THE PHYSICS OF FLUIDS VOLUME 4, NUMBER 3 MARCH, 1961

Stability of a Stokesian Fluid in Couette Flow

WILLIAM P. GRAEBEL
The University of Michigan, Ann Arbor, Michigan
(Received August 4, 1960; revised manuscript received December 15, 1960)

The stability of a Stokesian fluid (Reiner-Rivlin fluid) in Couette motion is examined and shown to depend on the Taylor number as well as a further dimensionless parameter which is proportional to the coefficient of cross viscosity. The method of Chandrasekhar is used for small values of this parameter. It is found that for fluids with a positive coefficient of cross viscosity, the critical Taylor number can be appreciably smaller than for the corresponding flow of a Newtonian fluid.

I. INTRODUCTION

PROBLEMS in the flow of non-Newtonian fluids have in recent years received a growing amount of attention, both theoretically and experimentally. The mathematical description of such a fluid is provided by the relation between the stress tensor and the various kinematic and thermodynamic variables, this relation being called the constitutive equation. The choice as to which relation is best suited to describe a particular fluid is not at all a simple one, as certain non-Newtonian effects may be shown theoretically to be common to a widely varying assortment of constitutive equations,[1] and even experimental work can lead to conflicting results.[2] A prime difficulty here has been the lack of sufficient analytical solutions of distinctive enough natures with which experimental results could be compared, for in the few flows hitherto examined the departures from the results for Newtonian fluids were in many cases difficult to measure.

Accordingly, in an effort to contribute to the understanding of some of these non-Newtonian fluids, the present paper analyzes the stability of a non-Newtonian fluid in Couette flow using a constitutive equation first proposed by Reiner.[3] Such a fluid has been shown by Reiner to exhibit normal stress effects in Couette flow (see references 1 and 2), and by Ericksen[4] to behave peculiarly when flowing through noncircular conduits, and thus some of its characteristics are fairly well known. Rheologically it lies close to the Newtonian fluid, in that the constitutive equation may be derived mathematically by starting with the same assumptions required of a Newtonian fluid excepting linearity. Nevertheless, as will be seen, the extent of the departure from the Newtonian case and from what might physically be expected is very pronounced.

Prior to publishing this work, the author came upon a previous paper by Jain[5] which treats the same problem by a variational technique. The conclusion which Jain draws is contrary to the author's own. Examination of Jain's work (which at best can give an upper bound on the critical Taylor number) reveals several errors which would affect the numerical results. Further, Jain gives only one point on the neutral stability curve and does not exploit his results further. For these reasons an additional treatment of the problem is considered important.

II. FORMULATION OF THE DYNAMIC EQUATIONS

The most general isotropic relation between stresses and rates of deformation for a visco-inelastic fluid has been shown to be given by

$$\tau_i^i = (-p + A^{(0)})\,\delta_i^i + 2A^{(1)}\,d_i^i + 2A^{(2)}\,d_k^i\,d_i^k, \quad (1)$$

where d_i^i is the rate-of-deformation tensor defined by

$$d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \quad (2)$$

and δ_i^i is the Kronecker delta.[6] The subscript after the comma indicates covariant differentiation with

[1] See, for instance, J. G. Oldroyd, Proc. Roy. Soc. (London) **A200**, 523 (1950); F. H. Garner and A. H. Nissan, Nature **158**, 634 (1946); K. Weissenberg, Proc. 1st Intern. Rheological Congr. Amsterdam, I, 29, 46, 1948; *ibid.* **II**, 114; J. M. Burgers, Proc. Acad. Sci. Amsterdam **51**, 787 (1948); M. Mooney, J. Colloid Sci. **6**, 96 (1951), for various constitutive equations, all of which describe fluids exhibiting the normal stress effect described by K. Weissenberg, Nature **159**, 310 (1947).

[2] See M. Reiner, *Handbuch der Physik* (Springer-Verlag, Berlin, 1958), Vol. 6, p. 516, for a brief account of one such controversy.

[3] M. Reiner, Am. J. Math. **67**, 350 (1945).

[4] J. L. Ericksen, Quart. Appl. Math. **14**, 318 (1956).

[5] M. K. Jain, J. Sci. and Engr. Research **1**, 195 (1957).

[6] C. Truesdell, J. Ratl. Mech. Anal. **1**, 125 (1952), credits Reiner (reference 3) with first stating this law. In deriving Eq. (1) Reiner assumes the stress tensor to be a polynomial in the rate of deformation tensor. The validity of this assumption was first shown by R. S. Rivlin [Proc. Roy. Soc. (London) **A193**, 260 (1948)], and later by J. Serrin, J. Math. and Mech. **8**, 459 (1959), who assume that the stress components are arbitrary functions of the rate of deformation tensor.

respect to that coordinate. A fluid governed by Eq. (1) has been variously called a Stokesian fluid or a Reiner-Rivlin fluid. If the fluid is considered to be incompressible so that the pressure is not connected with any of the thermodynamic coordinates, and if none of the boundary conditions involve the pressure, $A^{(0)}$ will always appear with the pressure and may thus be absorbed in it with no loss of generality, since what is called "pressure" in an incompressible fluid is somewhat arbitrary. For a Newtonian fluid, obviously $A^{(1)}=\mu$, $A^{(2)}=0$. It might also be noted that this constitutive equation sometimes appears in the literature in the disguised form

$$\tau_j^i = (-p + A^{(0)})\,\delta_j^i + 2\mu_k^i\,d_j^k\,, \tag{3}$$

where

$$\mu_k^i = A^{(1)}(\delta_k^i + d_k^i A^{(2)}/A^{(1)}) \tag{4}$$

is called the viscosity tensor. Truesdell[7] has suggested on dimensional grounds that the ratio $A^{(2)}/A^{(1)}$ (for constant $A^{(1)}$, $A^{(2)}$) be called the natural time of the fluid.

Equation (1) together with the equations of motion and continuity

$$\tau^{ik}_{,k} + \rho X^i = \rho\left(\frac{\partial v^i}{\partial t} + v^i_{,j}v^j\right), \tag{5}$$

$$d_i^i = v^i_{,i} = 0, \tag{6}$$

constitute the field equations for the problem. Upon substituting Eq. (1) into Eq. (5) and using Eqs. (2) and (6), the result is obtained

$$-g^{ij}\frac{\partial p}{\partial x^j} + 2\frac{\partial A^{(1)}}{\partial x^k}d^{ik} + A^{(1)}\overline{\nabla}^2 v^i$$
$$+2\frac{\partial A^{(2)}}{\partial x^k}g_{lm}\,d^{lk}\,d^{mi} + A^{(2)}g_{lm}\,[d^{mi}\overline{\nabla}^2 v^l$$
$$+ 2\,d^{mi}_{,k}\,d^{lk}] + \rho X^i = \rho\left[\frac{\partial v^i}{\partial t} + v^j v^i_{,j}\right], \tag{7}$$

where g^{ij} is the metric tensor and $\overline{\nabla}^2 = _{,st}g^{st}$ is the generalized Laplacian.

In general, $A^{(1)}$ and $A^{(2)}$ can be functions of the invariants of the rate of deformation tensor. Frequently in the literature, $A^{(1)}$ and $A^{(2)}$ are taken as constants. Under this assumption, $A^{(2)}$ is called the "cross viscosity." The validity of this assumption is debatable until more is understood about the rheological properties of such a fluid, although certainly it should be valid at least for small rates of deformation. In the following analysis, $A^{(1)}$ and $A^{(2)}$ will be considered constant, for even under this restriction some light can be thrown on the behavior of non-Newtonian fluids.

[7] C. Truesdell, reference 6.

III. PRIMARY FLOW (STEADY COUETTE FLOW)

For steady Couette flow, $v^1 = v^3 = 0$, $v^2 = v^2(x^1)$. Therefore in terms of the contravariant components in cylindrical polar coordinates,

$$d^{12} = d^{21} = \tfrac{1}{2}(\partial v^2/\partial x^1), \tag{8}$$

and all other components of d^{ij} are zero.

Equation (7) then becomes

$$-\frac{\partial p}{\partial x^1} + \frac{1}{2}\frac{\partial}{\partial x^1}\left\{A^{(2)}\left(x^1\,\frac{\partial v^2}{\partial x^1}\right)^2\right\} = -\rho x^1(v^2)^2,$$
$$\frac{\partial}{\partial x^1}\left(A^{(1)}\,\frac{\partial v^2}{\partial x^1}\right) + \frac{3A^{(1)}}{x^1}\frac{\partial v^2}{\partial x^1} = 0, \tag{9}$$
$$-\partial p/\partial x^3 - \rho g = 0.$$

If the inner cylinder of radius R_1 is rotating with an angular velocity Ω_1 and the outer cylinder of radius R_2 is rotating with an angular velocity Ω_2, Eq. (9) along with the condition of no slip at the boundaries yields[8]

$$v^2 = \Omega_1[A - B(R_1/x^1)^2], \tag{10}$$

$$\partial p/\partial x^1 = \rho x^1(\Omega_1)^2\{[A - B(R_1/x^1)^2]^2 - (8B^2A^{(2)}/\rho R_1^2)(R_1/x^1)^6\}, \tag{11}$$

where

$$A = \frac{\Omega_2/\Omega_1 - (R_1/R_2)^2}{1 - (R_1/R_2)^2}, \tag{12}$$

$$B = \frac{\Omega_2/\Omega_1 - 1}{1 - (R_1/R_2)^2}. \tag{13}$$

It is seen that all of the stress components except $\tau_3^3 = -p$ are exactly the same as for the Newtonian fluid. If $A^{(2)}$ is positive, the pressure is increased over that in a Newtonian fluid, the increase being greatest on the inner cylinder, giving rise to normal stress effects. Thus, theoretically at least, it is possible to determine $A^{(2)}$ from pressure measurements. In practice, the cross-viscosity term in Eq. (11) can be expected to be much smaller than the other terms, and the error due to the cylinders being of finite length would make any conclusions drawn on

[8] Rivlin, Truesdell, and Serrin (references 6) among others have previously presented the solution described in this section.

such an experiment questionable. A solution for the stability problem exhibiting a different parameter is thus much to be desired.

IV. THE STABILITY EQUATIONS

As in previous stability analyses, it is assumed that the flow develops disturbances of infinitesimal amplitude (so that the resulting equations may be linearized), and that these disturbances are rotationally symmetric. With e^{ij}, u^i, and p' denoting, respectively, the rate of deformation tensor, the velocity vector, and the pressure for the disturbance, Eq. (7) becomes, after linearization,

$$\begin{aligned} -g^{ii}\frac{\partial p'}{\partial x^i} + A^{(1)}\overline{\nabla}^2 u^i + \frac{A^{(2)}B\Omega_1 R_1^2}{(x^1)^3}\Big\{2g_{22}\frac{\partial e^{2i}}{\partial x^1} \\ + \delta_1^i(g_{22}\overline{\nabla}^2 u^2 - 10x^1 e^{12}) \\ + \delta_2^i(\overline{\nabla}^2 u^1 - 2e^{11}/x^1 + 6x^1 e^{22})\Big\} \\ = \rho\Big\{\frac{\partial u^i}{\partial t} - 2\Omega_1\ \delta_1^i[A - B(R_2/x^2)^2]x^1 u^2 \\ + 2\Omega_1\ \delta_2^i A u^1/x^1\Big\}, \end{aligned} \tag{14}$$

where

$$g_{22} = (x^1)^2, \qquad e^{11} = \partial u^1/\partial x^1, \qquad e^{22} = u^1/(x^1)^3,$$

$$e^{12} = \tfrac{1}{2}(\partial u^2/\partial x^1),\ e^{23} = \tfrac{1}{2}(\partial u^2/\partial x^3), \tag{15}$$

$$\overline{\nabla}^2 u^i = \nabla^2 u^i - u^1\ \delta_1^i/(x^1)^2 + \frac{2}{x^1}\frac{\partial u^2}{\partial x^1}\ \delta_2^i, \tag{16}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^1\,\partial x^1} + \frac{1}{x^1}\frac{\partial}{\partial x^1} + \frac{\partial^2}{\partial x^3\,\partial x^3}. \tag{17}$$

The velocities can be assumed to be periodic in x^3 and to have an exponential time factor. Introduce the following quantities:

$$\begin{aligned} u^1 &= R_1\Omega_1 u(r)\sin(\lambda x^3/R_1)\exp(\sigma\Omega_1 t), \\ x^1u^2 &= R_1\Omega_1 v(r)\sin(\lambda x^3/R_1)\exp(\sigma\Omega_1 t), \\ u^3 &= R_1\Omega_1 w(r)\cos(\lambda x^3/R_1)\exp(\sigma\Omega_1 t), \\ p' &= \rho\Omega_1^2 R_1^2 q(r)\sin(\lambda x^3/R_1)\exp(\sigma\Omega_1 t), \\ r &= x^1/R_1, \qquad S = A^{(2)}/2\rho R_1^2, \\ L &= \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}, \quad \beta = (R_2 - R_1)/R_1, \\ R &= \rho\Omega_1 R^2/A^{(1)}, \qquad \alpha = \Omega_2/\Omega_1 - 1. \end{aligned} \tag{18}$$

Then Eq. (14) becomes

$$\begin{aligned} \begin{Bmatrix} -2(A - B/r^2)v \\ 2Au \\ 0 \end{Bmatrix} = \begin{Bmatrix} -dq/dr \\ 0 \\ -\lambda q \end{Bmatrix} + \frac{1}{R}\begin{Bmatrix} (L - \lambda^2 - \sigma R)u \\ (L - \lambda^2 - \sigma R)v \\ \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \lambda^2 - \sigma R\right)w \end{Bmatrix} \\ + 2SB\begin{Bmatrix} (2L - \lambda^2)(v/r^2) \\ (1/r^2)(L - \lambda^2)u \\ (\lambda/r^2)\left(\frac{d}{dr} - \frac{1}{r}\right)v \end{Bmatrix}. \end{aligned} \tag{19}$$

The velocity in the x^3 direction can be found from continuity to be given in terms of u by

$$\lambda w = (d/dr + 1/r)u. \tag{20}$$

Upon eliminating q and w, the equations to be solved become

$$\begin{aligned} (L - \lambda^2 - \sigma R)(L - \lambda^2)u \\ = 2R\lambda^2[(A - B/r^2)v + SB(L - \lambda^2)(v/r^2)] \end{aligned} \tag{21}$$

and

$$(L - \lambda^2 - \sigma R)v = 2R[Au - (SB/r^2)(L - \lambda^2)u]. \tag{22}$$

The boundary conditions are still no slip at the cylinders, hence

$$u = v = du/dr = 0 \quad \text{at} \quad r = 1 \quad \text{and} \quad r = 1 + \beta. \tag{23}$$

V. THE STABILITY PROBLEM FOR SMALL SPACING

When

$$R_2 - R_1 \ll \tfrac{1}{2}(R_2 + R_1), \tag{24}$$

Eqs. (21) and (22) may be simplified somewhat and the resulting equations can then be solved in the manner of the Taylor problem. By letting

$$\begin{aligned} \zeta &= (r - 1)/\beta, & R' &= \beta^2 R, \\ D &= d/d\zeta = \beta\, d/dr, & S' &= S/2\beta^3, \\ k &= \beta\lambda, & T &= -4A(R')^2, \end{aligned} \tag{25}$$

Eqs. (21) and (22) become, to the first order in β,

$$\begin{aligned} (D^2 - k^2 - \sigma R')(D^2 - k^2)u \\ = [1 + \alpha\zeta + S'\alpha(D^2 - k^2)]v. \end{aligned} \tag{26}$$

$$(D^2 - k^2 - \sigma R')v = -Tk^2 u, \tag{27}$$

if S' is of the order of one, or

$$(D^2 - k^2 - \sigma R')(D^2 - k^2)u = S'\alpha(D^2 - k^2)v, \quad (28)$$

$$(D^2 - k^2 - \sigma R')v = -Tk^2[1 - 2S'\beta(D^2 - k^2)]u, \quad (29)$$

if $\beta S'$ is of the order of one.

Case 1: S′ of the Order of One

The set of Eqs. (26) and (27) differ from the Taylor problem by only one term. One would expect then that the method of Chandrasekhar[9] would be well suited to the determination of the eigenvalues.

With

$$v = \sum A_m \sin m\pi\zeta, \quad (30)$$

$$D_m = (m\pi)^2 + k^2, \quad (31)$$

Eq. (26) becomes

$$(D^2 - k^2 - \sigma R')(D^2 - k^2)u = \sum A_m[1 + \alpha\zeta - S'\alpha\, D_m] \sin m\pi\zeta, \quad (32)$$

and thus, for neutral stability,

$$u = \sum (A_m/D_m^2)\{(1 - S'\alpha\, D_m + \alpha\zeta) \sin m\pi\zeta + (4m\pi\alpha/D_m) \cos m\pi\zeta + (B_m + C_m\zeta) \sinh k\zeta + (F_m + G_m\zeta) \cosh k\zeta\}. \quad (33)$$

This is substituted into Eq. (27) to obtain

$$\sum (A_m\, D_m/k^2T) \sin m\pi\zeta = \sum (A_m/D_m^2)\{(1 - S'\alpha\, D_m + \alpha\zeta) \sin m\pi\zeta + (4m\pi\alpha/D_m) \cos m\pi\zeta + (B_m + C_m\zeta) \sinh k\zeta + (F_m + G_m\zeta) \cosh k\zeta\}. \quad (34)$$

The B_m, C_m, F_m, and G_m are determined from the boundary conditions. They are found to be

$$B_m = m\pi (\sinh^2 k - k^2)^{-1}\{(1 - D_mS'\alpha) \cdot [k + (-1)^m \sinh k] + (4\alpha/D_m) \cdot [\cosh k - (-1)^m][\sinh k - (-1)^m k] + (-1)^m\alpha \sinh k\}, \quad (35)$$

$$C_m = m\pi (\sinh^2 k - k^2)^{-1}\{(1 - D_mS'\alpha) \cdot [\cosh k - (-1)^m][\sinh k + (-1)^m k] + (4\alpha k \sinh k/D_m)[\sinh k - (-1)^m k] + \alpha(-1)^m(k \cosh k - \sinh k)\}, \quad (36)$$

$$F_m = -4m\pi\alpha/D_m, \quad (37)$$

$$G_m = m\pi (\sinh^2 k - k^2)^{-1} \cdot \{-\sinh k(1 - D_mS'\alpha)[\sinh k + (-1)^m k] - (4\alpha k/D_m)[\cosh k - (-1)^m] \cdot [\sinh k - (-1)^m k] - (-1)^m k\alpha \sinh k\}. \quad (38)$$

Upon putting these into Eq. (34), multiplying by $\sin n\pi\zeta$, and integrating from zero to one, the result

$$0 = \sum_m A_mE_{mn}/D_m^2, \qquad n = 1, 2, \cdots, \quad (39)$$

is obtained, where after much reduction,

$$E_{mn} = a_{mn}(1 - D_m\alpha S') - \delta_{mn}\, D_m^3/2k^2T + b_{mn}\alpha, \quad (40)$$

$$a_{mn} = \tfrac{1}{2}\, \delta_{mn} + (2kmn\pi^2/D_n^2) \cdot (\sinh^2 k - k^2)^{-1}[(-1)^m \sinh k + k] \cdot [1 + (-1)^{m+n}][1 - (-1)^m \cosh k], \quad (41)$$

$$b_{mn} = \tfrac{1}{4}\, \delta_{mn} - 2mn[1 - (-1)^{m+n}] \cdot [(m^2 - n^2 + \delta_{mn})^{-2}\, \pi^{-2} + 2\, \pi^2\, D_m^{-1}\, D_n^{-1} + 2\, D_m^{-1}(m^2 - n^2 + \delta_{mn})^{-1}] + (-1)^m\, 2kmn\, \pi^2\, D_n^{-2}[(-1)^n \sinh k - k]^{-1}\{[(-1)^n - \cosh k] + 4k\, D_m^{-1} \sinh k[1 - (-1)^{m+n}]\}. \quad (42)$$

The eigenvalues are determined by setting the determinant of the E_{mn} to zero. Where the first approximation holds, this yields

$$T = \frac{(\pi^2 + k^2)^3}{k^2[1 - 8k\pi^2(1 + \cosh k)/(\sinh k + k)(\pi^2 + k^2)^2][1 + \alpha/2 - (\pi^2 + k^2)\alpha S']}. \quad (43)$$

Plots of the neutral stability curves are shown in Figs. 1 through 5. For the ranges of the parameters shown on these figures, the first and second approximations agree to within 3% if $\alpha > -1.5$, the agreement improving as S' increases.

The present case, S' of the order of one, is the

[9] S. Chandrasekhar, Mathematika 1, 5 (1954).

most interesting from a physical point of view, in that it allows direct comparison with the ordinary Newtonian fluid. The effect of the cross viscosity on the stability of the flow is more remarkable than one might expect. If the pressure gradient in the primary

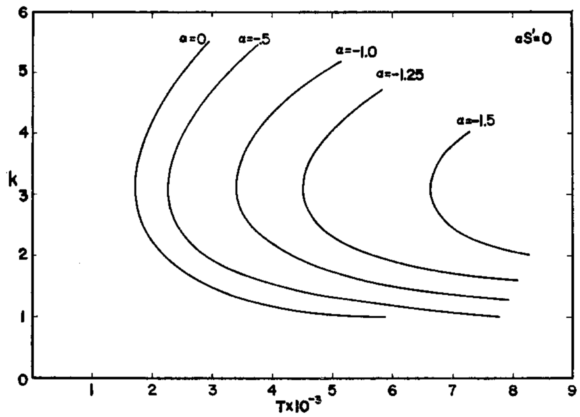

FIG. 1. Neutral stability curves for various values of α, with $\alpha S' = 0$. (G. I. Taylor's case).

flow [given by Eq. (11)] is written in terms of the dimensionless parameters, it becomes for small spacing

$$\partial p/\partial x^1 \approx \rho R_1(\Omega_1)^2\{(1+\alpha\zeta)^2 - 8\alpha^2\beta S'\}. \tag{44}$$

Hence the contribution of the non-Newtonian effects to the stresses in the primary flow for, say, $\alpha = -1$, $S' = 0.1$, is of the order of β, a negligible quantity. The corresponding critical Taylor number is, however, found to be reduced by a factor of 7 over the Newtonian case. This large effect on the stability parameters thus cannot be explained by the cross-viscosity effect on the primary flow, but must be due instead to the added components of disturbance stress which can arise in the flow. This is shown by Eqs. (33), (35), (36), and (38), where it is seen that the magnitudes of each of the terms in the expression for the radial velocity is increased over the Newtonian case when $\alpha S'$ is negative.

To see what the relative magnitudes of the two

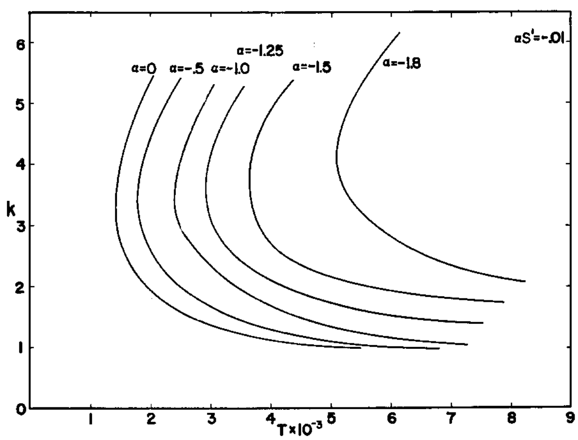

FIG. 2. Neutral stability curves for various values of α, with $-\alpha S' = 0.01$.

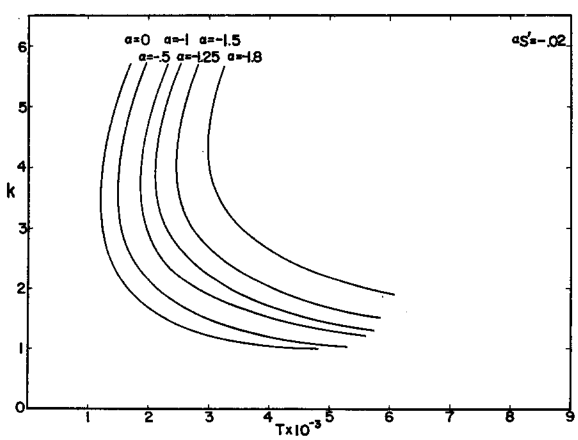

FIG. 3. Neutral stability curves for various values of α, with $-\alpha S' = 0.02$.

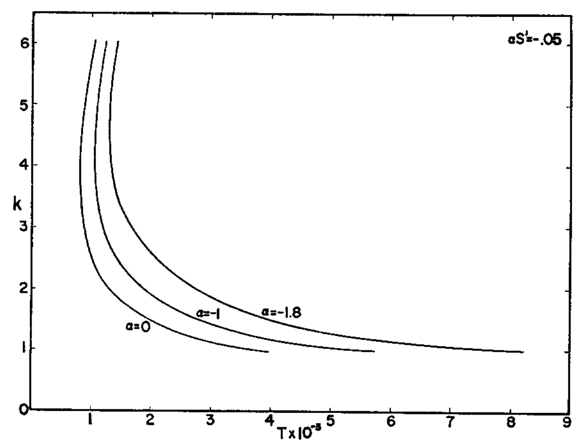

FIG. 4. Neutral stability curves for various values of α, with $-\alpha S' = 0.05$.

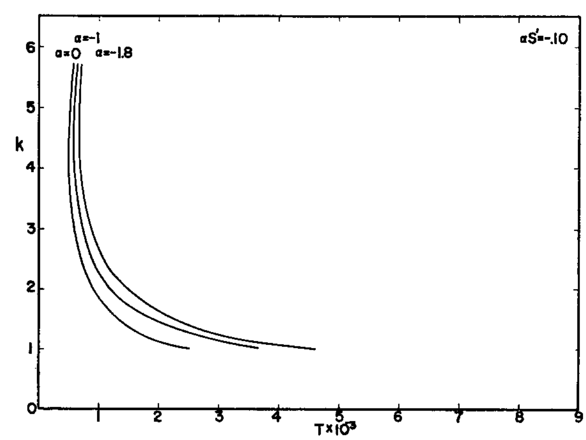

FIG. 5. Neutral stability curves for various values of α, with $-\alpha S' = 0.10$.

viscosities would have to be, it is a simple matter to show that

$$A^{(2)}/A^{(1)} = 4\beta/S'R'\Omega_1. \tag{45}$$

The cross viscosity then can be much smaller than the Newtonian viscosity, and the departures from Newtonian flow will still be significant.

Equations (26) and (27), with neglect of the linear term in ζ, can be used to obtain Eq. (31) in Jain's paper, with the result that $Q = -S'\alpha T$. Jain's result for $Q = 0.01$ then implies a very small value of S', hence a fluid with a very small cross viscosity. Further, there is an error in sign in his Eq. (52), and no less than four errors in his Eq. (53). These errors, along with the fact that a variational approach can at best give an upper bound, are the probable reason for the disagreement between Jain's conclusion and the present author's.

Case 2: $S'\beta$ of the Order of One

The set of equations, Eqs. (28) and (29), turns out to be the most amenable to solution. In fact, for the case of neutral stability ($\sigma = 0$), they may be solved exactly in finite form. By solving Eq. (29) for $(D^2 - k^2)v$ and putting this into Eq. (28), one obtains an equation in u,

$$[(D^2 - k^2)^2 - 2(S')^2\beta\alpha Tk^2(D^2 - k^2) + S'\alpha Tk^2]u = 0. \tag{46}$$

For convenience the following substitutions will be used:

$$\eta = \zeta - \tfrac{1}{2}, \qquad a = S'\alpha Tk^2, \qquad b = 2S'\beta,$$
$$n_1^2 = k^2 + \tfrac{1}{2}\{ab - [(ab)^2 - 4a]^{\frac{1}{2}}\},$$
$$n_2^2 = k^2 + \tfrac{1}{2}\{ab + [(ab)^2 - 4a]^{\frac{1}{2}}\}. \tag{47}$$

Then

$$u = A\sinh n_1\eta + B\sinh n_2\eta + C\cosh n_1\eta + D\cosh n_2\eta, \tag{48}$$

and

$$u = du/d\eta = 0 \quad \text{at} \quad \eta = \pm\tfrac{1}{2}. \tag{49}$$

On applying the boundary conditions, a set of four homogeneous equations in A, B, C, and D is obtained. For a nontrivial solution to exist, the determinant of the coefficients must equal zero. After some simplification this yields

$$0 = [n_2\sinh(\tfrac{1}{2}n_1)\cosh(\tfrac{1}{2}n_2) - n_1\sinh(\tfrac{1}{2}n_2)\cosh(\tfrac{1}{2}n_1)] \cdot [n_2\cosh(\tfrac{1}{2}n_1)\sinh(\tfrac{1}{2}n_2) - n_1\cosh(\tfrac{1}{2}n_2)\sinh(\tfrac{1}{2}n_1)]. \tag{50}$$

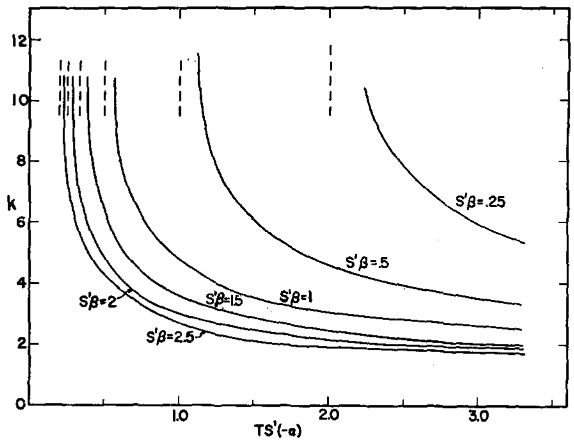

FIG. 6. Neutral stability curves for $S'\beta$ of the order of 1.

This may be interpreted as

$$(2/n_1)\tanh(\tfrac{1}{2}n_1) = (2/n_2)\tanh(\tfrac{1}{2}n_2) \tag{51}$$

for anti-symmetric disturbances, and

$$\tfrac{1}{2}n_1\tanh(\tfrac{1}{2}n_1) = \tfrac{1}{2}n_2\tanh(\tfrac{1}{2}n_2) \tag{52}$$

for symmetric disturbances.

If n_1 and n_2 are both real, the only possibility of satisfying either of these is $n_1 = n_2$, or $a = 4/b^2$. However, by comparison with what occurs in a Newtonian fluid, it can be expected that a will be large and also, if α is negative, a will be negative also. Hence n_1 will be imaginary. Equations (51) and (52) then become, with $m_1^2 = -n_1^2$,

$$\tfrac{1}{2}n_2\tanh(\tfrac{1}{2}n_2) = -\tfrac{1}{2}m_1\tan(\tfrac{1}{2}m_1), \tag{53}$$

$$(2/n_2)\tanh(\tfrac{1}{2}n_2) = (2/m_1)\tan(\tfrac{1}{2}m_1). \tag{54}$$

For a given b and k, Eq. (53) yields a lower value of a than Eq. (54), hence it can be expected that symmetrical disturbances are most likely to occur. Curves of k vs $-\alpha S'T$ are plotted for various values of $S'\beta$ in Fig. 6. The critical value of the Taylor number is found to be

$$T_{\text{critical}} = -\beta/2\alpha(\beta S')^2 \tag{55}$$

at infinitely large k. Since $\beta S'$ has been assumed to be of the order of one, this implies a critical Taylor number of the order of $\beta \ll 1$. This result agrees qualitatively with the results for case 1, in the sense that a tendency for the neutral stability curves to flatten out and yield low critical Taylor numbers at large values of k is verified. The occurence of instability at large k physically means that the convection cells which form are very short and flat in shape. Such results do not occur in Newtonian fluids, and more would have to be known about the rheological properties of the fluids under consideration to decide whether these results are of physical significance.

VI. CONCLUSIONS

For a fluid with a positive coefficient of cross viscosity, it is seen that the effect of the non-Newtonian terms is definitely destabilizing, the rate of change in the critical Taylor number being greatest in the range $0 < \alpha S' < -0.1$. This destabilization effect is due primarily to the additional cross-viscosity terms which augment the disturbance stresses when $\alpha S' < 0$. While the calculations have not been carried out for $\alpha S' > 0$, the tendency for stabilization is clearly indicated. It may even be conjectured that there is some positive value of $\alpha S'$ for which the flow would be completely stable.

ACKNOWLEDGMENTS

The author is indebted to Dr. C. S. Yih for suggesting the investigation which led to this work, as well as for valuable discussions during its pursuance.

This work was supported under contract with the Office of Ordnance Research, U. S. Army.

THE PHYSICS OF FLUIDS VOLUME 4, NUMBER 3 MARCH, 1961

Turbulent Flow in a Circular Pipe with Porous Wall

S. W. Yuan and E. W. Brogren
The University of Texas, Austin, Texas
(Received October 11, 1960)

A theoretical investigation of an isothermal turbulent flow in a porous wall pipe with fluid injection or suction at the wall has been made. An exact solution of the Reynolds equations, reduced to ordinary nonlinear differential equations with appropriate boundary conditions, is obtained. The axial velocity distribution is expressed as functions of velocity through the porous wall, the axial pressure gradient and the mixing length proportionality constant K. Experimental data are used in conjunction with the solution to calculate the values of K over a range of injection-to-main-stream velocity ratio. Good agreement is obtained between the present velocity profiles and such data.

INTRODUCTION

Surfaces in the neighborhood of high-temperature gases can be appreciably reduced if a poor conductive barrier can be maintained between the surface and the hot gases. One successful means of accomplishing this purpose is that of mass transfer cooling. Such process can be realized by the use of porous surface through which the coolant is forced into the high-temperature stream. The effect of fluid injection at the wall on the flow distribution must be known before the prediction of heat transfer of such a flow can be made.

The effect of fluid injection at the wall on isothermal and nonisothermal laminar flow of a fluid in a porous wall pipe has been investigated by Yuan and Finkelstein.[1,2] Although the above investigations yield considerable basic knowledge on the laminar pipe flow, they do not apply the flow in the turbulent state which occurs in most engineering problems. For this reason, further exploratory study of the effect of coolant injection through a porous wall pipe in the velocity and temperature distributions of a fully developed turbulent pipe flow was made.[3] In the above approximate solution, the axial velocity distribution obtained is independent of the distance in the flow direction. Hence, this solution is valid only for a pipe with a small length-to-diameter ratio. Similar problems in laminar and turbulent flow of a fluid in channels with porous walls have also been investigated by Berman[4] and Yuan.[5,6]

In the present study the Reynolds equations in cylindrical coordinates have been reduced to ordinary differential equations for the case in which the flow through the porous wall varies as an exponential function of x/R. A perturbation method

[1] S. W. Yuan and A. B. Finkelstein, Trans. Am. Soc. Mech. Engrs. **78**, 719 (1956).
[2] S. W. Yuan and A. B. Finkelstein, Jet Propulsion **28**, 178 (1958).
[3] S. W. Yuan and L. Galowin, Proc. 9th Intern. Congr. Appl. Mech. **II**, 331 (1957).
[4] A. S. Berman, J. Appl. Phys. **24**, 1232 (1953).
[5] S. W. Yuan, J. Appl. Phys. **27**, 267 (1956).
[6] S. W. Yuan, J. Math. and Phys. **38**, 166 (1959).