THE PHYSICS OF FLUIDS VOLUME 15, NUMBER 6 JUNE 1972

Microscopic Theory of Conductivity of a Weakly Ionized Plasma

W. R. CHAPPELL

Joint Institute for Laboratory Astrophysics and Department of Physics and Astrophysics, University of Colorado, Boulder, Colorado 80302

AND

R. H. WILLIAMS

Department of Physics, University of Michigan, Ann Arbor, Michigan 48104

(Received 14 October 1971; final manuscript received 2 February 1972)

Through the use of the microscopic density functions in μ space the conductivity for a weakly ionized plasma can be expressed in terms of a function describing single particle diffusion in the plasma. This diffusion function or transition probability has been shown to obey a linear Boltzmann equation. As a result the approximate solution of the conductivity problem can be accomplished either by modeling the collision term in the kinetic equation or by a stochastic modeling of the diffusion function itself. The two approaches are seen to be completely equivalent.

I. INTRODUCTION

In a recent paper[1] [hereafter, referred to as (I)] we developed kinetic equations for the self and distinct parts of the electron density cross-correlation function of a weakly ionized gas. These kinetic equations were then solved by assuming the BGK model and expressions were obtained for the self and total electron cross-correlation functions.

We would like to demonstrate that this approach can be used to calculate the conductivity of a weakly ionized gas and thereby provide a link between the methods of Dougherty[2] and others[3] and the method recently proposed by Hagfors.[4]

The method used by Hagfors[4] involves a transition probability describing the diffusion of particles in μ space. He emphasizes that such a treatment involving the motion of individual particles in the gas is important for many problems where the results depend intimately on the detailed kinetic behavior of the plasma. On the other hand, Hagfors asserts that the BGK model[5] used by Dougherty[2] and others[3] is outside the scope of his approach.

We will show that, in fact, the approach taken by Hagfors can be interpreted in such a way that the BGK and other relaxation models easily fit into the theory. Thus, it is equally valid to take Hagfors' point of view and construct stochastic models of the transition probability or to construct kinetic equations in which the collisions with neutrals are modeled in some manner.

The reason for this connection between the different approaches is simply that the transition probability defined by Hagfors is directly related to the $W_{1,1}$ function introduced by Rostoker.[6] For a dilute gas of neutral particles this quantity has been shown to obey a linear kinetic equation.[7–10] In (I) we showed that for a weakly ionized gas this quantity obeyed (in lowest order) an equation of the Boltzmann form with a collision operator which is of the Boltzmann form but which is linear.

As a result the conductivity problem can be reduced to solving this linear kinetic equation and the assumption of various collision models provides one method of solution. That this is the case is to be expected because of the standard method of using kinetic equations to calculate the linear response of the system to an applied field and also because of the work of Weinstock[11] which shows that any autocorrelation function (in this case the current autocorrelation) can be related to a one-particle kinetic equation. On the other hand, because of the direct relation to the transition probability used by Hagfors one is free to attack this problem from the point of view of constructing stochastic models for this quantity directly.

II. MICROSCOPIC RESPONSE OF A PLASMA TO AN EXTERNAL ELECTRIC FIELD

We begin by calculating the linear response of the plasma to an electric field from a point of view that is truly microscopic and deals explicitly with individual particle motion. The exact dynamics of a plasma can be expressed in terms of the familiar microscopic phase density

$$\hat{f}_\alpha(X, t) = \sum_{i=1}^{N_\alpha} \delta[X - X_i^\alpha(t)], \tag{1}$$

where the delta function is actually a six-dimensional delta function in μ space and $X_i^\alpha(t) = [\mathbf{R}_i^\alpha(t), \mathbf{v}_i^\alpha(t)]$ is the phase-space position occupied by the ith particle of the αth species at time t. We will consider a three-species system consisting of electrons ($\alpha = e$), ions ($\alpha = i$), and neutrals ($\alpha = 0$).

From this density function $\hat{f}_\alpha$ we can construct several quantities of interest. In particular, the autocorrelation functions

$$f_{\alpha\beta}(X, t; X', t') = \langle \hat{f}_\alpha(X, t) \hat{f}_\beta(X', t') \rangle \tag{2}$$

for $\alpha, \beta = 0, i, e$ are of particular interest in calculating particle density, current density, and electric field autocorrelation functions. Rostoker[6] has introduced the quantities $W_{1,1}{}^\alpha(X, t; X', t')$ and $W_{1,2}{}^{\alpha\beta}(X, t; X', t')$

through the equation

$$f_{\alpha\beta}(X,t;X',t')=\delta_{\alpha,\beta}n_\alpha W_{1,1}{}^{\alpha}(X,t;X',t') + n_\alpha n_\beta W_{1,2}{}^{\alpha\beta}(X,t;X',t'), \quad (3)$$

where n_α is the particle density of species α,

$$W_{1,1}{}^{\alpha}(X,t;X',t') = n_\alpha{}^{-1}\sum_{i=1}^{N_\alpha}\langle\delta[X-X_i{}^\alpha(t)]\delta[X'-X_i{}^\alpha(t')]\rangle, \quad (4)$$

and

$$W_{1,2}{}^{\alpha\beta}(X,t'X',t')=(n_\alpha n_\beta)^{-1}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}(1-\delta_{\alpha,\beta}\delta_{i,j}) \times\langle\delta[X-X_i{}^\alpha(t)]\delta[X'-X_j{}^\beta(t')]\rangle. \quad (5)$$

In (I) we derived kinetic equations for $f_{\alpha e}$, $W_{1,1}{}^e$, and $W_{1,2}{}^{\alpha e}$, where $\alpha=e$, i. These equations are readily extended to $f_{\alpha\beta}$, $W_{1,1}{}^\alpha$, and $W_{1,2}{}^{\alpha\beta}$ where $(\alpha,\beta)=e$, i. In particular, for $W_{1,1}{}^\alpha$ and $f_{\alpha\beta}$ we obtain the results

$$\left(\frac{\partial}{\partial t}+L_1{}^{(0)}(1)\right)W_{1,1}{}^\alpha(1,t;1',0) = J[\phi_0(\mathbf{v}_2)W_{1,1}{}^\alpha(1,t;1',0)] \quad (6)$$

and

$$\left(\frac{\partial}{\partial t}+L_1{}^{(0)}(1)\right)f_{\alpha\beta}(1,t;1',0)+\frac{n_\alpha q_\alpha}{m_\alpha}\mathbf{E}_\beta(\mathbf{R}_1,t) \cdot\frac{\partial}{\partial\mathbf{v}_1}\phi_\alpha(\mathbf{v}_1)=J[\phi_0(\mathbf{v}_2)f_{\alpha\beta}(1,t;1',0)], \quad (7)$$

where

$$L_1{}^{(0)}(1)=\mathbf{v}_1\cdot\frac{\partial}{\partial\mathbf{R}_1}, \quad (8)$$

where $1=X_1$, $1'=X_1'$, $(\alpha,\beta)=e$, or i, q_α is the charge of a particle of species α, $\phi_0(\mathbf{v}_2)$ and $\phi_\alpha(\mathbf{v}_1)$ are the Maxwell–Boltzmann distributions for the neutrals and species $\alpha(=e,i)$, respectively, and $J[\ ]$ is the usual Boltzmann collision operator. The quantity $E_\beta(\mathbf{R}_1,t)$ is given by

$$\nabla\cdot\mathbf{E}_\beta(\mathbf{R}_1,t)=4\pi\sum_{\alpha=e,i}q_\alpha\int d\mathbf{v}_1 f_{\alpha\beta}(1,t;1',0). \quad (9)$$

We could proceed from here to obtain expressions for the conductivity since it is known[12] that the conductivity can be expressed in terms of the current density autocorrelation function and is, therefore, directly related to the quantity $f_{\alpha\beta}$. The choice of various models for the charged particle-neutral collision term would then give approximate expressions for the conductivity.

Instead, however, we will use an approach similar to Hagfors' which in an explicit way involves the dependence of the conductivity on the individual particle motions. In order to do this we will use "approximate" equations for the microscopic densities $f_\alpha(X,t)$. By approximate, we refer to the difficulty posed by the singular nature of the exact phase space densities. Thus, the validity of such approximations can only be discussed in terms of smoothing operations (e.g., averaging) which are assumed to occur later. Such equations have been used to treat both fully ionized plasmas[13,14] and gases of neutral particles.[9,15] In the case of a fully ionized gas the use of the linearized Vlasov equation for $\hat{f}_\alpha$ has been shown to be equivalent to the random phase approximation.[13,14] Since the variables X_1' and t' act only as parameters in Eq. (7), we choose the kinetic equation for $\hat{f}_\alpha$ as

$$\left(\frac{\partial}{\partial t}+L_1{}^{(0)}(1)\right)\hat{f}_\alpha(1,t)+\frac{n_\alpha q_\alpha}{m_\alpha}\hat{\mathbf{E}}(\mathbf{R}_1,t)\cdot\frac{\partial}{\partial\mathbf{v}_1}\phi_\alpha(\mathbf{v}_1) = J[\phi_0(\mathbf{v}_2)\hat{f}_\alpha(1,t)], \quad (10)$$

where

$$\nabla\cdot\hat{\mathbf{E}}(\mathbf{R}_1,t)=4\pi\sum_\alpha q_\alpha\int d\mathbf{v}_1\hat{f}_\alpha(1,t). \quad (11)$$

The term on the right-hand side then represents the correction to the usual random phase approximation equation[1] due to the presence of neutral particles.

Since Eq. (10) is linear in $\hat{f}_\alpha$, we can write it in the form

$$\frac{\partial\hat{f}_\alpha}{\partial t}+L\hat{f}_\alpha=0. \quad (12)$$

If we were to apply a small external electric field $\mathbf{E}_0(r,t)$ to the plasma, the equation for $\hat{f}_\alpha$ would become

$$\frac{\partial\hat{f}_\alpha}{\partial t}+L\hat{f}_\alpha+\frac{q_\alpha}{m_\alpha}\mathbf{E}_0\cdot\frac{\partial\hat{f}_\alpha}{\partial\mathbf{v}}=0. \quad (13)$$

We can now solve for the linear *microscopic* response in the usual manner by writing

$$\hat{f}_\alpha=\hat{f}_\alpha{}^0+\hat{f}_\alpha{}', \quad (14)$$

so that

$$\frac{\partial\hat{f}_\alpha{}^0}{\partial t}+L\hat{f}_\alpha{}^0=0 \quad (15)$$

and

$$\frac{\partial\hat{f}_\alpha{}'}{\partial t}+L\hat{f}_\alpha{}'+\frac{q_\alpha}{m_\alpha}\mathbf{E}_0(\mathbf{r},t)\cdot\frac{\partial\hat{f}_\alpha{}^0}{\partial\mathbf{v}}=0. \quad (16)$$

Equation (15) expresses the fact that

$$\hat{f}_\alpha{}^0=\sum_{i=1}^{N_\alpha}\delta[X-X_i{}^{\alpha0}(t)], \quad (17)$$

where $X_i{}^{\alpha0}(t)$ is the field-free trajectory in μ space.

We then solve Eq. (16) formally as

$$\hat{f}_\alpha{}'(X,t)=\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{t}d\tau\exp[-L(t-\tau)] \times\mathbf{E}_0(\mathbf{r},\tau)\cdot\frac{\partial}{\partial\mathbf{v}}\hat{f}_\alpha{}^0(X,\tau). \quad (18)$$

From the delta function form of $\hat{f}_\alpha^0$ we see that

$$\frac{q_\alpha}{m_\alpha}\mathbf{E}_0(\mathbf{r},\tau)\cdot\frac{\partial\hat{f}_\alpha^0(X,\tau)}{\partial\mathbf{v}}=-\sum_\beta\sum_{j=1}^{N_\beta}\frac{q_\beta}{m_\beta}\mathbf{E}_0(\mathbf{r}_j^{\beta0}(\tau),\tau)\cdot\frac{\partial}{\partial\mathbf{v}_j^{\beta0}(\tau)}\hat{f}_\alpha^0(X,\tau) \quad (19)$$

(using the fact that the $\beta\neq\alpha$ terms are zero). We are interested in calculating the ensemble averaged microscopic density $f_\alpha=\langle\hat{f}_\alpha\rangle$. From Eqs. (18) and (19) we obtain

$$f_\alpha=\int d\Gamma\, f_N(\Gamma)\hat{f}_\alpha(X,t)$$
$$=\sum_{j=1}^{N_\beta}\int_{-\infty}^{t}dt\int d\Gamma\, f_N(\Gamma)\frac{q_\beta}{m_\beta}\mathbf{E}_0[\mathbf{r}_j^{\beta0}(\tau),\tau]\cdot\frac{\partial}{\partial\mathbf{v}_j^{\beta0}(\tau)}\exp[-L(t-\tau)]\hat{f}_\alpha^0(X,\tau), \quad (20)$$

where Γ is the Γ-space variable having $\sum_\alpha 6N_\alpha=N$ components and $f_N(\Gamma)$ is the equilibrium ensemble. Since $d\Gamma$ and $f_N(\Gamma)$ are invariant under the field-free motion of the particles, on integration by parts we obtain

$$f_\alpha=(KT)^{-1}\sum_\beta\sum_{j=1}^{N_\beta}\int_{-\infty}^{t}d\tau\int d\Gamma f_N(\Gamma)q_\beta\mathbf{E}_0[\mathbf{r}_j^{\beta0}(\tau),\tau]\cdot\mathbf{v}_j^{\beta0}\exp[-L(t-\tau)]\hat{f}_\alpha^0(X,\tau), \quad (21)$$

where T is the temperature and K is Boltzmann's constant. Finally, using the delta function form of $\hat{f}_\beta^0$ and Eq. (15) we obtain

$$f_\alpha(X,t)=(KT)^{-1}\sum_\beta\int_{-\infty}^{t}d\tau\int d\Gamma\int dX'f_N(\Gamma)\times q_\beta\mathbf{E}_0(\mathbf{r}',\tau)\cdot\mathbf{v}'\hat{f}_\beta^0(X',\tau)\hat{f}_\alpha^0(X,t)$$
$$=(KT)^{-1}\sum_\beta\int_0^\infty d\tau\int dX'q_\beta\mathbf{v}'\cdot\mathbf{E}_0(\mathbf{r}',t-\tau)\times\langle\hat{f}_\alpha^0(X,t)\hat{f}_\beta^0(X',t-\tau)\rangle \quad (22)$$
$$=(KT)^{-1}\sum_\beta\int_0^\infty d\tau\int dX'q_\beta\mathbf{v}'\cdot\mathbf{E}_0(\mathbf{r}',t-\tau)\times f_{\alpha\beta}^0(X,t;X',t-\tau),$$

where $f_{\alpha\beta}^0$ is the quantity first introduced in Eq. (3). It is proportional to the joint probability of finding a particle of type β at X' at time $t-\tau$ and a particle of type α at X, at time τ later.

The conductivity is obtained by calculating the current density $\mathbf{J}(\mathbf{r},t)$ produced by the external electric field. From Eq. (22) this is given by

$$J(\mathbf{r},t)=\sum_\alpha q_\alpha\int d\mathbf{v}\,\mathbf{v}f_\alpha(\mathbf{r},\mathbf{v},t)$$
$$=(KT)^{-1}\sum_{\alpha,\beta}q_\alpha q_\beta\int_0^\infty d\tau\int dX'\int d\mathbf{v}\times\mathbf{v}\mathbf{v}'\cdot\mathbf{E}_0(\mathbf{r}',t-\tau)f_{\alpha\beta}^0(X,t;X',t-\tau) \quad (23)$$
$$=(KT)^{-1}\int_0^\infty d\tau\int d\mathbf{r}'\langle\mathbf{J}(\mathbf{r},t)\mathbf{J}(\mathbf{r}',t-\tau)\rangle\cdot\mathbf{E}_0(\mathbf{r}',t-\tau),$$

where $\langle\mathbf{J}(\mathbf{r},t)\mathbf{J}(\mathbf{r}',t-\tau)\rangle$ is the current autocorrelation function. If we then perform a Fourier transform in space and time, we obtain

$$\mathbf{J}(\mathbf{k},\omega)=\boldsymbol{\sigma}(\mathbf{k},\omega)\cdot\mathbf{E}_0(\mathbf{k},\omega), \quad (24)$$

where

$$\boldsymbol{\sigma}(\mathbf{k},\omega)=(KT)^{-1}\int_0^\infty d\tau\exp(i\omega\tau)\int d\mathbf{R}\exp(-i\mathbf{k}\cdot\mathbf{R})\times\langle\mathbf{J}(\mathbf{R},\tau)\mathbf{J}(0,0)\rangle \quad (25)$$
$$=(KT)^{-1}\sum_{\alpha,\beta}\int_0^\infty d\tau\exp(i\omega\tau)\int d\mathbf{R}\exp(-i\mathbf{k}\cdot\mathbf{R})\times q_\alpha q_\beta\int d\mathbf{v}\int d\mathbf{v}'\mathbf{v}\mathbf{v}'f_{\alpha\beta}^0(\mathbf{R},\mathbf{v},\tau;0,\mathbf{v}',0).$$

The expression given in the first line of Eq. (25) is the familiar result of linear response theory.[12]

We can put our result in the form obtained by Hagfors by noting that from Eqs. (3), (4), (5), and (15) we obtain

$$f_{\alpha\beta}^0(X,\tau;X',0)=\exp(-L\tau)[\delta_{\alpha,\beta}n_\alpha W_{1,1}^\alpha(X,0;X',0)+n_\alpha n_\beta W_{1,2}(X,0;X',0)] \quad (26)$$
$$=\exp(-L\tau)[\delta_{\alpha,\beta}\delta(X-X')\phi_\alpha(\mathbf{v})+n_\alpha n_\beta F_{\alpha\beta}(X,X')],$$

where $F_{\alpha\beta}(X,X')$ is the usual two-particle (one time) distribution function. Since the variable X' is unaffected by the operator L, and since $F_{\alpha\beta}$ is even in $\mathbf{v}'$, we obtain

$$\boldsymbol{\sigma}(\mathbf{k},\omega)=(KT)^{-1}\sum_\alpha\int_0^\infty d\tau\exp(i\omega\tau)\int d\mathbf{R}\exp(-i\mathbf{k}\cdot\mathbf{R})\times n_\alpha q_\alpha^2\mathbf{v}\mathbf{v}'W_{1,1}^\alpha(\mathbf{R},\mathbf{v},\tau;0,\mathbf{v}',0). \quad (27)$$

But, we can write

$$\mathbf{W}_{1,1}^\alpha(\mathbf{R},\mathbf{v},\tau;0,\mathbf{v}',0)=W_+(\mathbf{R},\mathbf{v},\tau\,|\,0,\mathbf{v}',0)\phi_\alpha(\mathbf{v}), \quad (28)$$

where W_+ is the transition probability introduced by

Hagfors. It is the probability that if a particle is at the phase space point $(0, \mathbf{v}')$, time 0, it will be found at $(\mathbf{R}, \mathbf{v})$ at time τ. This completes the connection between the work described in (I), the linear response theory, and Hagfors' derivation. All that remains is the calculation of $W_{1,1}^{\alpha}$ or W_{+}. Since we know that $W_{1,1}^{\alpha}$ satisfies a kinetic equation, we can try to solve this equation or we can use the interpretation as a transition probability and take the approach used by Hagfors.

III. CALCULATION OF $W_{1,1}^{\alpha}$

In fact, the construction of various models for $W_{1,1}^{\alpha}$ or W_{+} implies a choice of model of the collision term $J[\phi_0 W_{1,1}^{\alpha}]$ which appears in the kinetic equation for $W_{1,1}^{\alpha}$ [cf. Eq. (6)]. The simplest possible models are the free particle models

$$\frac{\partial W_{1,1}^{\alpha}}{\partial t}+\mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{R}_1} W_{1,1}^{\alpha}=0 \qquad (29)$$

and the simple relaxation term

$$\frac{\partial W_{1,1}^{\alpha}}{\partial t}+\mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{R}_1} W_{1,1}^{\alpha}=-\nu_{\alpha 0}W_{1,1}^{\alpha}. \qquad (30)$$

A considerably better model is the BGK model

$$\frac{\partial}{\partial t}+\mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{R}_1} W_{1,1}^{\alpha}(1, t)$$

$$=-\nu_{\alpha 0}\left[W_{1,1}^{\alpha}(1, t)-\phi_\alpha(\mathbf{v}_1)\int d\mathbf{v}_1 W_{1,1}^{\alpha}(1, t)\right]. \qquad (31)$$

This equation is easily solved to obtain

$$\sigma(\mathbf{k}, \omega)=\sum_{\alpha}\beta q_{\alpha 2}^2\left\{i\int d\mathbf{v}_1\frac{\mathbf{v}_1\mathbf{v}_1\phi_\alpha(\mathbf{v}_1)}{\omega-\mathbf{k}\cdot\mathbf{v}_1+i\nu_{\alpha 0}}-[F_\alpha(\mathbf{k}, \omega)]^{-1}\right.$$
$$\left.\times\left[\int d\mathbf{v}_1\frac{\mathbf{v}_1\phi_\alpha(\mathbf{v}_1)}{\omega-\mathbf{k}\cdot\mathbf{v}_1+i\nu_{\alpha 0}}\right]\left[\int d\mathbf{v}_1'\frac{\mathbf{v}_1'\phi_\alpha(\mathbf{v}_1')}{\omega-\mathbf{k}\cdot\mathbf{v}'+i\nu_{\alpha 0}}\right]\right\}, \qquad (32)$$

where

$$F_\alpha(\mathbf{k}, \omega)=1-i\nu_{\alpha 0}\int d\mathbf{v}[\phi_\alpha(\mathbf{v})/(\omega-\mathbf{k}\cdot\mathbf{v}+i\nu_{\alpha 0})]. \qquad (33)$$

This is the result obtained by Dougherty.[2] This model has also been called the strong collision model by Rautian and Sobel'man[16] who discussed it in connection with the collisional narrowing of spectral lines. In (I) this model was used to calculate the self- and total-electron-density autocorrelation functions.

The effect of weak collisions can be modeled by a Fokker–Planck term giving

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{R}_1}\right) W_{1,1}^{\alpha}(1, 1', t)$$

$$=\nu_{\alpha 0}\frac{\partial}{\partial \mathbf{v}_1}\cdot\mathbf{v}_1 W_{1,1}^{\alpha}+q\frac{\partial^2}{\partial \mathbf{v}_1^2} W_{1,1}^{\alpha}. \qquad (34)$$

The solution of this equation by standard means[16] gives the Brownian motion model for $W_{1,1}$ proposed by Hagfors.[4] Rautian and Sobel'man in their work on spectral line shapes have considered the case where the collision term is modeled by a sum of the BGK and Fokker–Planck terms.[16] The Lorentz model presents another attractive possibility for describing weakly ionized plasmas. On the other hand, one could as well choose to bypass the kinetic equation and model W_{+} directly as proposed by Hagfors.[4]

IV. CONCLUSION

The case of a plasma dominated by charged particle-neutral collisions can be treated in the same manner as a collisionless plasma. Such quantities as the conductivity can be related to the microscopic motions and through these to a function $W_{1,1}$ describing the diffusion of charged particles in the plasma. This function has been shown to obey a linear, Boltzmann-like equation. The final solution can be approached via a solution of the kinetic equation or a direct modeling of this function. Although it is true that the final description of the particle motion is probabilistic, the same can be said of the kinetic equation itself since for long times[1] it represents a Markovian behavior. The primary difficulty lies in the choice of model either for the kinetic equation or the transition probability described by the equation. The assumption of a model for W_{+} or $W_{1,1}$ implies the assumption of a model for the kinetic equations for these functions and vice versa.

ACKNOWLEDGMENTS

This research was supported in part by the Advanced Research Projects Agency, The Department of Defense and was monitored by U.S. Army Research Office-Durham, Durham, North Carolina 27706, under Contract No. DA-31-124-ARO-D-139.

[1] R. H. Williams and W. R. Chappell, Phys. Fluids **14**, 591 (1971).
[2] J. P. Dougherty, J. Fluid Mech. **16**, 126 (1963).
[3] R. M. Lewis and J. B. Keller, Phys. Fluids **5**, 1248 (1962).
[4] T. Hagfors, Phys. Fluids **13**, 2790 (1970).
[5] P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).
[6] N. Rostoker, Phys. Fluids **7**, 479, 491 (1964).
[7] J. M. J. van Leeuwen and S. Yip, Phys. Rev. **149**, A1138 (1965).
[8] D. Montgomery, Phys. Fluids **12**, 804 (1969).
[9] F. L. Hinton, Phys. Fluids **13**, 857 (1970).
[10] W. R. Chappell, J. Stat. Phys. **2**, 267 (1970).
[11] J. Weinstock, Phys. Rev. **139**, A388 (1965).
[12] S. Bekefi, *Radiation Processes in Plasmas* (Wiley, New York, 1966), p. 108.
[13] W. R. Chappell, J. Math. Phys. **8**, 298 (1967).
[14] J. K. Percus and G. Yevick, Phys. Rev. A **2**, 1526 (1970).
[15] M. Bixon and R. Zwanzig, Phys. Rev. **187**, 267 (1969).
[16] S. G. Rautian and I. I. Sobel'man, Usp. Fiz. Nauk **90**, 209 (1966) [Sov. Phys. Usp. **9**, 701 (1967)].