

Behavior of Gas Bubbles in Viscoelastic Materials in a Creep Process

Wen-Jei Yang and C. Y. Liang

Department of Mechanical Engineering, The University of Michigan, Ann Arbor, Michigan 48104

(Received 24 May 1971)

This paper investigates the dynamic behavior of a spherical bubble situated in viscoelastic (or elasticoviscous) materials in a creep process. Both the diffusion of the dissolved gas in the material and the thermodynamic behavior of the gas inside the bubble are taken into account. The generalized model of the Kelvin or Maxwell type which consists of the spring, dashpot, and Kelvin units is used to describe the creep behavior of the material under combined stresses. The growth or shrinkage of the gas bubble may be induced by a change in the dissolved gas concentration in the material, a change in the external load applied to the material, or a combination of both. Numerical results are obtained for simpler models and the effects of their parameters on the radius-time history of a bubble shrinking in an undersaturated solution or growing in an oversaturated solution are disclosed.

INTRODUCTION

The stability and dynamic behavior of gas bubbles in a liquid or liquid-gas solution have been the subjects of comprehensive investigations in the last two decades. The applications of these studies include cavitation, boiling, electrolysis, aeration processes, etc. Under certain circumstances, gases may be found in the form of bubbles or pockets in metals at high temperature, polymer melts, and biological tissues. For example, the swelling of fissionable materials during irradiation is known to be caused by the generation and subsequent growth of inert gases dissolved in the material.¹ Inert gases are injected into polymer melts in plastic foaming processes.² The formation of decompression bubbles in tissues and other organs is known to be the cause of decompression sickness.³ The nomenclature used in this paper is given in the Appendix.

Yang⁴ has analyzed the stability of gas bubbles in deformable solids with dissolved gas. The gas bubble in the solid solution grows or shrinks as the solution is oversaturated or undersaturated. As a result, the creep of the material takes place under the influence of the gas pressure in the bubble. The stress-strain rate relation in the secondary or steady-state creep stage is employed:

$$\dot{\epsilon}_{rr} = -(\tau_{\theta\theta} - \tau_{rr})^a / A, \quad (1)$$

where $\dot{\epsilon}_{rr}$ is the creep rate, $\tau_{\theta\theta} - \tau_{rr}$ is the principal shear stress applied to the spherically symmetrical system, and a and A are the experimental constants for a given material. The bubble-dynamics equation valid for the second creep stage is found to be

$$\dot{R} = \frac{R}{2A} \left(\frac{3}{2a} \right)^a \left(\tau_{\infty} + P - \frac{2\sigma}{R} \right)^a, \quad (2)$$

where τ_{∞} (not taken into consideration in Ref. 4) is an external stress applied on the material at $r = \infty$. The effects of time-dependent generation of heat and/or mass on the dynamic behavior of the bubble are examined by Yang.⁵ These effects are included in the $P(t)$ term.

The secondary creep stage represents the strain behavior under constant load over a wide time period of the entire strain-time history. The strain or creep rate remains almost constant during this stage. Equation (1) has been recommended to describe the stress-strain re-

lation in the stage for many metals at elevated temperature. Although Eq. (1) may also describe the creep behavior of a viscoelastic material over a fraction of the entire creep-time history, a more elaborate form of the stress-strain relation should be employed so that most of the characteristics of creep curves are described.

Many linear models have been proposed to describe the stress-strain relation of a viscoelastic material.⁶⁻¹³ They are the special cases of the generalized Kelvin and Maxwell models which consist of the spring, dashpot, and Kelvin units (a spring and a dashpot being arranged in parallel) in parallel and series, respectively. The rheological equation of state of any model of the generalized Kelvin or Maxwell type has the form (written for a spherical symmetrical system under the principal stresses of $\tau_{rr} - \tau_{\theta\theta}$)

$$\sum_{k=0}^n \lambda_k D^k (\tau_{rr} - \tau_{\theta\theta}) = \sum_{i=0}^l \mu_i D^i \epsilon_{rr} \quad (3)$$

in which λ 's and μ 's are the rheological constants and D is the derivative operator. The value of λ_0 is always unity, while that of μ_0 is nonzero for the Kelvin or solid type and zero for the Maxwell or fluid type.

The most commonly employed simpler models are the four-element (fluid) model

$$(1 + \lambda_1 D + \lambda_2 D^2)(\tau_{rr} - \tau_{\theta\theta}) = (\mu_1 + \mu_2 D) \dot{\epsilon}_{rr} \quad (4)$$

and the standard linear model

$$(1 + \lambda_1 D)(\tau_{rr} - \tau_{\theta\theta}) = (\mu_0 + \mu_1 D) \dot{\epsilon}_{rr}. \quad (5)$$

Equation (4) with $\mu_2 = \lambda_2 = 0$ expresses the two-parameter fluid model, while that with $\lambda_2 = 0$ expresses the three-parameter one.

In this paper, the dynamic equation of a gas bubble situated in a viscoelastic material is derived using the stress-strain relation of the generalized model. The growth or shrinkage of the bubble may be induced by a change in the dissolved gas concentration in the material, the application of an external load on the material, or by both. The bubble-dynamics equation is reduced to more tractable forms for the three- and four-parameter models. Numerical results are obtained for simpler models to examine the effects of the parameters μ_0 , μ_1 ,

μ_2 and λ_1 on the dynamic characteristics of the gas bubble.

ANALYSIS

Let us suppose that at the initial time $t = 0$, a spherical gas bubble of radius R_0 is situated in an elastic material in which the dissolved gas concentration is uniform and equal to C_∞ . The system is at constant temperature and the dissolved gas concentration under saturated state at this temperature is C_s . Then, according to $C_s < C_\infty$ or $C_s > C_\infty$, the gas bubble in the solution will grow or shrink by diffusion.

For convenience in the analysis, the center of the gas bubble is taken as the origin of a spherical polar coordinate system. At any time $t > 0$ when the bubble radius is R , the dissolved gas concentration C at a point in the solution at a distance r from the origin can be found from the mass diffusion equation

$$\frac{\partial C}{\partial t} = \frac{\alpha}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial C}{\partial r} \right), \tag{6}$$

where α is the coefficient of diffusion of the gas in the material. The appropriate initial and boundary conditions are

$$C(r, 0) = C_\infty, \tag{7}$$

$$C(R, t) = C_s, \quad C(\infty, t) = C_\infty. \tag{8}$$

With the transformation of

$$u = r(C - C_\infty) \text{ and } \xi = r - R$$

Eq. (6) becomes

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial \xi^2}.$$

The initial and boundary conditions are

$$u(r, 0) = 0,$$

$$u(0, t) = R(C_s - C_\infty).$$

The solution can be found from Ref. 14 to be

$$u(r, t) = R(C_s - C_\infty) \operatorname{erfc} \left[\frac{(r - R)}{2(\alpha t)^{1/2}} \right]. \tag{9}$$

Therefore, the mass flow rate across the bubble surface is

$$\dot{m} = 4\pi R^2 \alpha \left(\frac{\partial C}{\partial r} \right)_{r=R} = 4\pi R^2 \alpha (C_s - C_\infty) \left(\frac{1}{R} + \frac{1}{(\pi \alpha t)^{1/2}} \right). \tag{10a}$$

The mass flow m and its time derivatives of higher order are

$$m = 4\pi \alpha (C_s - C_\infty) \int_0^t [R + R^2/(\pi \alpha t)^{1/2}] dt + m_0, \tag{10b}$$

$$\dot{m} = 4\pi \alpha (C_s - C_\infty) [\dot{R} + (2R\dot{R} - R^2/2t)/(\pi \alpha t)^{1/2}], \tag{10c}$$

⋮

where m_0 is the initial mass of the gas in the bubble.

The equation of state for the gas inside the bubble is

$$\frac{4}{3} \pi R^3 P = m \bar{R} T \text{ or } P = 3m \bar{R} T / 4\pi R^3, \tag{11}$$

where \bar{R} is the gas constant.

Creeping of the material may be caused by the gas pressure developed in the bubble as a result of mass diffusion across the bubble surface, or by an external stress $\tau_\infty(t)$ applied on the material, or a combination of both. The distribution of the stresses thus developed around the bubble in the homogeneous and isotropic material is spherically symmetrical. In this case the non-diagonal components of the stress tensor are equal to zero and $\tau_{\theta\theta} = \tau_{\phi\phi} = -\frac{1}{2}\tau_{rr}$. The distribution of the stresses is governed by the laws of conservation of mass and momentum and the stress-strain rate relation such as Eqs. (3), (4), or (5), subject to the appropriate boundary conditions. The definition of strain rate is

$$\dot{\epsilon}_{rr} = \frac{dv_r}{dr}, \tag{12}$$

where v_r is the radial component of the deformation rate \vec{v} . The continuity equation is

$$\operatorname{div} \vec{v} = \frac{dv_r}{dr} + 2\frac{v_r}{r} = 0. \tag{13}$$

The stress-equilibrium equation is

$$\frac{d\tau_{rr}}{dr} + \frac{2(\tau_{rr} - \tau_{\theta\theta})}{r} = 0. \tag{14}$$

The boundary conditions are

$$r = R: \tau_{rr} = -P + 2\sigma/R, \tag{15a}$$

$$r = \infty: \tau_{rr} = \tau_\infty(t), \tag{15b}$$

where σ is the coefficient of surface tension. τ_∞ is equal to zero if the stress is induced solely under the influence of the gas pressure in the bubble.

Equation (13) is integrated to give

$$v_r = (R/r)^2 \dot{R}. \tag{16}$$

Equation (16) is then substituted into Eq. (12). Upon integration, it yields

$$\epsilon_{rr} = -\frac{2}{3} r^{-3} (R^3 - R_0^3) + \epsilon_{rr0}, \tag{17a}$$

where ϵ_{rr0} is the initial local strain. The time derivatives of ϵ_{rr} are

$$\dot{\epsilon}_{rr} = -2R^2 \dot{R} / r^3, \tag{17b}$$

$$\ddot{\epsilon}_{rr} = -(2R/r^3)(2\dot{R}^2 + R\ddot{R}), \tag{17c}$$

$$\ddot{\dot{\epsilon}}_{rr} = -(4\dot{R}/r^3)(\dot{R} + 2R\ddot{R}) - (2R/r^3)(2\dot{R}\ddot{R} + R\ddot{\dot{R}}), \tag{17d}$$

⋮

Now, Eqs. (14) and (17a) are combined with the general stress-strain rate relation (3) to eliminate the $\tau_{rr} - \tau_{\theta\theta}$ and ϵ_{rr} terms. It yields

$$\sum_{k=0}^n \lambda_k D^k \left(\frac{\partial \tau_{rr}}{\partial r} \right) = \frac{4}{3r^4} \sum_{i=0}^j \mu_i D^i (R^3 - R_0^3) - \frac{2}{r} \mu_0 \epsilon_{rr0},$$

μ_0 is zero for any model of the Maxwell or fluid type, while $\mu_0 \epsilon_{rr0}$ is equal to $\tau_{rr0} - \tau_{\theta\theta 0}$ according to Eq. (3), where τ_{rr0} and $\tau_{\theta\theta 0}$ are the initial local stresses. The last equation is then integrated with respect to r from

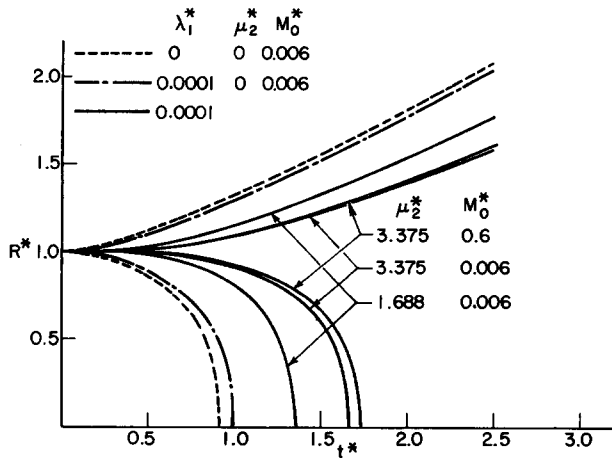


FIG. 1. Dynamic behavior of gas bubbles in viscoelastic (one-, two-, and three-parameter models) materials for $\mu_1^* = 3.38$, $\sigma^* = 0$, $C^* = 0.5$ for growth and -0.5 for shrinkage.

$r = R$ to $r = \infty$ with the aid of the boundary condition (15) and Eq. (14). One obtains

$$(4/9R^3) \sum_{i=0}^j \mu_i D^i (R^3 - R_0^3) = \sum_{k=0}^n \lambda_k D^k (\tau_\infty + P - 2\sigma/R) + \Delta\tau_{rr0}, \tag{18}$$

where $-\Delta\tau_{rr0} = \tau_{rr}(\infty, 0) - \tau_{rr}(R_0, 0)$. This is the general form of the bubble-dynamics equation in viscoelastic or elasticoviscous materials. Equation (18) is subject to the initial conditions $R(0) = R_0$, $\dot{R}(0) = 0$, and others which have to be determined by the application of Eq. (3).

The mode of growth or collapse may be classified into the following three cases:

(i) The case where C_∞ is not equal to C_s . When C_∞ is less than C_s , mass transfer takes place from a gas bubble to the solution eventually resulting in the shrinkage of the bubble. On the other hand, if C_∞ is greater than C_s , the dissolved gas will migrate to a gas bubble and result in the expansion of the bubble. With the substitution of Eq. (11), Eq. (18) reduces to

$$\frac{4}{9x} \sum_{i=0}^j \mu_i D^i (x - 1) = \sum_{k=0}^n \lambda_k D^k \left(\tau_\infty + \frac{3m\bar{R}T}{4\pi x R_0^3} - \frac{2\sigma}{R} \right) + \Delta\tau_{rr0}, \tag{19}$$

where $x = (R/R_0)^3$. With the substitution of Eq. (10) for m and its time derivatives, Eq. (19) takes the form of an integro-differential equation.

(ii) Without mass transfer, $C_s = C_\infty$. In the absence of mass transfer, the gas inside the bubble can be assumed to undergo a reversible polytropic process, i. e., $P(t) = P_0(R/R_0)^\gamma$, where γ is the polytropic exponent. For an isothermal process, γ is unity, while for an adiabatic process, γ is equal to the ratio of specific heats. In the actual situation, the thermal condition lies between the isothermal and adiabatic limits. Equation (18) can be written as

$$\frac{4}{9x} \sum_{i=0}^j \mu_i D^i (x - 1) = \sum_{k=0}^n \lambda_k D^k \left(\tau_\infty + P_0 x^\gamma - \frac{2\sigma}{R} \right) + \Delta\tau_{rr0}. \tag{20}$$

Both Eqs. (19) and (20) are nonlinear with variable coefficients.

(iii) Zero normal stress in the solution at the bubble surface. When the gas pressure is balanced with surface tension, i. e., $P = 2\sigma/R$, the normal stress in the material at the bubble surface becomes negligibly small. Equation (18) can then be expressed as

$$\frac{4}{9x} \sum_{i=0}^j \mu_i D^i (x - 1) = \sum_{k=0}^n \lambda_k D^k \tau_\infty + \Delta\tau_{rr0}. \tag{21}$$

The equation which is linear with variable coefficients shows that the mechanism of bubble growth or collapse is stress controlled.

RESULTS AND DISCUSSION

Equation (21), valid for the case in which bubble behavior is stress controlled, can be solved for an exact solution when it is a lower-order equation:

(a) Order 1. For the standard linear model of Eq. (5), one obtains the solution

$$x = e^{-G(t)} + (\mu_0/\mu_1) e^{-G(t)} \int_0^t G^{G(t)} dt, \tag{22}$$

where

$$G(t) = (\mu_0/\mu_1) \int_0^t [(1 - 9/4\mu_0)(\tau_\infty + \lambda_1 \dot{\tau}_\infty + \Delta\tau_{rr0})] dt.$$

(b) Order 2. Equation (21) reads

$$\ddot{x} + (\mu_1/\mu_2) \dot{x} - (9/4\mu_2)(\tau_\infty + \lambda_1 \dot{\tau}_\infty + \lambda_2 \ddot{\tau}_\infty + \Delta\tau_{rr0}) x = 0 \tag{23a}$$

for the four-parameter fluid model and

$$\ddot{x} + (\mu_1/\mu_2) \dot{x} + (\mu_0/\mu_2)[(1 - 9/4\mu_0) \times (\tau_\infty + \lambda_1 \dot{\tau}_\infty + \lambda_2 \ddot{\tau}_\infty + \Delta\tau_{rr0})] x = \mu_0/\mu_2 \tag{23b}$$

for the five-parameter solid model. The solutions of Eq. (23) are readily available when τ_∞ takes typical functions of time, such as a step function.

If the gas inside the bubble undergoes a reversible isothermal process, $\gamma = 1$, Eq. (20) for the standard linear model with the neglect of surface tension takes the form of the Riccati equation

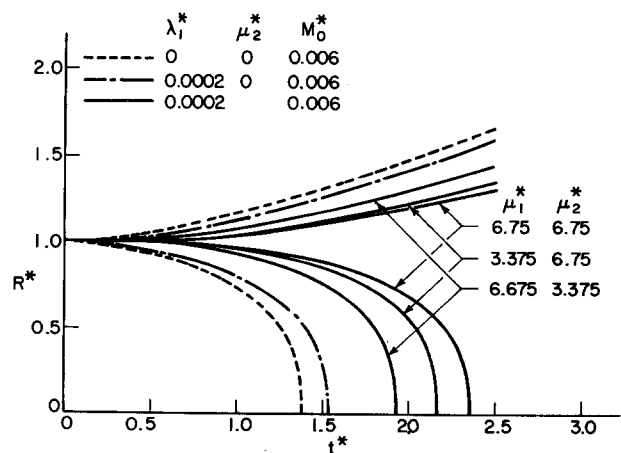


FIG. 2. Dynamic behavior of gas bubbles in viscoelastic (one-, two-, and three-parameter models) materials for $\mu_1^* = 6.75$, $\sigma^* = 0$, $C^* = 0.5$ for growth and -0.5 for shrinkage.

$$\dot{x} - (9P_0/4\mu_1)x^2 + [(\mu_0/\mu_1 - 9/4\mu_1) \times (\tau_\infty + \Delta\tau_{rr0} + \lambda_1\dot{\tau}_\infty)]x = \mu_0/\mu_1 \quad (24)$$

Numerical results are obtained for the mass diffusion, controlled growth, and shrinkage of gas bubbles by integrating Eq. (19) without the terms involving τ_∞ . Figures 1 and 2 depict the results of the two- and three-parameter fluid models, while those of the standard linear solid model are illustrated in Fig. 3. The dimensionless parameters are defined as

$$t^* = \alpha t/R_0^2, \quad \lambda_k^* = \lambda_k(\alpha/R_0^2)^k, \quad \mu_i^* = (\mu_i/\bar{R}T_0C_\infty)(\alpha/R_0^2)^i, \\ m_0^* = 3m_0/4\pi R_0^3C_\infty, \quad C^* = 1 - C_s/C_\infty, \quad \sigma^* = \sigma/\bar{R}T_0C_\infty R_0 \quad (25)$$

It is disclosed from Figs. 1-3 that a gas bubble will grow or shrink faster as the magnitude of λ_1^* , μ_0^* , μ_1^* , or μ_2^* decreases; a bubble grows faster but collapses slower as the magnitude of m_0^* increases. Growth or shrinkage occurs according as the material is oversaturated with the dissolved gas (i.e., $C_\infty > C_s$) or undersaturated (corresponding to $C_\infty < C_s$). When the material is saturated ($C_s = C_\infty$), the bubble will remain the same size in the absence of surface tension. The effects of surface tension are found to be very small except during a short time interval near the complete collapse of the bubble.

CONCLUSION

The general form of the bubble-dynamics equation in viscoelastic or elasticoviscous materials in a creep process is derived utilizing the generalized model of the Kelvin or Maxwell type to describe the rheological characteristics of the materials. The growth or shrinkage of a gas bubble may be induced by a change in the concentration of the dissolved gas C_∞ , a change in the external stress τ_∞ , or both. The analysis leads to

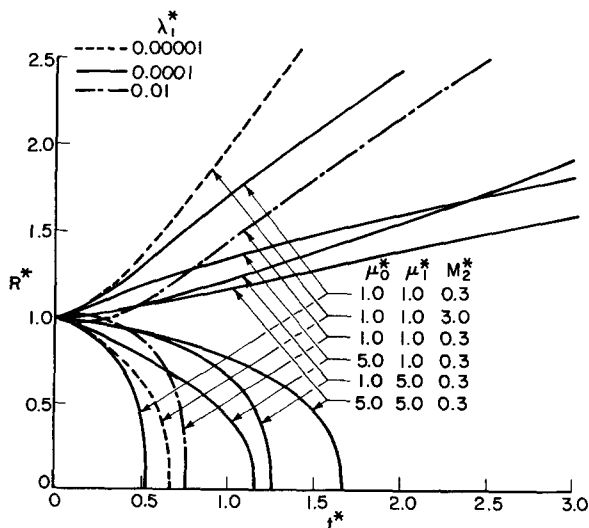


FIG. 3. Dynamic behavior of gas bubbles in elasticoviscous (three-parameter solid model) materials for $\sigma^* = 0$, $C^* = 0.5$ for growth and -0.5 for shrinkage.

a classification of the growth or collapse mode into three categories, (i) where the mode is controlled by both mass transfer and the external load applied to the material, (ii) where both the external load and the thermal behavior of the gas inside the bubble are of comparable importance, and (iii) where the external load is the sole controlling factor. The factors affecting bubble behavior are identified.

An exact solution of the bubble-dynamics equation can be obtained for the second and third modes when the model is simple and the equation is of lower order. The effect of surface tension on bubble behavior is entirely negligible except near its complete collapse. Numerical results obtained for the first mode indicate that a gas bubble grows or shrinks faster as the magnitude of the dimensionless rheological constants λ^* and/or μ^* decreases. However, an increase in the initial mass of the gas inside the bubble m_0^* results in faster growth but slower collapse.

APPENDIX: NOMENCLATURE

- A rheological constant;
- a rheological constant;
- C concentration of the gas dissolved in viscoelastic materials (molar density of the solid-gas solution); C_s , at saturation; C_∞ , at a distance from the bubble; C_s^* , C_s/C_∞ ;
- D time-derivative operator;
- i integer;
- j integer;
- k integer;
- m mass (in moles) of the gas in the bubble; m_0 initial value; m_0^* , m_0 in dimensionless form as defined by Eq. (25),
- n integer;
- P gas pressure in the bubble;
- \bar{R} bubble radius; R_0 , initial radius;
- \bar{R} gas constant;
- r radial coordinate system with the origin at the bubble center;
- T gas temperature in the bubble;
- t time; t^* , dimensionless time defined as $\alpha t/R_0^2$;
- u $r(C - C_\infty)$;
- v_r radial component of the deformation vector \bar{v} ;
- α mass diffusivity of the dissolved gas in viscoelastic materials;
- γ polytropic exponent;
- ϵ_{rr} strain; $\dot{\epsilon}_{rr}$, creep rate; ϵ_{rr0} , initial strain;
- θ spherical coordinate;
- λ_k rheological constants; λ_1^* , λ_1 in dimensionless form as defined by Eq. (25);
- μ_i rheological constants; μ_i^* , μ_i in dimensionless form as defined by Eq. (25);
- σ surface tension;
- τ stress; τ_{rr} and $\tau_{\theta\theta}$, normal components in the direction of r and θ , respectively; $\tau_\infty = \tau_{rr}(\infty, t)$ radial stress at a distance from the bubble; τ_{rr0} , initial radial stress; $\Delta\tau_{rr0} = \tau_{rr}(R_0, 0) - \tau_{rr}(\infty, 0)$.

Superscripts

., . . first and second time derivatives, respectively.

Subscripts

- 0 at $t = 0$;
 s saturation;
 ∞ at infinity.

¹V. M. Agranovich, E. Y. Mikhin, and L. P. Semenov, *At. Energy*, 15, 1140 (1963).

²R. S. Swanson, *Plastics Technology* (McKnight and McKnight, Bloomington, Ill., 1965).

³H. D. Van Liew and M. P. Hlastala, *Respirat. Physiol.* 7, 111 (1969).

⁴W. J. Yang, *J. Chem. Phys.* 56, 1610 (1972).

⁵W. J. Yang, *Japan J. Appl. Phys.* 11, No. 1 (1972).

⁶H. W. Hayden, W. G. Moffatt, and J. Wulff, *The Structure and Properties of Materials, Vol. III Mechanical Behavior*

(Wiley, New York, 1965).

⁷W. Flugge, *Viscoelasticity* (Blaisdell, Waltham, Mass., 1967).

⁸D. R. Bland, *The Theory of Linear Viscoelasticity* (Pergamon, New York, 1960).

⁹J. A. H. Hult, *Creep in Engineering Structures* (Blaisdell, Waltham, Mass., 1966).

¹⁰P. D. Pitchie, *Physics of Plastics* (Van Nostrand, Princeton, N. J., 1965).

¹¹L. E. Nielsen, *Mechanical Properties of Polymers* (Reinhold, New York, 1962).

¹²V. H. Frankel and A. H. Burstein, *Orthopaedic Biomechanics* (Lea and Febiger, Philadelphia, 1970).

¹³J. H. DuBois and F. W. John, *Plastics* (Reinhold, New York 1967).

¹⁴H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (Oxford U. P., London, 1959).

Stored Energy Function of Rubberlike Materials Derived from Simple Tensile Data*

T. J. Peng and R. F. Landel

Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California 91103

(Received 17 November 1971)

An explicit formulation to obtain the stored energy function W from simple tension experiments alone, based on the Valanis-Landel separable symmetric stored energy function, $W(\lambda_1, \lambda_2, \lambda_3) = w(\lambda_1) + w(\lambda_2) + w(\lambda_3)$, is developed. For a simple extension stress-strain law of the form $\lambda\sigma = E\epsilon$, the analytical formula of W in the limited range $1 \leq \lambda \leq 2.5$ is found to be

$$W = E \sum_{i=1}^3 [\lambda_i - 1 - \ln \lambda_i - \frac{1}{6} (\ln \lambda_i)^2 + \frac{1}{18} (\ln \lambda_i)^3 - \frac{1}{216} (\ln \lambda_i)^4],$$

where the limitation originates with the stress-strain law. The expression is used to verify the validity of the Valanis-Landel postulation through prediction of the stress-strain behavior in multiaxial deformations.

INTRODUCTION

The theory of finite elasticity shows that the stress-strain behavior of a hyperelastic material under any type of deformation field can be expressed in terms of the strain energy function W . The strain energy function W is expressed in terms of the strain tensor γ_{ij}^2 , where $i, j = 1, 2, \text{ and } 3$. When the material is isotropic, W can be represented in terms of three principal invariants, I_1, I_2, I_3 ; locally $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_2 = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2$, and $I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2$, where λ 's denote the principal extension ratios. For a homogeneous deformation, the λ 's of course refer to the specimen boundary displacements, too. Thus the stress-strain relation can be expressed in terms of three partial derivatives, $\partial W / \partial I_1 \equiv W_1$, $\partial W / \partial I_2 \equiv W_2$, and $\partial W / \partial I_3 \equiv W_3$, which are regarded as three material parameters.

In the investigations of the mechanical properties of rubberlike materials at finite deformations, there are several authors¹ who have applied the elasticity theory in attempts to find an analytical formula of the strain energy function W in terms of W_1 and W_2 by assuming the materials to be isotropic and incompressible ($I_3 = 1$). To evaluate W_1 and W_2 , stress-strain data are needed as a function of I_1 at constant I_2 , and vice versa. The evaluation of W_1 and W_2 requires highly accurate data, especially at relatively small strains,¹ since the equations that furnish the relations between stress and strain to determine W_1 and W_2 are of a form such that small experimental errors lead to amplified errors in W_1 and

W_2 . Hence, producing the information which is necessary for a complete description of the large deformation behavior of elastomers thus becomes a formidable task experimentally. (Recent data²⁻⁵ have shown that the dependence of the partial derivatives W_1 and W_2 on I_1 and I_2 is complex.)

In order to circumvent the difficulty, a theory⁶ has been recently proposed for incompressible solids in which W is assumed to be a separable symmetric function of the λ 's:

$$W(\lambda_1, \lambda_2, \lambda_3) = w(\lambda_1) + w(\lambda_2) + w(\lambda_3), \quad (1)$$

where the symmetric function satisfies all symmetry relations imposed by isotropy.⁷ This hypothesis for W makes it very simple to evaluate W from experiments carried out under pure shear deformation.

Applying this approach, Valanis and Landel⁶ evaluated $dw/d\lambda_i$ graphically and then approximated the results to obtain the following empirical equation for W :

$$W = 2\mu \sum_{i=1}^3 \lambda_i (\ln \lambda_i - 1), \quad (2)$$

where μ is shear modulus. However, this equation is not quite satisfactory to describe the mechanical behavior of rubber; moreover, there is some uncertainty in the data obtained from pure shear deformation. Subsequent experiments by Kawabata and co-workers² and Dickie and Smith⁸ appear to confirm the basic postulate. At the same time their results show more clearly the