On some properties of solutions of Helmholtz equation [a)]

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We give a new method to prove results of the following type. Let: $(\nabla^2 + k^2)u = 0$ in $D_R = \{x \mid |x| \geqslant R\}$, $k^2 > 0$. (1) If $u \in L^2(D_R)$, then $u \equiv 0$ in D_R. (2) If $|x|^m u(x) \to 0$ as $|x| \to \infty$, $x_1^2 + \cdots + x_{N-1}^2 \leqslant c x_N^{-2p}, p > 0$, $m = 1, 2, 3, \ldots$, $|x|(\partial u/\partial |x| / - iku) \underset{|x| \to \infty}{\longrightarrow} 0$ then $u \equiv 0$ in D_R.

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1. INTRODUCTION

Some of the above results were proved by a different method in Refs. 1 and 2, but the new method of the proof is of interest in our opinion. We start with the following theorem.

Theorem 1: Let

$$(\nabla^2 + k^2)u = 0 \quad \text{in } D_R, \quad k^2 > 0. \tag{1}$$

and

$$u \in L^2(D_R). \tag{2}$$

Then $u \equiv 0$ in D_R.

Proof. From (1) and (2) it follows that (see Appendix)

$$\nabla u \in L^2(D_R) \tag{3}$$

and

$$u(x) = \int_{S_R} \left\{ g^+(x,t,k) \frac{\partial u}{\partial N} - u \frac{\partial g^+(x,t,k)}{\partial N} \right\} dt = \int_{S_R} \left(g^- \frac{\partial u}{\partial N} - u \frac{\partial g^-}{\partial N} \right) dt, \tag{4}$$

where N is the unit normal to the sphere $S_R = \{x : |x| = R\}$ directed outside of D_R,

$$g^{\pm}(x,y,k) = \frac{\exp(\pm ik|x-y|)}{4\pi|x-y|},$$
$$|x-y| = (r^2 - 2r|y|\cos\widehat{xy} + |y|^2)^{1/2}, \quad r = |x|. \tag{5}$$

Now the main idea can be explained. We analytically continue functions (4) on the complex plane $z = r\exp(i\psi)$ (see also Refs. 3–5). From (4) it follows that ($\omega = x|x|^{-1}$)

$$u(x) = u(r,\omega) = u(z,\omega) = \frac{\exp(ikz)}{z} f_1(z,\omega) = \frac{\exp(-ikz)}{z} f_2(z,\omega), \tag{6}$$

where $f_1(z,\omega)$ and $f_2(z,\omega)$ are analytic in z for $|z| > R$ and bounded near infinity. Thus

$$f_j(z,\omega) = \sum_{s=0}^{\infty} f_{js}(\omega) z^{-s}, \quad j = 1,2. \tag{7}$$

But in this case (6) implies that $f_1 = f_2 \equiv 0$. Indeed, if $z = iy$, $y \to +\infty$ then $([\exp(ikz)]/z) f_1(z,\omega)$ in (6) goes to zero exponentially, while $([\exp(-ikz)]/z) f_2(z,\omega)$ goes to infinity exponentially unless $f_2 \equiv 0$. Thus $u(z,\omega) \equiv 0$, $u(r,\omega) \equiv 0$ in D_R.

Let us show how the idea works in a different problem.

2. RATE OF DECREASE OF SOLUTIONS TO HELMHOLTZ EQUATION

Theorem 2: Let (1) hold, u satisfies the radiation condition and

$$|x|^m u(x) \to 0 \quad \text{as} \quad x \to \infty, \quad \rho \leqslant c|x_3|^{-p}$$
$$c = \text{const} > 0, \quad 0 \leqslant p, \quad m = 1,2,3,\ldots, \tag{8}$$

where $\rho = (x_1^2 + x_2^2)^{1/2}$, then $u \equiv 0$ in D_R.

Proof: Since u satisfies (1) and the radiation condition, we can use the first equality in (4) and the third equality in (6). If $p = 0$ the condition (8) says that $u(x)$ decreases faster than any negative power of $|x|$ at infinity in the cylinder $\rho \leqslant c$. From this and (6) it follows that $f_1 \equiv 0$ for ω directed along the axis x_3. By shifting the origin a little, we conclude that $f_1 = 0$ along any ray in the cylinder $\rho \leqslant c$. Thus $u \equiv 0$ in this cylinder and by unique continuation theorem for solutions of homogeneous elliptic equations $u \equiv 0$ in D_R. If $p > 0$ our argument is a little more complicated. In this case let us write the equation $\rho = c|x_3|^{-p}$ in the spherical coordinates: $r^{p+1}\cos^p\theta \cdot \sin\theta = c$. For large r the angle θ is near 0, and $\theta = \theta(r)$ is an analytic bounded function of cr^{-1-p} for large r. Let us prove that $u = 0$ in the body $r^{p+1}\cos^p\theta\sin\theta \leqslant c$. From this and the unique continuation theorem we conclude that $u \equiv 0$ in D_R. Let us take in (4) $x = (r, \theta_b(r))$, where $\theta_b(r)$ is constructed as $\theta(r)$ but instead of c we use $0 < b < c$. For simplicity we shall write $\theta(r)$ instead of $\theta_b(r)$ in what follows. Then $\omega = \omega(r)$ and $\omega(r)$ is an analytic and bounded function of the argument br^{-1-p} for large r. From this it follows that $f_1(r,\omega(r))$ will be analytic and bounded near infinity on an appropriate Riemann surface (which by the way will be finite-sheeted for rational p. Since we can always find a rational number $p_1 > p$ such that the body $r^{1+p}\cos^p\theta \sin\theta = b$ contains the body $r^{1+p_1}\cos^{p_1}\theta\sin\theta = b$, we can consider only finite-sheeted Riemannian surfaces). If $f_1(z)$ decreases faster than any negative power of z on such a surface, $f_1 \equiv 0$. Thus $u(x) = 0$ on any curve $r^{p+1}\cos^p\theta\sin\theta = b \leqslant c$ and we conclude that $u \equiv 0$ in D_R.

3. GENERALIZATIONS

(1) We can consider general elliptic equations with constant coefficients in $\mathbb{R}^N$.

(2) It is possible to consider the case when $u(x)$ is a solution of (1) in a domain with infinite boundary.

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1. APPENDIX

Lemma 1: From (1) and (2) inclusion (3) follows. Let

$$g(r) = \int_R^r dr\, r^2 \int_{S_1} |u(r, \omega)|^2\, d\omega,$$

$g(r)$ increases monotonically and $g(\infty) < \infty$,

$$g'(r) = r^2 \int_{S_1} |u(r,\omega)|^2\, d\omega \geqslant 0, g'' = 2r \int_{S_1} |u|^2\, d\omega + 2r^2 \int_{S}^{\prime} \frac{\partial u(r,)}{\partial r} u(r,\omega)\, d\omega.$$

Here we assume without any loss of generality that u is a real valued function [since the coefficients of Eq. (1) are real]. If $g''(r_n) \to 0$ as $r_n \to \infty$, then

$$\int_{S_{r_n}} \frac{\partial u}{\partial r} u\, dS_{r_n} \to 0 \quad \text{as} \quad z_n \to \infty. \tag{A1}$$

But from (1) it follows that

$$\int_{R \leq |x| \leq r_n} |\nabla u|^2\, dx = k^2 \int_{R \leq |x| \leq r_n} |u|^2\, dx + \int_{S_{r_n}} u \frac{du}{dr} dS_{r_n} + \int_{S_R} u \frac{\partial u}{\partial N} dS_R. \tag{A2}$$

From (A1), (A2), and (2) we get (3). If $g''(r)$ does not go to zero whatever sequence $r_n \to \infty$ we choose, then $|g''| \geqslant \epsilon > 0$ for all $r \geqslant R_1 \geqslant R$. If $g'' \geqslant \epsilon$, then $g'(r) \to +\infty$. This is impossible because of (2). If $g'' \leqslant -\epsilon$, then $g'(r) \to -\infty$. Again this is impossible because of (2). This completes the proof.

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