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COLLEGE OF ENGINEERING
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Technical Report

AN ELEMENTARY REVIEW OF THE SCATTERING OF PHOTONS BY HIGHLY IONIZED PLASMAS

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INTRODUCTION

In this report we develop the theory of the scattering of photons by equilibrium plasmas sufficiently highly ionized that the presence of neutrals can be neglected. Of course, this is an old problem[1,2,3] to which we bring nothing new so far as analysis is concerned, nor from which we extract no new results. However, we do bring under one roof so to speak all of the details of the analysis so that hopefully the complete argument can be traced from start to finish with minimal recourse to outside literature. Furthermore, the theory will be developed systematically so that schemes for generating higher order approximations (if ever necessary) will be clearly delineated at each stage in the argument. The point at issue here is that, in the past, this calculation has either been dealt with in semi-intuitive terms[1,3] which is attractive to the lowest order of approximation but leaves the question of generalization in obscurity; or as incidental to the general problem of the kinetic theory of plasmas,[2,4] in which case the essential simplicity of the analysis rarely emerges for the uninitiated reader. Thus the primary burden of this report is the presentation of a systematic and self-contained analysis of this specific class of experiments.

The problem at hand quite naturally divides into two almost independent, parts. In the first part (to be discussed in detail in Section I) we are exclusively concerned with the establishment of a relationship between the observed distribution of scattered photons and the dynamical and statistical characteristics of the scattering plasma. Here we depart somewhat from the more conventional treatments referenced above in that we characterize the result of the experiments in terms of a photon scattering cross-section rather than an electric field intensity. This forces us (at least initially) into the context of quantum, rather than classical mechanics. We do not take this path because we anticipate any pecularily quantum effects to show up in the usable results. (Though, as we shall see, one effect of substantial importance in principle does emerge from the present analysis which does not emerge from the classical analysis of the electric field intensity.) We do it because it is surely right—the observations consist in the counting of photons rather than in the measurement of an electric field—and because, in our opinion, the analysis is simpler and more clarifying. We find, of course, in the sense of well-defined approximations, that the cross-section is simply related to the autocorrelation of the electron density fluctuations in the plasma. In this process, we also establish the sense in which a classical calculation of this autocorrelation function is legitimate for our purposes.

In the second section we devote our attention entirely to the calculation of the electron density autocorrelation function. For this purpose we exploit a scheme of analysis most recently described in great generality in

Reference 4, and developed quite extensively earlier by Klimontovich.[5] Though, as indicated, this approach to the kinetic theory of plasmas has received considerable attnetion; it has not, to our knowledge, been applied explicitly to the calculation of the statistical function needed here.

Here again we find that the problem at hand subdivides into two parts which may be dealt with independently. The first part has to do with the determination of the way in which plasma dynamics influence the autocorrelation function; whereas the second part explores the influence of plasma statistics. Of course this is a separation into two calculational parts, not physical parts, since the plasma statistics depend in detail on plasma dynamics.

In Section III, we briefly discuss the results (they need be discussed only briefly since they have been explored in qualitative and quantitative detail elsewhere[1,2,3]). In concluding this section we point out some of the related areas of unfinished business which we feel require further study.

I. DERIVATION OF THE CROSS-SECTION

The results of scattering experiments are conveniently correlated in terms of the scattering cross-section; or alternatively the so-called scattering function to which the cross-section is simply related. The idealized experiment consists of a monoenergetic, monodirectional photon beam incident upon a sample of plasma, and a photon detector capable of counting photons selectively within small energy and angular ranges. We will assume that detection is confined to angular regions well out of the incident beam. We can define a cross-section to describe the experiment as the effective area that a charged particle in the sample presents to an incident photon for the scattering of that photon into a prescribed small solid angle and energy range. More analytically, we may write

$$\sigma(\omega,\underline{\Omega};\ \omega'\underline{\Omega}')d\omega'd\Omega' = \frac{T(\omega,\Omega;\ \omega',\Omega')d\omega'd\Omega'}{N \times \text{Incident Flux}}\ , \qquad (I.1)$$

where $T(\omega,\underline{\Omega};\ \omega',\underline{\Omega}')d\omega'd\Omega'$ is the transition probability per unit time that the sample will scatter a photon of frequency ω traveling in direction $\underline{\Omega}$ into the frequency range $d\omega'$ about ω' and the angular range $d\underline{\Omega}'$ about the direction $\underline{\Omega}'$, N is the number of scatterers in the sample, and the incident flux is the number of photons crossing a square centimeter per sec in the incident beam. Thus our main task here is the calculation of the transition probability per unit time.

Our system, consisting of the scattering sample and the radiation field, is characterized by a Hamiltonian

$$H = H^R + H^P + H^{PR}\ , \qquad (I.2)$$

where H^R represents the energy of the free radiation field, H^P the energy of the plasma, and H^{PR} the energy of interaction between the radiation field and the plasma. The density operator for the system satisfies the Liouville Equation,

$$\frac{\partial D}{\partial \tau} = \frac{i}{\hbar}[D,H]\ , \qquad (I.3)$$

and the diagonal elements of D in an arbitrary representation specify the probability of finding the system in a particular state corresponding to the chosen representation at a given instant. If appropriate complete orthonormal sets of eigenstates of H^R and H^P span the space of H and are put together to form product eigenstates for the whole system, i.e.,

$$H^R|\eta> \;=\; E_\eta|\eta> \;,$$

$$H^P|n> \;=\; E_n|n> \;,$$

and

$$|n\eta> \;=\; |n>|\eta> \;; \qquad (I.4)$$

then the diagonal elements of D in the representation $|n\eta>$ are the probabilities of finding the plasma in the state $|n>$ and the radiation field in the state $|\eta>$. Now according to (I.3),

$$D(t+\tau) \;=\; U(\tau)D(t)U^\dagger(\tau) \;, \qquad (I.5)$$

where U is the time evolution operator represented by

$$U(\tau) \;=\; e^{-i\tau H/\hbar} \;. \qquad (I.6)$$

It then follows that the probability of finding the system in the state $|n'\eta'>$ at time, $t+\tau$, is given by

$$D_{n'\eta',n'\eta'}(t+\tau) \;=\; \sum_{\eta''n''\eta'''n'''} U_{n'\eta',n''\eta''}(\tau)D_{n''\eta'',n'''\eta'''}(t)U^\dagger_{n'''\eta''',n'\eta'} \;. \qquad (I.7)$$

If we specify that at time t, the system is definitely in the state $|n\eta>$, then

$$D_{n''\eta'',n'''\eta'''}(t) \;=\; 0,\; n''\eta'' \neq n'''\eta''' \;,$$

$$D_{n''\eta'',n''\eta''}(t) \;=\; 0,\; n''\eta'' \neq n\,\eta \;,$$

and

$$D_{n\eta,n\eta}(t) \;=\; 1 \;. \qquad (I.8)$$

Thus the probability of finding the system in the state $|n'\eta'>$ at time $t+\tau$, given that it was in the state $|n\eta>$ at t is

$$\begin{aligned} D_{n'\eta',n'\eta'}(t+\tau) & \\ &= U_{n'\eta',n\eta}(\tau)U^\dagger_{n\eta,n'\eta'}(\tau) \\ &= |U_{n'\eta',n\eta}(\tau)|^2 \\ &= |(e^{-i\tau H/\hbar})_{n'\eta',n\eta}|^2 \;. \end{aligned} \qquad (I.9)$$

Then the probability per unit time that the system will undergo a transition from state $|n\eta>$ to state $|n'\eta'>$ is

$$T_{n'\eta',n\eta} = \frac{1}{\tau} \left|(e^{-i\tau H/\hbar})_{n'\eta',n\eta}\right|^2 . \qquad (I.10)$$

The expression (I.10) for the transition probability is quite general. However for very small τ it is usually neither experimentally nor calculationally very accessible. In the cases of present interest, for τ large compared to photon-charged particle interaction times, it is found that T is independent of τ and that simple approximate formulas representing it are easily obtained. The most familiar, and one that is quite adequate for our purposes here is

$$T_{n'\eta',n\eta} = \frac{2\pi}{\hbar} \left|<n'\eta'|H^{PR}|n\eta>\right|^2 \delta(E_{n'}+E_{\eta'}-E_n-E_\eta) . \qquad (I.11)$$

To proceed further, we require an explicit operator representation for the Hamiltonian, H. Since relativistic effects are not expected to be important here, we may take it to be[6]

$$H = \int \frac{E^2+H^2}{8\pi} d^3x + \sum_\sigma \frac{\left(\underline{p}^\sigma - \frac{e_\sigma}{c} A(\underline{x}^\sigma)\right)^2}{2m_\sigma} + V^P . \qquad (I.12)$$

The index σ is a particle index, and the sum runs over all charged particles in the system. The first term in (I.12) is the energy of the free radiation field (i.e., H^R), and V^P is the coulomb potential energy of interaction of the particles in the plasma with each other. In general, $\underline{A}(\underline{x}^\sigma)$ would describe interactions between plasma particles via transverse fields, as well as their interactions with the transverse radiation field imposed from the outside (the incident photon beam as well as any other externally applied fields that might be present). For the purposes of this initial study, we will neglect transverse interactions between plasma particles and assume no external fields except the incident photon beam. It is convenient to rewrite the second term in (I.12) as

$$\sum_\sigma \frac{(\underline{p}^\sigma)^2}{2m_\sigma} - \sum_\sigma \frac{e_\sigma}{2m_\sigma c} (\underline{p}^\sigma \cdot \underline{A}(\underline{x}^\sigma)$$

$$+ A(\underline{x}^\sigma) \cdot \underline{p}^\sigma) + \sum_\sigma \frac{e_\sigma^2}{2m_\sigma c^2} \left|\underline{A}(\underline{x}^\sigma)\right|^2$$

$$= T^P - \int d^3x\, \underline{A}(\underline{x})\circ\sum_\sigma \frac{e_\sigma}{2m_\sigma c}\, (\underline{p}^\sigma\delta(\underline{x}-\underline{x}^\sigma)$$

$$+ \delta(\underline{x}-\underline{x}^\sigma)\underline{p}^\sigma)$$

$$+ \int d^3x\, |\underline{A}(\underline{x})|^2 \sum_\sigma \frac{e_\sigma^2}{2m_\sigma c^2}\, \delta(\underline{x}-\underline{x}^\sigma)$$

$$= T^P - \frac{1}{2c}\int d^3x\, \underline{A}(\underline{x})\circ\underline{J}(\underline{x})$$

$$+ \int d^3x\, |\underline{A}(\underline{x})|^2 \sum_A \frac{e_A^2}{2m_A c^2} \sum_\sigma^{N_A} \delta(\underline{x}-\underline{x}^\sigma) \quad . \tag{I.13}$$

Here we have introduced T^P to represent the kinetic energy of the plasma particles, and $\underline{J}(\underline{x})$ to represent the current density operator, i.e.,

$$J(\underline{x}) = \sum_\sigma \left[\frac{e_\sigma}{m_\sigma}\, \underline{p}^\sigma\delta(\underline{x}-\underline{x}^\sigma) + \delta(\underline{x}-\underline{x}^\sigma)\, \frac{e_\sigma}{m_\sigma}\, \underline{p}^\sigma\right] . \tag{I.14}$$

In the last term in (I.13) we have taken explicit account of the fact that there may be many kinds of charged particles (electrons and several types of ions) in the system. We have labeled the kinds of particles with the index A. Then the symbol $\sum_\sigma^{N_A}$ implies a sum over all particles of the A^{th} kind. Introducing the density operator for particles of the A^{th} kind as

$$\rho^A(\underline{x}) = \sum_\sigma^{N_A} \delta(\underline{x}-\underline{x}^\sigma) \quad , \tag{I.15}$$

and noting that

$$H^P = T^P + V^P \quad , \tag{I.16}$$

we see that we may finally write

$$H = H^R + H^P + H^{PR} \quad ,$$

with

$$H^{PR} = -\frac{1}{2c}\int d^3x\, \underline{A}(\underline{x})\cdot\underline{J}(\underline{x})$$

$$+\int d^3x\,|\underline{A}(\underline{x})|^2 \sum_A \frac{e_A^{\;2}}{2m_A c^2}\, \rho^A(\underline{x}) \;. \tag{I.17}$$

We turn now to the task of calculating the matrix element in Eq. (I.11). In order that this matrix element correspond to a scattering transition, we calculate only

$$\langle n'\eta'_{k'\lambda'}+1\eta'_{k\lambda}-1|H^{PR}|n\eta_{k'\lambda'}\eta_{k\lambda}\rangle \;. \tag{I.18}$$

That is, we demand that the number of photons of wave vector $\underline{k}$ (momentum $\hbar\underline{k}$) and polarization λ in the initial state, $\eta_{\lambda k}$, be decreased by one; and the number of wave vector $\underline{k}'$ and polarization λ', $\eta_{k'\lambda'}$, be increased by one. Since the operator $\underline{A}$ appearing in H^{PR} is a linear function of the photon creation and destruction operators, it is apparent that only the term quadratic in $\underline{A}$ in (I.17) will contribute to (I.18). That is, if we let

$$H^{PR2} = \int d^3x\,|\underline{A}(\underline{x})|^2 \sum_A \frac{e_A^{\;2}}{2m_A c^2}\, \rho^A(\underline{x}) \;, \tag{I.19}$$

then we need only calculate

$$\langle n'\eta'_{k'\lambda'}+1\eta_{k\lambda}-1|H^{PR2}|n\eta_{k'\lambda'}\eta_{k\lambda}\rangle \;. \tag{I.20}$$

Recalling (I.4), we see that (I.20) may be rewritten as

$$\sum_A \frac{e_A^{\;2}}{2m_A c^2}\int d^3x \langle\eta'_{k'\lambda'}+1\eta'_{k\lambda}-1|\underline{A}^2(\underline{x})|\eta_{k'\lambda'}\eta_{k\lambda}\rangle$$

$$(x)\;\;\langle n'|\rho^A(\underline{x})|n\rangle \;. \tag{I.21}$$

The operator $\underline{A}(\underline{x})$ may be written explicitly as a function of photon creation and destruction operators as

$$\underline{A}(\underline{x}) = \sum_{k\lambda}\sqrt{\frac{2\pi\hbar c}{L^3 k}}\; e^{-i\underline{k}\cdot\underline{x}}[\underline{\epsilon}_\lambda(\underline{k})\alpha^\dagger_\lambda(\underline{k}) + \underline{\epsilon}_\lambda(-\underline{k})\alpha_\lambda(-\underline{k})] \;. \tag{I.22}$$

The expansion, (I.22), is in terms of the complete, orthonormal fourier functions in a large, finite cubical region of edge-length L. The discrete values taken on by the components of the vectors, $\underline{k}$, are specified by imposing periodic boundary conditions, i.e.,

$$e^{-ik_j L/2} = e^{+ik_j L/2} ,$$

or

$$e^{ik_j L} = 1 , \qquad (I.23)$$

which implies that

$$k_j = \frac{2\pi}{L} n_j, \quad n_j = 0, \pm 1, \pm 2 \ldots \qquad (I.24)$$

The $\underline{\epsilon}$'s appearing in (I.22) are the unit polarization vectors specified by

$$\underline{\epsilon}_1(\underline{k}) \cdot \underline{\epsilon}_1(\underline{k}) = \underline{\epsilon}_2(\underline{k}) \cdot \underline{\epsilon}_2(\underline{k}) = 1 ,$$

$$\underline{\epsilon}_1(\underline{k}) \cdot \underline{\epsilon}_2(\underline{k}) = 0 ,$$

$$\underline{\epsilon}_1(\underline{k}) \cdot \underline{k} = \underline{\epsilon}_2(\underline{k}) \cdot \underline{k} = 0 . \qquad (I.25)$$

The quantities $\alpha^+_\lambda(\underline{k})$ and $\alpha_\lambda(\underline{k})$ are the creation and destruction operators for photons of wavevector $\underline{k}$ and polarization λ respectively. Making use of the facts that

$$\alpha_\lambda(\underline{k}) \, |\eta_{k\lambda}> = \sqrt{\eta_{k\lambda}} \, |\eta_{k\lambda} - 1> ,$$

$$\alpha^+_\lambda(\underline{k}) \, |\eta_{k\lambda}> = \sqrt{\eta_{k\lambda}+1} \, |\eta_{k\lambda}+1> , \qquad (I.26)$$

it is easy to show that the radiation matrix element in (I.21) becomes

$$<\eta'_{k'\lambda'}+1 \ \eta'_{k\lambda}-1 | \underline{A}^2(\underline{x}) | \eta_{k'\lambda'} \ \eta_{k\lambda}>$$

$$= \frac{4\pi\hbar c}{L^3 \sqrt{kk'}} \sqrt{\eta_{k\lambda}(\eta_{k'\lambda'}+1)} \, \underline{\epsilon}_{\lambda'}(\underline{k}') \cdot \underline{\epsilon}_\lambda(\underline{k}) \, e^{i\underline{x}\cdot(\underline{k}-\underline{k}')} . \qquad (I.27)$$

Entering this result into (I.21), we find that (I.20) becomes

$$<n'\eta'_{k'\lambda'}+1 \ \eta'_{k\lambda}-1 | H^{PR2} | n\eta_{k'\lambda'} \ \eta_{k\lambda}>$$

$$= \sum_A \frac{e_A^2}{2m_A c^2} \frac{4\pi\hbar c}{L^3 \sqrt{kk'}} \, \underline{\epsilon}_{\lambda'}(\underline{k}') \cdot \underline{\epsilon}_\lambda(\underline{k}) \sqrt{\eta_{k\lambda}(\eta_{k'\lambda'}+1)}$$

$$(x) \int d^3x <n' | \rho^A(\underline{x}) | n> e^{i\underline{x}\cdot(\underline{k}-\underline{k}')}$$

$$= \sum_A \frac{2\pi\hbar e_A^2}{m_A c L^3 \sqrt{kk'}} \sqrt{\eta_{k\lambda}(\eta_{k'\lambda'}+1)} \cdot \underline{\epsilon}_\lambda(\underline{k}) \cdot \underline{\epsilon}_{\lambda'}(\underline{k}') <n' | \bar{\rho}^A(\underline{\kappa}) | n> , \qquad (I.28)$$

where we have introduced the fourier transform of the density operator

$$\bar{\rho}^A(\underline{\kappa}) = \int d^3x\, e^{i\underline{x}\cdot\underline{\kappa}} \rho^A(\underline{x}), \quad \underline{\kappa} = \underline{k}-\underline{k}' \,. \qquad (I.29)$$

We observe that the terms in Eq. (I.28) are inversely proportional to the mass of the scattering particle. Since the mass of the ions is >1800 times the mass of the electrons, we surmise that all of the observable scattering will be by the electrons and approximate (I.28) by

$$\frac{2\pi\hbar e^2}{m_e c L^3 \sqrt{kk'}} \sqrt{\eta_{k\lambda}(\eta_{k'\lambda'}+1)}\; \underline{\epsilon}_\lambda(\underline{k})\cdot\underline{\epsilon}_{\lambda'}(\underline{k}')\langle n'|\bar{\rho}^e(\underline{\kappa})|n\rangle \,. \qquad (I.30)$$

We now display the transition probability in Eq. (I.11), which represents the probability per unit time that the plasma will undergo a transition from state $|n\rangle$ to state $|n'\rangle$ while scattering a photon from state $(\underline{k}\lambda)$ to state $(\underline{k}'\lambda')$ as

$$T_{n'n}(\underline{k}',\underline{k}) = \frac{2\pi}{\hbar}\frac{(2\pi)^2\hbar^2 e^4}{m_e^2 c^2 L^6 kk'} \eta_{k\lambda}(\eta_{k'\lambda'}+1)$$

$$(x)\; |\underline{\epsilon}_\lambda(\underline{k})\cdot\underline{\epsilon}_{\lambda'}(\underline{k}')|^2 |\langle n'|\bar{\rho}^e(\underline{\kappa})|n\rangle|^2$$

$$(x)\; \delta(E_{n'}+\hbar ck'-E_n-\hbar ck) \,. \qquad (I.31)$$

At this point we must sum over unobserved final states and average over the probabilities of finding the system in the specified initial states. Furthermore we must also sum over a small spread of final photon wave vectors, i.e.,

$$\sum_{\underline{k}'\epsilon d^3k'} T_{n'n}(\underline{k}',\underline{k}) \Rightarrow \left(\frac{L}{2\pi}\right)^3 d^3k'\; T_{n'n}(\underline{k}',\underline{k}) \,. \qquad (I.32)$$

Multiplying the right hand side of (I.32) by $D_{n\eta,n\eta}$, performing the sums indicated above, and changing variables according to $\omega = ck$ and $\underline{\Omega} = \underline{k}/k$ we obtain for the numerator of Eq. (I.1)

$$T(\omega,\underline{\Omega};\, \omega',\underline{\Omega}')d\omega' d\Omega'$$

$$= \frac{c}{L^3} r_e^2 \frac{\omega'}{\omega} d\omega' d\Omega' \sum_{\lambda\lambda'} f(\underline{k},\lambda)\,[f(\underline{k}'\lambda')+1]$$

$$(x)\; |\underline{\epsilon}_\lambda(\underline{k})\cdot\underline{\epsilon}_{\lambda'}(\underline{k}')|^2 \sum_{nn'} |\langle n'|\bar{\rho}^e(\underline{\kappa})|n\rangle|^2 P_n$$

$$(x)\ \delta(\omega_{n'}+\omega'-\omega_n-\omega),\ r_e \equiv \frac{e^2}{m_e c^2}. \tag{I.33}$$

In going from (I.31) and (I.32) to (I.33) we have approximated

$$D_{n\eta,n\eta} \simeq D_{nn}D_{\eta\eta}, \tag{I.34}$$

and then have identified D_{nn} as P_n, which at this stage in the analysis is the probability of finding the plasma in the state $|n\rangle$ and may be changing with time in an arbitrary manner. We have also approximated

$$\sum_{\eta} D_{\eta\eta}\eta_{k\lambda}(\eta_{k'\lambda'}+1)\,|\underline{\epsilon}_{\lambda}(\underline{k})\cdot\underline{\epsilon}_{\lambda'}(\underline{k}')|^2$$

$$\simeq \sum_{\lambda\lambda'} f(\underline{k}\lambda)\,[f(\underline{k}'\lambda')+1]\,|\underline{\epsilon}_{\lambda}(\underline{k})\cdot\underline{\epsilon}_{\lambda'}(\underline{k}')|^2, \tag{I.35}$$

which involves the replacement of an average of a product by the product of the averages. We have also assumed that the detection process for counting scattered photons does not discriminate between polarization states. Note that $f(\underline{k},\lambda)$ is the expected number of photons in the incident beam having wave-vector $\underline{k}$ and polarization λ. If now we assume further that the incident photons are randomly polarized, we have $f(\underline{k},\lambda) = 1/2\ f(\underline{k})$. Furthermore the incident beam intensity is just $cf(\underline{k})$. Finally we assume that stimulated scattering is negligible in these experiments so that

$$[f(\underline{k}'\lambda')+1] \simeq 1. \tag{I.36}$$

Introducing these assumptions into (I.33) and dividing by the number of electrons in the plasma and the incident beam intensity as prescribed by Eq. (I.1), we find for the differential cross-section

$$\sigma(\omega,\underline{\Omega};\ \omega',\underline{\Omega}')\,d\omega' d\Omega'$$

$$= \frac{1}{2N_e} r_e \sum_{\lambda\lambda'} |\underline{\epsilon}_{\lambda}(\underline{k})\cdot\underline{\epsilon}_{\lambda'}(\underline{k}')|\ \frac{\omega'}{\omega}\ d\omega' d\Omega'$$

$$(x) \sum_{nn'} |\langle n'|\bar{\rho}^{e}(\underline{k})|n\rangle|^2 P_n \delta(\omega_{n'}+\omega'-\omega_n-\omega). \tag{I.37}$$

We have also introduced the notation, $\omega_n = E_n/\hbar$. The quantity

$$\frac{1}{2} r_e^2 \sum_{\lambda\lambda'} |\underline{\epsilon}_\lambda(\underline{k}) \circ \underline{\epsilon}_{\lambda'}(\underline{k}')|^2 = \frac{1}{2} r_e^2 (1 + \cos^2 \theta) , \tag{I.38}$$

where r_e is the classical radius of the electron and $\cos \theta = \underline{k} \circ \underline{k}'/kk'$. Equation (I.38) is the Thomson cross-section which we will henceforth designate as $\sigma_T(\theta)$. Thus finally we may write (I.37) as

$$\sigma(\omega, \underline{\Omega};\ \omega', \underline{\Omega}') = \frac{\omega'}{\omega} \sigma_T(\theta)$$

$$(x)\ \frac{1}{N_e} \sum_{nn'} |\langle n'|\bar{\rho}^e\ (\underline{\kappa})|n\rangle|^2 \delta(\omega_{n'} + \omega' - \omega_n - \omega) P_n . \tag{I.39}$$

At this point we introduce the notation $\omega - \omega' = \Delta\omega$, and the scattering function defined by

$$S(\underline{\kappa}, \Delta\omega) = \frac{1}{N_e} \sum_{nn'} |\langle n'|\bar{\rho}^e\ (\underline{\kappa})|n\rangle|^2 P_n \delta(\omega_{n'} - \omega_n - \Delta\omega) . \tag{I.40}$$

The cross-section then reads

$$\sigma(\underline{\omega}, \underline{\Omega};\ \omega' \underline{\Omega}') = \frac{\omega'}{\omega} \sigma_T(\theta) S(\underline{\kappa}, \Delta\omega) , \tag{I.41}$$

and evidently the remainder of our task is the calculation of the scattering function which depends only on the statistical and dynamical properties of the scattering system.

We observe at this point that we have largely accomplished the end pointed toward in this section, i.e., that of relating the cross-section to a well-defined statistical function dependent solely upon the parameters and dynamics of the scattering sample. However, before proceeding to the next section in which we describe a classical calculation of the autocorrelation of the electron density fluctuations; we must give some consideration to the sense in which the function defined in (I.40) can be calculated in classical terms at all. To facilitate further discussion here, we will assume that the plasma is in the thermodynamic state, i.e.,

$$P_n = z^{-1} e^{-E_n/\theta} , \tag{I.42}$$

where θ is the plasma temperature in units of energy. If this condition is not fulfilled, then the question of approximate classical calculations must be reviewed for any given case. Given the condition (I.42), it is easy to show that

$$S(\underline{\kappa}, -\Delta\omega) = e^{-\hbar\Delta\omega/\theta} S(\underline{\kappa}, \Delta\omega) , \qquad (I.43)$$

which, upon reflection, is seen to be necessary if the cross-section given in (I.41) is to satisfy detailed balance. However, as we will see later, the classical calculation yields a scattering function that is symmetric under the transformation $\Delta\omega \to -\Delta\omega$. Thus we are alerted to the necessity of examining with some care the nature of the connection between (I.40) and any classical approximation thereto. Following VanHove[7] we rewrite (I.40) according to

$$S(\underline{\kappa}, \Delta\omega) = \frac{1}{2\pi N_e} \int dte^{-it\Delta\omega}$$

$$(x) \sum_{nn'} P_n e^{it(\omega_{n'} - \omega_n)} \langle n' | \bar{\rho}^e | n \rangle^*$$

$$(x) \langle n' | \bar{\rho}^e | n \rangle$$

$$= \frac{1}{2\pi N_e} \int dte^{-it\Delta\omega} \sum_{nn'} P_n \langle n | \bar{\rho}^{e\dagger} e^{itH^P/\hbar} | n' \rangle$$

$$(x) \langle n' | \bar{\rho}^e e^{-itH^P/\hbar} | n \rangle$$

$$= \frac{1}{2\pi N_e} \int dte^{-it\Delta\omega} \sum_{n} P_n \langle n | \bar{\rho}^{e\dagger} e^{itH^P/\hbar} \bar{\rho}^e e^{-itH^P/\hbar} | n \rangle$$

$$= \frac{1}{2\pi N_e} \int dte^{-it\Delta\omega} \chi(\underline{\kappa}, t) , \qquad (I.44)$$

where we have defined

$$\chi(\underline{\kappa}, t) \equiv \sum_{n} P_n \langle n | \rho^{-e\dagger}(\underline{\kappa}, o) \bar{\rho}^e (\underline{\kappa}, t) | n \rangle . \qquad (I.45)$$

Again, since we are considering the thermodynamic state of the plasma, we have

$$P_n = z^{-1} e^{-E_n/\theta}$$

$$= z^{-1} \langle n | e^{-H^P/\theta} | n \rangle$$

$$= D_{nn} , \qquad (I.46)$$

and also

$$D_{nn'} = 0, \quad n \neq n' . \qquad (I.47)$$

Thus, for this case, we may write

$$\chi(\underline{\kappa},t) = \mathrm{Tr}\ \bar{\rho}^{e\dagger}(\underline{\kappa},o)\bar{\rho}^{e}(\underline{\kappa},t)D , \qquad (I.48)$$

or simply

$$\chi(\underline{\kappa},t) = \langle\bar{\rho}^{e\dagger}(\underline{\kappa},o)\bar{\rho}^{e}(\underline{\kappa},t)\rangle_T , \qquad (I.49)$$

where $\langle\ \rangle_T$ means thermal average. With χ expressed as in (I.49), the interpretation of $S(\underline{\kappa},\Delta\omega)$ as the space-time fourier transform of the electron density fluctuations becomes evident.

According to Eq. (I.49), χ is the thermal average of a certain function of the dynamical variables of the plasma which satisfy the quantum mechanical equations of motion. The question as to how this is to be related to a calculation of a thermal average of the same function of the system dynamical variables assuming they satisfy the classical equations of motion is highly non-trivial and has been carefully examined elsewhere.[8,9] We will not discuss this problem in detail here, because the difference between the quantum and classical interpretation does not appear to be quantitatively significant for the interpretation of the experiments of interest here. We emphasize it, however, because it is important in principle and because in some future circumstance it may be important in practice.

II. CLASSICAL DERIVATION OF THE SCATTERING FUNCTION

In this section we turn our attention to the derivation of the scattering function as displayed in Eqs. (I.44) and (I.45). As indicated in Eqs. (I.12) and (I.13), we consider the plasma Hamiltonian to be

$$H^P = T^P + V^P . \tag{II.1}$$

Also as discussed in Section I, we will assume that the dynamical variables of the plasma obey the classical equations of motion, i.e.,

$$\overset{\circ}{x}{}^{\sigma}_{j} = \frac{\partial H}{\partial p^{\sigma}_{j}} = \{x^{\sigma}_{j}, H\} , \tag{II.2a}$$

$$\overset{\circ}{p}{}^{\sigma}_{j} = -\frac{\partial H}{\partial x^{\sigma}_{j}} = \{p^{\sigma}_{j}, H\} . \tag{II.2b}$$

The symbol { } means Poisson bracket, and for A any function of the system dynamical variables

$$\{A,H\} \equiv \sum_{\sigma} \left(\frac{\partial A}{\partial x^{\sigma}_{j}} \frac{\partial H}{\partial p^{\sigma}_{j}} - \frac{\partial A}{\partial p^{\sigma}_{j}} \frac{\partial H}{\partial x^{\sigma}_{j}} \right) . \tag{II.3}$$

To facilitate manipulations later on, we introduce the Poisson bracket operator L, so defined that

$$LA \equiv \{H,A\} . \tag{II.4}$$

Thus it follows that

$$\overset{\circ}{A} = -LA , \tag{II.5}$$

and

$$A(t) = e^{-tL} A(o) , \tag{II.6}$$

where the operator e^{-tL} is defined by its power series expansion, and

$$L^2 A \equiv \{H, \{H,A\}\} ,$$

$$L^3 A \equiv \{H, \{H, \{H,A\}\}\} , \tag{II.7}$$

etc.

Let $D(\underline{x}^1 \ldots \underline{x}^N \underline{p}^1 \ldots \underline{p}^N, t)$ be the Liouville function for the system, i.e.,

$$D(\underline{x}^\sigma, \underline{p}^\sigma, t) \prod_\sigma^N d^3x^\sigma d^3p^\sigma$$

$\equiv D d\tau =$ The probability of finding the system in a state in which particle (1) is in $d^3x^1 d^3p^1$, particle (2) is in $d^3x^2 d^3p^2$, etc. Then D satisfies the Liouville equation,[10]

$$\frac{\partial D}{\partial t} = LD \ , \qquad \text{(II.8)}$$

which has the solution

$$D(t) = e^{tL} D(o) \ . \qquad \text{(II.9)}$$

The time dependent, configuration space electron density operators whose fourier transforms appear in (I.45) are defined by (recall (I.15))

$$\rho^e(\underline{x}, t) = \sum_\sigma^{N_e} \delta(\underline{x} - \underline{x}^\sigma(t))$$

$$= e^{-tL} \sum_\sigma^{N_e} \delta(\underline{x} - \underline{x}^\sigma(o)) \ . \qquad \text{(II.10)}$$

We will carefully distinguish between the dynamical variables of the system—the $\underline{x}^\sigma$'s and $\underline{p}^\sigma$'s which are the phase variables of the particles and are generally inaccessible to observation—and the laboratory variables, $(\underline{x}, \underline{p}, t)$ which are under control of the observer. Any quantity which depends upon the system variables we will call an operator. The average of an operator will of course depend only upon laboratory variables. For example, the measurable, time-dependent electron density in the plasma is given by

$$n^e(\underline{x}, t) = \int d\tau \rho^e(\underline{x}, o) D(t)$$

$$= \int d\tau \rho^e(\underline{x}, t) D(o) \ . \qquad \text{(II.11)}$$

The quantity which we must calculate is $\chi(\underline{\kappa}, t)$ as given in (I.49), but which we here identify as

$$\chi(\underline{\kappa}, t) = \int d\tau \bar{\rho}^{e*}(\underline{\kappa}, o) \bar{\rho}^e(\underline{\kappa}, t) D_T \ , \qquad \text{(II.12)}$$

where D_T is the thermodynamic Liouville function

$$D_T = Z^{-1} e^{-H^P/\theta} \ , \qquad \text{(II.13a)}$$

$$z = \int d\tau e^{-H^P/\theta} . \qquad (II.13b)$$

The asterisk appearing in (II.12) implies complex conjugation. Again we will use the shorthand notation

$$\int d\tau \rho^{-e^*}(\underline{\kappa},o)\bar{\rho}^e(\underline{\kappa},t)D_T$$

$$\equiv \langle\rho^{-e^*}(\underline{\kappa},o)\bar{\rho}^e(\underline{\kappa},t)\rangle . \qquad (II.14)$$

The quantity in (II.14) may also be displayed as

$$\langle\rho^{-e^*}(\underline{\kappa},o)\bar{\rho}^e(\underline{\kappa},t)\rangle$$

$$= \int d^3x d^3x' e^{i\underline{\kappa}\cdot(\underline{x}-\underline{x}')}\langle\rho^e(\underline{x}',o)\rho^e(\underline{x},t)\rangle , \qquad (II.15)$$

as well as

$$\int d^3x d^3v d^3x' d^3v' e^{i\underline{\kappa}\cdot(\underline{x}-\underline{x}')}$$

$$(x) \langle g^e(\underline{x}',\underline{v}',o)g^e(\underline{x},\underline{v},t)\rangle , \qquad (II.16)$$

where the phase-space-density operator g^e is defined by

$$g^e(\underline{x},\underline{v},t) = e^{-tL}\sum_{\sigma}^{N_e} \delta(\underline{x}-\underline{x}^\sigma)\delta(\underline{v}-\underline{v}^\sigma) . \qquad (II.17)$$

The present calculational scheme is directed toward the calculation of the average appearing in Eq. (II.16).

If the Hamiltonian of the plasma is taken to be

$$H^P = \sum_{\sigma}^{N_e+N_i} \frac{(\underline{p}^\sigma)^2}{2m_\sigma} + \frac{1}{2}\sum_{\sigma,\alpha}^{N_e} V^{ee}(|\underline{x}^\sigma-\underline{x}^\alpha|)$$

$$+ \frac{1}{2}\sum_{\sigma,\alpha}^{N_i} V^{ii}(|\underline{x}^\sigma-\underline{x}^\alpha|) + \sum_{\sigma,\alpha}^{N_e N_i} V^{ei}(|\underline{x}^\sigma-\underline{x}^\alpha|) , \qquad (II.18)$$

then it is a straight-forward matter to show that $g^e(\underline{x},\underline{v},t)$ satisfies the equation

$$\frac{\partial g^e}{\partial t} + v_j \frac{\partial g^e}{\partial x_j} - \frac{1}{m_e}\frac{\partial g^e}{\partial v_j}\int d^3x'd^3v' \frac{\partial V^{ee}(|\underline{x}-\underline{x}'|)}{\partial x_j} g^e(\underline{x}',\underline{v}')$$

$$-\frac{1}{m_e}\frac{\partial g^e}{\partial v_j}\int d^3x'd^3v' \frac{\partial V^{ei}(|\underline{x}-\underline{x}'|)}{\partial x_j} g^i(\underline{x}',\underline{v}') = 0 . \qquad (II.19)$$

Since the equation for the electron density operator depends upon the ion density operator, it is evident that an equation for the latter is needed also. It is trivially obtained from Eq. (II.19) by merely interchanging $e \rightleftarrows i$. Now let the average of g^e be f^e, i.e.,

$$f^e(\underline{x},\underline{v},t) = \langle g^e(\underline{x},\underline{v},t)\rangle , \qquad (II.20)$$

and further let the fluctuation operator, δg^e, be defined by

$$g^e(\underline{x},\underline{v},t) \equiv f^e(\underline{x},\underline{v},t) + \delta g^e(\underline{x},\underline{v},t) . \qquad (II.21)$$

The equation for f^e is trivially obtained from (II.19), and by differencing it and Eq. (II.19) one obtains an equation for δg^e, i.e.,

$$\frac{\partial \delta g^e}{\partial t} + v_j \frac{\partial \delta g^e}{\partial x_j} - \frac{1}{m_e}\frac{\partial f^e}{\partial v_j}\int d^3x'd^3v' \frac{\partial V^{ee}}{\partial x_j}\delta g^e(\underline{x}'\underline{v}')$$

$$-\frac{1}{m_e}\frac{\partial \delta g^e}{\partial v_j}\int d^3x'd^3v' \frac{\partial V^{ee}}{\partial x_j} f^e(\underline{x}'\underline{v}')$$

$$-\frac{1}{m_e}\frac{\partial f^e}{\partial v_j}\int d^3x'd^3v' \frac{\partial V^{ei}}{\partial x_j}\delta g^i(\underline{x}'\underline{v}')$$

$$-\frac{1}{m_e}\frac{\partial \delta g^e}{\partial v_j}\int d^3x'd^3v' \frac{\partial V^{ei}}{\partial x_j} f^i(\underline{x}'\underline{v}')$$

$$= \frac{1}{m_e}\int d^3x'd^3v' \frac{\partial V^{ee}}{\partial x_j}\frac{\partial}{\partial v_j}[\delta g^e(\underline{xv})\delta g^e(\underline{x}',\underline{v}') - \langle\delta g^e(\underline{xv})\delta g^e(\underline{x}',\underline{v}')\rangle]$$

$$+ \frac{1}{m_e}\int d^3x'd^3v' \frac{\partial V^{ei}}{\partial x_j}\frac{\partial}{\partial v_j}[\delta g^e(\underline{xv})\delta g^i(\underline{x}',\underline{v}') - \langle\delta g^e(\underline{xv})\delta g^i(\underline{x}',\underline{v}')\rangle] . \qquad (II.22)$$

Again, the equation for δg^i is obtained from (II.22) by interchanging $e \rightleftarrows i$.

At this point, as approximate equation for the fluctuation operators is obtained by neglecting the terms quadratic in these operators. (A substantial discussion of the possible meaning of this approximation is presented in Reference 4.) Equation (II.22), and its counterpart for the ion fluctuations, is then further simplified by assuming that the target plasma is in the thermodynamic state, and hence that the singlet densities, f^e and f^i, are independent of space and time and are Maxwellian functions of velocity. In view of the approximation and the assumption the equation for the fluctuation operator becomes

$$\frac{\partial \delta g^e}{\partial t} + v_j \frac{\partial \delta g^e}{\partial x_j} - \frac{1}{m_e} \frac{\partial f^e}{\partial v_j} \int d^3x' d^3v' \frac{\partial V^{ee}}{\partial x_j} \delta g^e(\underline{x}'\underline{v}')$$

$$- \frac{1}{m_e} \frac{\partial f^e}{\partial v_j} \int d^3x' d^3v' \frac{\partial V^{ei}}{\partial x_j} \delta g^i(\underline{x}'\underline{v}') = 0. \qquad \text{(II.23)}$$

Our task now is to solve this linear, homogeneous equation (actually the coupled pair of equations for the electron and ion fluctuation operators) to obtain the electron fluctuation operator at time t in terms of its initial value and that of ion fluctuation operator. Then, by virtue of (II.21), it is seen that the calculation of (II.16) is thereby reduced to a calculation of the static two-particle correlation functions.

The solution of (II.23) and its companion equation is accomplished easily by means of a Fourier transformation in configuration space and a Laplace transformation on the time. Rather than introduce special symbols for the transformed functions, we will let the arguments of the functions specify the space in which they are to be evaluated, i.e.,

$$\delta g^e(\underline{x},\underline{v},s) = \int_0^\infty dt e^{-st} \delta g^e(\underline{x},\underline{v},t) , \qquad \text{(II.24a)}$$

$$\delta g^e(\underline{\kappa},\underline{v},s) = \int d^3x e^{i\underline{\kappa}\cdot\underline{x}} \delta g^e(\underline{x},\underline{v},s) . \qquad \text{(II.24b)}$$

Performing the indicated transformations on the equations for the fluctuation operators, one obtains

$$\begin{aligned}
&(s - i\underline{\kappa}\cdot\underline{v})\delta g^e(\underline{\kappa},\underline{v},s) \\
&+ \frac{iV^{ee}(\underline{\kappa})\underline{\kappa}\cdot\underline{\nabla}_v f^e}{m_e} \int d^3v' \delta g^e(\underline{\kappa},\underline{v}',s) \\
&+ \frac{iV^{ei}(\underline{\kappa})\underline{\kappa}\cdot\underline{\nabla}_v f^e}{m_e} \int d^3v' \delta g^i(\underline{\kappa},\underline{v}',s) \\
&= \delta g^e(\underline{\kappa},\underline{v},o) , \qquad \text{(II.25a)}
\end{aligned}$$

and

$$(s-i\underline{\kappa}\cdot\underline{v})\delta g^i(\underline{\kappa},\underline{v},s)$$

$$+\ \frac{iV^{ii}(\underline{\kappa})\underline{\kappa}\cdot\underline{\nabla}_v f^i}{m_i}\int d^3v'\delta g^i(\underline{\kappa},\underline{v}',s)$$

$$+\ \frac{iV^{ie}(\underline{\kappa})\underline{\kappa}\cdot\underline{\nabla} f^i}{m_i}\int d^3v'\delta g^e(\underline{\kappa},\underline{v}',s)$$

$$= \delta g^i(\underline{\kappa},\underline{v},o)\ . \tag{II.25b}$$

Dividing both equations by $(s-i\underline{\kappa}\cdot\underline{v})$, integrating over velocity, and introducing

$$h^{e,i}(\underline{\kappa},s) \equiv \int d^3v\delta g^{e,i}(\underline{\kappa},\underline{v},s)\ , \tag{II.26}$$

we obtain the coupled pair of algebraic equations,

$$(1+\Lambda_{ee})h^e + \Lambda_{ei}h^i = \int \frac{d^3v\delta g^e(\underline{\kappa},\underline{v},o)}{s-i\underline{\kappa}\cdot\underline{v}}\ ,$$

$$(1+\Lambda_{ii})h^i + \Lambda_{ie}h^e = \int \frac{d^3v\delta g^i(\underline{\kappa},\underline{v},o)}{s-i\underline{\kappa}\cdot\underline{v}}\ . \tag{II.27}$$

In (II.27) we have introduced the notation,

$$\Lambda_{rk} \equiv \int \frac{i\underline{\kappa}\cdot\underline{\nabla}_v f^r V^{rk}(\underline{\kappa})}{m_r(s-i\underline{\kappa}\cdot\underline{v})}\, d^3v\ . \tag{II.28}$$

It is now a simple matter to obtain

$$h^e(\underline{\kappa},s) = \frac{1+\Lambda_{ii}}{\epsilon}\int\frac{\delta g^e(\underline{\kappa},\underline{v},o)d^3v}{s-i\underline{\kappa}\cdot\underline{v}}$$

$$-\ \frac{\Lambda_{ei}}{\epsilon}\int\frac{\delta g^i(\underline{\kappa},\underline{v},o)d^3v}{s-i\underline{\kappa}\cdot\underline{v}}\ , \tag{II.29}$$

where we have introduced

$$\epsilon(\underline{\kappa},s) \equiv 1 + \Lambda_{ee}(\underline{\kappa},s) + \Lambda_{ii}(\underline{\kappa},s)\ , \tag{II.30}$$

and have made use of the fact that

$$\Lambda_{ee}\Lambda_{ii} = \Lambda_{ei}\Lambda_{ie} . \qquad (II.31)$$

Finally, inverting the Laplace transformation and noting Eq. (II.26), we display

$$\int d^3v\delta g^e(\underline{\kappa},\underline{v},t)$$

$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds e^{st}\left[\frac{1+\Lambda_{ii}}{\epsilon}\int\frac{\delta g^e(\underline{\kappa},\underline{v},o)d^3v}{s-i\underline{\kappa}\cdot\underline{v}}\right.$$

$$\left.-\frac{\Lambda_{ei}}{\epsilon}\int\frac{\delta g^i(\underline{\kappa},\underline{v},o)d^3v}{s-i\underline{\kappa}\cdot\underline{v}}\right] . \qquad (II.32)$$

Recall now Eq. (II.16), which, according to (II.21), we may rewrite as

$$\chi(\underline{\kappa},t) = \int d^3xd^3x'd^3v'd^3v\; e^{i\kappa\cdot(\underline{x}-\underline{x}')}$$

$$(x)\;[f^e(\underline{v})f^e(\underline{v}')+\langle\delta g^e(\underline{x}',\underline{v}',o)\delta g^e(\underline{x},\underline{v},t)\rangle]$$

$$= (2\pi)^3n^eN^e\delta(\underline{\kappa})$$

$$+ \int d^3v'\langle\delta g^{e*}(\underline{\kappa},\underline{v}',o)\int d^3v\delta g^e(\underline{\kappa},\underline{v},t)\rangle . \qquad (II.33)$$

Entering (II.32) into (II.33) we obtain

$$\chi(\underline{\kappa},t) = (2\pi)^3n^eN^e\delta(\underline{\kappa})$$

$$+ \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds e^{st}\int\frac{d^3vd^3v'}{s-i\underline{\kappa}\cdot\underline{v}}$$

$$(x)\left[\frac{1+\Lambda_{ii}}{\epsilon}\langle\delta g^{e*}(\underline{\kappa},\underline{v}',o)\delta g^e(\underline{\kappa},\underline{v},o)\rangle\right.$$

$$\left.-\frac{\Lambda_{ei}}{\epsilon}\langle\delta g^{e*}(\underline{\kappa},\underline{v}',o)\delta g^i(\underline{\kappa},\underline{v},o)\rangle\right] . \qquad (II.34)$$

In Eqs. (II.33) and (II.34) we have introduced the notation

$$n^e = \int d^3vf^e(\underline{v}) , \qquad (II.35a)$$

and

$$N^e = n^e \times (\text{Volume of scattering plasma}) \, . \qquad \text{(II.35b)}$$

At this point it is desirable to estimate the thermal averages appearing in Eq. (II.34). This, however, is not a task that we need to undertake here, since it has been accomplished elsewhere. For example, Montgomery and Tidman[11] describe a calculation suitable to our purposes here in complete detail. Their results for the correlation and cross-correlation functions in phase space are

$$\begin{aligned} &\langle \delta g^e(\underline{x}',\underline{v}',o)\,\delta g^e(\underline{x},\underline{v},o)\rangle \\ &= \delta(\underline{x}-\underline{x}')\,\delta(\underline{v}-\underline{v}')\,f^e(\underline{v}) \\ &- f^e(\underline{v})f^e(\underline{v}')\,\frac{e^2}{\theta}\,\frac{e^{-|\underline{x}-\underline{x}'|/\lambda_D}}{|\underline{x}-\underline{x}'|} \, , \end{aligned} \qquad \text{(II.36a)}$$

and

$$\begin{aligned} &\langle \delta g^e(\underline{x}',\underline{v}',o)\,\delta g^i(\underline{x},\underline{v},o)\rangle \\ &= f^e(\underline{v}')f^i(\underline{v})\,\frac{z_i e^2}{\theta}\,\frac{e^{-|\underline{x}-\underline{x}'|/\lambda_D}}{|\underline{x}-\underline{x}'|} \, , \end{aligned} \qquad \text{(II.36b)}$$

where the Debye length, λ_D, is given by

$$\lambda_D^{-2} = \frac{4\pi n^e e^2}{\theta} + \frac{4\pi n^i z_i^2 e^2}{\theta} \, . \qquad \text{(II.37)}$$

Quite generally, in order that there be charge neutrality in the thermodynamic plasma,

$$n^e = z_i n^i \equiv n. \qquad \text{(II.38)}$$

Thus we may display (II.37) as

$$\lambda_D^{-2} = \frac{4\pi n e^2}{\theta}(1+z_i) \, . \qquad \text{(II.39)}$$

The extra (singular) term in (II.36a) relative to (II.36b) arises because Montgomery and Tidman calculate an electron-electron correlation function from which the singular term has been explicitly removed. The Fourier transforms of (II.36a) and II.36b), which are the quantities required in (II.34), are

$$\langle \delta g^{e*}(\underline{\kappa},\underline{v}',o)\delta g^{e}(\underline{\kappa},\underline{v},o)\rangle$$

$$\equiv \int d^3x d^3x' e^{i\underline{\kappa}\cdot(\underline{x}-\underline{x}')}\langle \delta g^{e}(\underline{x}',\underline{v}',o)\delta g^{e}(\underline{x},\underline{v},o)\rangle$$

$$= N^e\delta(\underline{v}-\underline{v}')M^e(\underline{v})$$

$$- M^e(\underline{v})M^e(\underline{v}')nN^e \frac{e^2}{\theta} \int d^3R \frac{e^{i\underline{\kappa}\cdot\underline{R}-R/\lambda_D}}{R}$$

$$= N^e\delta(\underline{v}-\underline{v}')M^e(\underline{v})$$

$$- N^eM^e(\underline{v})M^e(\underline{v}') \frac{(4\pi ne^2/\theta)\lambda_D^2}{1+(\kappa\lambda_D)^2}, \qquad \text{(II.37a)}$$

and

$$\langle \delta g^{e*}(\underline{\kappa},\underline{v}',o)\delta g^{e}(\underline{\kappa},\underline{v},o)\rangle$$

$$= N^eM^e(\underline{v}')M^i(\underline{v}) \frac{(4\pi ne^2/\theta)\lambda_D^2}{1+(\kappa\lambda_D)^2}, \qquad \text{(II.37b)}$$

where

$$M^{e,i}(\underline{v}) = f^{e,i}(\underline{v})/n^{e,i}. \qquad \text{(II.38)}$$

Entering (II.37a) and (II.37b) into (II.34) and neglecting the forward scattering term (term containing $\delta(\underline{\kappa})$), we find that

$$\chi(\underline{\kappa},t) = \frac{N^e}{2\pi\kappa}\int_{\sigma-i\infty}^{\sigma+i\infty} ds e^{st}\left[\left(\frac{1+\Lambda_{ii}}{\epsilon}\right)(1-X)\Phi^e\left(-\frac{is}{\kappa}\right)\right.$$

$$\left. - \frac{\Lambda_{ei}}{\epsilon} X\Phi^i\left(-\frac{is}{\kappa}\right)\right], \qquad \text{(II.39)}$$

where we have defined

$$\Phi^{e,i}(z) = \int_{-\infty}^{\infty} \frac{du M^{e,i}(u)}{u-z}, \qquad \text{(II.40)}$$

$$M^{e,i}(u) = \left(\frac{m_{e,i}}{2\pi\theta}\right)^{1/2} e^{-m_{e,i}u^2/2\theta} , \qquad (II.41)$$

and

$$X = \frac{(4\pi ne^2/\theta)\lambda_D^2}{1+(\kappa\lambda_D)^2}$$

$$= \frac{1}{(1+z_i)[1+(\kappa\lambda_D)^2]} . \qquad (II.42)$$

Recall now Eq. (I.44) which relates the scattering function to the quantity appearing in Eq. (II.39). Since the scattering cross-section is proportional to the scattering function and the function of proportionality is explicitly real, the scattering function must also be real. According to (I.44), this implies that

$$\chi^*(\underline{\kappa},t) = \chi(\underline{\kappa},-t) . \qquad (II.43)$$

That this is so is readily shown from the original definition of χ, Eq. (I.45), as well as from its classical approximation, Eq. (II.12). However, the property (II.43) is not at all evident in the formula (II.39). To circumvent a possible difficulty here we note that the scattering function may trivially be displayed in manifestly real form as

$$S(\underline{\kappa},\Delta\omega) = \frac{1}{4\pi N^e} \int dt e^{-i\Delta\omega t} [\chi(\underline{\kappa},t)+\chi^*(\underline{\kappa},-t)]$$

$$= \frac{1}{2\pi N^e} \text{Re} \int dt e^{-i\Delta\omega t} \chi(\underline{\kappa},t) . \qquad (II.44)$$

For temporary convenience, we define

$$Q(\underline{\kappa},s) \equiv \frac{1+\Lambda_{ii}}{\epsilon}(1-X)\Phi^e\left(-\frac{is}{\kappa}\right) - \frac{\Lambda_{ei}}{\epsilon} X \Phi^i\left(-\frac{is}{\kappa}\right) , \qquad (II.45)$$

and then introduce (II.39) into (II.44) to obtain

$$S(\underline{\kappa},\Delta\omega) = \frac{\text{Re}}{(2\pi)^2\kappa}\int_{-\infty}^{\infty} dt e^{-i\Delta\omega t} \int_{\sigma-i\infty}^{\sigma+i\infty} ds e^{st} Q(\underline{\kappa},s) . \qquad (II.46)$$

Let $s = \sigma+i\nu$, so that $ds = id\nu$ and Eq. (II.46) becomes

$$S(\underline{\kappa},\Delta\omega) = \frac{\text{Re } i}{(2\pi)^2\kappa}\int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} dt e^{-it(\Delta\omega-\nu+i\sigma)}$$

$$(x)\; Q(\underline{\kappa},\sigma+i\nu)$$

$$= \frac{\text{Re } i}{2\pi\kappa}\int_{-\infty}^{\infty} d\nu\delta(\Delta\omega-\nu+i\sigma)Q(\underline{\kappa},\sigma+i\nu)$$

$$= \text{Re } \frac{i}{2\pi\kappa} \lim_{\sigma\to 0+} Q(\underline{\kappa},\sigma+i\Delta\omega) \; . \qquad \text{(II.47)}$$

At this point it is helpful to give explicit attention to the functions Λ_{ii} and Λ_{ei} appearing in (II.45). Recalling (II.28) and (II.40), we see that we have

$$\Lambda_{ii} = - \frac{4\pi n^i z_i^2 e^2}{\kappa^2\theta}\left[\frac{is}{\kappa}\,\Phi^i\left(-\frac{is}{\kappa}\right)-1\right] , \qquad \text{(II.48a)}$$

$$\Lambda_{ei} = + \frac{4\pi n^e z_i e^2}{\kappa^2\theta}\left[\frac{is}{\kappa}\,\Phi^e\left(-\frac{is}{\kappa}\right)-1\right] , \qquad \text{(II.48b)}$$

and

$$\Lambda_{ee} = - \frac{4\pi n^e e^2}{\kappa^2\theta}\left[\frac{is}{\kappa}\,\Phi^e\left(-\frac{is}{\kappa}\right)-1\right] . \qquad \text{(II.48c)}$$

Recalling the limit required in Eq. (II.47), we observe that the function Φ becomes

$$\lim_{\sigma\to 0+} \Phi^{e,i}\left(-\frac{is}{\kappa}\right)$$

$$= \int_{-\infty\cap}^{\infty} \frac{du\, M^{e,i}(u)}{u-\Delta\omega/\kappa} \equiv \Phi^{e,i}\left(\frac{\Delta\omega}{\kappa}\right)$$

$$= P\int_{-\infty}^{\infty} \frac{du\, M^{e,i}(u)}{u-\Delta\omega/\kappa} - i\pi M^{e,i}\left(\frac{\Delta\omega}{\kappa}\right) . \qquad \text{(II.49)}$$

The function Φ is the so-called plasma dispersion function, and has been extensively studied and tabulated by Fried and Conte.[12] Using (II.49), (II.38), and (II.39), we find that Eq. (II.47) becomes

$$S(\underline{\kappa}, \Delta\omega) = \mathrm{Re}\, \frac{i}{2\pi\kappa} \frac{1}{\epsilon} \left[(1-X)\Phi^e \left(\frac{\Delta\omega}{\kappa}\right) \left\{ 1 \right.\right.$$

$$\left. + \frac{z_i}{1+z_i} \frac{1}{(\kappa\lambda_D)^2} \left(1 + \frac{\Delta\omega}{\kappa} \Phi^i \left(\frac{\Delta\omega}{\kappa}\right)\right)\right\}$$

$$\left. + \frac{z_i}{1+z_i} \frac{1}{(\kappa\lambda_D)^2} \left(1 + \frac{\Delta\omega}{\kappa} \Phi^e \left(\frac{\Delta\omega}{\kappa}\right)\right) X\Phi^i \left(\frac{\Delta\omega}{\kappa}\right)\right], \qquad \text{(II.50)}$$

where

$$\epsilon = 1 + \frac{1}{1+z_i} \frac{1}{(\kappa\lambda_D)^2} \left(1 + \frac{\Delta\omega}{\kappa} \Phi^e \left(\frac{\Delta\omega}{\kappa}\right)\right)$$

$$+ \frac{z_i}{1+z_i} \frac{1}{(\kappa\lambda_D)^2} \left(1 + \frac{\Delta\omega}{\kappa} \Phi^i \left(\frac{\Delta\omega}{\kappa}\right)\right). \qquad \text{(II.51)}$$

The final step in the calculation, that of taking the real part of the expression in (II.50), is straight forward but tedious. Since this report is concerned primarily with a careful and minimally cluttered review of principles, rather than with maximum generality, we assume now that the ionic component in the plasma is singly ionized, i.e., $z_i = 1$. In this event, useful simplification is achieved in formulas (II.42), (II.50), and (II.51); and, after a large amount of trivial manipulation, the scattering function is ultimately reduced to

$$S(\underline{\kappa}, \Delta\omega) = \frac{1}{2\kappa|\epsilon|^2} \left[\left| \left\{ 1 + \frac{1}{2(\kappa\lambda_D)^2} \left(1 + \frac{\Delta\omega}{\kappa} \Phi^i \left(\frac{\Delta\omega}{\kappa}\right)\right)\right\}\right|^2 M^e \left(\frac{\Delta\omega}{\kappa}\right) \right.$$

$$\left. + \left| \frac{1}{2(\kappa\lambda_D)^2} \left(1 + \frac{\Delta\omega}{\kappa} \Phi^e \left(\frac{\Delta\omega}{\kappa}\right)\right)\right|^2 M^i \left(\frac{\Delta\omega}{\kappa}\right)\right]. \qquad \text{(II.52)}$$

This is the result obtained and extensively discussed in References (1, 2, and 3). In particular, Bernstein et. al.[3] have presented many approximation formulas for (II.52) as well as numerous graphs depicting the behavior of the scattering function as $\underline{\kappa}$ and $\Delta\omega$ are varied. (Of course the cross-section is to be obtained from (II.52) according to (I.41).) Our purposes here and in the next section will be adequately served by the presentation of two extremely simplified approximations to Equ. (II.52). Let

$$\Phi^{e,i}\left(\frac{\Delta\omega}{\kappa}\right) = \Phi_R^{e,i}\left(\frac{\Delta\omega}{\kappa}\right) + i\Phi_I^{e,i}\left(\frac{\Delta\omega}{\kappa}\right), \qquad (II.53)$$

be the decomposition of the plasma dispersion function into its real and imaginary parts. Then, according to (II.49),

$$\Phi_R^{e,i}\left(\frac{\Delta\omega}{\kappa}\right) = P\int_{-\infty}^{\infty} \frac{du\ M^{e,i}(u)}{u-\Delta\omega/\kappa},$$

$$\Phi_I^{e,i}\left(\frac{\Delta\omega}{\kappa}\right) = -\pi\ M^{e,i}\left(\frac{\Delta\omega}{\kappa}\right). \qquad (II.54b)$$

Thus the imaginary part is an explicit function of its argument. As mentioned above, the real part has been extensively approximated and tabulated. Two limiting forms of the latter which we will make use of here are:

$$\sqrt{\frac{m_{e,i}}{2\theta}}\ \frac{\Delta\omega}{\kappa} \ll 1,$$

$$\Phi_R^{e,i}\left(\frac{\Delta\omega}{\kappa}\right) \sim -2\ \frac{\Delta\omega}{\kappa}\ \frac{m_{e,i}}{2\theta}\left[1 - \frac{2}{3}\ \frac{m_{e,i}}{2\theta}\left(\frac{\Delta\omega}{\kappa}\right)^2 + \frac{4}{15}\left(\frac{m_{e,i}}{2\theta}\right)^2\left(\frac{\Delta\omega}{\kappa}\right)^4 - \ldots\right]; \qquad (II.54a)$$

and

$$\sqrt{\frac{m_{e,i}}{2\theta}}\ \frac{\Delta\omega}{\kappa} \gg 1,$$

$$\Phi_R^{e,i}\left(\frac{\Delta\omega}{\kappa}\right) \sim -\ \frac{\kappa}{\Delta\omega}\left[1 + \frac{\theta}{m_{e,i}}\left(\frac{\kappa}{\Delta\omega}\right)^2 + \frac{3\theta^2}{m_{e,i}^2}\left(\frac{\kappa}{\Delta\omega}\right)^4 + \ldots\right].$$

In the first, and simplest, case corresponding to the approximation (II.54a) (which is suitable for examining the scattered spectrum in the region of small energy transfers) we find that

$$\frac{\Delta\omega}{\kappa}\ \Phi^{e,i}\left(\frac{\Delta\omega}{\kappa}\right) \sim 0, \qquad (II.55)$$

so that

$$\epsilon \sim \frac{1+(\kappa\lambda_D)^2}{(\kappa\lambda_D)^2}, \tag{II.56}$$

and

$$S(\underline{\kappa}, \Delta\omega) \sim \frac{1}{2\sqrt{2}\kappa}\frac{1}{v_e}\left[\frac{(\kappa\lambda_D)^2}{1+(\kappa\lambda_D)^2}\right]^2 \left[\left(1 + \frac{1}{2(\kappa\lambda_D)^2}\right)^2 e^{-(\Delta\omega/\kappa)^2/2v_e^2} + \left(\frac{1}{2(\kappa\lambda_D)^2}\right)^2 \sqrt{\frac{m_i}{m_e}}\, e^{-(\Delta\omega/\kappa)^2/2v_i^2}\right], \tag{II.57}$$

where $v_{e,i} \equiv \sqrt{\theta/m_{e,i}}$ is the thermal speed of the electrons or ions in the plasma

In the limit corresponding to (II.54b), and in the range of frequency shifts in the vicinity of the electron plasma frequency, we have

$$M^e\left(\frac{\Delta\omega}{\kappa}\right) \gg M^i\left(\frac{\Delta\omega}{\kappa}\right),$$

$$\Delta\omega \sim \omega_e ,$$

$$\Delta\omega \gg \omega_i ,$$

$$\epsilon_R \sim 1 - \left(\frac{\omega_e}{\Delta\omega}\right)^2 \left[1 + \frac{3\theta}{m_e}\left(\frac{\kappa}{\Delta\omega}\right)^2\right]$$

$$\simeq 1 - (\omega_e/\Delta\omega)^2 ,$$

$$\epsilon_I \sim -\frac{\pi(\Delta\omega/\kappa)}{2(\kappa\lambda_D)^2} M^e\left(\frac{\Delta\omega}{\kappa}\right),$$

and

$$S(\underline{\kappa}, \Delta\omega) \sim \frac{M^e\left(\frac{\Delta\omega}{\kappa}\right)/2\kappa}{[1-(\omega_e/\Delta\omega)^2]^2 + \left[\frac{\pi(\Delta\omega/\kappa)}{2(\kappa\lambda_D)^2} M^e\left(\frac{\Delta\omega}{\kappa}\right)\right]^2} \tag{II.58}$$

III. COMPARISON OF THE THEORY WITH EXPERIMENT AND GENERAL DISCUSSION

A recent experiment reported by Chan and Nodwell[13] provides a striking qualitative, if not in fact quantitative, confirmation of the theory reviewed in the previous sections of this report. The details of their experiment will not be dwelt upon here. It is sufficient to note that they specifically looked for the kind of structure implied in Eq. (II.58) which corresponds to the inelastic scattering of photons associated with energy transfer to the collective electron oscillatory modes of the plasma. They saw peaks in the scattered spectrum, symmetrically displaced from $\Delta\omega = 0$, at wave length shifts of approximately 38 angstroms. Such a wave length shift corresponds roughly to a frequency shift of $(3/2) \times 10^{13}$ radians per second. The electron density which is needed to provide an electron plasma frequency of this magnitude is approximately 5×10^{16} electrons/cm^3. Chan and Nodwell actually fitted formula (II.52) to their experimental data employing the electron density as a fitting parameter. They found a quite satisfactory fit with an electron density of 6×10^{16} electrons/cm^3. This experiment is therefore a beautiful illustration of collective electron behavior in the plasma.

Earlier experiments[14] have mainly been successful in exploring the scattered spectrum in the vicinity of the incident frequency. For these experiments, the approximate formula (II.57) is qualitatively applicable. It is seen that, if

$$\frac{\sqrt{m_i/m_e}}{4(\kappa\lambda_D)^4}$$

is sufficiently large, the scattering function is proportional to

$$e^{-(\Delta\omega/\kappa)^2/2v_i^2}$$

which yields a very sharp peak about $\Delta\omega = 0$ with a half-width determined by the ion thermal motion instead of that of the electrons which are actually doing the scattering. This sharpening of the main peak is observed, and is illustrative of electron-ion collective behavior in the plasma.

Throughout the whole of this review, the emphasis has been upon the delineation of a systematic and reproducible argument leading to an interpretation of the observation of photon scattering by fully ionized plasmas. Many approximations have been made; usually with some care as to what the approximation is, but never with much regard as to what the approximations mean or as to their range of validity. One approximation is that of using ordinary time dependent perturbation theory to lowest order only in developing the relationship between the cross-section and the scattering function. This one is rarely worried much about in the context of the present problem. Another approximation is incurred in the calculation of the scattering function according to the rules of classical mechanics, when in fact it is actually related to the scattering experiment according to the rules of quantum mechanics. So far as this writer knows, this approximation has not even been noted in the study of plasmas, though it has long been a matter of concern to the students of neutron diffraction.[8,9] As pointed out earlier, this approximation is probably of no quantitative concern in connection with the interpretation of any plasma scattering experiment performed so far. However, it is vitally important qualitatively. The point is that the cross-section displayed in Eqs. (I.40) and (I.41) satisfies the principle of detailed balance, but not so the one obtained from (I.41) and (II.52). Thus if the latter approximation to the cross-section were to be used in a study of radiation thermalization in a large volume of plasma, erroneous conclusions would be anticipated. However, the method of correcting the error introduced by this approximation is explained and specified in References 8 and 9.

Finally there is the approximation made in going from Eq. (II.22) to (II.23). If, within the present scheme, a next higher order calculation were to be attempted, it would be found that the scattering function would contain additional (to those already displayed above) terms dependent upon static three-particle correlations. The next approximation would add terms dependent upon four-particle static correlations, etc. The approximation that we have used here has been the subject of much study. The conclusion seems to be that the present calculation is limited in applicability to plasmas in which the number of electrons in a Debye sphere is large. Therefore it would be enormously interesting to measure the scattering function in plasmas for which the number of electrons in a Debye sphere is of the order of or less than one.

In the interest of simplicity and clarity, this review has also introduced a number of assumptions about the scattering plasma which may make the results presented here unrealistic relative to some experiments. In particular, the assumption of the thermodynamic plasma should be questioned with care.

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