Relation between Time Symmetry and Reflection Symmetry of Turbulent Fluids

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An assumption of statistical reflection symmetry is proposed for certain turbulent flows. On the basis of this assumption it is shown that some space-time velocity correlations characterizing the flow are even functions of the time delay between the correlation measurements. In particular the second- and one of the fourth-order velocity correlations are symmetric in this relative time.

THE idea of using statistical symmetry conditions to simplify the treatment of problems in the theory of turbulence was first proposed by Taylor in 1935.[1] He suggested that certain types of turbulence might be considered to be statistically homogeneous and isotropic, ideas which were borne out by experiment. On the basis of these hypotheses, for instance, the second-order space-time velocity correlation can be written,[2]

$$Q_{ij}(\mathbf{r}; T, t) = \overline{v_i(\mathbf{r}', t')v_j(\mathbf{r}'', t'')}$$
$$= Q_1(r; t', t'')r_ir_j + Q_2(r; t', t'')\delta_{ij}, \qquad (1)$$

with $\mathbf{r} = \mathbf{r}'' - \mathbf{r}'$, $t = t'' - t'$, and $T = (t' + t'')/2$, and where $v_i(\mathbf{r}', t')$ is the ith component of the velocity at $\mathbf{r}'$, t'. Here $v_x = v_1$ and similarly for other components. The functions Q_1 and Q_2 are scalar functions of the scalars r, T, t. The bar indicates ensemble averaging. If in addition to these spatial assumptions, one also supposes that the turbulence is statistically stationary, the correlations will not depend upon the absolute time T, but only upon the relative, t. Results similar to these can be written down for correlations other than the second-order velocity correlation of Eq. (1).

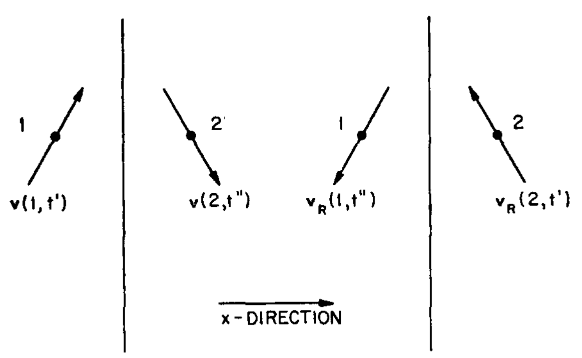

FIG. 1. A member of the ensemble is shown opposite its mirror image, a_R. The line represents the plane of symmetry.

These considerations have not explicitly involved the use of reflection symmetry properties, except insofar as the requirement has been made that tensor correlations formed from true vectors (velocities, for example) be true tensors and not have pseudo-tensor properties. It is the purpose of this paper to show that by fixing reflection symmetry restrictions upon the ensemble from which the correlations are calculated, some correlations characterizing the turbulence field can be shown to be even functions of the relative time t.

We return to the general problem where the assumptions of statistical homogeneity, isotropy, and stationarity do not necessarily apply. Statistical reflection symmetry is now assumed. It is supposed that for every given member of the ensemble (velocity and pressure given as a function of space and time) there is a single mirror image for all times. The equations of mechanics of course allow this possibility. Then we imagine, for clarity, that a replica of the original ensemble is made and placed by the original ensemble in such a way that each member of the original has its mirror image opposite it. In Fig. 1 a member a of the ensemble is shown with its reflection a_R opposite it. In that figure the plane of symmetry is the $y - z$ plane and the two points 1 and 2 are located on the same perpendicular to that plane and lie the same distance from it. The quantities $\mathbf{v}$ and $\mathbf{v}_R$ are the velocity and the reflected value of the velocity at the indicated space-time points.

The procedure to be utilized is the following. A function G of the velocities at the points 1 and 2 and the times t' and t'' is averaged over the original ensemble. This average is then compared with the average of the same function G over the reflection of the original ensemble (which is really the original ensemble reordered) but with the difference that

[1] G. I. Taylor, Proc. Roy. Soc. (London) A151, 421 (1935).
[2] See G. K. Batchelor, *Homogeneous Turbulence* (Cambridge University Press, New York, 1956).

the velocities in the second case are to be evaluated at the point 1 and the time t'' and the point 2 and the time t'. It is noted that the sign of the relative time, $t = t'' - t'$, has been changed in the second average. Then if the two averages can be shown to be equal, the average (correlation) is an even function of the relative time.

To begin, it follows from the construction of the mirror image ensemble that

$$\mathbf{v}_R(1, t'') = [-v_x(2, t''), v_y(2, t''), v_z(2, t'')]$$

and (2)

$$\mathbf{v}_R(2, t') = [-v_x(1, t'), v_y(1, t'), v_z(1, t')].$$

Then the function G of the velocities to be used in averaging over the original ensemble is

$$G[v_x(1, t'), v_y(1, t'), v_z(1, t'); \quad v_x(2, t''), v_y(2, t''), v_z(2, t'')], \tag{3}$$

and the function for the second averaging is

$$G_R[v_{xR}(1, t''), v_{yR}(1, t''), v_{zR}(1, t''); \quad v_{xR}(2, t'), v_{yR}(2, t'), v_{zR}(2, t')]. \tag{4}$$

Using (2), G_R can be written

$$G_R[-v_x(2, t''), v_y(2, t''), v_z(2, t''); \quad -v_x(1, t'), v_y(1, t'), v_z(1, t')]. \tag{5}$$

Finally, then, if the correlation being considered is such that

$$G_R = G, \tag{5a}$$

that is, the expression (5) equals the expression (3), the correlation is an even function of the relative time t. This is true even for *nonstationary* stochastic processes. The same sort of considerations can be applied to correlations involving functions other than the velocity.

Some examples of interest in the theory of turbulence will now be considered. For incompressible fluid flow the second-order velocity correlation of Eq. (1) can be expressed in terms of the longitudinal velocity correlation Q when it is supposed that the turbulence is homogeneous and isotropic, by the relation[2]

$$Q_{ij}(\mathbf{r}; T, t) = -\frac{1}{2r}\frac{\partial Q(r; T, t)}{\partial r} r_i r_j + \left[Q(r; T, t) + \frac{r}{2}\frac{\partial Q(r; T, t)}{\partial r}\right]\delta_{ij}, \tag{6}$$

where the longitudinal correlation is defined by

$$Q(r; T, t) = \overline{v_x(\mathbf{r}', t')v_x(\mathbf{r}' + r\hat{\imath}, t'')}, \tag{7}$$

with $\hat{\imath}$ the unit vector in the x direction. Thus the expression (3) is in this case

$$G = v_x(1, t')v_x(2, t''), \tag{8}$$

where the point 1 is determined by the vector $\mathbf{r}'$ and the point 2 by $\mathbf{r}' + r\hat{\imath}$. Also we have, under the reflection symmetry assumption,

$$G_R = [-v_x(2, t'')][-v_x(1, t')], \tag{9}$$

which is the same as (8). Hence,

$$Q(r; T, t) = Q(r; T, -t) \tag{10}$$

and

$$Q_{ij}(r; T, t) = Q_{ij}(r; T, -t), \tag{11}$$

so that the second-order velocity correlation is symmetric in the relative time.* It should be noted that to establish Eq. (11) it is assumed that the ensemble has reflection symmetry about every plane.

When one considers the third-order velocity correlation, $Q_{i;jk}$, it is found that on the basis of the assumption of reflection symmetry, no time symmetry property can be established. The fourth-order velocity correlation is defined by

$$Q_{ij;kl} = \overline{u_i(\mathbf{r}', t')u_j(\mathbf{r}', t')u_k(\mathbf{r}'+\mathbf{r}, t'+t)u_l(\mathbf{r}'+\mathbf{r}, t'+t)}. \tag{12}$$

This correlation can be expressed in terms of five generating scalar functions under the assumptions of homogeneity and isotropy. It is possible to find enough combinations of velocity components which satisfy the relation (5a) to show that each one of the scalar generating functions is symmetric in the relative time, if the ensemble has the above proposed reflection symmetry. Consequently, the fourth-order velocity correlation is symmetric in the relative time,

$$Q_{ij;kl}(\mathbf{r}; T, t) = Q_{ij;kl}(\mathbf{r}; T, -t). \tag{13}$$

In conclusion it is seen that on the basis of a reflection symmetry assumption for the ensemble

*It is not difficult to show, by considering the correlation of y components, that even for compressible flows the two functions Q_1 and Q_2 of Eq. (1) are symmetric in t under the reflection assumption and thus Q_{ij} is symmetric for such flows.

of turbulent flows, certain of the more symmetric correlations are even functions of the relative time, even for nonstationary processes. In particular it has been shown that the second- and one of the fourth-order velocity correlations are even functions of the relative time. It would be of interest to check the reflection symmetry assumption through an experimental observation of the time symmetry properties of the correlations.

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