THE RADIATION PATTERN OF AN ELECTRIC LINE CURRENT ENCLOSED BY AN AXIALLY SLOTTED PLASMA SHEATH - II

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ABSTRACT

A circular uniform plasma sheath of complex conductivity is enclosing a unit electric line source at the origin. The plasma sheath has an infinite axial slot of rectangular cross section. It is shown that the slot supports propagating modes which transport power to the external space. Radiation is calculated for a narrow plasma slot that supports the propagation only of the lower order mode. It is shown that radiation in the forward direction is comparable to that of the unit line source radiating in the free space as long as the lower order mode is propagating.

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INTRODUCTION

When the plasma sheath enclosing the electric line source is of a substantial thickness it is well known that for signal frequency ω less than plasma frequency ω_p negligible amount of power leaks through the sheath. If one opens a slot in the plasma sheath as shown in Fig. 1, then it appears that the slot will act as a waveguide in transporting the power from the source in the cavity to the outside free space. The power transfer in the waveguide takes place via the propagating modes. In this report the radiation of a unit electric line source is calculated on a basis that exploits the waveguide properties of the plasma slot. This is a continuation of a previous study (Olte, 1965) in which the integral equation approach was used to solve for the plasma current. That method was found to be useful for cases where at least some radiation leaked through the sheath. For the parameters of the present study negligible radiation leaks through the sheath itself and under these conditions the integral equation method becomes too unwieldy.

We are dealing with a two dimensional problem in which the non-zero components for electric field are $E_z(r, \phi)$ and for the magnetic field $H_r(r, \phi)$ and $H_\phi(r, \phi)$. Thus we have to solve for Transverse Electric Modes in the plasma waveguide. This is done in Section II for the parallel face plasma waveguide. The properties of the Symmetric Transverse Electric Modes are derived and eigenvalues calculated for both loss-less and slightly lossy plasma. A wedge waveguide is a natural one to consider in a cylindrical coordinate system. However, for a finite plasma frequency it does not support uncoupled radial modes. Therefore we selected the parallel face guide because it is the only configuration that leads to a uniform waveguide for a finite plasma frequency.

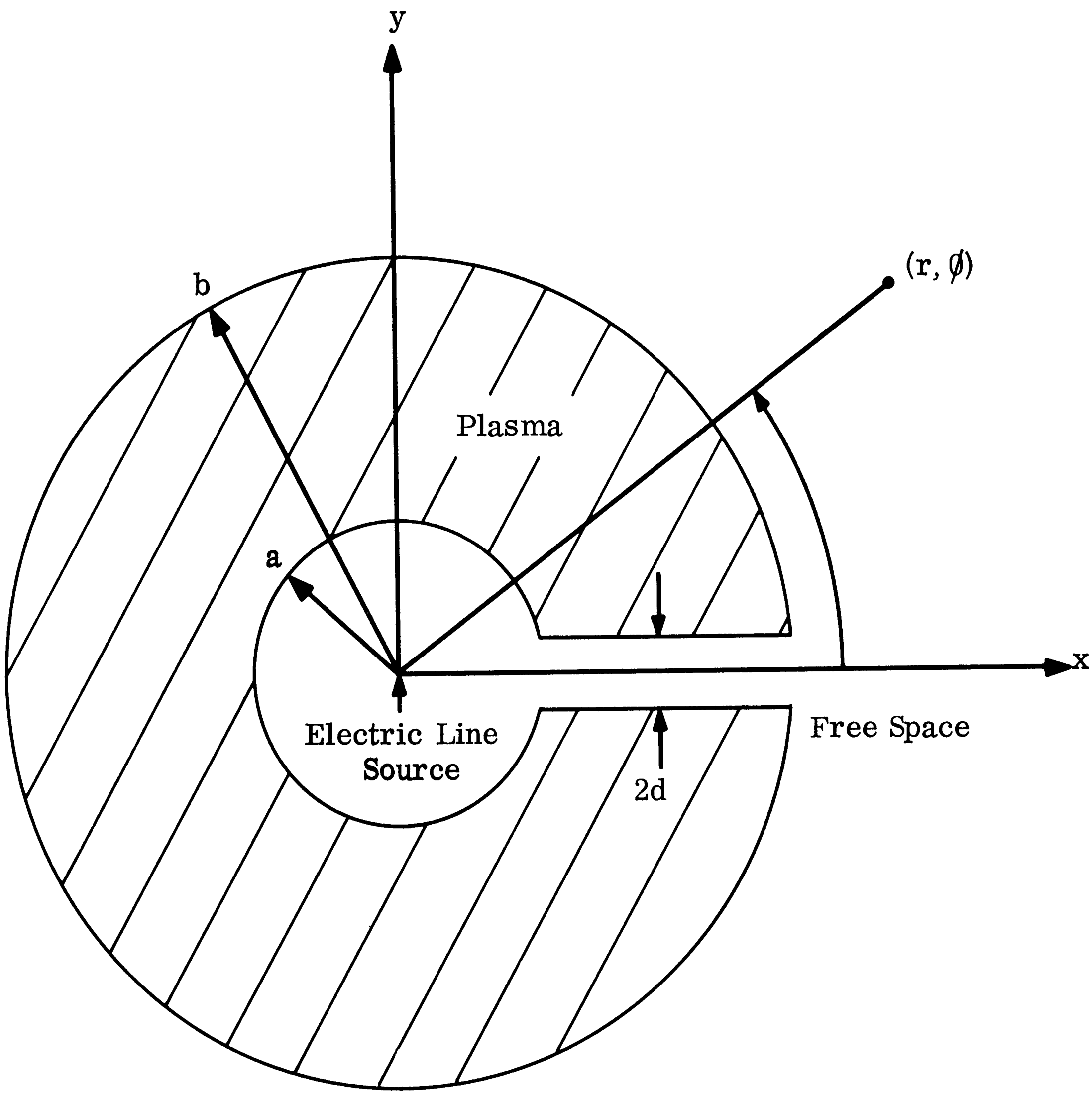

FIGURE 1: THE CONFIGURATION OF THE PROBLEM

Since we are primarily interested in narrow plasma slots we may limit the plasma guide problem to a single propagating mode. The power transport out of the cavity is then via this mode. Knowing that aperture antennas are efficient radiators we may neglect the reflection at the outside aperture and hence we have to calculate only the amplitude of the outwardly propagating mode. This is the most difficult part of the problem. The approximation that opening a narrow slot leaves the magnetic field in the cavity unchanged is the basis for the calculation of the outwardly traveling mode amplitude. This is done in Section III.

Knowing the electric field in the external plasma slot aperture we may easily compute the radiation field by using the Green's Function for the perfectly conducting cylinder. This entails the assumption that the electric field is negligible for the rest of the cylinder surface. This analysis is carried out in Section IV and leads to the practical results of this study, i.e., to the radiation power density in the free space as a function of the plasma sheath electron density, collision frequency, radius and thickness, and the width of the plasma guide. The study concludes with a discussion of the results and the main conclusions of this report.

The rational MKS system of units is used and the time dependence $e^{j\omega t}$ is assumed.

II

SYMMETRIC TRANSVERSE ELECTRIC MODES IN A PARALLEL PLASMA SLAB WAVEGUIDE

The waveguide is formed by two semi-infinite plasma slabs separated a distance 2d. We introduce a cartizian coordinate system (x, y, z) with the y - axis normal to the plasma slabs and the origin at the half way point between them. For the electromagnetic parameters we take the permeability and permittivity of free space (i.e., $\mu = \mu_o$ and $\epsilon = \epsilon_o$) and the electrical conductivity of the plasma

$$\sigma = \frac{\epsilon_o \omega_p^2}{\nu + j\omega} \quad . \tag{1}$$

The plasma, collision, and signal frequencies are ω_p, ν, and ω. The TM-modes will have the following non-zero field components for propagation along the x - axis: $E_z(x, y)$, $H_x(x, y)$, $H_y(x, y)$. Each of these field components satisfies the scalar wave equation. For the symmetric TE-modes we then have

$$H_x(x, y) = Ae^{-Ky} e^{-\gamma x}, \quad y>d \tag{2a}$$

$$= B \sin (K_o y) e^{-\gamma x}, \quad -d<y<d \tag{2b}$$

$$= -Ae^{Ky} e^{-\gamma x}, \quad y<-d \; . \tag{2c}$$

From the Maxwell equations we obtain that

$$H_y(x, y) = - \frac{\gamma}{\gamma^2 + \kappa_o^2} \frac{\partial}{\partial y} H_x(x, y) \; , \quad -d<y<d \tag{3a}$$

$$= - \frac{\gamma}{\gamma^2 + \kappa^2} \frac{\partial}{\partial y} H_x(x, y) \; , \quad y< d \tag{3b}$$

and

$$E_z(x,y) = \frac{j\omega\mu_o}{\gamma^2 + \kappa_o^2} \frac{\partial}{\partial y} H_x(x,y) \ , \ -d<y<d \tag{4a}$$

$$= \frac{j\omega\mu_o}{\gamma^2 + \kappa^2} \frac{\partial}{\partial y} H_x(x,y) \ , \ y>|d| \tag{4b}$$

where $\kappa_o = \omega\sqrt{\mu_o \epsilon_o}$ and $\kappa = \kappa_o\sqrt{1 - j\frac{\sigma}{\omega\epsilon_o}}$. Since (1) has to satisfy the scalar wave equation we have

$$\gamma^2 + \kappa_o^2 = K_o^2 , \tag{5}$$

and

$$\gamma^2 + \kappa^2 = -K^2. \tag{6}$$

From the boundary conditions at the plasma interfaces we obtain

$$\frac{A}{K} e^{-Kd} - \frac{B}{K_o} \cos(K_o d) = 0 \tag{7a}$$

$$A e^{-Kd} - B \sin(K_o d) = 0 . \tag{7b}$$

For a non-trivial solution of (7) we require that its determinant must vanish and thus we obtain

$$\frac{K}{K_o} = \tan K_o d . \tag{8}$$

Eliminating γ between (5) and (6) we obtain

$$K^2 = j\frac{\sigma}{\omega\epsilon_o}\kappa_o^2 - K_o^2 \tag{9}$$

and thus (8) may be written as

$$\left[j\frac{\sigma}{\omega\epsilon_o}\left(\frac{\kappa_o}{K_o}\right)^2 - 1 \right]^{1/2} = \tan K_o d . \tag{10}$$

The appropriate roots of this equation determine the eigenvalues K_{o1}, K_{o2}, K_{o3}, . . . of the TE modes. In order to further discuss (10) we introduce

$$K_o d \equiv u \tag{11}$$

$$\frac{1}{\pi} \kappa_o d \equiv h = \frac{2d}{\lambda_o} \tag{12}$$

$$\frac{\nu}{\omega} \equiv Y \tag{13}$$

$$\frac{\omega_p}{\omega} \equiv X , \tag{14}$$

then (10) becomes

$$\left[\frac{(\pi h X)^2}{(1 - j Y) u^2} - 1 \right]^{1/2} = \tan u \ . \tag{15}$$

For the loss-less plasma (Y = 0) we expect the eigenvalues to be real. We treat this case first.

2.1 The Loss-Less Case

In the loss-less case we may write (15) as

$$f(hX, u) = \tan u \tag{16}$$

where

$$f(hX, u) \equiv \left[\left(\frac{\pi h}{u} X \right)^2 - 1 \right]^{1/2} . \tag{17}$$

The intersections of tan u with f (h X, u) are shown in Fig. 2. The abscisas of the intersections u_1, u_2, u_3, . . . are the real roots of (16). We see that for $0 < hX < 1$ we have only one real root: u_1; for $1 < hX < 2$ we have two real roots: u_1 and u_2; etc. Further we see that as $X \rightarrow \infty$ the number of real roots becomes infinite and they are given by $u_n = (2n - 1)\pi/2$,

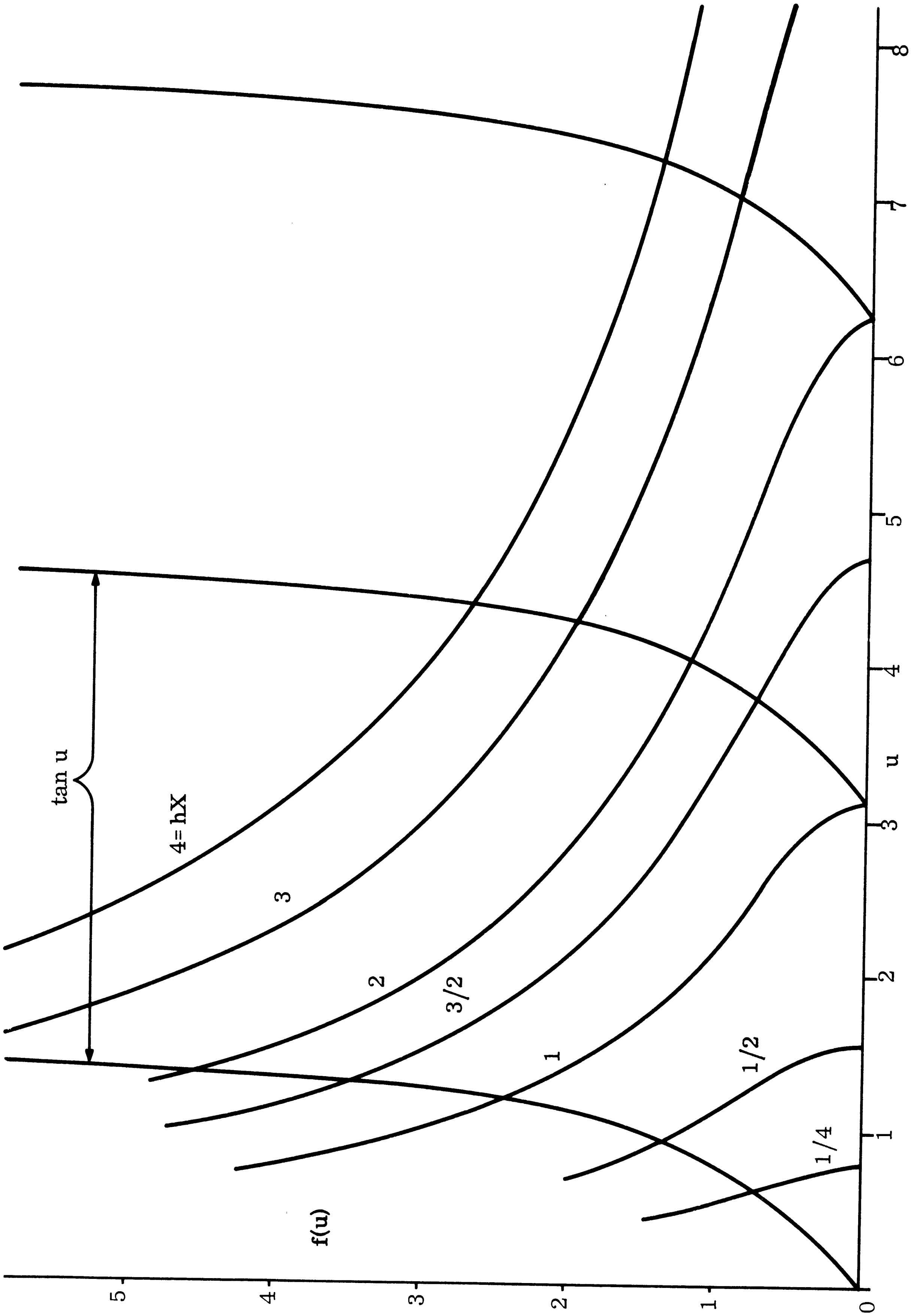

FIG. 2: GRAPHICAL SOLUTION FOR REAL ROOTS OF EQUATION (17).

n = 1, 2, 2, ... This limit corresponds to the perfect conductor parallel plate guide.

Whenever the tan u may be approximated by a single term in the series, then we may solve for the roots of (16) algebraically. By a simple analysis we obtain

$$u_1 \simeq \pi h X \quad , \quad h X << 1 \tag{18}$$

and

$$u_1 \simeq \frac{\pi}{2} - \frac{1}{2 h X} \quad , \quad h X >> 1 \ . \tag{19}$$

From the last two formulas and Figure 2 we record in Table I some of the lower order roots that we will make use of later on .

TABLE I: THE LOWEST ORDER REAL ROOTS OF (17)

h X	u_1	u_2	u_3
0	0		
$\frac{1}{4}$	0.63		
$\frac{1}{2}$	0.93		
1	1.18		
$\frac{3}{2}$	1.30	3.78	
2	1.35	4.03	
3	1.42	4.23	7.02
4	1.45	4.36	7.23

The eigenvalue for the TM_n mode is given by

$$K_{on} = \frac{u_n}{d} \tag{20}$$

and hence from (5) the propagation constant

$$\gamma_n = \left[\left(\frac{u_n}{d}\right)^2 - \kappa_o^2\right]^{1/2} . \tag{21}$$

From (9) and (20) we obtain

$$K_n = \left[j \frac{\sigma}{\omega \epsilon_o} \kappa_o^2 - \left(\frac{u_n}{d}\right)^2\right]^{1/2} \tag{22}$$

and for the loss-less case

$$K_n = \left[(\kappa_o X)^2 - \left(\frac{u_n}{d}\right)^2\right]^{1/2} . \tag{23}$$

We let

$$\gamma_n = \alpha_n + j \beta_n \tag{24}$$

and from (21) obtain for $\kappa_o < u_n / d$

$$\alpha_n = \left[\left(\frac{u_n}{d}\right)^2 - \kappa_o^2\right]^{1/2} ; \quad \beta_n = 0 , \tag{25}$$

and for $\kappa_o > u_n / d$

$$\alpha_n = 0 ; \quad \beta_n = \left[\kappa_o^2 - \left(\frac{u_n}{d}\right)^2\right]^{1/2} . \tag{26}$$

In the first case we say the mode is beyond cut-off, in the second case the mode is propagating. The mode is at cut-off when

$$\kappa_o = \frac{u_n}{d} . \tag{27}$$

Using (12) we re-write (27) as

$$\pi h = u_n(h, X) \tag{28}$$

where we have indicated that the root u_n depends on h and X. Notice from (12) that h measures the guide width in terms of the free space wavelength. On the h-X plane equation (28) defines the common boundary between the cut-off region and the propagation region for the n-th mode.

Also from Fig. 2 and the subsequent discussion it is clear that the n-th mode will exist either as a propagating or a non-propagating mode only when

$$h X > (n - 1) , \quad n = 1, 2, 3, \ldots \tag{29}$$

The parabola

$$h X = (n - 1) \tag{30}$$

on the h - X plane is the boundary between the regions of existence and non-existence for the n-th mode.

In Fig. 3 we present these curves for the two lowest order modes, i.e., the first and the second mode (n = 1 and n = 2, respectively). We are concerned only with the first quadrant of the h - X plane, because for obvious physical reasons $h \geqslant 0$ and $X \geqslant 0$. For the first mode we have from (30) that $h X = 0$ and thus the first mode exists for any h and X values. The boundary curve between the beyond cut-off and propagating regions for the first mode is given by (28) with n = 1. It is shown as the solid curve in Fig. 3. For the second mode we have from (30) for the non-existence boundary the parabola $h X = 1$ which we have plotted in Fig. 3 as a dotted curve. For any (h, X) below this curve the second mode does not exist; above it the mode exists. The existence region is split into the beyond cut-off and the propagating regions by a curve from (28) with n = 2. This curve starts at (1, 1) and curves to the right and sharply upward. Viewing Fig. 3 one should consider it as two independent figures superimposed in order to save space. The solid curve refers only to the first mode, and the dotted curves refer only to the second mode.

Next we examine the penetration of the mode fields into the plasma. For this purpose we substitute (12) in (23) to obtain

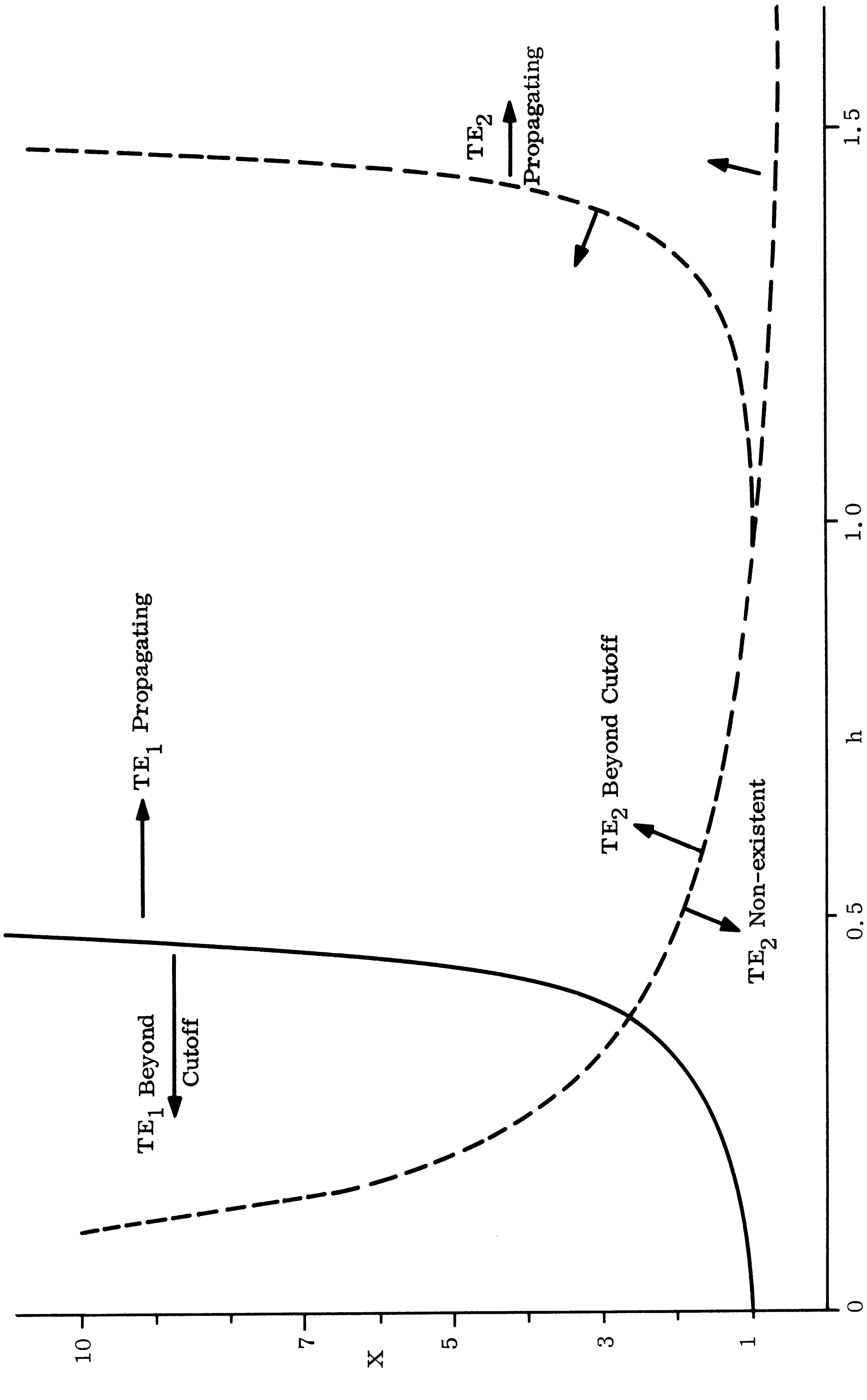

FIG. 3: REGIONS OF PROPAGATION, BEYOND CUTOFF AND NON-EXISTENCE FOR TWO LOWER ORDER TRANSVERSE ELECTRIC MODES.

$$K_n = \kappa_o X \left[1 - \left(\frac{u_n}{\pi h X}\right)^2\right]^{1/2} . \tag{31}$$

The n-th mode skin depth δ_n we define in the usual manner by setting

$$K_n \delta_n = 1 . \tag{32}$$

As a fraction of the free space wavelength λ_o the skin depth δ_n then is given by

$$\delta_n = \frac{\lambda_o}{2\pi X} \left[1 - \left(\frac{u_n}{\pi h X}\right)^2\right]^{-1/2} . \tag{33}$$

From (33) and Table I we observe that for the first mode when $hX > 1$ we have

$$\delta_1 \simeq \frac{\lambda_o}{2\pi X} \tag{34}$$

and for the second mode when $hX > 4$ δ_2 is also well approximated by (34). We conclude that for any appreciable X the field penetration in the plasma is small for the various modal fields.

2.2 The Lossy Case

For the lossy plasma

$$u_n \rightarrow u_n' + j u_n'' , \quad n = 1, 2, 3, \ldots \tag{35}$$

where u_n' and u_n'' are both real because the medium making up the waveguide is passive. The set of roots (35) satisfies (15). Since

$$K_{on} = \frac{1}{d}\left[u_n' + j u_n''\right] = \frac{\kappa_o}{\pi h}\left[u_n' + j u_n''\right] \tag{36}$$

we obtain from (5) for the n-th mode propagation constant

$$\gamma_n = \alpha_n + j\beta_n \tag{37}$$

where

$$\alpha_n = \kappa_o (-a_n + b_n)^{1/2} \tag{38}$$

$$\beta_n = \kappa_o (a_n + b_n)^{1/2} \tag{39}$$

with

$$a_n = \frac{1}{2}\left[1 + \left(\frac{u_n''}{\pi h}\right)^2 - \left(\frac{u_n'}{\pi h}\right)^2\right] \tag{40}$$

$$b_n = \left\{\frac{1}{4}\left[1 + \left(\frac{u_n''}{\pi h}\right)^2 - \left(\frac{u_n'}{\pi h}\right)^2\right]^2 + \left[\frac{u_n' u_n''}{(\pi h)^2}\right]^2\right\}^{1/2} . \tag{41}$$

From (22) by using (1), (12), (13), (14) and (35) we obtain

$$K_n = K_n' + jK_n'' \tag{42}$$

where

$$K_n' = \kappa_o \left[-g_n + q_n\right]^{1/2} \tag{43}$$

$$K_n'' = \kappa_o \left[g_n + q_n\right]^{1/2} \tag{44}$$

with

$$g_n = \frac{1}{2}\left[\left(\frac{u_n'}{\pi h}\right)^2 - \frac{X^2}{1 + Y^2} - \left(\frac{u_n''}{\pi h}\right)^2\right] \tag{45}$$

$$q_n = \frac{1}{2}\left\{\left[\left(\frac{u_n'}{\pi h}\right)^2 - \frac{X^2}{1 + Y^2} - \left(\frac{u_n''}{\pi h}\right)^2\right]^2 + \left[\frac{Y X^2}{1 + Y^2} - \frac{2 u_n' u_n''}{(\pi h)^2}\right]^2\right\}^{1/2} . \tag{46}$$

The skin depth of the n-th mode now is given by

$$\delta_n = \frac{1}{K_n'} . \tag{47}$$

To discuss these formulas requires many more cases to be considered as well as much more numerical work than for the loss-less case. For these reasons we limit the discussion to a slightly lossy plasma, i.e, $Y << 1$. For this condition we expect $u''/u' << 1$ and hence from (15) by equating the real and imaginary parts we obtain

$$\left[\left(\frac{\pi h}{u'} X\right)^2 - 1\right]^{1/2} \simeq \tan u' \tag{48}$$

$$u'' \simeq \frac{1}{2} Y \frac{u'}{1+C} \tag{49}$$

with

$$C = \frac{1}{\pi h X} \left(\frac{u'}{\cos u'}\right)^2 \left[1 - \left(\frac{u'}{\pi h X}\right)^2\right]^{1/2} . \tag{50}$$

Of most practical interest is the case of the first mode. From (48) we observe that

$$u_1' \simeq u_1 \quad \text{of (17)}, \tag{51}$$

i.e., the u_1' to a good approximation is the same as for the loss-less case which we have already calculated. As an easy second step we compute from (49)

$$u_1'' \simeq \frac{1}{2} Y \frac{u_1}{1+C_1} \tag{52}$$

with

$$C_1 = \frac{1}{\pi h X} \left(\frac{u_1}{\cos u_1}\right)^2 \left[1 - \left(\frac{u_1}{\pi h X}\right)^2\right]^{1/2} . \tag{53}$$

Numerical studies indicate that the approximate formulas are valid for $Y<0.1$. The propagation constant and the skin depth for the slightly lossy plasma are only slightly modified as compared with the loss-less plasma case. The largest differences are noticable at the cut-off.

2.3 The Mode Fields and Power Flow

From (2) to (7) we obtain for the field components of n-th mode propagating in the positive x-direction

$$H_x^{(n)}(x,y) = B_n \sin(K_{on} d)\, e^{-K_n(y-d)}\, e^{-\gamma_n x} , \quad y > d \tag{54a}$$

$$= B_n \sin(K_{on} y)\, e^{-\gamma_n x} , \quad -d < y < d \tag{54b}$$

$$= -B_n \sin(K_{on} d)\, e^{K_n(y+d)}\, e^{-\gamma_n x} . \quad y < -d \quad . \tag{54c}$$

$$H_y^{(n)}(x,y) = -B_n \frac{\gamma_n}{K_{on}} \cos(K_{on} d)\, e^{-K_n(y-d)}\, e^{-\gamma_n x} , \quad y > d \tag{55a}$$

$$= -B_n \frac{\gamma_n}{K_{on}} \cos(K_{on} y)\, e^{-\gamma_n x} , \quad -d < y < d \tag{55b}$$

$$= -B_n \frac{\gamma_n}{K_{on}} \cos(K_{on} d)\, e^{K_n(y+d)}\, e^{-\gamma_n x} , \quad y < -d . \tag{55c}$$

$$E_z^{(n)}(x,y) = -\frac{j\omega\mu_o}{\gamma_n} H_y^{(n)}(x,y) . \tag{56}$$

The modes propagating in the negative x-direction are obtained by substituting in the above expressions $B_n \rightarrow B_n^1$ and $\gamma_n \rightarrow -\gamma_n$. The time average power flow per unit length, $P_n(x)$, for the mode propagating in the positive x-direction we compute from

$$P_n(x) = -\frac{1}{2} \mathrm{Re} \int_{-\infty}^{\infty} E_z^{(n)}(x,y)\, H_y^{(n)*}(x,y)\, dy$$

$$= \frac{1}{2} \omega\mu_o \beta_n \left| \frac{B_n}{K_{on}} \right|^2 \left\{ \frac{\left| \cos(K_{on} d) \right|^2}{K_n'} + \right.$$

$$\left. d \left[\frac{\sin(2K_{on}' d)}{2K_{on}' d} + \frac{\sinh(2K_{on}'' d)}{2K_{on}'' d} \right] \right\} e^{-2\alpha_n x} \tag{57}$$

III

THE CAVITY COUPLING TO THE PLASMA WAVEGUIDE

A plasma cavity of diameter 2a is formed in Fig. 1, by letting $d \rightarrow 0$. Then for $b > a$ and X sufficiently greater than one the electromagnetic fields to a very good order of accuracy are given by

$$E_z(r,\phi) = \frac{1}{4}\omega\mu_o \left[-H_o^{(2)}(\kappa_o r) + D_1 J_o(\kappa_o r)\right], \quad 0 < r < a \tag{58a}$$

$$= \frac{1}{4}\omega\mu_o D_2 H_o^{(2)}(\kappa r), \quad r > b \tag{58b}$$

$$H_\phi(r,\phi) = \frac{1}{4} j\kappa_o \left[-H_1^{(2)}(\kappa_o r) + D_1 J_1(\kappa_o r)\right], \quad 0 < r < a, \tag{59a}$$

$$= \frac{1}{4} j\kappa D_2 H_1^{(2)}(\kappa r), \quad r > a, \tag{59b}$$

$$H_r(r,\phi) = 0 \tag{60}$$

From the continuity of the fields at $r = a$ we obtain

$$J_o(\kappa_o a) D_1 - H_o^{(2)}(\kappa a) D_2 = H_o^{(2)}(\kappa_o a) \tag{61a}$$

$$\kappa_o J_1(\kappa_o a) D_1 - \kappa H_1^{(2)}(\kappa a) D_2 = \kappa_o H_1^{(2)}(\kappa_o a) \tag{61b}$$

and hence the cavity fields are specified with the computing of

$$D_1 = \frac{\kappa_o H_o^{(2)}(\kappa a) H_1^{(2)}(\kappa_o a) - \kappa H_1^{(2)}(\kappa a) H_o^{(2)}(\kappa_o a)}{\kappa_o H_o^{(2)}(\kappa a) J_1(\kappa_o a) - \kappa H_1^{(2)}(\kappa a) J_o(\kappa_o a)}. \tag{62}$$

Opening a waveguide in the plasma sheath as shown in Fig. 1 the plasma cavity excites the waveguide. For a finite X only a finite number of modes may be excited of which only a few may propagate. The others are beyond cut-off. Restricting the guide width to $h < 1$ we observe from Fig. 3 that only the lowest order mode may propagate, i.e. TE_1. This, then, is the mode which transports the power from the cavity to the free space. The object of this section is to compute the amplitude of the outward traveling TE_1 mode. The mode is expected to radiate without suffering appreciable reflection at the external plasma guide aperture. Therefore we neglect the reflected TE_1 mode in the further discussions.

Any exact formulation of the problem to compute the amplitude of the outward traveling TE_1 mode leads to a mathematical morass. Instead, we shall compute the mode amplitude by relying on physical perception for the approximations. Narrow waveguides opening in the cavity tend to leave the magnetic field at the cavity wall undisturbed. We may match in some sense this field to the transverse magnetic field of the outward traveling TE_1 mode by requiring that the circumferential integral is continuous across the waveguide aperture. From (55b) and (59a) we then obtain

$$\frac{1}{2}\, j\phi_1 \kappa_o \left[-H_1^{(2)}(\kappa_o a) + D_1 J_1(\kappa_o a)\right] =$$

$$-B_1 \frac{\gamma_1}{K_{o1}} \int_{-\phi_1}^{\phi_1} d\phi \, \cos\left[K_{o1}\, a \sin\phi\right] e^{-\gamma_1 a \cos\phi} \tag{63}$$

where

$$\phi_1 = \arcsin \frac{h\lambda_o}{2a} \quad . \tag{64}$$

Performing the integration for the case $(h\lambda_o)/(2a) << 1$ we obtain

$$B_1 \simeq -\frac{1}{2} \frac{j\,\kappa_o u_1^2 D_1'}{h\lambda_o \gamma_1 \sin u_1} e^{\gamma_1 a} \tag{65}$$

where

$$D_1' \equiv -H_1^{(2)}(\kappa_o a) + D_1 J_1(\kappa_o a) \cdot \tag{66}$$

Substituting (62) in (63) we obtain

$$D_1' = \frac{j \frac{2\kappa}{\pi \kappa_o a} H_1^{(2)}(\kappa a)}{\kappa_o H_o^{(2)}(\kappa a) J_1(\kappa_o a) - \kappa H_1^{(2)}(\kappa a) J_o(\kappa_o a)} \cdot \tag{67}$$

When the Hankel asymptotic forms hold then

$$D_1' \sim \frac{\frac{\kappa}{\kappa_o}\sqrt{\frac{2}{\pi \kappa_o a}}}{-\sin(\kappa_o a - \frac{\pi}{4}) + j\frac{\kappa}{\kappa_o}\cos(\kappa_o a - \frac{\pi}{4})} , \tag{68}$$

$$\kappa_o a >> 1, \quad \kappa a >> 1 .$$

For a plasma with $X > 1$ and $Y << 1$ the time average power per unit length picked up by the TE_1 mode at the plasma guide inside aperture $(x = a)$ we compute from (57) as

$$P_1(a) \simeq \frac{1}{16} \omega\mu_o \beta_1 (h\lambda_o)^3 \left|\frac{B_1}{u_1}\right|^2 \left[1 + \frac{\sin 2u_1}{2u_1}\right] e^{-2\alpha_1 a} . \tag{69}$$

In the above formula we have essentially neglected the fields which penetrate the plasma through the guide walls. The time average power per unit length for the electric line source radiating in the free space we easily compute as

$$P_o = \frac{1}{8}\omega\mu_o . \tag{70}$$

It is convenient to normalize (69) with respect to (70), i.e., we express the time average power picked up by the plasma guide as a fraction of the line source radiation in the free space

$$P_1^{(n)}(a) \equiv \frac{P_1(a)}{P_o} \quad . \tag{71}$$

We compute that

$$P_1^{(n)}(a) \simeq \frac{1}{2}\,\beta_1\,(h\lambda_o)^3 \left|\frac{B_1}{u_1}\right|^2 \left[1+\frac{\sin 2u_1}{2u_1}\right] e^{-2\alpha_1 a} \quad . \tag{72}$$

Substituting (65) in (72) we obtain for the case of $\phi_1 = \arcsin(\frac{d}{a}) \simeq \frac{d}{a}$, $Y << 1$, and $X > 1$

$$P_1^{(n)}(a) \simeq \frac{\pi}{4}\,\frac{h\,\beta_1\,\kappa_o}{\alpha_1^2+\beta_1^2}\left(\frac{u_1}{\sin u_1}\right)^2 \left[1+\frac{\sin 2u_1}{2u_1}\right] \left|D_1'\right|^2 \quad . \tag{73}$$

Since u_1, α_1, and β_1 are functions of h, X, and Y we cannot discuss (73) without some numerical work. In Fig. 4 we have plotted curves of $P_1^{(n)}(a)$ as a function of X with h as a parameter and Y = 0.1. The plasma cavity radius is kept at one free space wavelength, i.e., $a = 1\lambda_o$. For h = 0.25, 0.30, 0.35 and 0.4 the curves start at about 0.02 for X = 1, peak towards 1.0 and then break and fall abruptly. The break is because the mode becomes beyond cut-off. When h = 0.5, then the mode approaches cut-off only as $X \rightarrow \infty$. When $h > \frac{1}{2}$, there is no cut-off and also no peaking as may be seen for the case h = 1. Cut-off occurs when $|\gamma_1|$ is at minimum; in the loss-less case $|\gamma_1|$ minimum is at zero. The peaking is caused by the transverse magnetic field of TE_1 modes decreasing to a low value at the cut-off as can be judged from (55). In fact the transverse magnetic field would be zero at the cut-off for the loss-less plasma guide. Because of this modal field behavior one can see from our approximate scheme (63) that B_1 is peaked when $|\gamma_1|$ is at a minimum. We expect (63) to be more approximate at the cut-off than elsewhere and therefore have

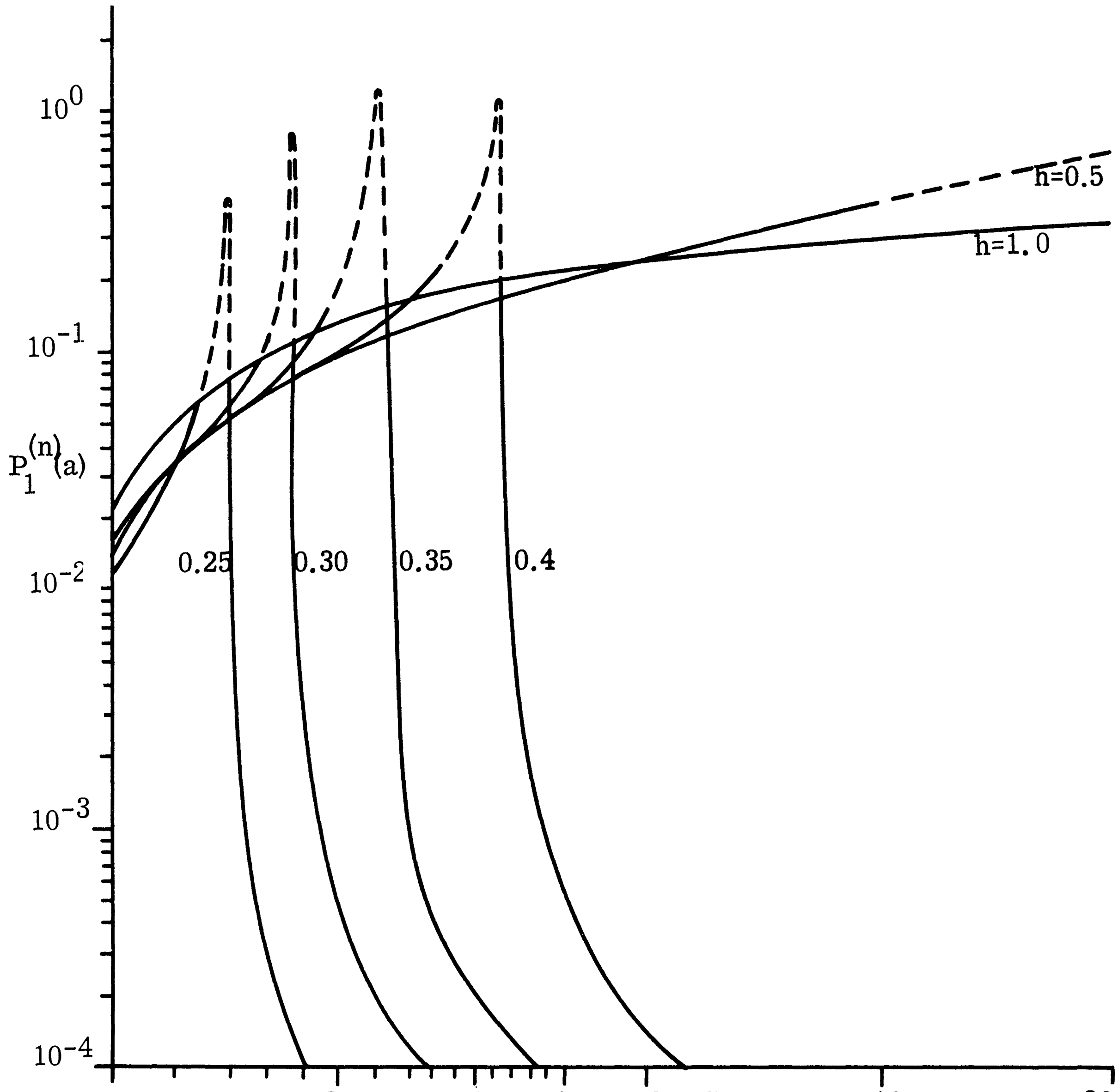

FIG. 4: NORMALIZED CAVITY COUPLING TO TE_1 MODE AS FUNCTION OF X WITH GUIDEWIDTH h a PARAMETER AND $a = 1\lambda_o$, $Y = 0.1$.

indicated those portions of the curves by a dotted line. The curves would be only slightly modified for the loss-less plasma (Y = 0). One general observation may be made from Figure 4: for the guide width considered the guide picks up from the cavity when the TE_1 mode is propagating on the order of one tenth of the line source radiation in the free space, when the mode is non-propagating then the guide picks up essentially no time average power. Hence we may conclude that for the non-propagating condition the guide isolates the cavity source from the free space and for the propagating condition on the other hand we may expect a close coupling in at least some free space directions.

IV

THE RADIATION

For $r \geqslant b$ the non-zero field components are $E_z(r,\phi)$, $H_r(r,\phi)$, $H_\phi(r,\phi)$. From Green's Second Identity and the Maxwell's equations we obtain

$$E_z(r,\phi) = \int_{-\pi}^{\pi} \left[E_z^{(1)}(b,\phi';r,\phi)\, H_\phi(b,\phi') - H_\phi^{(1)}(b,\phi';r,\phi) E_z(b,\phi') \right] b d\phi' \tag{74}$$

where $E_z^{(1)}$ and $H_\phi^{(1)}$ are the fields associated with the unit electric line source at (r,ϕ). For the line source in the presence of a perfectly conducting cylinder

$$E^{(1)}(b,\phi';r,\phi) = 0 \tag{75}$$

and hence from (74)

$$E_z(r,\phi) = - \int_{-\pi}^{\pi} H_\phi^{(1)}(b,\phi';r,\phi)\, E_z(b,\phi')\, b d\phi' . \tag{76}$$

It is an easy matter to find (Harrington, 1961)

$$E_z^{(1)}(r',\phi';r,\phi) = \frac{1}{4}\omega\mu_o \left[-H_o^{(2)}(\kappa_o |\bar{r}' - \bar{r}|) + \sum_{n=-\infty}^{\infty} \frac{J_n(\kappa_o b)}{H_n(\kappa_o b)} H_n^{(2)}(\kappa_o r)\, H_n^{(2)}(\kappa_o r')\, e^{jn(\phi' - \phi)} \right] \tag{77}$$

and for $r' < r$

$$H_\phi^{(1)}(b,\phi';r,\phi) = -\frac{1}{2\pi b} \sum_{n=-\infty}^{\infty} \frac{H_n^{(2)}(\kappa_o r)}{H_n^{(2)}(\kappa_o b)} e^{jn(\phi' - \phi)} . \tag{78}$$

We substitute (78) in (76) and interchanging summation with integration obtain

$$E_z(r,\phi) = \sum_{n=-\infty}^{\infty} \frac{a_n H_n^{(2)}(\kappa_o r)}{H_n^{(2)}(\kappa_o b)} e^{-jn\phi} \tag{79}$$

with

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi' \; E_z(b,\phi') \; e^{jn\phi'} \tag{80}$$

From (79) and the Maxwell's equations we derive the free space radiation field

$$E_z(r,\phi) \sim F(\phi) \sqrt{\frac{2}{\pi \kappa_o r}} \; e^{-j(\kappa_o r - \pi/4)} \tag{81}$$

$$H_\phi(r,\phi) \sim -\sqrt{\frac{\epsilon_o}{\mu_o}} \; E_z(r,\phi) \tag{82}$$

$$H_r(r,\phi) \sim 0 \tag{83}$$

where

$$F(\phi) = \sum_{n=-\infty}^{\infty} \frac{j^n a_n}{H_n^{(2)}(\kappa_o b)} e^{-jn\phi} \tag{84}$$

The radiation normalized with respect to the unit electric line source radiation in the free space we define by

$$W(\phi) = \frac{\frac{1}{2}\sqrt{\frac{\epsilon_o}{\mu_o}} \left| E_z(r,\phi) \right|^2}{\frac{P_o}{2\pi r}} \;, \quad r >> b \;. \tag{85}$$

Substituting (70) and (81) in (85) we obtain

$$W(\phi) = \frac{16 \, \epsilon_o}{\mu_o \kappa_o^2} \left| F(\phi) \right|^2 \tag{86}$$

We define the cylinder aperture gain function $G(\phi)$ by

$$G(\phi) = \frac{\left|F(\phi)\right|^2}{\int_0^{2\pi} \left|F(\phi)\right|^2 d\phi} . \tag{87}$$

Both of the last two formulas are of physical significance: $W(\phi)$ tells us the extent to which the plasma guide re-establishes the radiation as compared to unit electric line source radiation in the free space. $G(\phi)$ provides us with information on the focusing properties of the plasma guide external aperture; it is the radiation power pattern normalized with respect to an isotopic radiator.

In this section so far the formulas are exact. In order to compute $F(\phi)$ we have to find the Fourier coefficients a_n as defined by (80) which requires the knowledge of $E_z(b,\phi')$. We introduce the approximation

$$E_z(b,\phi') = E_z^{(1)}(b\cos\phi', \ b\sin\phi') \ , \quad -\phi_o < \phi' < \phi_o \ , \tag{88a}$$

$$= 0 \ , \quad \phi_o < \phi' < 2\pi - \phi_o \tag{88b}$$

with

$$\phi_o = \arcsin\left(\frac{h\lambda_o}{2b}\right) . \tag{90}$$

That is to say, we approximate the tangential electric field in the external guide aperture by the electric field of the incident TE_1 mode, and for the remainder of the cylinder surface we set the tangential electric field equal to zero. From (56), (80) and (88) we then obtain

$$a_n \simeq \frac{j\omega\mu_o B_1}{2\pi K_{o1}} \int_{-\phi_o}^{\phi_o} d\phi \cos\left[K_{o1}\, b\sin\phi\right] e^{-\gamma_1 b\cos\phi + jn\phi} \tag{91}$$

For a narrow slot we may approximate (91) to

$$a_n \simeq \frac{j\omega\mu_o B_1 \phi_o}{2\pi K_{o1}} \left[\frac{\sin (K_{o1} b + n)\phi_o}{(K_{o1} b + n)\phi_o} + \frac{\sin (K_{o1} b - n)\phi_o}{(K_{o1} b - n)\phi_o} \right] e^{-\gamma_1 b} . \quad (92)$$

We observe that for (91) as well as for (92)

$$a_n = a_{-n} . \quad (93)$$

Substituting (65) in (92) and the resulting equation in (84) we obtain

$$F(\phi) \simeq \frac{\omega\mu_o \phi_o \kappa_o u_1 D_1'}{8\pi\gamma_1 \sin u_1} e^{-\gamma_1 (b-a)} \sum_{n=0}^{\infty} \epsilon_n j^n \frac{a_n' \cos n\phi}{H_n^{(2)}(\kappa_o b)} , \quad (94)$$

with

$$a_n' = \frac{\sin (K_{o1} b + n)\phi_o}{(K_{o1} b + n)\phi_o} + \frac{\sin (K_{o1} b - n)\phi_o}{(K_{o1} b - n)\phi_o} \quad (95)$$

and

$$\epsilon_n = 1, \quad n = 1$$

$$= 2, \quad n = 1, 2, 3, \ldots$$

Substituting (94) in (86) we have

$$W(\phi) \simeq \left(\frac{\phi_o u_1}{2\pi \sin u_1} \right)^2 \left| \frac{\kappa_o}{\gamma_1} D_1' \right|^2 e^{-2\alpha_1 (b-a)} \left| \sum_{n=0}^{\infty} \epsilon_n j^n \frac{a_n' \cos n\phi}{H_n^{(2)}(\kappa_o b)} \right|^2 . \quad (96)$$

From (94) and (87) we calculate

$$G(\phi) \simeq \frac{\left| \sum_{n=0}^{\infty} \epsilon_n j^n \frac{a_n' \cos n\phi}{H_n^{(2)}(\kappa_o b)} \right|^2}{\sum_{n=0}^{\infty} \epsilon_n \left| \frac{a_n'}{H_n^{(2)}(\kappa_o b)} \right|^2} \quad (97)$$

Assuming that the TE_1 mode radiates without reflections we also may calculate $W(\phi)$ from

$$W(\phi) \simeq P_1^{(n)}(a) \; e^{-2\alpha_1(b-a)} \, G(\phi) \qquad (98)$$

This form is particularly convenient for describing the process of radiation. According to (98) the radiation depends on three factors: the power coupled from the line source into the TE_1 mode, the guide attenuation, and the gain function of the cylinder aperture. Equations (96) and (98) are not exactly equivalent forms of $W(\phi)$ approximation. In (96) the radiation is computed from the assumption that in the aperture the electric field is that of the incident TE_1 mode while in (98) we assume that all incident power is radiated. We expect both forms to agree at least to within a few db when the TE_1 mode is propagating in the guide. This was confirmed by computations. When the waveguide is beyond cut-off (96) predicts less attenuation of the free space radiation than (98). We feel that the former in this case is a better approximation than the latter. However, before we proceed with the discussion of (96) we return to (98).

The $P_1^{(n)}(a)$ factor we have already presented in Fig. 4 and discussed in Section II. The guide attenuation factor $\exp 2\alpha_1\lambda_o$ is shown in Fig. 5 as a function of X for Y = 0.1 and the guide width h as a parameter. The sharp drop-off in the curves indicates that the waveguide there is beyond cut-off. Thus the first two factors in (98) indicates that when the waveguide is beyond cut-off that it will very effectively isolate the cavity from the free space, unless there happens to be a large cavity resonnance. The last factor in (98) is the gain function $G(\phi)$ of the axial slot excited cylinder of radius b. We have shown $G(\phi)$ in Fig. 6 for the case $b = 3\lambda_o$, Y = 0.1, X = 4, and h as a parameter. We see that we have only a main forward lobe and insignificant back-radiation. The essentially smooth single lobe gain function

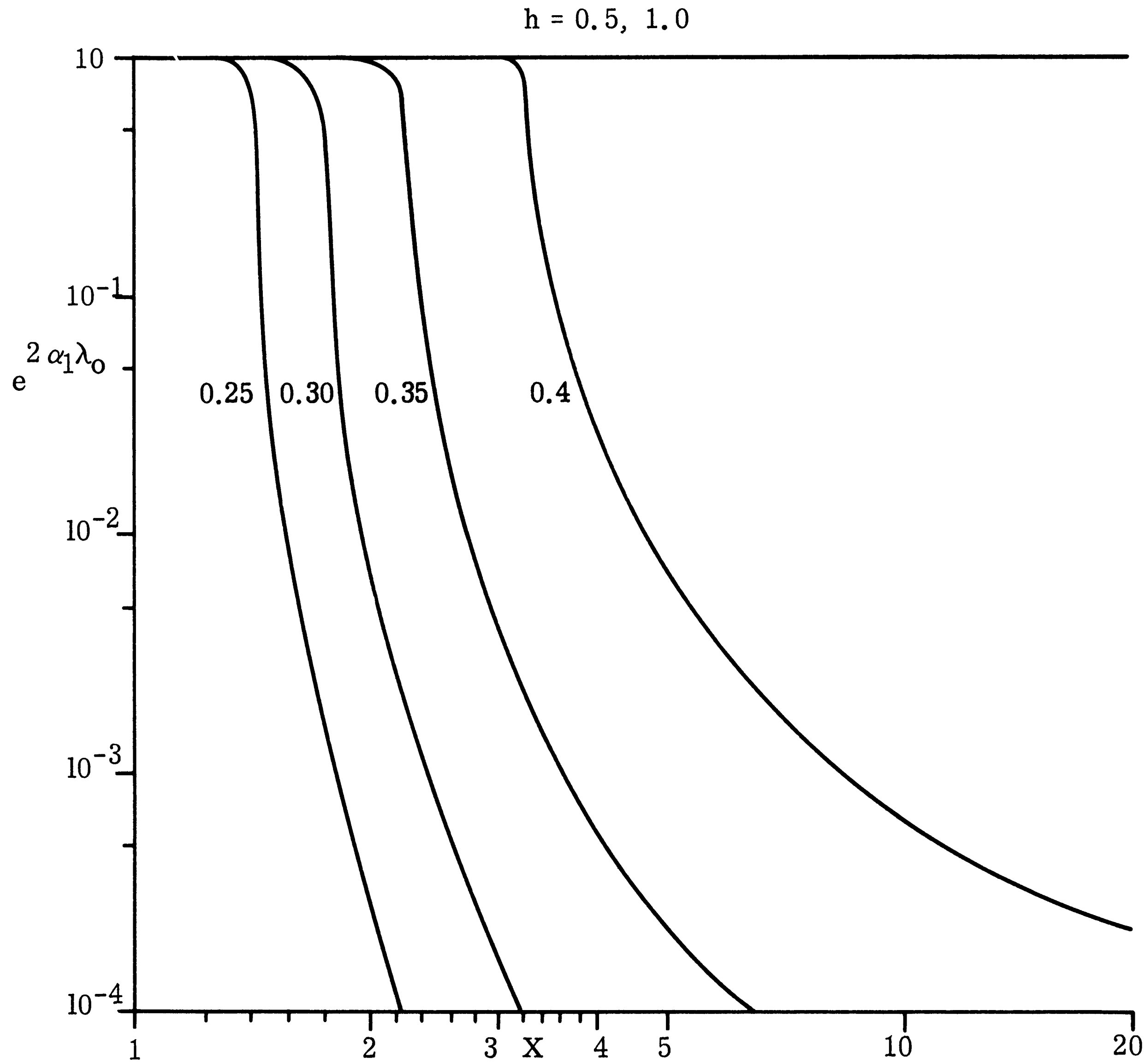

FIG. 5: POWER ATTENUATION IN A GUIDE 1 λ_o LONG AS A FUNCTION OF X WITH GUIDEWITH h a PARAMETER AND Y = 0.1.

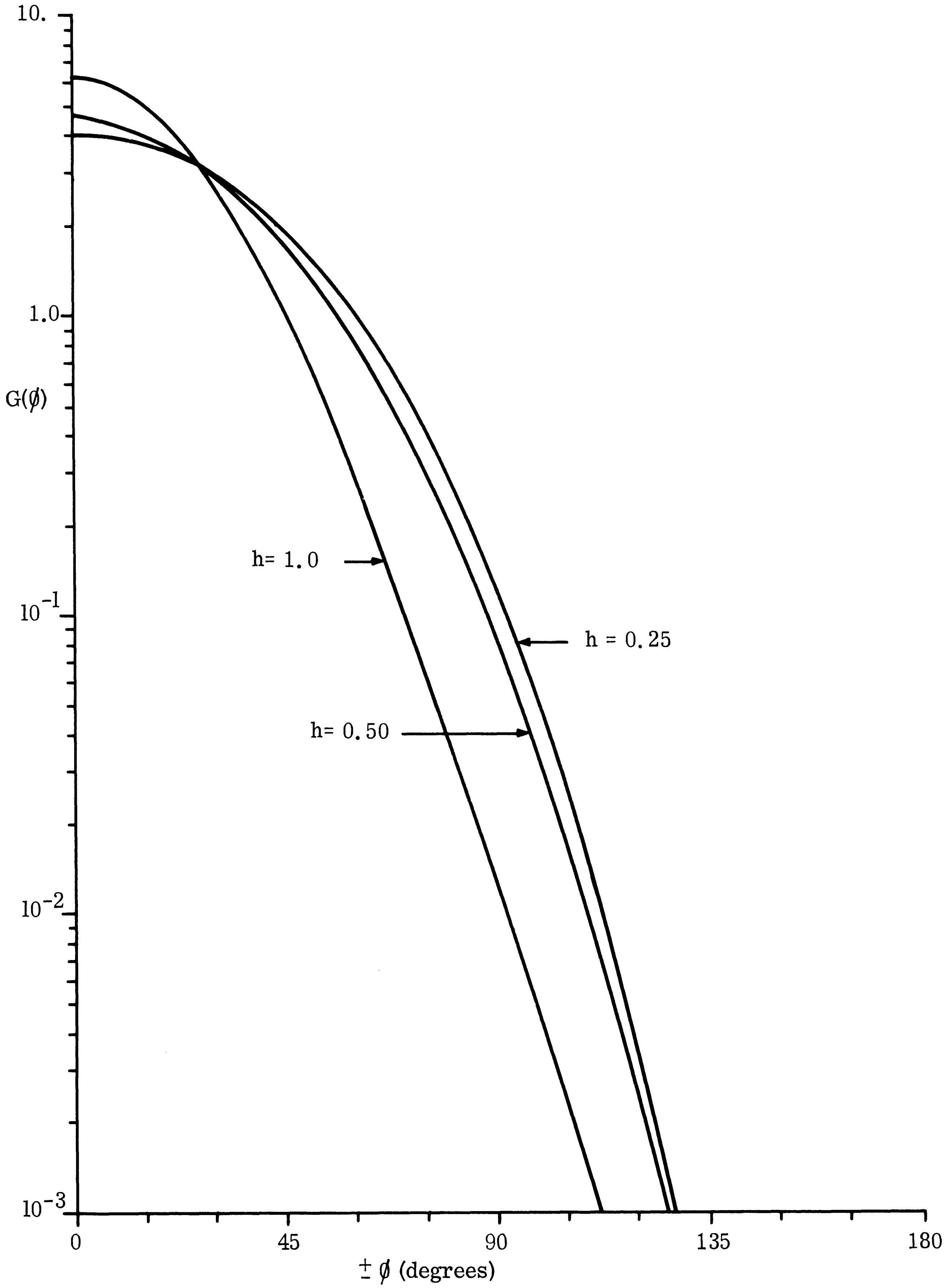

FIG. 6: GAIN FUNCTION OF $3\lambda_0$ RADIUS CYLINDER WITH GUIDE WIDTH h A PARAMETER

(i.e. the normalized radiation power pattern) arises from the three factors

1. insignificant radiation through the plasma sheath,
2. the slot width no larger than one wavelength with only the principal mode propagating,
3. all the field components are continuous at the plasma cylinder surface.

The gain is not a sensitive function either of the cylinder radius b or X for $b > 2\lambda_o$ and $Y < 0.1$. From the 3 factors of (98) an appreciation is gained for the source coupling in the various direction of the free space. Further we see that for practical purposes it is sufficient to give the radiation only in the forward direction i.e. $\phi = 0$.

In Fig. 7 we present the normalized radiation in the forward direction W(0) as computed from (96) as a function of X and the guide width h a parameter, the cylinder radius $b = 2\lambda_o$, and $Y = 0.1$. The normalization is with respect to the unit electric line source radiation in the free space. Thus $W(0) = 1$ means that the plasma slot has re-established the radiation in the forward direction to the free space radiation level. The curves are shown dotted where the approximation is poor because of the $P_1^{(n)}(a)$ behavior as discussed in Section II. The W(0) curves exhibit the decoupling of the source from the free space when $h < 0.5$ and X gets sufficiently large. Whereas when $h > 0.5$ no decoupling will occur for any X. The surprising feature is that the radiation decreases as X decreases to unity. However for no X will the radiation in the forward direction be less than 10 db below the free space case when the TE_1 mode is propagating. In Figure 8 we have presented W(0) for the same case as in Fig. 7, except $b = 3\lambda_o$. The only essential difference with the preceeding Figure is that the decoupling curves are steeper. The other preceeding comments apply to this case as well.

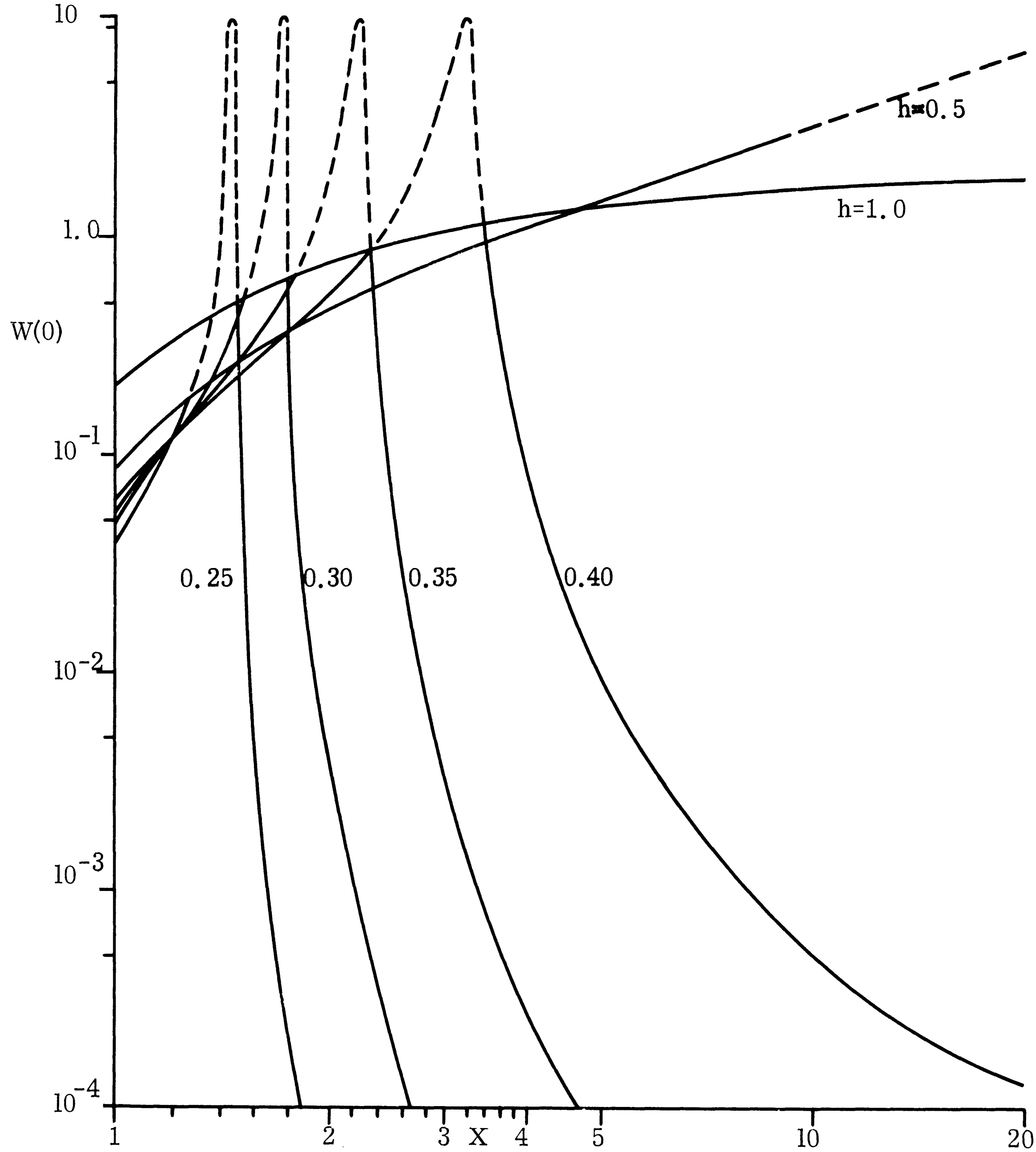

FIG. 7: NORMALIZED RADIATION IN FORWARD DIRECTION OF UNIT ELECTRIC LINE SOURCE EXCITED SLOTTED PLASMA SHEATH AS A FUNCTION OF X WITH SLOT WIDTH h a PARAMETER AND Y=0.1, $a=1\lambda_o$, $b=2\lambda_o$.

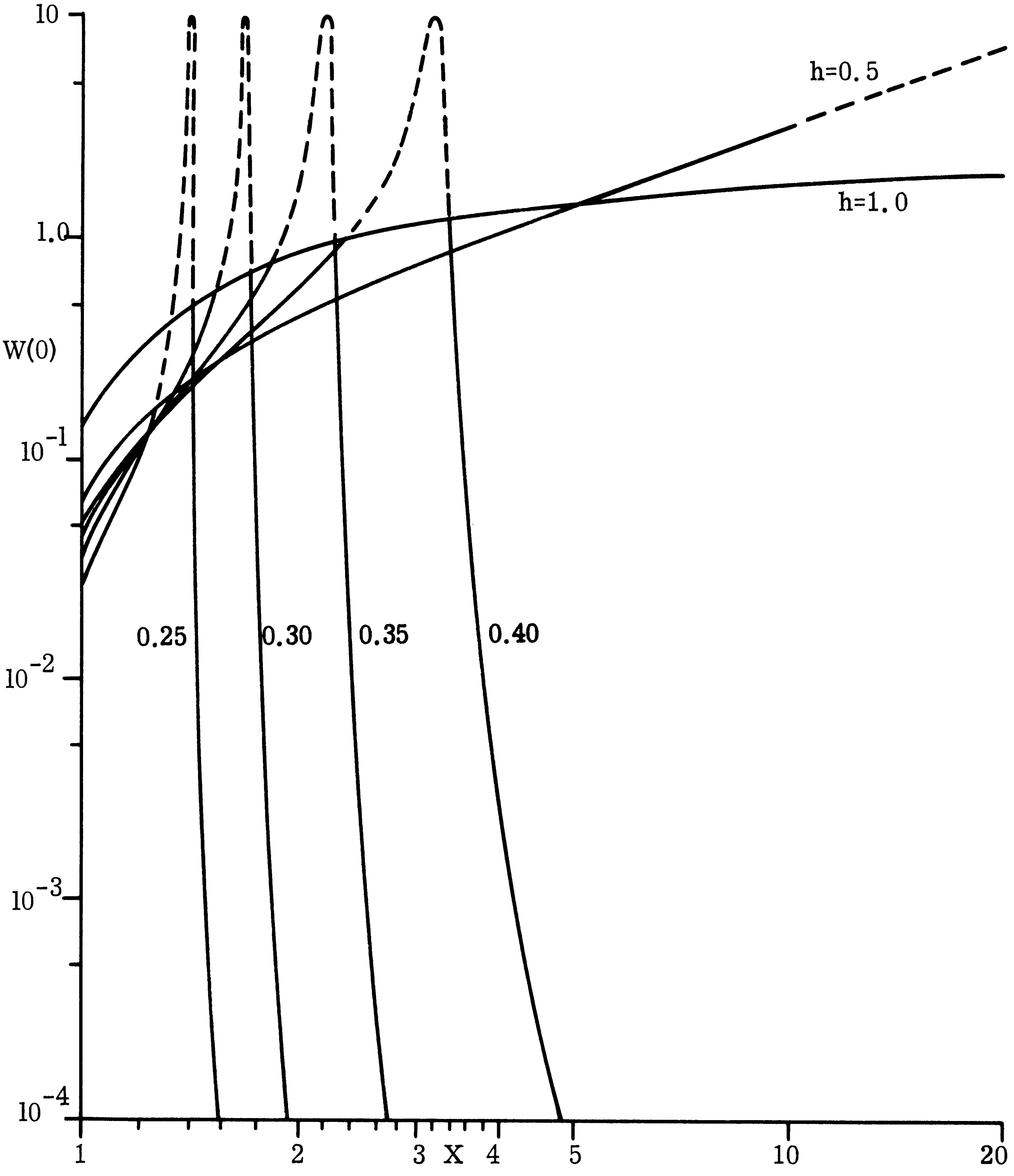

FIG. 8: NORMALIZED RADIATION IN FORWARD DIRECTION OF UNIT ELECTRIC LINE SOURCE EXCITED SLOTTED PLASMA SHEATH AS A FUNCTION OF X WITH SLOT WIDTH h a PARAMETER AND Y=0.1, $a=1\lambda_o$, $b=3\lambda_o$.

V

DISCUSSION AND CONCLUSIONS

The normalized forward radiation W(0) as shown in Figures 7 and 8 may be compared for some cases with the radiation from the integral equation method (Olte, 1965) as given in Figs. 10 and 11 of that report respectively. Although the plasma parameters correspond exactly for the two cases the slots are of different nature: a parallel wall slot in this study and a 36^o wedge slot in the previous study for which calculations were carried out. The 36^o wedge slot inside opening is $0.616\lambda_o$ wide. The one half wavelength wide slot $(h=\frac{1}{2})$ in the present case is the closest we may approach the 36^o wedge slot. When $b = 2\lambda_o$ the integral equation method yields 0.09 (0.15) for $X = 1$ and 0.125 (0.098) for $X = \sqrt{2}$, and for $b = 3\lambda_o$ we obtain 0.07 (0.23) for $X = 1$. In the brackets we have included the corresponding numbers for $h = 0.5$ parallel wall slot. The agreement is as good as can be expected under the circumstances and lends some support to the present analysis.

In a study of this kind a number of approximations were necessary. Of these the assumption that the principal mode radiates without appreciable reflection at the plasma waveguide external aperture is justified on the basis of our knowledge that apertures are efficient radiators. The most difficult part in this study is to compute the power coupled from the source into the principal mode. To carry out the computation we made the assumption that opening the plasma slot leaves the cavity magnetic field unchanged. For narrow slots this perhaps is not a bad assumption. For wider slots some correction should be necessary. Some further work in the cavity coupling is justified. The other approximations may be numerically justified in view of the quantities we wanted to compute.

In the case studied the plasma slot was opened by reducing the electron density to zero. In a practical case we may succeed only in depressing the plasma frequency in the plasma slot below the signal radian frequency. Is this enough? How wide should the waveguide slot be under these conditions? The study may be easily extended to this case. However, here we limit the discussion to a few observations about the lower order mode under these conditions. Replacing κ_o in (5) by

$$\kappa_1 = \kappa_o \sqrt{1 - j\,\frac{\sigma_1}{\omega\epsilon_o}} \tag{99}$$

where

$$\sigma_1 = \frac{\epsilon_o \omega_{p1}^2}{\nu_1 + j\omega} \tag{100}$$

we obtain for (15)

$$\left\{\left(\frac{\pi h}{u}\right)^2 \left[\frac{X^2}{1 - jY} - \frac{X_1^2}{1 - jY_1}\right] - 1\right\}^{1/2} = \tan u \tag{101}$$

where the new variables

$$X_1 \equiv \frac{\omega_{p1}}{\omega} \quad ; \quad Y_1 = \frac{\nu_1}{\omega} \; . \tag{102}$$

For the loss-less case (101) simplifies to

$$\left\{\left(\frac{\pi h}{u} X_r\right)^2 - 1\right\}^{1/2} = \tan u \tag{103}$$

where

$$X_r^2 = X^2 - X_1^2 \; . \tag{104}$$

Restricting to case $X > X_1$ we observe that (103) is of the same form as (16) and thus the results of Table I apply if we replace X by X_r

in the first column. The root u_1 is reduced as compared to the κ_o case slot. The propagation constant for the lower order mode in the loss-less case now will be given by

$$\alpha_1 = \kappa_o \left\{ \left(\frac{u_1}{\pi h}\right)^2 + X_1^2 - 1 \right\}^{1/2}, \quad \beta_1 = 0 \quad ; \left(\frac{u_1}{\pi h}\right)^2 + X_1^2 > 1, \quad (105)$$

$$\alpha_1 = 0 \, , \quad \beta_1 = \kappa_o \left\{ 1 - \left[\left(\frac{u_1}{\pi h}\right)^2 + X_1^2 \right] \right\}^{1/2} ; \left(\frac{u_1}{\pi h}\right)^2 + X_1^2 < 1, \quad (106)$$

the non-propagating and propagating case, respectively. From (106) we conclude that the mode is propagating when

$$h > \frac{u_1/\pi}{\sqrt{1 - X_1^2}} \qquad (107)$$

The root u_1 assumes a maximum of $\pi/2$ as $X \rightarrow \infty$ and thus the mode is propagating for any X as long as

$$h > \frac{0.5}{\sqrt{1 - X_1^2}} \qquad (108)$$

As one may expect the plasma slot for the lower order mode to propagate for any X should be at least one half slot medium wavelength wide, which is an obvious generalization of our study of a special case, i.e. $X_1 = 0$. Examination of the other equations reveals that the other modifications because of non-zero X_1 will be minor compared to the impact of X_1 on the minimum slot width required in order to establish radiation in the forward direction.

If we do not depress in the slot the plasma frequency below the radian signal frequency, i.e. if $X_1 > 1$, then as indicated by (105) the attenuation constant α_1 will be large for any slot width and a very substantial waveguide attenuation of the radiation will result. For the loss-less plasma case this is explicitly given in (105), and for a lossy plasma may be computed from (21) with κ_0 replaced by κ_1 and the appropriate root selected from (101).

We summarize the main conclusions of this study:

1) When the slot of the uniform plasma sheath may be approximated as a parallel face slot then it assumes the properties of a parallel face waveguide that supports under some conditions propagating modes. These propagating modes play very dominant role in the transfer of power from the source to the outside free space. In fact the regions of blackout can be delinated solely on the basis of non-propagation of all modes in the plasma slot waveguide. When the modes are propagating they suffer no attenuation in the case of loss-less plasma. It has been shown that the lower order mode at least suffers negligible attenuation also for low loss plasma.

2) It has been shown that symmetrically excited plasma slot of the cylindrical sheath produces a broad forward lobe: 3 db width of the order of 75^o for $\frac{1}{2}$ wavelength slot, and of the order of 50^o for one wavelength slot. The polarization of the electric field is entirely tangential to the cylinder surface and the leakage through the plasma sheath is assumed to be insignificant. No significant back radiation exists.

3) This study extends to a larger class of parameters the conclusion from the integral equation method that the slot effectively re-establishes radiation in the forward direction. For plasma slot wider than $\frac{1}{2}$ wavelength the radiation in the forward direction is returned to the no-plasma sheath levels, and at least to within -10 db for narrower slots as long as the lower order plasma waveguide mode is propagating and the plasma is of low loss type.

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