A phase cell approach to Yang–Mills theory. II. Analysis of a mode

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Properties of a mode announced in a previous paper are proved. This involves some complicated calculations in linear algebra, and observation of the structure of a function of several complex variables.

I. A'_1 AND A'_2, LINEAR ALGEBRA

$A'_i(p)$ is given by (3.12) of Ref. 1. We will consider a mode at level zero, specified by bond assignments at level zero, having only one bond assignment different from zero. We choose the bond to be the bond in the $+1$ direction from the origin, and the assignment to it to be unity. By invariance under certain interchanges of coordinate axes it is sufficient to find A'_1 and A'_2 to determine A'_i.

With i fixed, (3.12) of Ref. 1 is a matrix product, ABC, with A, B, C, respectively, a (1×6) matrix, (6×6) matrix, and (6×1) matrix. The coordinates of the six-dimensional space are labeled by oriented plaquette directions, for which we choose the ordering

$$(1,2),\ (2,3),\ (3,4),\ (1,3),\ (1,4),\ (2,4). \tag{1.1}$$

We find it convenient to introduce certain special notation and matrices. We let

$$f_i \equiv (e^{-ip_i} - 1) \tag{1.2}$$

and D be the 6×6 diagonal matrix with

$$D_{(i,j),(i,j)} \equiv f_i f_j \tag{1.3}$$

(here labeling rows and columns of D by the associated oriented plaquette directions). We let

$$\langle h(p)\rangle \equiv \sum_n \frac{1}{(p+2\pi n)^2} \prod_{i=1}^{4} \left| \frac{e^{ip_i}-1}{(p_i+2\pi n_i)} \right|^2 h(p+2\pi n), \tag{1.4}$$

where h is any function of $p = (p_1,p_2,p_3,p_4)$.

We now detail the matrices occurring in (3.12) of Ref. 1. We first view the (6×1) matrix

$$\beta_\gamma e^{i\hat{\gamma}\cdot p} \equiv v_1 \equiv \bar{f}_1^{-1} \bar{D} w_1, \tag{1.5}$$

$$w_1 \equiv (1,0,0,1,1,0)^T, \tag{1.6}$$

where the bars indicate complex conjugates. We next view the left (1×6) matrix, two different vectors depending on whether one is studying A'_1 or A'_2. With r_L defined by

$$r_L \equiv -\frac{i}{(2\pi)^2} \prod_k \left(\frac{e^{-ip_k}-1}{p_k} \right), \tag{1.7}$$

we have

$$\tilde{P}_1 = r_L (1/p_1) w_1^T D \tag{1.8}$$

and

$$\tilde{P}_2 = r_L (1/p_2) w_2^T D \tag{1.9}$$

with

$$w_2 \equiv (-1,1,0,0,0,1)^T. \tag{1.10}$$

At this point we may write (3.12) of Ref. 1 as

$$A'_i(p) = (1/p^2) \tilde{P}_i M^{-1} v_1. \tag{1.11}$$

Recall M is singular; and M^{-1} is defined on suitable vectors as $\lim_{\epsilon\to 0^+} (\epsilon + M)^{-1}$.

Substituting (1.5) and (1.8) or (1.9) into (1.11) we get

$$A'_i = (-1/p^2) r_L \bar{f}_1^{-1} (1/p_i) w_i^T \langle M_0\rangle^{-1} w_1, \quad i = 1,2, \tag{1.12}$$

where

$$M = \bar{D}\,\langle M_0\rangle D \tag{1.13}$$

and the inverse of $\langle M_0\rangle$ taken in the same sense as the inverse of M. We now display M_0, itself, in its full glory:

$$\begin{pmatrix}
1/p_1^2 + 1/p_2^2 & -1/p_2^2 & 0 & 1/p_1^2 & 1/p_1^2 & -1/p_2^2 \\
-1/p_2^2 & 1/p_2^2 + 1/p_3^2 & -1/p_3^2 & 1/p_3^2 & 0 & 1/p_2^2 \\
0 & -1/p_3^2 & 1/p_3^2 + 1/p_4^2 & -1/p_3^2 & 1/p_4^2 & 1/p_4^2 \\
1/p_1^2 & 1/p_3^2 & -1/p_3^2 & 1/p_1^2 + 1/p_3^2 & 1/p_1^2 & 0 \\
1/p_1^2 & 0 & 1/p_4^2 & 1/p_1^2 & 1/p_1^2 + 1/p_4^2 & 1/p_4^2 \\
-1/p_2^2 & 1/p_2^2 & 1/p_4^2 & 0 & 1/p_4^2 & 1/p_2^2 + 1/p_4^2
\end{pmatrix}.$$

Here $\langle M_0\rangle$ is a positive Hermitian matrix, and w_1 is in its range. We deal with the limit

$$\lim_{\epsilon\to 0^+} (\langle M_0\rangle + \epsilon)^{-1} w_1. \tag{1.14}$$

This limit may be taken by the following procedure. Let P be the (orthogonal) projection onto the range of $\langle M_0\rangle$, S. Let $m = P\langle M_0\rangle P$, a strictly positive Hermitian matrix. We invert m, in S, writing it as m^{-1}. One has for the result of (1.14)

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$m^{-1}w_1$. (1.15)

The observations of this paragraph reduce the computations of (1.12) to calculations in a three-dimensional vector space. We note that S is spanned by w_1, w_2, and $w_3 = (0,0,1,0,1,1)^T$.

We first present the results of the computation in (1.15):

$$\langle M_0\rangle^{-1}w_1 = (1/\mathscr{D})(4bc+4bd+8cd, 4bd-4cd,$$
$$4bc-4bd, 4bc+8bd+4cd,$$
$$8bc+4bd+4cd, 4bc-4cd)^T, \tag{1.16}$$

where

$$a = \langle 1/p_1^2\rangle,\quad b = \langle 1/p_2^2\rangle,$$
$$c = \langle 1/p_3^2\rangle,\quad d = \langle 1/p_4^2\rangle, \tag{1.17}$$

and

$$\mathscr{D} = 16\sum_k \prod_{j\neq k}\langle 1/p_j^2\rangle. \tag{1.18}$$

Here $\mathscr{D}$ has arisen as the determinant of the 3×3 matrix involved in the computation of m^{-1}.

Collecting our results and substituting into (1.12) we easily find

$$A_1' = \Big(-r_L\bar{f}_1^{-1}\frac{1}{p^2}\Big)\frac{1}{p_1}\frac{16}{\mathscr{D}}\Big(\Big\langle\frac{1}{p_2^2}\Big\rangle\Big\langle\frac{1}{p_3^2}\Big\rangle$$
$$+\Big\langle\frac{1}{p_2^2}\Big\rangle\Big\langle\frac{1}{p_4^2}\Big\rangle+\Big\langle\frac{1}{p_3^2}\Big\rangle\Big\langle\frac{1}{p_4^2}\Big\rangle\Big), \tag{1.19}$$

$$A_2' = \Big(-r_L\bar{f}_1^{-1}\frac{1}{p^2}\Big)\frac{1}{p_2}\frac{16}{\mathscr{D}}\Big(-\Big\langle\frac{1}{p_3^2}\Big\rangle\Big\langle\frac{1}{p_4^2}\Big\rangle\Big). \tag{1.20}$$

II. A_i' AND $X(p)$

From (1.19) and (1.20), and the invariance under interchange of coordinate directions—except for the special one-direction—we find our expression for $A_i'(p)$,

$$A_i' = \Big(-r_L\bar{f}_1^{-1}\frac{1}{p^2}\Big)\frac{1}{p_i}\frac{16}{\mathscr{D}}l_i, \tag{2.1}$$

$$l_1 = \Big\langle\frac{1}{p_2^2}\Big\rangle\Big\langle\frac{1}{p_3^2}\Big\rangle+\Big\langle\frac{1}{p_2^2}\Big\rangle\Big\langle\frac{1}{p_4^2}\Big\rangle+\Big\langle\frac{1}{p_3^2}\Big\rangle\Big\langle\frac{1}{p_4^2}\Big\rangle, \tag{2.2}$$

$$l_i = -\prod_{j\neq 1,i}\Big\langle\frac{1}{p_j^2}\Big\rangle,\quad i\neq 1. \tag{2.3}$$

A little study of these expressions for the region near $p_1\sim p_2\sim p_3\sim p_4\sim 0$ shows that the A_i' have a singularity in the complex four-dimensional region on the surface $p^2=0$, of course hitting the real axis. We define $A_i^N(p)$, a gauge transformation of $A_i'(p)$, by

$$A_i^N(p) = A_i'(p) + p_iX(p), \tag{2.4}$$

with

$$X(p) = \Big(-r_L\bar{f}_1^{-1}\frac{1}{(p^2)^2}\Big)\frac{1}{\langle 1/p_1^2\rangle}\frac{1}{(1+p^2)^s}, \tag{2.5}$$

where s is an arbitrary integer, sufficiently large. The analytic properties of $A_i^N(p)$ will be studied in the next section.

III. ANALYTIC PROPERTIES OF $A_i^N(p)$

The analysis we follow herein is analogous to the similar analysis due to Gawedzki and Kupiainen in the appendix of Ref. 2. We take the analytic extensions of the expressions in Sec. II from real p to complex p. For example,

$$\bar{f}_i \to e^{ip_i}-1$$

and

$$\Big|\frac{e^{-ip_i}-1}{p_i+2\pi n_i}\Big|^2 \to \frac{e^{-ip_i}-1}{p_i+2\pi n_i}\,\frac{e^{ip_i}-1}{p_i+2\pi n_i}.$$

We make the preliminary observation that f_i, D, $\mathscr{D}$, and $\langle\cdot\rangle$ are periodic functions of p, invariant under

$$p\to p+2\pi n \tag{3.1}$$

(n is a four-vector with integer components). We note that r_L, p^2, and p_i are not periodic functions of p.

Because of the periodicity mentioned in the last paragraph it is natural to divide our results into the following three theorems.

Theorem 3.1 (Local analyticity): There is an $\epsilon_0>0$ such that in the domain, $\mathscr{D}_L$, specified by

$$-\pi\leqslant \operatorname{Re}p_j\leqslant\pi,\quad |\operatorname{Im}p_j|<\epsilon_0, \tag{3.2}$$

$A_i^N(p)$ is analytic.

Theorem 3.2 (Global analyticity): $A_i^N(p)$ is analytic in the domain, $\mathscr{D}_G$, specified by

$$|\operatorname{Im}p_j|<\epsilon_0. \tag{3.3}$$

Theorem 3.3 (Boundedness): Within the domain, $\mathscr{D}_B$, specified by

$$|\operatorname{Im}p_j|<\epsilon_0/2, \tag{3.4}$$

$A_i^N(p)$ satisfies bounds of the form

$$|A_i^N(p)|<c\prod_j\frac{1}{(|p_j|+1)}\frac{1}{|p^2|+1}. \tag{3.5}$$

In Theorems 3.2 and 3.3 the ϵ_0 is as defined in Theorem 3.1.

Theorems 3.2 and 3.3 are easy consequences of the form of $A_i^N(p)$ and Theorem 3.1. We will devote our attention entirely to the proof of Theorem 3.1, which is carried out in Sec. V.

IV. CONCLUSIONS

Equations (3.13)–(3.15) of Ref. 1 follow directly from Theorems 3.2 and 3.3 of the last section by standard techniques.

V. LOCAL ANALYTICITY

We must study $A_i^N(p)$ on $\mathscr{D}_L$ [of (3.2)]. We write $A_i^N(p)$ from (2.1)–(2.5),

$$A_i^N(p) = -r_L\bar{f}_1^{-1}\frac{1}{p^2}\Big[\frac{1}{p_i}\frac{16}{\mathscr{D}}l_i+\frac{p_i}{p^2}\frac{1}{\langle 1/p_1^2\rangle}\frac{1}{(1+p^2)^s}\Big]. \tag{5.1}$$

In studying analyticity we may remove the factor $-r_L$ and replace $\bar{f}_1^{-1}$ by p_1^{-1}. We thus study

$$a_i(p)=\frac{1}{p_1}\frac{1}{p^2}\left[\frac{1}{p_i}\frac{16}{\mathscr{D}}l_i+\frac{p_i}{p^2}\frac{1}{\langle 1/p_1^2\rangle}\frac{1}{(1+p^2)^s}\right]. \quad (5.2)$$

We introduce some notation

$$p_j^2\langle 1/p_j^2\rangle \equiv r_0(p)e_j(p)(1/p^2), \quad (5.3)$$

where $e_j(0)=1$ and

$$r_0(p)=\prod_{i=1}^{4}\left|\frac{e^{-ip_i}-1}{p_i}\right|^2. \quad (5.4)$$

It will be sufficient to study $a_1(p)$ and $a_2(p)$. With the notation of (5.3) and (5.4) we have

$$r_0(p)a_1(p)=\frac{1}{\Sigma_k p_k^2\Pi_{j\neq k}e_j}(p_4^2e_2e_3+p_3^2e_2e_4+p_2^2e_3e_4)$$
$$+\frac{1}{p^2}\frac{p_1^2}{e_1}\frac{1}{(1+p^2)^s}, \quad (5.5)$$

$$r_0(p)a_2(p)=\frac{-p_2p_1}{\Sigma_k p_k^2\Pi_{j\neq k}e_j}e_3e_4+\frac{p_2p_1}{p^2}\frac{1}{e_1}\frac{1}{(1+p^2)^s}. \quad (5.6)$$

We now write $e_i(p)$ as

$$e_i(p)=1+\delta_i(p), \quad (5.7)$$

near $p=0$, δ_i is small. In (5.5) we substitute (5.7) and set $\delta_i=0$ to get

$$r_0(p)a_1^0(p)\equiv 1+\frac{p_1^2}{p^2}\left(\frac{1}{(1+p^2)^s}-1\right) \quad (5.8)$$

and

$$r_0(p)a_2^0(p)\equiv\frac{p_2p_1}{p^2}\left(\frac{1}{(1+p^2)^s}-1\right). \quad (5.9)$$

Writing

$$a_i(p)=a_i^0(p)+R_i(p), \quad (5.10)$$

we see from (5.8) and (5.9) that $a_i^0(p)$ are analytic in $\mathscr{D}_L$. This is the main "algebraic miracle" involved in showing local analyticity of $A_i^N(p)$. This algebraic miracle motivated, of course, our choice of $X_i(p)$, which, however, was far from unique. We turn our attention to $R_i(p)$. By showing the analyticity in $\mathscr{D}_L$ of R_i we will have completed the proof. (We have already used the analyticity in $\mathscr{D}_L$ of r_L, $p_1\bar{f}_1^{-1}$, and r_0^{-1}.)

All hinges on certain properties of e_i and δ_i in $\mathscr{D}_L$, which we now pursue.

VI. PROPERTIES OF THE $e_i(p)$ IN $\mathscr{D}_L$

(1) $e_i(p)$ is analytic, (6.1)

(2) $\delta_i(p)=p_i^2p^2h_i(p)$ with $h_i(p)$ analytic, (6.2)

(3) $e_i^{-1}(p)$ is analytic, (6.3)

(4) $\left[\frac{1}{\Sigma_k p_k^2\Pi_{j\neq k}e_j}-\frac{1}{p^2}\right]$ is analytic. (6.4)

It is quite immediate that (6.1)–(6.4) imply the analyticity of $R_i(p)$. We are faced with our final task, the proof of these properties. (1) and (2) follow from the form of $e_i(p)$ upon inspection. It is only (3) and (4) that must be studied.

We write $e_1(p)$ in elaborate fashion [$e_1(p)$ and $e_i(p)$, $i\neq 1$, have the same properties],

$$e_1(p)=1+p^2\sum_\rho\left[\chi_\rho(1)\prod_{i\notin\rho}p_i^2K_\rho(p)\right.$$
$$\left.+\chi_{\rho c}(1)p_1^2\prod_{i\notin\rho}p_i^2K_\rho^1(p)\right], \quad (6.5)$$

$$K_\rho(p)=\sum_{n\sim\rho}\frac{1}{(p+2\pi n)^2}\prod_{i\notin\rho}\frac{1}{(p_i+2\pi n_i)^2}, \quad (6.6)$$

$$K_\rho^1(p)=\sum_{n\sim\rho}\frac{1}{(p+2\pi n)^2}\prod_{i\notin\rho}\frac{1}{(p_i+2\pi n_i)^2}$$
$$\times\frac{1}{(p_1+2\pi n_1)^2}, \quad (6.7)$$

where

(a) ρ is a proper subset of (1,2,3,4);

(b) $\chi_\rho(1)=\begin{cases}1 & 1\in\rho,\\ 0 & 1\notin\rho,\end{cases}$

$\chi_{\rho c}(1)=\begin{cases}0 & 1\in\rho,\\ 1 & 1\notin\rho;\end{cases}$

(c) $n\sim\rho$ means that $\rho=\{i,n_i=0\}$.

We easily see, for p in $\mathscr{D}_L$, that

$$|K_\rho(p)|<m,\quad |K_\rho^1(p)|<m, \quad (6.8)$$

for some fixed m. And that

$$|\mathrm{Arg}\,K_\rho(p)|<\epsilon',\quad |\mathrm{Arg}\,K_\rho^1(p)|<\epsilon', \quad (6.9)$$

where ϵ' can be made arbitrarily small by choosing ϵ_0, of (3.2), suitably small. Again we have

$$|\mathrm{Arg}\,p_i^2|<\epsilon''\quad\text{if } |p_i|^2>\bar\epsilon,$$
$$|\mathrm{Arg}\,p^2|<\epsilon''\quad\text{if } |p|^2>\bar\epsilon, \quad (6.10)$$

where ϵ'' and $\bar\epsilon$ can be fixed arbitrarily small, choosing ϵ_0 small enough. Picking ϵ', ϵ'', and $\bar\epsilon$ small enough, and using (6.8), we see $e_1(p)$ is invertible in $\mathscr{D}_L$ and so analytic implying (3), (6.3). [The terms on the right side of (6.5), individually, will either be small, or have small argument.]

We turn to the study of (4), (6.4). We write the brackets in (6.4) as

$$\frac{1}{p^2(1+g)}-\frac{1}{p^2}=\frac{1}{p^2}\frac{g}{1+g}, \quad (6.11)$$

this relation defining g. Property (4) will be proved by showing

(5) g/p^2 is analytic, (6.12)

(6) $(1+g)^{-1}$ is analytic. (6.13)

For g we find the formula

$$g=\sum_k\frac{p_k^2}{p^2}\left[\prod_{j\neq k}(1+\delta_j)-1\right]. \quad (6.14)$$

From (6.2) we see that (5) [(6.12)] holds. The proof that $(1+g)$ is invertible, and so analytic, follows quite immediately from the proof above that $e_1(p)$ is invertible.

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