Homing Navigation in the Presence of Longitudinal Accelerations

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LIST OF SYMBOLS

α	Target heading angle
β	Angle of target-to-interceptor line of sight
$\dot{\beta}_d$	Desired rate of rotation of the line of sight
θ	Interceptor heading angle
K	Navigation constant
R	Target-to-interceptor range
T	Time remaining until interception
V_m	Interceptor speed
V_t	Target speed

I

INTRODUCTION AND SUMMARY

Methods of homing guidance to be used in navigating interceptors to interception have been widely studied. Because of the complexity of the problem, it has generally been desirable to make use of two basic simplifying assumptions:

1. Both the target and the interceptor have constant speed.
2. The interceptor has a speed advantage over its target.

The mathematical models obtained with the aid of these restrictions have a desirable tractability and are often reasonably good approximations to the true situation. However, in order to attain results which are useful in certain important cases, both of these simplifying assumptions must be abandoned. To give one of the many examples which lead to this conclusion: the nature of certain interceptions can easily be such that a very short time is available to the defense; such interceptions frequently require the interceptor to accelerate all or nearly all of the way.[1]

In this report, techniques for navigating in the presence of longitudinal accelerations of the target and interceptor are developed.

1.1 PROPORTIONAL NAVIGATION

Homing guidance systems generally use proportional navigation, in which the rate of rotation of the interceptor heading, ($\dot{\theta}$), is made proportional to the rate of rotation of the line of sight, ($\dot{\beta}$):

$$\dot{\theta} = K\dot{\beta} . \qquad (1.1\text{-}1)$$

It is shown in Section II that in the presence of continuous and moderately large longitudinal acceleration of either the target or the interceptor, conventional proportional navigation becomes impracticable.

[1]Classified applications of the theory developed in this paper can be found in the following SECRET University of Michigan Reports: UMR-91 and UMR-111.

1.2 GUIDANCE CRITERIA

A guidance criterion is defined herein as that rate of rotation of the target-to-interceptor line of sight which would result in a straight-line path. Such a path is always available to the interceptor if interception is possible. For interception with a speed disadvantage there are ordinarily two such paths, but all of the following can be made to refer to either path.

In this paper, the guidance criterion is denoted by $\dot{\beta}_d$. For example, the guidance criterion for the constant-speed case is simply $\dot{\beta}_d = 0$. An integral equation giving the guidance criterion for arbitrary behavior of target and interceptor speeds is derived, and it is shown that this equation can be written in closed form (Sec. III). The resulting equations are dependent upon the time remaining until interception, but in practical cases this time is not generally a known quantity. Certain approximations, however, can be made in terms of available quantities; in this paper approximations for constant target and constant interceptor accelerations are derived.

1.3 NAVIGATION CONSTANT

The proportionality factor, K, (Eq. 1.1-1) is usually taken to be a constant, but it has been observed frequently that it may be necessary to program the value of K during the course of an interception. The results of the present study (Sec. 4.1) strongly imply that in the presence of longitudinal accelerations and a speed disadvantage, the navigation constant, K, should be replaced by:

$$K_1 = \frac{2\dot{R}k_1}{V_m \cos(\theta - \beta)} \qquad (1.3\text{-}1)$$

where k_1 is a constant.

1.4 NAVIGATION TECHNIQUE

It was stated above that for a constant-speed interceptor and a constant-velocity target, the exact guidance criterion is $\dot{\beta}_d = 0$. The interceptor measures the rate of rotation of the target-to-interceptor line of sight. Whenever this rate departs from zero (implying the presence of longitudinal or lateral accelerations) a correction of heading proportional to the departure is called for. This correction is achieved by means of the guidance equation:

$$\dot{\theta} = K\dot{\beta}. \tag{1.4-1}$$

Similarly, if the target or interceptor is not expected to have constant speed, the interceptor may be guided along an approximately straight-line course by:

$$\dot{\theta} = K_1(\dot{\beta} - \dot{\beta}_d) \tag{1.4-2}$$

(Sec. 4.4). The special case of Equation (1.4-2) in which $\dot{\beta}_d = 0$ and K_1 is constant is simply Equation (1.4-1), the ordinary proportional navigation equation.

1.5 THE CONSTANT-SPEED ANALOGY

When the navigation factor, K_1, is chosen as above, the rate of rotation of the line of sight can be obtained explicitly as a function of the relative range. For correct guidance criteria this rate is shown to be independent of the assumed behavior of the interceptor and target speeds (Sec. 3.3). The correct guidance criterion is generally not available to the interceptor. The effect of an error in the selection of $\dot{\beta}_d$ is also shown to be independent of the type of guidance used (Sec. 4.5).

These facts imply that, to a good approximation, a variable-speed interception may be studied in terms of an appropriately chosen constant-speed case. A case in which both target and interceptor are subject to constant longitudinal acceleration is compared with the corresponding constant-speed interception in Section IV. Figures 3 and 4 in Section 4.7 represent interceptor lateral acceleration and trajectory comparisons for these cases and show good agreement.

II

THE EFFECT OF LONGITUDINAL ACCELERATIONS ON CONVENTIONAL GUIDANCE

It is well known that an interception can be achieved by guiding an interceptor in such a way that the line of sight between it and its target does not rotate. Conventional homing guidance systems make use of this principle, reducing the rate of rotation of the line of sight to zero by mechanizing the equation

$$\dot{\theta} = K\dot{\beta}, \tag{2-1}$$

where θ is the interceptor heading, K is the navigation constant, and β is the direction of the interceptor-to-target line of sight. This method of guidance is called proportional navigation; it is discussed in detail in Reference 2.

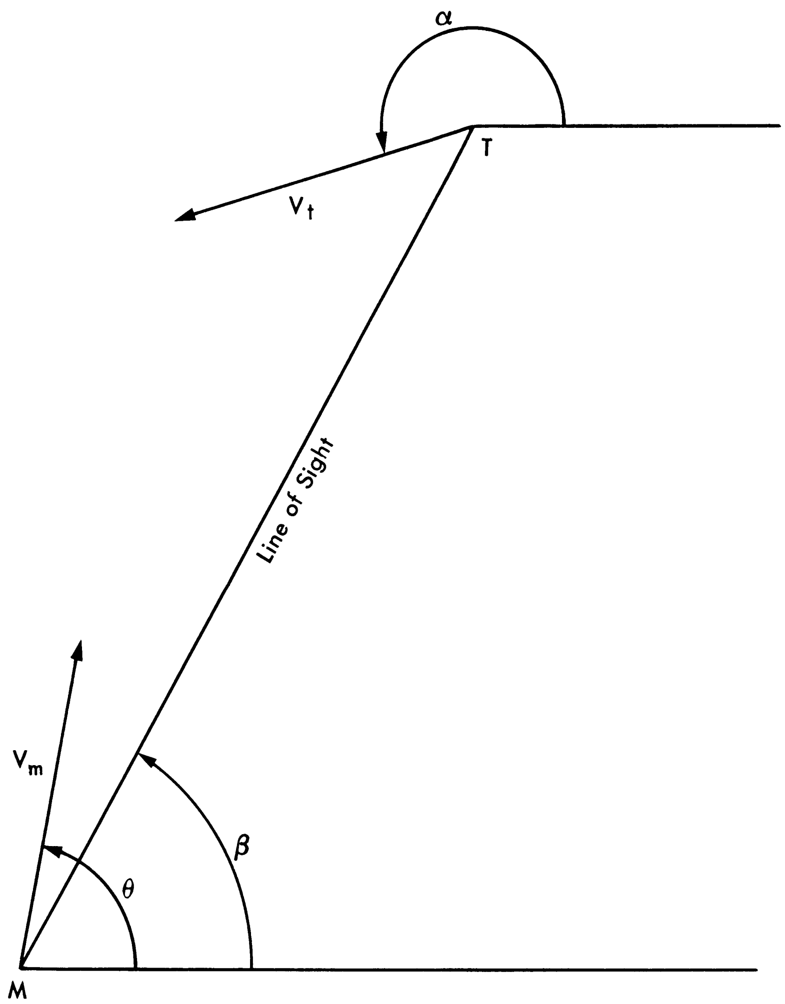
If both the target and the interceptor travel at constant speed and the target flies a straight-line path, the interceptor path defined by the non-rotating line of sight is also a straight line. In practice, significant departures from these idealized conditions are tolerable and are compensated for by interceptor maneuvers. If the target or the interceptor undergo large longitudinal accelerations, however, the lateral acceleration capability of the interceptor may very easily be saturated in attempting to maintain a non-rotating line of sight.

For a non-rotating line of sight,

$$R\dot{\beta} = V_t \sin(\alpha - \beta) - V_m \sin(\theta - \beta) = 0\ . \tag{2-2}$$

The geometry and notation used in this equation are shown in Figure 1. Differentiating Equation (2-2) with respect to time and solving for the lateral acceleration of the interceptor, $V_m\dot{\theta}$, yields:

$$V_m\dot{\theta} = \frac{\dot{V}_t \sin(\alpha - \beta) - \dot{V}_m \sin(\theta - \beta) + V_t \dot{\alpha} \cos(\alpha - \beta)}{\cos(\theta - \beta)}. \tag{2-3}$$



M, T — Interceptor, Target

β — Angle of Line of Sight

V_m, V_t — Interceptor, Target Speeds

θ — Interceptor Heading Angle

α — Target Heading Angle

FIG. 1 GEOMETRICAL CONFIGURATION AND NOTATION

Equations (2-2) and (2-3) show that, if the rate of rotation of the target-to-interceptor line of sight is to be maintained at zero, longitudinal acceleration on the part of either the target or the interceptor leads to lateral acceleration of the interceptor. From structural considerations, this lateral acceleration can be sufficient to make proportional navigation unusable.

Even if the lateral acceleration given by Equation (2-3) is only moderately large, the use of proportional navigation in the presence of non-negligible longitudinal accelerations leads to difficulties. Equation (2-1) shows that no lateral acceleration is commanded when the rate of rotation of the line of sight is zero; Equation (2-3) however, implies that if there are longitudinal accelerations, continuous lateral acceleration is required to maintain the rate of rotation of the line of sight at zero. In the derivation of Equation (2-3), the navigation technique (the actual method by which the line of sight is maintained non-rotating) is not considered. This equation would apply only with an infinitely large navigation constant, K. If K has any finite value, it is impossible to attain a non-rotating line of sight under the above conditions, and the acceleration requirements on the interceptor are greater and occur later in the trajectory than those given by Equation (2-3).

III

GENERAL CONDITIONS FOR INTERCEPTION

A straight-line path to interception is always available to an interceptor unless interception is impossible. A guidance criterion is defined to be the rate of rotation of the target-to-interceptor line of sight which corresponds to such a path and is denoted by $\dot{\beta}_d$. If the target has a speed advantage, two such paths usually exist, but in the following it is not necessary to distinguish between them; all of the following may be applied to either case.

3.1 THE CONSTANT-VELOCITY CASE

In the presence of longitudinal accelerations, interception requires a more refined guidance criterion than the usual non-rotating line of sight. The approach used to obtain improved criteria can be demonstrated by considering the constant-velocity case.

For the target-interceptor configuration shown in Figure 1, if the target and interceptor fly along straight-line paths at constant speed, the following equations define an interception:

$$V_t T \sin(\alpha - \beta) = V_m T \sin(\theta - \beta) \qquad (3.1\text{-}1)$$

$$R = \left[V_m \cos(\theta - \beta) - V_t \cos(\alpha - \beta)\right] T \qquad (3.1\text{-}2)$$

where T is the time remaining until interception. The following relationships are obtained from the geometry and do not depend on any assumptions about target or missile motion:

$$R\dot{\beta} = V_t \sin(\alpha - \beta) - V_m \sin(\theta - \beta) \qquad (3.1\text{-}3)$$

$$\dot{R} = V_t \cos(\alpha - \beta) - V_m \cos(\theta - \beta). \qquad (3.1\text{-}4)$$

Introducing Equations (3.1-3) and (3.1-4) in Equations (3.1-1) and (3.1-2) gives:

$$R\dot{\beta} = 0 \text{ and} \qquad (3.1\text{-}5)$$

$$R + \dot{R}T = 0 \qquad (3.1\text{-}6)$$

Equation (3.1-5) states that the line of sight should be non-rotating--the conventional guidance criterion. Equation (3.1-6) defines the time remaining until interception.

3.2 GENERAL EQUATIONS

To obtain general equations analogous to Equations (3.1-1) and (3.1-2), arbitrary target and interceptor motions are resolved into components perpendicular to and parallel to the line of sight, and the condition that the target and the interceptor coincide at time T is imposed; viz.,

$$\int_0^T \left[V_t \sin\left(\alpha - \beta_o\right) - V_m \sin\left(\theta - \beta_o\right) \right] dt = 0 \qquad (3.2\text{-}1)$$

$$R = \int_0^T \left[V_m \cos\left(\theta - \beta_o\right) - V_t \cos\left(\alpha - \beta_o\right) \right] dt \qquad (3.2\text{-}2)$$

where β_o is the initial value of β. By use of Equations (3.1-3) and (3.1-4), the above equations become:

$$\int_0^T \left[R\dot{\beta} \cos\left(\beta - \beta_o\right) + \dot{R} \sin\left(\beta - \beta_o\right) \right] dt = 0 \qquad (3.2\text{-}3)$$

$$R + \int_0^T \left[\dot{R} \cos\left(\beta - \beta_o\right) - R\dot{\beta} \sin\left(\beta - \beta_o\right) \right] dt = 0 . \qquad (3.2\text{-}4)$$

Integration yields:

$$R \sin\left(\beta - \beta_o\right) \Big|_0^T = 0 \qquad (3.2\text{-}5)$$

$$R + R \cos\left(\beta - \beta_o\right) \Big|_0^T = 0 . \qquad (3.2\text{-}6)$$

Expanding $R\sin(\beta - \beta_o)$ and $R\cos(\beta - \beta_o)$ in Maclaurin series and evaluating the above equations gives the general equations for interception; viz.,

$$0 = R\dot{\beta}T + (1/2)T^2\left[R\ddot{\beta} + 2\dot{R}\dot{\beta}\right] + (1/6)T^3\left[R\left(\dddot{\beta} - \dot{\beta}^3\right) + 3\ddot{R}\dot{\beta} + 3\dot{R}\ddot{\beta}\right] + \cdots \tag{3.2-7}$$

$$0 = R + \dot{R}T + (1/2)T^2\left[\ddot{R} - R\dot{\beta}^2\right] + (1/6)T^3\left[----\right] + \cdots. \tag{3.2-8}$$

These equations are analogous to Equations (3.1-5) and (3.1-6) for arbitrary target and interceptor motions. The coefficients of Equations (3.2-7) and (3.2-8) have a direct physical interpretation. In Equation (3.2-7) the coefficients of T, $T^2/2$, $T^3/6$, and higher powers of T are composed of the projections perpendicular to the line of sight of target and interceptor velocity, acceleration, third derivative, and higher derivatives, respectively. Comparable relationships hold for Equation (3.2-8) for the projections along the line of sight. Evaluation of the coefficients shows that if there are no target or interceptor lateral accelerations, Equations (3.2-7) and (3.2-8) may be replaced by:

$$0 = \sum_{k=0}^{\infty} \frac{\left[V_t^{(k)}\sin(\alpha - \beta) - V_m^{(k)}\sin(\theta - \beta)\right] T^{k+1}}{(k+1)!} \tag{3.2-9}$$

$$0 = R + \sum_{k=0}^{\infty} \frac{\left[V_t^{(k)}\cos(\alpha - \beta) - V_m^{(k)}\cos(\theta - \beta)\right] T^{k+1}}{(k+1)!} \tag{3.2-10}$$

where $V^{(k)}$ is the kth derivative of V with respect to time. In general, for any case in which all the derivatives of target or interceptor motion higher than a fixed order are zero, Equations (3.2-7) and (3.2-8) become finite sums. In the rest of this report, the target is assumed to be flying a straight-line path, unless the contrary is stated explicitly.

3.3 GUIDANCE CRITERIA

Equations (3.2-9) and (3.2-10) will be used to derive guidance criteria; that is, rates of rotation of the line of sight which exist along a straight-line interceptor path. The presence of derivatives of motion in Equations (3.2-9) and (3.2-10) makes the criteria depend directly on time-to-go. Because time-to-go is not readily available to the interceptor, approximations to these criteria, better suited to practical guidance, will also be derived.

If there are no target or interceptor accelerations, the coefficients of T^2 and higher powers of T in Equations (3.2-7) and (3.2-8) are zero, and these equations reduce to Equations (3.1-5) and (3.1-6), the defining equations of conventional non-rotating line-of-sight guidance.

For the more general cases, multiplication of Equation (3.2-9) by R gives:

$$0 = \sum_{k=0}^{N} R \left[\frac{V_t^{(k)} \sin(\alpha - \beta) - V_m^{(k)} \sin(\theta - \beta)}{(k+1)!} \right] T^{k+1} \qquad (3.3\text{-}1)$$

where N is the greater of p and q, which are respectively the orders of the highest ordered non-zero derivatives of target and interceptor velocities. It is easy to prove by mathematical induction that

$$R \left[V_t^{(k)} \sin(\alpha - \beta) - V_m^{(k)} \sin(\theta - \beta) \right] = \frac{d^k}{dt^k} \left(R^2 \dot{\beta} \right)$$

$$- \sin(\theta - \alpha) \sum_{j=0}^{k-2} \frac{d^j}{dt^j} \left(V_m V_t^{(k-j-1)} - V_t V_m^{(k-j-1)} \right). \qquad (3.3\text{-}2)$$

Substituting Equation (3.3-2) into Equation (3.3-1):

$$0 = \sum_{k=0}^{N} \frac{1}{(k+1)!} T^{k+1} \frac{d^k}{dt^k} \left(R^2 \dot{\beta} \right) - \sin(\theta - \alpha) \sum_{k=2}^{N} \frac{1}{(k+1)!} T^{k+1}$$

$$\sum_{j=0}^{k-2} \frac{d^j}{dt^j} \left(V_m V_t^{(k-j-1)} - V_t V_m^{(k-j-1)} \right). \qquad (3.3\text{-}3)$$

A solution of this differential equation is

$$\dot{\beta} = \frac{1}{R^2} \sum_{i=2}^{p+q+1} C_i T^i \qquad (3.3\text{-}4)$$

where the coefficients C_i are constant. That is, in general, with a finite number of non-vanishing derivatives of both target and interceptor speeds, the guidance criterion which gives rise to a straight-line interceptor path is a polynomial in T whose coefficients are inversely proportional to the square of the target-to-interceptor range.

The solution given by Equation (3.3-4) is not a practical guidance criterion because it would require continuous computation of the range and the time-to-go. Criteria which do not involve this shortcoming are derived below for the constant-acceleration case. Such criteria probably provide sufficient improvement over proportional navigation to make guidance in the presence of longitudinal accelerations feasible.

In the constant-acceleration case, Equation (3.3-4) becomes:

$$\dot{\beta} = C\left(\frac{T^2}{R^2}\right). \tag{3.3-5}$$

Equation (3.3-5) is the exact guidance criterion when the second and higher derivatives of velocity are zero. To use this equation in homing guidance, it is necessary to compute T. In order to eliminate this computation, it is desirable to obtain an approximation to the exact guidance criterion which uses measurable quantities. The quantity R/T is the mean closing speed over the remainder of the flight. Therefore, $\frac{R^2}{T^2}$ can be approximated as:

$$\frac{R^2}{T^2} \sim \dot{R}\dot{R}_f \tag{3.3-6}$$

where $\dot{R}_f$ is the final closing rate, a constant during the course of a straight-line interception. Substituting Equation (3.3-6) in Equation (3.3-5):

$$\dot{\beta} = \frac{C}{\dot{R}\dot{R}_f} = \frac{C'}{\dot{R}} \tag{3.3-7}$$

which is an approximation to the guidance criterion (Eq. 3.3-5) for the constant acceleration case. The constant C' can be evaluated as follows: Differentiating Equation (3.1-3) and substituting Equation (3.1-4) with the condition that target and interceptor lateral accelerations are zero yields:

$$R\ddot{\beta} + 2\dot{R}\dot{\beta} = \dot{V}_t \sin(\alpha - \beta) - \dot{V}_m \sin(\theta - \beta) \,. \tag{3.3-8}$$

Evaluating Equation (3.3-8) at interception (R = 0) yields:

$$\dot{\beta}_f = \frac{\dot{V}_t \sin(\alpha - \beta)_f - \dot{V}_m \sin(\theta - \beta)_f}{2\dot{R}_f} = \frac{C'}{\dot{R}_f} \tag{3.3-9}$$

$$C' = \frac{\dot{V}_t \sin(\alpha - \beta)_f - \dot{V}_m \sin(\theta - \beta)_f}{2} \tag{3.3-10}$$

The usability of Equations (3.3-7) and (3.3-10) is illustrated in Figure 2 by a numerical example. The target was assumed to fly a straight-line path with constant deceleration of 133 ft/sec^2, while the missile was subjected to a constant acceleration of 67 ft/sec^2 and required to fly a straight line. The target speed varied from 3000 to 1000 ft/sec, and the interceptor speed varied from 500 to 1500 ft/sec. Figure 2 shows the actual $\dot{\beta}$ and $C/\dot{R}$ as functions of time. It is seen that these quantities are in very good agreement; the maximum error involved in using Equation (3.3-7) over a full 15-second period is about 4 per cent. That is, if instead of a straight-line course the interceptor had flown a course defined by the approximate guidance criterion of Equation (3.3-7), it would have required virtually no lateral acceleration.

It is possible to derive another approximation to $\dot{\beta}_d$ which is not as accurate as the foregoing but which is particularly adaptable to practical use. The coefficient of T^2 in Equation (3.2-8) consists of projections of acceleration along the line of sight. Usually this term is small; this is equivalent to having a nearly constant closing rate. Introducing Equation (3.1-6) into Equation (3.2-7) shows that:

$$\ddot{\beta} \sim 0. \tag{3.3-11}$$

The approximation to $\dot{\beta}_d$ is obtained by setting $\ddot{\beta} = 0$ in Equation (3.3-8):

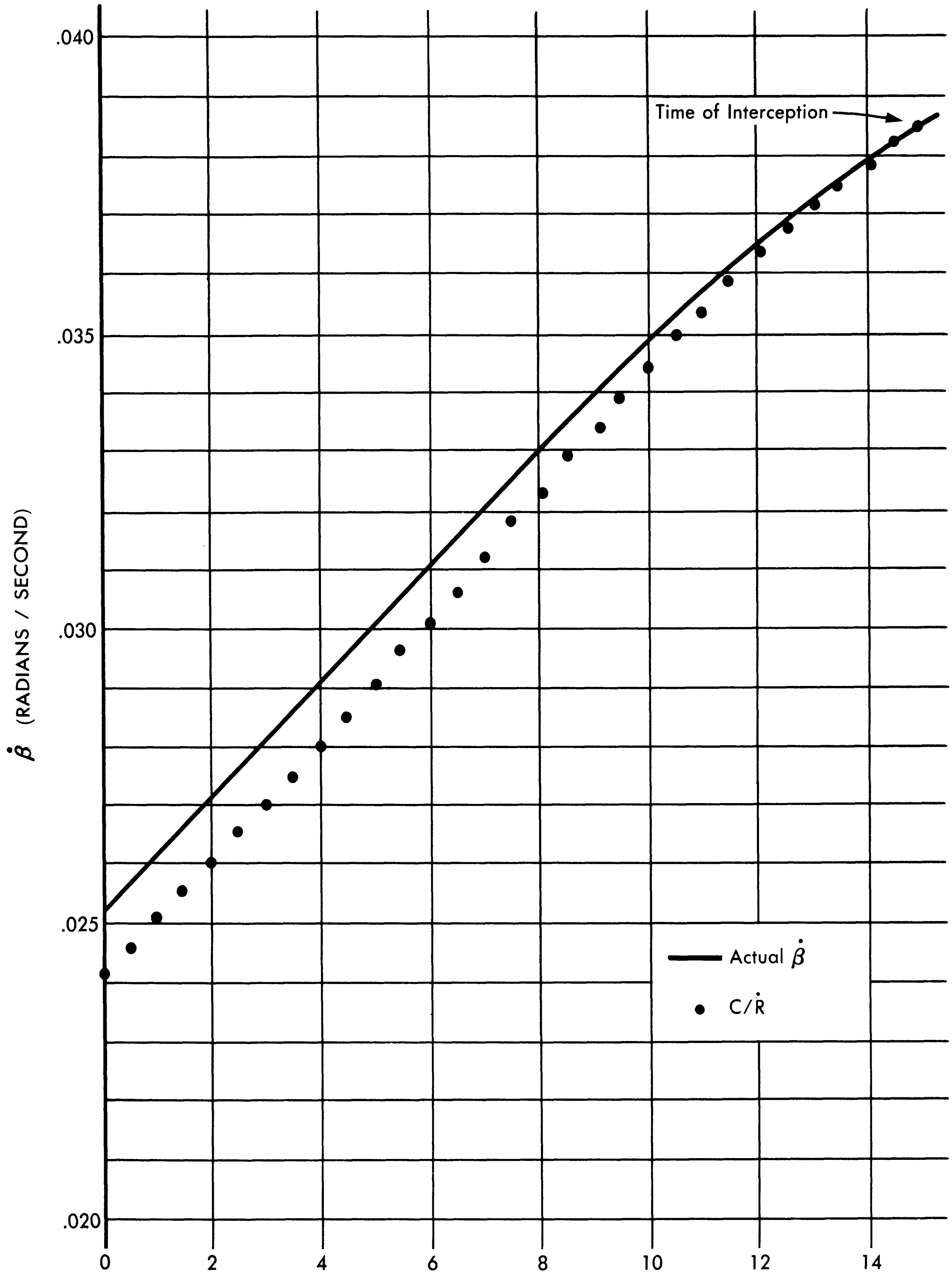


FIG. 2 COMPARISON OF $\dot{\beta}$ WITH C/$\dot{R}$ CONSTANT ACCELERATION INTERCEPTION

$$\frac{\dot{V}_t \sin(\alpha - \beta) - \dot{V}_m \sin(\theta - \beta)}{2\dot{R}} = \dot{\beta}_d \quad . \qquad (3.3\text{-}12)$$

This equation depends only on the instantaneous values of the parameters involved. Its physical interpretation becomes obvious upon multiplying it by R and using Equation (3.1-3). Equation (3.3-12) then becomes:

$$\left[V_t - \frac{1}{2}\dot{V}_t \frac{R}{\dot{R}}\right]\sin(\alpha - \beta) = \left[V_m - 1/2\dot{V}_m \frac{R}{\dot{R}}\right]\sin(\theta - \beta) \ . \qquad (3.3\text{-}13)$$

If $-\frac{R}{\dot{R}} = T$, this equation would be precisely the equation for a straight-line collision course with constant longitudinal accelerations. In particular, if Equation (3.3-12) is fullfilled, a straight-line interception will result, provided the longitudinal accelerations and closing rate remain unchanged. This condition cannot, of course, be expected. However, it is satisfied more and more closely as interception is approached.

In actual practice, neither the velocities nor the accelerations can be expected to be constant. The exact equation for the guidance criterion corresponding to an interception in which zero lateral acceleration is required of the interceptor, and in which target and interceptor undergo arbitrary longitudinal accelerations is derived as follows:

Equation (3.3-8) is a completely general equation, valid for arbitrary target and interceptor motion, subject only to the restriction that there are no lateral accelerations. Multiplying both sides of this equation by R leads to the exact differential of $R^2\dot{\beta}$:

$$\frac{d}{dt}\left(R^2\dot{\beta}\right) = R\left[\dot{V}_t \sin(\alpha - \beta) - \dot{V}_m \sin(\theta - \beta)\right] \quad . \qquad (3.3\text{-}14)$$

The requirement that an interception take place as imposed by integrating Equation (3.3-14) from 0 to τ, where τ is the time at which interception occurs. Or, changing variables so that the integration is over range:

$$R^2\dot{\beta} = \int_R^0 \frac{r}{\dot{r}}\left[\dot{V}_t \sin(\alpha - \beta) - \dot{V}_m \sin(\theta - \beta)\right] dr \quad . \qquad (3.3\text{-}15)$$

The preceding equation is an exact expression for the value of $\dot{\beta}$ along a straight-line trajectory to interception. If there are only a finite number of derivatives of the target and interceptor velocities, this equation can be integrated to give Equation (3.3-4). Although Equation (3.3-15) probably does not have general applicability as a guidance criterion, it is exact and any usable guidance criterion must be regarded as an approximation to this equation.

3.4 NAVIGATION

With the proper guidance criterion, an interceptor can, in theory, be made to follow an approximately straight-line course to its target. To achieve this course, the rate of change of interceptor heading is made proportional to the difference between the actual and the desired rates of rotation of the line of sight; that is, the following equation is mechanized:

$$\dot{\theta} = K\left(\dot{\beta} - \dot{\beta}_d\right) . \qquad (3.4\text{-}1)$$

Ordinary proportional navigation, then, is just a special case of Equation (3.4-1) in which $\dot{\beta}_d$ has the value of zero.

IV

ANALYTIC NAVIGATION STUDIES

4.1 MINIMUM NAVIGATION CONSTANT

The navigation constant, K, of Equation (3.4-1) must be sufficiently large that the rate of rotation of the line of sight, $\dot{\beta}$, will decrease in absolute value with time (Ref. 1). A lower bound for the navigation constant can be established by finding that value required to maintain $\dot{\beta}$ constant. Differentiating Equation (3.1-3) and setting $\ddot{\beta} = 0$ yields:

$$2\dot{R}\dot{\beta} - 2\dot{R}\dot{\beta}_d = -V_m \cos(\theta - \beta)\dot{\theta} \qquad (4.1\text{-}1)$$

where $\dot{\beta}_d$ is defined by Equation (3.3-12).

Substituting Equation (3.4-1) yields:

$$K_{min} = \frac{-2\dot{R}}{V_m \cos(\theta - \beta)} \quad . \qquad (4.1\text{-}2)$$

For any smaller value of K, the rate of rotation of the line of sight increases with time. For constant target and interceptor speeds and an interceptor speed advantage this equation can be shown to contain the result of Reference 3:

$$K_{min} = 2\left(1 + \frac{V_t}{V_m}\right).$$

If the interceptor has a speed advantage, the minimum navigation constant is no larger than four; this value can serve for all approach angles without introducing excessive sensitivity to noise.

If, on the other hand, the interceptor has a speed disadvantage, the minimum navigation constant may be arbitrarily large. As a consequence, it is desirable to replace the navigation constant, K, by a navigation factor, K_1, defined as follows:

$$K_1 = k_1 K_{min} = \frac{-2\dot{R}k_1}{V_m \cos(\theta - \beta)} . \quad (4.1\text{-}3)$$

It is clear that K_1 and k_1 cannot be constant simultaneously, except in trivial cases. The analysis which follows is based on the assumption that k_1 is constant. It is obvious that k_1 must be greater than or equal to unity; on the other hand, too large a value of K_1 leads to excessive acceleration.

4.2 THE KINEMATIC EQUATIONS

The navigation problem for an ideal interception in a single plane is completely described by Equations (3.1-3) and (3.1-4) and by Equation (2.1-1), the steering equation.

In the constant-speed case, the steering equation serves to reduce $\dot{\beta}$ to zero, provided K is sufficiently large.

It is well known that the above set of three simultaneous differential equations is not directly integrable. However, when K is replaced by K_1 (Eq. 4.1-3), the resulting system can be integrated.

4.3 INTEGRATION OF KINEMATIC EQUATIONS

Replacing K by K_1, the steering equation becomes:

$$\dot{\theta} = \frac{-2\dot{R}k_1}{V_m \cos(\theta - \beta)} \dot{\beta} , \quad (4.3\text{-}1)$$

and the kinematic equations for the constant-speed case become Equations (3.1-3), (3.1-4) and (4.3-1). Dividing Equation (3.1-4) by Equation (3.1-3) yields:

$$\frac{\dot{R}}{R} = \frac{V_t \cos(\alpha - \beta) - V_m \cos(\theta - \beta)}{V_t \sin(\alpha - \beta) - V_m \sin(\theta - \beta)} \dot{\beta} \quad (4.3\text{-}2)$$

or

$$\frac{-\dot{R}}{R} = \frac{-V_t\cos(\alpha - \beta)\dot{\beta} + V_m\cos(\theta - \beta)\dot{\beta} - V_m\cos(\theta - \beta)\dot{\theta} + V_m\cos(\theta - \beta)\dot{\theta}}{V_t\sin(\alpha - \beta) - V_m\sin(\theta - \beta)} \quad . \tag{4.3-3}$$

Substituting Equation (4.3-1) in Equation (4.3-3) yields:

$$(2k_1 - 1)\frac{\dot{R}}{R} = \frac{d/dt(R\dot{\beta})}{R\dot{\beta}} \quad . \tag{4.3-4}$$

Integration of Equation (4.3-4) yields an explicit expression for the rate of rotation of the line of sight in terms of the relative range; viz.,

$$\dot{\beta} = \dot{\beta}_o(R/R_o)^{2k_1 - 2} \quad . \tag{4.3-5}$$

The lateral acceleration, $V_m\dot{\theta}$, of the interceptor is obtained immediately from this result and from Equation (4.3-1):

$$V_m\dot{\theta} = \frac{-2\dot{R}k_1}{\cos(\theta - \beta)}\dot{\beta}_o(R/R_o)^{2k_1 - 2} \quad . \tag{4.3-6}$$

This equation reveals the effect of the choice of k_1. If this parameter is smaller than unity, the lateral acceleration required of the interceptor tends to infinity as the interceptor approaches interception. A value of unity gives a constant $\dot{\beta}$ and hence an approximately constant lateral acceleration course. If the value of k_1 (Eq. 4.1-3) is greater than unity, the lateral acceleration required of the interceptor approaches zero as the interceptor approaches interception.

4.4 THE GENERAL CASE

Extension of the results of the preceding section to arbitrarily varying target and interceptor speeds is straightforward. The kinematic equations for this case are Equations (3.1-3), (3.1-4), and

$$\dot{\theta} = K_1\left(\dot{\beta} - \dot{\beta}_d\right) \qquad (4.4\text{-}1)$$

where the exact expression for $\dot{\beta}_d$ is obtained from Equation (3.3-15).

It is easily verifiable by logarithmic differentiation that the solution to these general kinematic equations is:

$$\left(\dot{\beta} - \dot{\beta}_d\right) = \left(\dot{\beta} - \dot{\beta}_d\right)_o \left(R/R_o\right)^{2k_1 - 2} \quad . \qquad (4.4\text{-}2)$$

Equation (4.4-2) obviously reduces to Equation (4.3-5) when $\dot{\beta}_d = 0$. The lateral acceleration of the interceptor in the general case is:

$$V_m\dot{\theta} = \frac{-2\dot{R}k_1}{\cos(\theta - \beta)}\left(\dot{\beta} - \dot{\beta}_d\right)_o \left(R/R_o\right)^{2k_1 - 2} \quad . \qquad (4.4\text{-}3)$$

The analogy between Equations (4.3-5) and (4.3-6) and Equations (4.4-2) and (4.4-3) suggests the study of the general navigation problem with the general guidance criterion of Equation (3.3-15) by interpretation of an appropriately chosen constant-speed case. Before applying this method it is necessary to consider errors in the guidance criterion. If these errors affect the validity of the analogy, the method must be abandoned.

4.5 GUIDANCE ERROR

The derivations in the previous sections apply to perfect guidance. Obviously, any practical guidance system must be based on the assumption of reasonably well behaved derivatives of interceptor and target velocities. It is necessary to keep in mind that the existence of "unanticipated" derivatives of the velocities results in unanticipated lateral accelerations (Sec. II). If a good choice of $\dot{\beta}_d$ is made, errors in the assumed velocities give rise to relatively small lateral accelerations, and the path of the interceptor remains nearly a straight line. When the time remaining until interception is small, departures from the predicted velocity behavior can be expected to be small, and in the closing phase of interception little or no lateral acceleration will be called for.

Errors in the prediction of the velocity are reflected as errors in the desired rate of rotation of the line of sight. Consider any error ϵ in $\dot{\beta}_d$ subject only to the restriction that ϵ may be expressed as a power series in R; i.e., $\epsilon = \Sigma a_i R^i$. Then the guidance equation becomes

$$\dot{\theta} = \frac{-2Rk_1}{V_m \cos(\theta - \beta)} \left(\dot{\beta} - \dot{\beta}_d - \sum a_i R^i \right) . \tag{4.5-1}$$

A solution to the system of equations defined by Equations (3.1-3), (3.1-4) and (4.5-1) is

$$\frac{R(\dot{\beta} - \dot{\beta}_d - \sum b_i R^i)}{R_o(\dot{\beta} - \dot{\beta}_d - \sum b_i R^i)_o} = \left(\frac{R}{R_o} \right)^{2k_1 - 1} . \tag{4.5-2}$$

Where b_i is related to a_i by:

$$b_i = \frac{2k_1 a_i}{2k_1 - 2 - i} \tag{4.5-3}$$

Rearranging Equation (4.5-2) gives:

$$\dot{\beta} - \dot{\beta}_d = \left(\dot{\beta} - \dot{\beta}_d \right)_o \left(\frac{R}{R_o} \right)^{2k_1 - 2} + \sum b_i R^i - \left(\sum b_i R_o^i \right) \left(\frac{R}{R_o} \right)^{2k_1 - 2} \tag{4.5-4}$$

Comparison of this equation with Equation (4.4-2) shows that an error ϵ in $\dot{\beta}_d$ adds a changing bias to the variation of $\dot{\beta}$ (and of course to the lateral acceleration) during flight.

At the end of the flight the bias in $\dot{\beta}$ (the difference between the actual and the correct $\dot{\beta}_d$) is just the value of the constant b_0. For any error which can be expressed as a power series in R, the terminal value of the bias in $\dot{\beta}$ is simply $\frac{k_1 a_o}{k_1 - 1}$. Similarly, the lateral acceleration always has a final value which involves only the first term of the power series for ϵ:

$$V_m \dot{\theta} = V_m K_1 \frac{k_1 a_o}{k_1 - 1} \quad . \tag{4.5-5}$$

These considerations show that the effect of an error in the guidance criterion, as reflected in the acceleration history of the interceptor, does not depend on the guidance criterion, but only on the error.

4.6 TARGET MANEUVER

The effect of a target maneuver may be regarded as simply the introduction of an error in the value of $\dot{\beta}_d$. If target maneuver is considered, Equation (3.3-15) becomes

$$\dot{\beta} = \frac{1}{R^2} \int_R^0 \frac{r}{\dot{r}} \left[\dot{V}_t \sin(\alpha - \beta) - \dot{V}_m \sin(\theta - \beta) + V_t \dot{\alpha} \cos(\alpha - \beta) \right] dr \quad . \tag{4.6-1}$$

The error in the guidance criterion due to maneuver may be written as:

$$\epsilon = \frac{1}{R^2} \int_R^0 \frac{V_t \dot{\alpha} \cos(\alpha - \beta)}{\dot{r}} r \, dr \quad . \tag{4.6-2}$$

In many cases, $V_t \dot{\alpha} \cos(\alpha - \beta)/\dot{r}$ is very nearly a constant and, within reasonable limitations, Equations (4.5-4) and (4.5-5) may be used to interpret the effect of target maneuver on the lateral acceleration and on the variation of $\dot{\beta}$.

4.7 THE CONSTANT-SPEED ANALOGY

The results of the preceding sections make it possible to carry out guidance studies yielding results of considerable generality, because the solutions to the kinematic equations are, to a great extent, independent of the target and missile longitudinal accelerations, the basis of the guidance criterion. There are two alternative applications of the results:

1. Ordinary proportional navigation may be studied in terms of the suggested method of navigation; that is, for situations in which K_1 can be expected to have a nearly constant value (short-range interceptions with favorable geometry), Equation (4.3-5) may be applied to ordinary proportional navigation.

2. Interceptions under conditions of arbitrary target and interceptor longitudinal accelerations may be studied by analogy with the constant-speed case.

Considerable caution must be used when the first alternative is to be employed. The results can easily be misleading when K_1 varies during the course of the interception. The second alternative has far more generality; to study a given variable-speed interception, it is only necessary to select a constant-speed case in which the mean target and interceptor speeds agree with those in the variable-speed case. The effects of errors in the guidance criterion in the variable-speed case have been shown to be dependent only on the guidance error (Sec. 4.5), just as in the constant-speed case. The agreement found in the results is generally excellent. For example, Figures 3 and 4 illustrate typical results of an application of the constant-speed analogy. In this example, the ratio of the mean speeds of the target and the interceptor was 1.66:1.

Two interceptions were calculated from the same initial geometry. In the first of these, constant speeds were assigned to both the target and the interceptor; in the second, the interceptor accelerated and the target decelerated at constant rates. The interceptor lateral accelerations are shown in Figure 3, and the trajectories are shown in Figure 4.

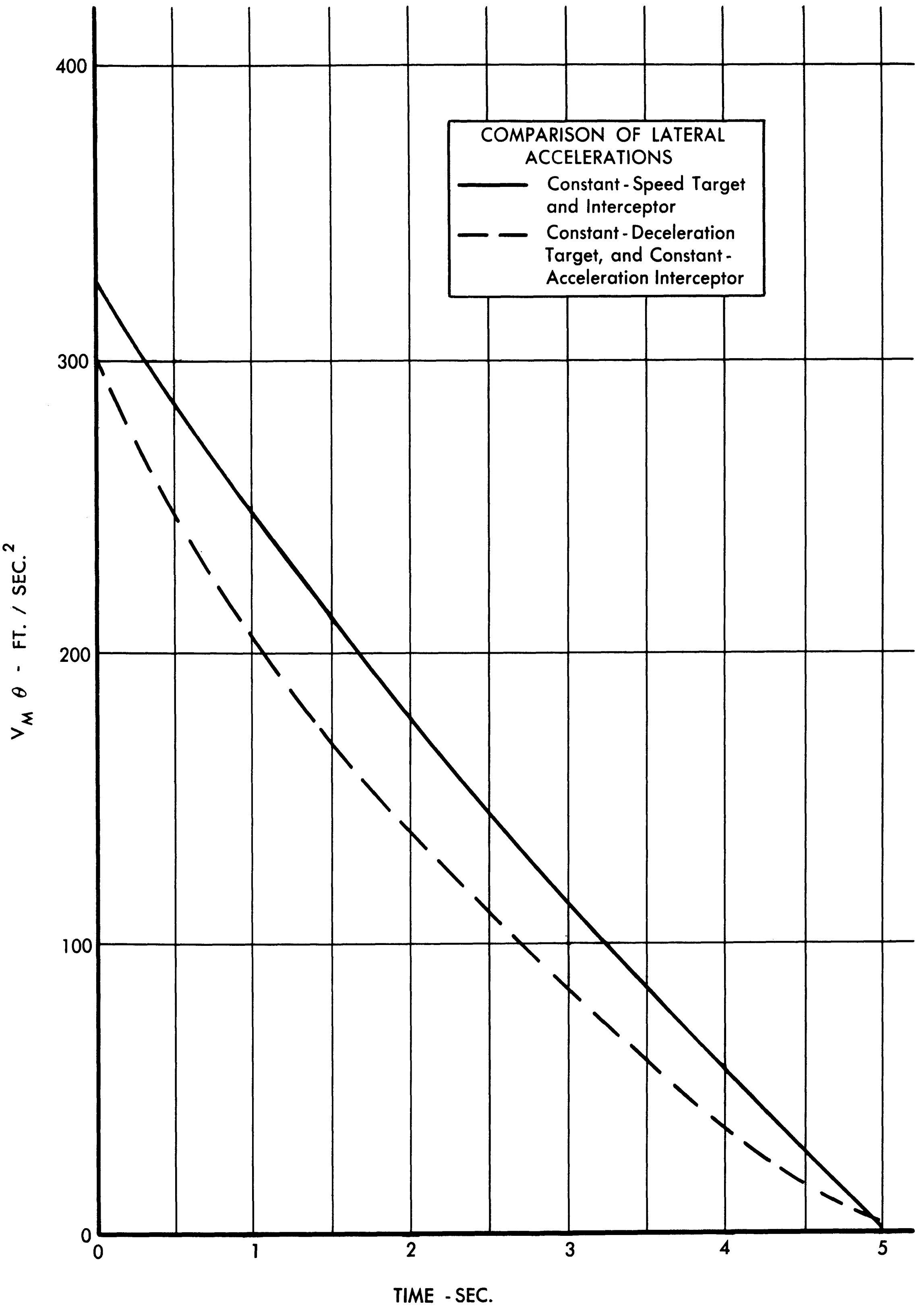


FIG. 3 INTERPRETATION OF CONSTANT ACCELERATION INTERCEPTION BY ANALOGY

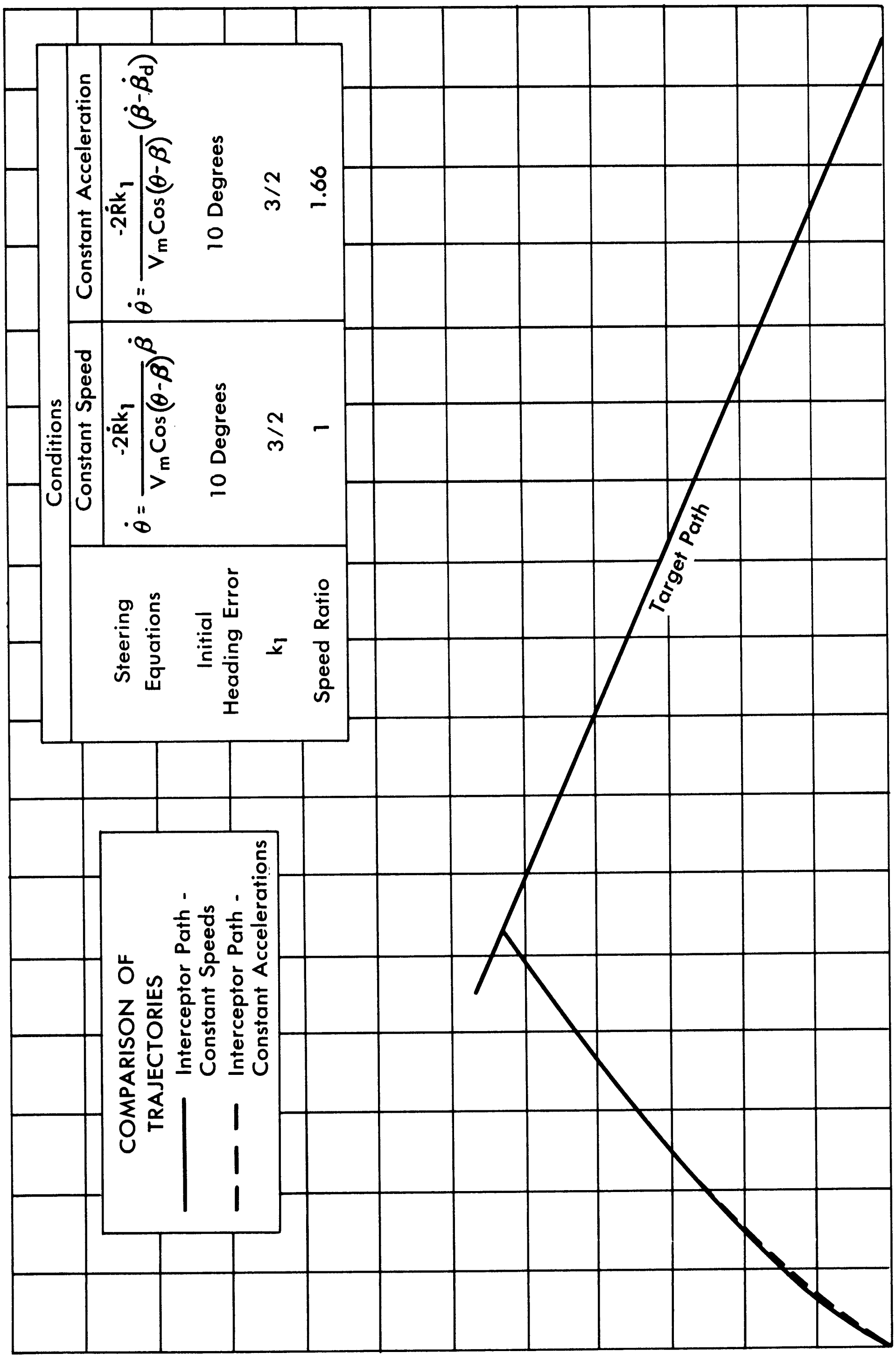


FIG. 4 INTERPRETATION OF CONSTANT ACCELERATION INTERCEPTION BY ANALOGY

The slight differences in lateral accelerations shown in Figure 3 would be still smaller if the lateral accelerations were plotted as functions of the relative range (R/R_o) rather than as functions of time.

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