case is (see app. I)

$$\xi = \int_0^{\tilde{x}} U \frac{p^\dagger}{p_e} d\tilde{x} \tag{6.1}$$

$$\eta = U\int_0^{y^\dagger} \rho^\dagger\, dy^\dagger$$

where $p_e = (\gamma - 1)/2\gamma$ and $U = \bar{u}_o/\bar{u}_e$; $\bar{u}_o$ is the velocity just outside the sublayer. This velocity is found by setting $\tilde{y} = 0$ in the composite solution for the outer part of the boundary layer.

$$\partial/\partial\tilde{x} = U\frac{p^\dagger}{p_e}\,\partial/\partial\xi + \eta_{\tilde{x}}\,\partial/\partial\eta$$

$$\partial/\partial y^\dagger = U\rho^\dagger\,\partial/\partial\eta$$

Since

$$y^\dagger_{\xi\eta} = (1/\rho^\dagger U)_\xi$$

we have

$$y^\dagger_\xi = \int_0^\eta (1/\rho^\dagger U)_\xi\, d\eta$$

Differentiating $\eta(\tilde{x}, y^\dagger)$ with respect to ξ,

$$\eta_{\tilde{x}} = -\eta_{y^\dagger} y^\dagger_\xi\, d\xi/d\tilde{x}$$

Hence

$$\eta_{\tilde{x}} = -\rho^\dagger U^2\frac{p^\dagger}{p_e}\int_0^\eta (1/\rho^\dagger/U)_\xi\, d\eta$$

For adiabatic flow of a thermally and calorically perfect gas over an insulated body, the density can be obtained from the integrated energy equation,

$$(u^\dagger U)^2 + \frac{2\gamma}{\gamma - 1}\frac{p^\dagger}{\rho^\dagger} = 1 \tag{6.2}$$

Also, the $\bar{x}$-momentum equation, 3.66, evaluated at the edge of the sublayer, yields (since $u^\dagger$ = 1 here)

$$-dp^\dagger/d\tilde{x} = \frac{2\gamma}{\gamma - 1}p^\dagger(1 - U^2)^{-1}U\,dU/d\tilde{x} \tag{6.3}$$

while, in terms of the Mach number M_o at the outer edge of the sublayer,

$$U^2 = \frac{\gamma - 1}{2}M_o^2\left(1 + \frac{\gamma - 1}{2}M_o^2\right)^{-1}$$
$$(1 - U^2)^{-1}U^{-1}\,dU/d\tilde{x} = M_o^{-1}\,dM_o/d\tilde{x} \tag{6.4}$$

Thus, the transformed equations are

$$u^\dagger_\xi + w_\eta = 0 \tag{6.5}$$

$$u^\dagger u^\dagger_\xi + wu^\dagger_\eta = \left(1 - u^{\dagger 2}\right)\dot{M}_o/M_o + u^\dagger_{\eta\eta} \tag{6.6}$$

where $\dot{M}_o$ denotes $dM_o/d\xi$, and where

$$w = -\rho^\dagger u^\dagger U\int_0^\eta (1/\rho^\dagger U)_\xi \, d\eta + \rho^\dagger \frac{p_e}{p^\dagger} v^\dagger \tag{6.7}$$

The boundary conditions are

$$u^\dagger(\xi, 0) = w(\xi, 0) = 0, \; u^\dagger(\xi, \infty) = 1 \tag{6.8}$$

6.2. SOLUTION BY THE METHOD OF INTEGRAL RELATIONS

To obtain the integral relations, we follow reference 22 and introduce a set of smoothing functions $f_k(u^\dagger)$ where $k = 1, 2, \ldots, N$. The f_k terms are defined to have the properties

$$\begin{gathered} f_k(0) = 1 \\ \lim_{u^\dagger \to 1} f_k(u^\dagger) = 0 \\ k = 1, 2, \ldots, N \end{gathered} \tag{6.9}$$

We multiply equation 6.5 by f_k and equation 6.6 by f'_k and add the results. Then we integrate on η from 0 to ∞, obtaining

$$\int_0^\infty \left[\frac{\partial}{\partial\xi}\left(f_k u^\dagger\right) + \frac{\partial}{\partial\eta}(f_k w)\right] d\eta = \frac{\dot{M}_o}{M_o}\int_0^\infty \left(1 - u^{\dagger 2}\right) f'_k \, d\eta + \int_0^\infty f'_k u^\dagger_{\eta\eta} \, d\eta \tag{6.10}$$

Introducing $\Theta = (\partial u^\dagger/\partial\eta)^{-1}$, changing the variable of integration from η to $u^\dagger$, and integrating by parts, we get

$$\frac{d}{d\xi}\int_0^1 f_k u^\dagger \Theta \, du^\dagger = \frac{\dot{M}_o}{M_o}\int_0^1 \left(1 - u^{\dagger 2}\right) f'_k \Theta \, du^\dagger - \frac{f'_k(0)}{\Theta_0} - \int_0^1 \frac{f''_k}{\Theta} du^\dagger \tag{6.11}$$

When $u^\dagger \to 1$, Θ becomes large. In the case of an adiabatic constant-pressure boundary layer, equation I-30 gives the velocity $\dot{g}(\beta)$ for $\beta \to \infty$. It follows that, as $\beta \to \infty$,

$$\ddot{g} \sim (\text{constant}) \exp(-\beta^2/2) \sim (\text{constant})\, \beta(1 - \dot{g})$$

and

$$\beta \sim (\text{constant}) [\log(1 - \dot{g})]^{1/2}$$

$$\ddot{g} \sim (\text{constant}) (1 - \dot{g})[-\log(1 - \dot{g})]^{1/2}$$

The same kind of behavior is expected when there is a pressure gradient, with η playing a similar role to β. Since $y^{\dagger} \propto \eta$ as $\eta \to \infty$ in the present notation,

$$\partial u^{\dagger}/\partial \eta \sim (\text{constant}) (1 - u^{\dagger})[-\log(1 - u^{\dagger})]^{1/2}$$

as $\eta = \infty$. The procedure in the present method [21] is to approximate the singularity in Θ by omitting the logarithmic factor. We assume instead that

$$\Theta = O[(1 - u^{\dagger})^{-1}]$$

as $u^{\dagger} \to 1$. To guarantee that the integrals exist, we let

$$f_k(u^{\dagger}) = (1 - u^{\dagger})^k \tag{6.12}$$

where $k = 1, 2, \ldots, N$, and we represent Θ by

$$\begin{aligned} \Theta &= (1 - u^{\dagger})^{-1} \sum_{m=0}^{N-1} a_m(\xi) u^{\dagger m} \\ \frac{1}{\Theta} &= (1 - u^{\dagger}) \sum_{m=0}^{N-1} b_m(\xi) u^{\dagger m} \end{aligned} \tag{6.13}$$

where the $a_m(\xi)$ and the $b_m(\xi)$ terms are related via

$$\sum_{m=0}^{N-1} b_m(\xi) u_k^{\dagger m} = \left[\sum_{m=0}^{N-1} a_m(\xi) u_k^{\dagger m} \right]^{-1} \tag{6.14}$$

where $u_k^{\dagger} = k/N$, and $k = 0, 1, \ldots, N - 1$. That is, the expressions for Θ and $1/\Theta$ are required to agree at N equally spaced values of $u^{\dagger}$.

When N = 1, we have

$$\Theta = (1 - u^{\dagger})^{-1} \Theta_0(\xi) \tag{6.15}$$

$$
\begin{aligned}
f_1 &= 1 - u^\dagger \\
f_1' &= -1 \\
f_1'' &= 0
\end{aligned}
\tag{6.16}
$$

and equation 6.11 becomes

$$
\frac{d}{d\xi}\int_0^1 u^\dagger \Theta_0 \, du^\dagger = -\frac{\dot{M}_o}{M_o}\int_0^1 (1 + u^\dagger)\Theta_0 \, du^\dagger + \frac{1}{\Theta_0}
$$

or

$$
\frac{d\Theta_0^2}{d\xi} + 6\frac{\dot{M}_o}{M_o}\Theta_0^2 = 4 \tag{6.17}
$$

Upon multiplication by the factor M_o^6, equation 6.17 can be integrated directly, giving

$$
\Theta_0 = 2M_o^{-3}\left(\int_{-\infty}^{\xi} M_o^6 \, d\xi\right)^{1/2} \tag{6.18}
$$

It follows from equation 4.29 and the definition of U that $U = O(1/\tilde{x})$ as $\tilde{x} \to -\infty$. If the velocity on any streamline in the sublayer decreases in this manner, then the distance of the streamline from the wall must increase linearly with $\tilde{x}$. The constant of integration in equation 6.18 has been chosen so that $y^\dagger$ obtained below in equation 6.22 has the required form for $\tilde{x} \to -\infty$. As pointed out following equation 6.35, this behavior appears to permit an upstream matching with Lighthill's [8] results.

Then from equations 6.2, 6.15, 6.18 and the relation $\Theta = (\partial u^\dagger/\partial \eta)^{-1}$ we obtain

$$
\partial u^\dagger/\partial \eta = \frac{(1 - u^\dagger)M_o^3}{2\left(\int_{-\infty}^{\xi} M_o^6 \, d\xi\right)^{1/2}} \tag{6.19}
$$

Integration on η and ξ fixed yields

$$
u^\dagger = 1 - \exp\left(-M_o^3\eta/2\zeta^{1/2}\right) \tag{6.20}
$$

where

$$\zeta = \int_{-\infty}^{\xi} M_o^6 \, d\xi$$
$$\xi = \int_0^x U \frac{p^\dagger}{p_e} dx \qquad (6.21)$$

The normal coordinate is found from

$$y^\dagger = \frac{\gamma - 1}{2\gamma p + U} \int_0^{u^\dagger} [1 - (u^\dagger U)^2] \Theta \, du$$
$$= \left(\frac{\gamma - 1}{\gamma p^\dagger U} \frac{\left(\int_{-\infty}^{\xi} M_o^6 \, d\xi\right)}{M_o^3}\right)^{1/2} \left[-(1 - U^2) \log (1 - u^\dagger) + U^2\left(u^\dagger + \frac{1}{2}U^2\right)\right] \qquad (6.22)$$

in the case N = 1. Since $d\xi = U\frac{p^\dagger}{p_e} d\tilde{x}$ and both U and M_o are of the order $1/\tilde{x}$ as $\tilde{x} \rightarrow -\infty$, it follows that $y^\dagger = O(\tilde{x})$ as $\tilde{x} \rightarrow -\infty$ for $u^\dagger$ = constant. Further evaluation of the properties in the sublayer requires numerical integration of the integrals for specified $M_o(\tilde{x})$.

When N = 2, the procedure for obtaining the integral relations is as follows. Equations 6.12 and 6.13 become

$$f_1 = (1 - u^\dagger)$$
$$f_1' = -1$$
$$f_1'' = 0$$
$$f_2 = (1 - u^\dagger)^2 \qquad (6.23)$$
$$f_2' = -2(1 - u^\dagger)$$
$$f_2'' = 2$$

$$\Theta = (1 - u^\dagger)^{-1}\left[\Theta_0(1 - 2u^\dagger) + \Theta_1 u^\dagger\right]$$
$$\frac{1}{\Theta} = (1 - u^\dagger)\left(\frac{1 - 2u^\dagger}{\Theta_0} + \frac{4u^\dagger}{\Theta_1}\right) \qquad (6.24)$$

and equation 6.11 becomes

$$\frac{d}{d\xi}\int_0^1 u^\dagger\left[\Theta_0(1 - 2u^\dagger) + \Theta_1 u^\dagger\right] du^\dagger = -\frac{\dot{M}_o}{M_o}\int_0^1 (1 + u^\dagger)\left[\Theta_0(1 - 2u^\dagger) + \Theta_1 u^\dagger\right] du^\dagger + \frac{1}{\Theta_0} \tag{6.25}$$

$$\frac{d}{d\xi}\int_0^1 u^\dagger(1 - u^\dagger)\left[\Theta_0(1 - 2u^\dagger) + \Theta_1 u^\dagger\right] du^\dagger = -\frac{2\dot{M}_o}{M_o}\int_0^1 \left(1 - u^{\dagger 2}\right)\left[\Theta_0(1 - 2u^\dagger) + \Theta_1 u^\dagger\right] du^\dagger$$

$$+ \frac{2}{\Theta_0} - 2\int_0^1 (1 - u^\dagger)\left(\frac{1 - 2u^\dagger}{\Theta_0} + \frac{4u^\dagger}{\Theta_1}\right) du^\dagger \tag{6.26}$$

so that one gets the differential equations

$$\Theta_0' + \frac{\dot{M}_o}{M_o}(9\Theta_0 + 7\Theta_1) = \frac{34}{\Theta_0} - \frac{32}{\Theta_1} \tag{6.27}$$

$$\Theta_1' + \frac{\dot{M}_o}{M_o}(4\Theta_0 + 6\Theta_1) = \frac{20}{\Theta_0} - \frac{16}{\Theta_1} \tag{6.28}$$

For a specified $M_o(\xi)$, equations 6.27 and 6.28 can be integrated numerically to find $\Theta_0(\xi)$ and $\Theta_1(\xi)$. The first of equations 6.24 gives, for ξ fixed,

$$\begin{aligned} d\eta &= \left[(\Theta_1 - 2\Theta_0)\left(\frac{1}{1 - u^\dagger} - 1\right) + \Theta_0\left(\frac{1}{1 - u^\dagger}\right)\right] du^\dagger \\ \eta &= -(\Theta_1 - \Theta_0)\log(1 - u^\dagger) - (\Theta_1 - 2\Theta_0)u^\dagger \end{aligned} \tag{6.29}$$

Again, the normal coordinate, $y^\dagger$, is recovered from equations 6.2 and the solution for Θ.

6.3. DISPLACEMENT THICKNESS

A displacement thickness of the sublayer may be defined by

$$\bar{\delta}_{SL^*} = \int_0^{\bar{\delta}_{SL}} \left(1 - \frac{\bar{\rho}\bar{u}}{\bar{\rho}_o\bar{u}_o}\right) d\bar{y} \tag{6.30}$$

where $\bar{\rho}_o$ and $\bar{u}_o$ are the density and velocity just outside the sublayer. Let $\delta_{SL}^* = \bar{\delta}_{SL}^*\big/R_w^{-1/2}L$. Then, in terms of the nondimensional stretched variables in the sublayer,

$$\delta_{SL}^* = R_w^{-1/4}\left[\int_0^\infty \left(1 - \frac{\rho^\dagger u^\dagger}{\rho_o}\right) dy^\dagger\right] \tag{6.31}$$

Changing the integration variable to $u^\dagger$, with $dy^\dagger = (1/\rho^\dagger U)\,d\eta$ and $d\eta = \Theta du^\dagger$, and using equation 6.2 and the equation of state to eliminate $\rho^\dagger$, we obtain

$$\rho^\dagger/\rho_o = (1 - U^2)[1 - (u^\dagger U)^2]^{-1} \tag{6.32}$$

$$1/\rho^\dagger = \frac{\gamma - 1}{2\gamma p^\dagger}[1 - (u^\dagger U)^2] \tag{6.33}$$

and

$$\delta^*_{SL} = R_w^{-1/4}\left(\frac{\gamma - 1}{2\gamma p^\dagger U}\right)\int_0^1 (1 - u^\dagger)(1 + u^\dagger)U^2\Theta du^\dagger \tag{6.34}$$

when $N = 1$, $\Theta = \Theta_0/(1 - u^\dagger)$ and where $\Theta_0(\xi)$ is given by equation 6.18. Thus, the integration in equation 6.34 yields

$$\delta^*_{SL} = R_w^{-1/4}\left(\frac{\gamma - 1}{\gamma p^\dagger}\right)\left(\frac{1 + \frac{1}{2}U^2}{U}\right)\frac{\left(\int_{-\infty}^{\tilde{x}} M_o^6 \frac{p^\dagger}{p_1} U\, d\tilde{x}\right)^{1/2}}{M_o^3} \tag{6.35}$$

The $N = 2$ result can be obtained in a similar manner.

Since $U = O(\tilde{x}^{-1})$ and $M_o = O(1/\tilde{x})$ as $\tilde{x} \to -\infty$, it follows from equation 6.35 that $\delta^*_{SL} = O\left(R_w^{-1/4}|\tilde{x}|\right)$ as $\tilde{x} \to -\infty$. Lighthill's [8] estimate of upstream influence is $(\bar{x}/L) - 1 = O\left(R_w^{-3/8}\right)$, i.e., $\tilde{x} = O\left(R_w^{1/8}\right)$. For $\tilde{x}$ of this order, the above result for displacement thickness may be expressed by

$$\bar{\delta}^*_{SL}/L = O\left(R_w^{-5/8}\right) \tag{6.36}$$

which is consistent with the sublayer thickness given by Lighthill.

7
CONCLUSIONS

In the present study we have developed a description of the acceleration of a laminar boundary layer approaching a sharp corner in the limit of large Reynolds number R_w and large external Mach number M_e, with $M_e R_w^{-1/2}$ tending to zero. Because of the large pressure gradient near the corner, the viscous effects are found to be confined to a sublayer thinner than the boundary-layer by a factor of the order $R_w^{-1/4}$. In a first approximation, the effects of the viscous sublayer are neglected, and the inviscid rotational equations govern the flow.

Numerical results are obtained for hypersonic, laminar, adiabatic flow of a perfect gas over a slender wedge or cone. The approach can be generalized to apply to nonadiabatic flows of real gases.

The outer inviscid flow in the accelerating layer is characterized by two distinguished limits. In a first outer limit, $\hat{x} = [(\bar{x}/L) - 1)]/M_e R_w^{-1/2}$ and $\tilde{y} = \bar{y}/R_w^{-1/2} L$ are held fixed, and the flow deflection angle is of order M_e^{-1} throughout the layer. The normal pressure gradient can be neglected in this case, and the governing equations in this limit are inviscid boundary-layer equations, which may be integrated directly.

In a second outer limit, $\tilde{x} = [(\bar{x}/L) - 1)]/R_w^{-1/2}$ and $\tilde{y} = \bar{y}/R_w^{-1/2} L$ are held fixed, and the flow deflection angle θ is of the order unity in the layer. The full inviscid equations govern the flow in this case. Now θ remains of the order M_e^{-1} at the outer edge of the layer, and in this limit the boundary condition $\theta(\tilde{x}, \delta) = 0$ is imposed.

In the first outer limit, a decrease in the pressure is associated with a corresponding decrease in the layer thickness since the changes in stream-tube area for the subsonic portion of the layer are dominant. This is designated as the subcritical condition. Eventually a further decrease in the pressure would cause the layer to become thicker instead of thinner. Further acceleration of the layer is accomplished by the sharp turning of the streamlines near the corner. This turning corresponds to the flow description in the second outer limit. This is designated as the supercritical condition, with $d\bar{\delta}^*/d\bar{p} < 0$. The critical point, where $d\bar{\delta}^*/d\bar{p} = 0$, occurs at a distance $o\left(M_e R_w^{-1/2} L\right)$ upstream from the corner.

This clear distinction between subcritical and supercritical flows arises as a result of taking the hypersonic limit, $M_e \to \infty$. Another result of the limit $M_e \to \infty$ is that the boundary-layer thickness is clearly defined and is equal to the displacement thickness. This is a useful simplification in our application of the method of integral relations to the system of equations in the second outer limit.

A composite expansion for the wall-pressure ratio, formed from the solutions in the first and second outer limits, compares reasonably well with Hama's experimental data for a wedge with $M_e = 4.02$. Hama's wall-pressure ratio data at three Reynolds numbers are correlated when plotted as a function of $(\bar{x} - L)/\bar{\delta}_1$, in agreement with the theory.

The most important second-order correction appears to be the sublayer effect, since $R_w^{-1/4} \approx 0.10$ in Hama's experiment. The governing equations in the sublayer are the boundary-layer equations with a pressure gradient. We are able to derive integral relations for the sublayer velocity and displacement thickness. Other higher order effects may, in principle, be considered in the analysis by studying a limit where $\hat{x} \to 0$, $\tilde{x} \to -\infty$, and $\tilde{x}/M_e^{1/5}$ is held fixed, and a limit where $\tilde{x} \to -\infty$ with $\tilde{x}/R_w^{1/8}$ held fixed.

In addition to the numerical solution in the second outer limit found with a single-strip application of the method of integral relations, the integral relations for a two-strip application are derived. In principle, any number of strips can be considered. However, it is necessary to carry out asymptotic expansions of the equations to the second order as $\tilde{x} \to -\infty$ in order to start the numerical integration technique, and the procedure for applying the initial conditions is not yet clear.

Another approach to solving the equations in the second outer limit might be to use the method of characteristics or a finite-difference method in the supersonic region, and a one-strip application of the method of integral relations for the portion of the flow that is initially subsonic.

Appendix I
COMPRESSIBLE LAMINAR BOUNDARY LAYER AT CONSTANT PRESSURE

As a preliminary to the analysis of boundary-layer acceleration at a corner, it is necessary to calculate the development of the boundary layer upstream from the interaction region. The velocity profile evaluated at the corner then provides the upstream boundary condition for the interaction calculation. For this reason and for the convenience of having the results available in notation consistent with other parts of this work, the solution of the constant-pressure laminar boundary layer on a wedge or cone, in the limit of large Mach and Reynolds numbers, is given here.

In this boundary-layer calculation, the following idealizations are made:

(1) Thermally and calorically perfect gas
(2) Unity Prandtl number
(3) Linear viscosity-temperature relation
(4) Adiabatic wall

However, these simplifications are not essential to the approach employed to analyze boundary-layer acceleration at a corner.

To establish order estimates for flow properties just outside the boundary layer on a wedge or cone, we shall consider the oblique-shock relations for high-speed flow past a slender wedge (see fig. 1). We are concerned with the limit

$$\begin{aligned} M_\infty &\to \infty \\ R_\infty &\to \infty \\ \tau &\to 0 \end{aligned} \tag{I-1}$$

If also

$$\frac{1}{M_\infty \tau} = O(1) \tag{I-2}$$

then it can be shown (for example, see Hayes and Probstein [1]) that

$$\frac{\bar{p}_e/\bar{p}_\infty - 1}{\gamma M_\infty^2 \tau^2} = \frac{\sigma}{\tau} = \frac{\gamma + 1}{4} + \left[\left(\frac{\gamma + 1}{4}\right)^2 + \frac{1}{M_\infty^2 \tau^2}\right]^{1/2} \tag{I-3}$$

and also

$$M_e^2 = \frac{(\sigma/\tau)M_\infty^2}{(\sigma/\tau - 1)\left[1 + \gamma M_\infty^2 \tau^2 (\sigma/\tau)\right]} \tag{I-4}$$

$$= O(\tau^{-2}) \tag{I-5}$$

Here $\overline{p}_e$ and M_e are the pressure and Mach number at the outer edge of the boundary layer, τ is the wedge half-angle, and σ is the shock-wave angle. Although equations I-3 and I-4 are applicable only for a wedge, equation I-5 is valid also for a slender cone.

A power-law viscosity-temperature relation of the form

$$\overline{\mu}/\overline{\mu}_w = (\overline{T}/\overline{T}_w)^\omega \tag{I-6}$$

is assumed. Since $\overline{T}_e/\overline{T}_\infty = O\left(M_\infty^2 \tau^2\right)$ and $\overline{T}_w/\overline{T}_e = O(\tau^{-2})$, we have the order estimates

$$R_e = O\left[(M_\infty \tau)^{-2\omega} R_\infty\right] \tag{I-7}$$

$$R_w = O\left[\tau^{2\omega+2} (M_\infty \tau)^{-2\omega} R_\infty\right] \tag{I-8}$$

where $R_e = \overline{\rho}_e \overline{u}_e L/\overline{\mu}_e$ and $R_w = \overline{\rho}_w \overline{u}_e L/\overline{\mu}_w$ are the Reynolds numbers based on the thermodynamic properties at the outer edge of the boundary layer and at the surface of the body, respectively. Later in the calculation, the exponent ω will be set equal to one. Since $\overline{T}/\overline{T}_w = O(1)$ for any point inside the boundary layer, $\overline{T}_w$ is a proper reference temperature. Then the order estimate for the boundary-layer thickness at the trailing edge of the body is

$$\overline{\delta}_1/L = O\left(R_w^{-1/2}\right) \tag{I-9}$$

Since from equations I-2 and I-3 we have $\sigma/\tau = O(1)$, the requirement that the boundary-layer thickness be much smaller than the shock layer thickness becomes

$$\tau^{-1} R_w^{-1/2} \rightarrow 0 \tag{I-10}$$

The limit expressed by equations I-1, I-2, and I-10 can be written in terms of local boundary-layer properties as

$$\begin{aligned} M_e &\rightarrow \infty \\ R_w &\rightarrow \infty \\ M_e R_w^{-1/2} &\rightarrow 0 \end{aligned} \tag{I-11}$$

It is appropriate, in view of the order estimate for the boundary-layer thickness given in equation I-9, to introduce the stretched coordinates

$$
\begin{aligned}
x &= \bar{x}/L \\
\tilde{y} &= \bar{y}/R_w^{-1/2}L \\
r &= \bar{r}/\tau L
\end{aligned}
\tag{I-12}
$$

Then, in the limit expressed in equations I-11, the leading terms in asymptotic representations for the x- and y-velocity components, pressure, and density are

$$
\begin{aligned}
\bar{u}/\bar{u}_e &\sim u(x, \tilde{y}) + \dots \\
\bar{v}/\bar{u}_e &\sim R_w^{-1/2} v(x, \tilde{y}) + \dots \\
\bar{p}/\bar{\rho}_w \bar{u}_e^2 &\sim (\gamma - 1)/2\gamma + \dots \\
\bar{\rho}/\bar{\rho}_w &\sim \rho(x, \tilde{y}) + \dots
\end{aligned}
\tag{I-13}
$$

In the first approximation for the hypersonic limit, the equations describing laminar boundary-layer flow become

$$(\rho u r^k)_x + (\rho v r^k)_{\tilde{y}} = 0 \tag{I-14}$$

$$\rho(u u_x + v u_{\tilde{y}}) = (\mu u_{\tilde{y}})_{\tilde{y}} \tag{I-15}$$

$$\rho = (1 - u^2)^{-1} \tag{I-16}$$

where k = 0 for a wedge, and k = 1 for a cone. For a linear viscosity-temperature dependence,

$$\mu = T = \rho^{-1} \tag{I-17}$$

where $\mu = \bar{\mu}/\bar{\mu}_w$ and $T = \bar{T}/\bar{T}_w$. Note that

$$\bar{u}_e^2 \Big/ \bar{q}_{max}^2 = \left(1 + \frac{2}{\gamma - 1} M_e^{-2}\right)^{-1} = 1 + O\left(M_e^{-2}\right) \tag{I-18}$$

in the hypersonic limit.

The differential equations I-14 and I-15 can be converted to the form for plane incompressible flow by the Dorodnitsyn transformation (see Belotserkovskii and Chushkin [22]), which, in this case, simplifies to

$$\xi = \int_0^x r^{2k}\, dx$$

$$\eta = r^k \int_0^{\tilde{y}} \rho \, d\tilde{y} \tag{I-19}$$

$$\partial/\partial x = r^{2k} \, \partial/\partial \xi + \eta_x \, \partial/\partial \eta$$

$$\partial/\partial \tilde{y} = \rho r^k \, \partial/\partial \eta$$

$$\eta_x = -(\tilde{y}_\xi / \tilde{y}_\eta) \, \xi_x$$

$$= \rho r^{3k} \int_0^{\eta} (\rho r^k)_\xi (\rho r^k)^{-2} \, d\eta$$

Integration of the continuity equation gives

$$\rho v r^k = - \int_0^{\eta} \left[(\rho u r^k)_\xi r^{2k} + \eta_x (\rho u r^k)_\eta \right] (\rho r^k)^{-1} \, d\eta$$

Then the resulting differential equations are

$$u_\xi + w_\eta = 0 \tag{I-20}$$

$$u u_\xi + w u_\eta = u_{\eta\eta} \tag{I-21}$$

where

$$w = \rho v r^{-k} + \rho u r^k \int_0^{\eta} (\rho r^k)_\xi (\rho r^k)^{-2} \, d\eta$$

The boundary conditions are

$$\left.\begin{aligned} u(0, \eta) &= 1 \\ u(\xi, 0) &= 0 \\ w(\xi, 0) &= 0 \\ u(\xi, \infty) &= 1 \end{aligned}\right\} \tag{I-22}$$

The system of equations I-20, I-21, and I-22 has a similarity solution originally found by Blasius (see Rosenhead [29]). Equation I-20 can be satisfied identically if we define a stream function, Ψ, by

$$\left.\begin{aligned} \Psi_\xi &= -w \\ \Psi_\eta &= u \end{aligned}\right\} \tag{I-23}$$

Then equation I-21 becomes

$$\Psi_\eta \Psi_{\xi\eta} - \Psi_\xi \Psi_{\eta\eta} = \Psi_{\eta\eta\eta} \tag{I-24}$$

In terms of a similarity variable

$$\beta = (2\xi)^{-1/2}\eta \tag{I-25}$$

the stream function has the form

$$\Psi = (2\xi)^{1/2} g(\beta) \tag{I-26}$$

and equation I-24 transforms to the ordinary differential equation

$$\dddot{g} + g\ddot{g} = 0 \tag{I-27}$$

where the dot denotes differentiation with respect to β. The boundary conditions are

$$\begin{aligned} g(0) &= 0 \\ \dot{g}(0) &= 0 \\ \dot{g}(\infty) &= 1 \end{aligned} \tag{I-28}$$

Rosenhead [29] tabulates the solution to equations I-27 and I-28 and gives

$$\ddot{g}(0) = 0.4696 \tag{I-29}$$

In the limit of large β,

$$\dot{g}(\beta) \sim 1 - 0.331(\zeta^{-1} - \zeta^{-3} + 3\zeta^{-5} - \ldots) \exp\left(-\frac{1}{2}\zeta^2\right) \tag{I-30}$$

where $\zeta = \beta - 1.21678$.

The values of our original variables are found from

$$x = [(2k + 1)\xi]^{\frac{1}{2k+1}} \tag{I-31}$$

$$\tilde{y} = (2\xi)^{1/2} r^{-k} \int_0^\beta (1 - \dot{g}^2)\, d\beta \tag{I-32}$$

Integrating by parts and using equation I-27, equation I-32 becomes

$$\tilde{y} = (2\xi)^{1/2} r^{-k} (\beta - g\dot{g} - \ddot{g} + 0.4696) \tag{I-33}$$

Also, $r = x$, $u = \dot{g}$, and $\rho = (1 - \dot{g}^2)^{-1}$, while the boundary-layer thickness is given by

$$\bar{\delta}/R_w^{-1/2}L = (2\xi)^{1/2}r^{-k} \lim_{\beta\to\infty} (\beta - g\dot{g} - \ddot{g} + 0.4696) = 1.68638(2\xi)^{1/2}r^{-k} \tag{I-34}$$

In the hypersonic limit, the boundary-layer thickness is finite and equal to the displacement thickness $\bar{\delta}^*$, where

$$\bar{\delta}^* = \int_0^{\bar{\delta}} \left(1 - \frac{\bar{\rho}\bar{u}}{\bar{\rho}_e\bar{u}_e}\right) d\bar{y} = \bar{\delta} - \int_0^{\bar{\delta}} (\bar{\rho}/\bar{\rho}_e) u\, d\bar{y}$$

Now $\bar{\rho}/\bar{\rho}_e = O\left(M_e^{-2}\right)$ in the hypersonic limit so that

$$\bar{\delta}/\bar{\delta}^* - 1 = o(1) \tag{I-35}$$

At the trailing edge, x = r = 1, and hence

$$\xi_1 = \frac{1}{2k + 1} \tag{I-36}$$

$$\bar{\delta}_1/R_w^{-1/2}L = 1.68638\left(\frac{2}{2k + 1}\right)^{1/2} \tag{I-37}$$

In the analysis of boundary-layer acceleration at a corner, it is necessary to generate the function $\dot{g}(\beta)$. This is carried out by transforming equation I-27 to a system of first-order equations and integrating numerically by the Runge-Kutta fourth-order method [21]. We introduce new dependent variables

$$\begin{aligned} y_1 &= g \\ y_2 &= \dot{g} \\ y_3 &= \ddot{g} \end{aligned} \tag{I-38}$$

and obtain the differential equations

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= -y_1 y_3 \end{aligned} \tag{I-39}$$

with the initial conditions

$$\begin{aligned} y_1(0) &= 0 \\ y_2(0) &= 0 \end{aligned} \tag{I-40}$$

$$y_3(0) = 0.4696$$

To start the integration process, we utilize the expansions of g, $\dot{g}$, and $\ddot{g}$ for small β:

$$g \sim \frac{1}{2}\alpha\beta^2 - \frac{1}{120}\alpha^2\beta^5 + \frac{11}{40320}\alpha^3\beta^8 - \ldots$$

$$\dot{g} \sim \alpha\beta - \frac{1}{24}\alpha^2\beta^4 + \frac{11}{5040}\alpha^3\beta^7 - \ldots \tag{I-41}$$

$$\ddot{g} \sim \alpha - \frac{1}{6}\alpha^2\beta^3 + \frac{11}{720}\alpha^3\beta^6 - \ldots$$

where $\alpha = 0.4696$.

Appendix II
APPROXIMATION BY THE METHOD OF INTEGRAL RELATIONS COMPARED WITH THE EXACT BLASIUS SOLUTION

The transformed relations for a boundary layer without a pressure gradient are equations I-20, I-21, and I-22. Since, in this case U = 1 and $\dot{M}_o = 0$ (see sec. 6), the integral relations, equations 6.11, become

$$\frac{d}{d\xi}\int_0^1 f_k u\Theta\, du = -\frac{f_k'(0)}{\Theta_o} - \int_0^1 \frac{f_k''}{\Theta} du \tag{II-1}$$

where k = 1, 2, . . . , N. When N = 1, we have

$$f_1 = 1 - u$$

$$f_1' = -1$$

$$f_1'' = 0$$

Assuming a similarity solution for u,

$$\Theta \equiv (\partial u/\partial\eta)^{-1} = 2\xi^{1/2}(1 - u)^{-1} \tag{II-2}$$

and

$$u = 1 - \exp\,(-\eta/2\xi^{1/2}) \tag{II-3}$$

$$u_\eta(0) = 0.5\xi^{-1/2} \tag{II-4}$$

A nondimensional displacement thickness is

$$\delta^* = \int_0^\infty (1 - u)\, d\eta = \int_0^1 (1 - u)\Theta\, du = 2.0\xi^{1/2} \tag{II-5}$$

When N = 2,

$$f_1 = 1 - u$$

$$f_1' = -1$$

$$f_1'' = 0$$

$$f_2 = (1 - u)^2$$

$$f_2' = -2(1 - u)$$

$$f_2'' = 2$$

$$\Theta = (1 - u)^{-1}[\Theta_0 + (\Theta_1 - 2\Theta_0)u] \tag{II-6}$$

$$\frac{1}{\Theta} = (1 - u)[B_0 + (B_1 - 2B_0)u] \tag{II-7}$$

Thus, from equation 6.14, $B_0 = \Theta_0^{-1}$, $B_1 = 4\Theta_1^{-1}$, and equation II-1 becomes

$$d\Theta_0/d\xi = 34\Theta_0^{-1} - 32\Theta_1^{-1} \tag{II-8}$$

$$d\Theta_1/d\xi = 20\Theta_0^{-1} - 16\Theta_1^{-1} \tag{II-9}$$

When we assume a form of solution similar to equation II-2,

$$\Theta_i = A_i \xi^{1/2} \tag{II-10}$$

where i = 0, 1. When substituted into equations II-8 and II-9, equation II-10 yields

$$A_0 = 2(17 - 16\lambda^{-1})^{1/2} = 3.1555 \tag{II-11}$$

$$A_1 = 2\lambda(17 - 16\lambda^{-1})^{1/2} = 3.4793 \tag{II-12}$$

where $\lambda = (13 + \sqrt{33})/17$. When N = 2, the result for the nondimensional wall shear stress is

$$u_\eta(0) = 0.317\xi^{-1/2}$$

while the displacement thickness is given by

$$\delta^* = 1.74\xi^{1/2}$$

Table V is a comparison of the results of the method of integral relations for N = 1 and 2 with the Blasius solution and the Pohlhausen approximation. It can be seen that when N = 2 the method of integral relations is comparable in accuracy with a Pohlhausen calculation. For a more complete assessment of the accuracy of the method of integral relations in a variety of problems, see reference 22.

TABLE V. COMPARISON OF SOLUTION BY THE INTEGRAL RELATIONS METHOD WITH THE BLASIUS BOUNDARY-LAYER SOLUTION AND A POHLHAUSEN CALCULATION

	Blasius Result	Method of Integral Relations		Pohlhausen Calculation [30]
		N = 1	N = 1	
$\xi^{1/2}u_\eta(0)$	0.332	0.5	0.317	0.343
$\xi^{-1/2}\delta^*$	1.729	2.0	1.74	1.752

Appendix III
REDUCTION OF HAMA'S WALL-PRESSURE DATA

In this appendix, the calculations carried out in reducing Hama's wall-pressure ratio data [6] (shown in fig. 4) to the form given in figure 5 are explained.

The Mach number given is $M_{e,i}$, and the wall pressure p_o is divided by $p_{e,i}$, where $M_{e,i}$ and $p_{e,i}$ are the values of Mach number and pressure at the surface which would be predicted by inviscid-flow theory. Also, the value of the Reynolds number R_e is specified.

There are three sets of experimental conditions:

$M_{e,i}$	R_e
4.02	2.16×10^5
3.15	1.34×10^5
2.35	1.97×10^5

We wish to express the results in terms of M_e, R_w and to calculate $\overline{\delta}_1$. We choose the static pressure orifice farthest upstream ($\overline{x}$ - L = -1.5 in.) as the reference point for measuring $\bar{p}_e$. Then $\bar{p}_o/\bar{p}_e$ can be calculated from the relation

$$\bar{p}_o/\bar{p}_e = (\bar{p}_o/\bar{p}_{e,i})/(\bar{p}_e/\bar{p}_{e,i}) \tag{III-1}$$

Since $|M_e - M_{e,i}|/M_{e,i}$ is considerably smaller than $|p_e - p_{e,i}|/p_{e,i}$, we will take $M_e = M_{e,i}$.

The Reynolds number based upon the thermodynamic properties at the wall is found from

$$R_w = (\bar{\rho}_w/\bar{\rho}_e)(\bar{\mu}_e/\bar{\mu}_w)R_e = \left(1 + \frac{\gamma - 1}{2}M_e^2\right)^{-(\omega+1)} R_e \tag{III-2}$$

and, in our calculation, $\omega = 1$ and $\gamma = 1.4$. The boundary-layer thickness is calculated from

$$\bar{\delta}_1 = \sqrt{2}(1.68638)R_w^{-1/2}L \tag{III-3}$$

(see app. I), with L = 4.783 in.

The results are as follows:

M_e	$\bar{p}_e/\bar{p}_{e,i}$	R_w	$\bar{\delta}_1$ (in.)
4.02	1.070	1.2×10^4	0.104
3.15	1.048	1.5×10^4	0.093
2.35	1.020	4.4×10^4	0.054

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13. ABSTRACT

An asymptotic description of the acceleration of a laminar hypersonic boundary layer approaching a sharp corner is obtained. The description assumes small interaction with the outer inviscid flow. Viscous forces are neglected except in a thin sublayer. The initial part of the expansion takes place over a distance $O(M_e\bar{\delta})$, where M_e is the external Mach number, and $\bar{\delta}$ is the boundary-layer thickness. Here the transverse pressure gradient is small, and a solution can be obtained analytically. Within a distance $O(\bar{\delta})$ from the corner, the effect of streamline curvature is essential, and a numerical solution is obtained by the method of integral relations for a single strip. The solution for surface pressure is compared with experimental results for a particular case, and an approximate velocity profile at the corner is calculated. Possibilities for improving the accuracy of the calculation, both by refining the numerical procedure and by including higher order effects, are considered.

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