that reflect the phase reactions given by Eqs. (1) and (2) and, denoted by the invariant planes in Fig. 4.

ACKNOWLEDGMENTS

We wish to thank Dr. H. W. Katz for useful discussions and T. D. Martin, D. W. Toivonon, and D. Schumacher for helping to prepare the samples and to perform the transport-property measurements. In addition, we are grateful to Dr. D. P. Spitzer for providing construction details of his thermal conductivity apparatus. We also wish to thank Mrs. L. Steeves for typing this manuscript.

JOURNAL OF APPLIED PHYSICS VOLUME 38, NUMBER 12 NOVEMBER 1967

Wave Excitation in Compressible Plasma and Equivalence Relations*

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(Received 20 December 1966; in final form 13 February 1967)

A general, unified solution of the wave excitation due to electric-current sources, magnetic-current sources, fluid-flux sources, and mechanical-body sources in a compressible plasma which may be anisotropic and inhomogeneous is presented. The Maxwell–Euler equations are reformulated through linear operator and generalized transform techniques into an equivalent matrix integral equation. The dispersion relation can be obtained from the kernel of the integral equation. When the medium is homogeneous, this integral equation has an ideal kernel and the explicit solution can be easily obtained. Equivalence relations between different types of sources are obtained from the forcing function of the integral equation, which can be employed to express the field excited by one type of source in terms of the field excited by another type of source. Generalized telegraphist's equations are also derived in due course.

Some dispersion curves, and asymptotic solutions for the radiation field from a point current source oriented in the direction of a constant magnetic field, are presented in graphical form. A proper ionospheric plasma is assumed for this calculation, which combines some of the results obtained for anisotropic cold-plasma problems on the one hand and some of the results obtained for isotropic warm-plasma problems on the other hand.

I. INTRODUCTION

The wave-excitation problem in an anisotropic compressible plasma is of current interest. This problem is just a part of complex phenomena in the fourth state of matter, plasma, and is related to the practical problems of communication through the shock-induced envelope of ionized gas surrounding the reentry vehicle, scattering cross section of missile trail or rocket exhaust, and many others.

The propagation of plane waves in a plasma has been studied extensively.[1–5] Recently, excitation problems in a plasma have attracted the attention of many investigators, and can be conveniently divided into three categories:

(1) Cold-plasma problems with or without a uniformly impressed constant magnetic field. In this type of work the plasma can be characterized by a tensor dielectric constant and the longitudinal acoustic type of wave does not come into picture. Typical examples of this type of problem are the works of Bunkin,[6] Kogelnik,[7] and Arbel and Felsen.[8]

(2) Compressible plasma without an externally applied constant magnetic field. With these assumptions Cohen[9] has shown that the field can be separated into two types of modes; one mode is transverse in nature and has all the fluctuating magnetic field, and another mode is longitudinal in nature and has all the fluctuating density field. The radiation of this longitudinal acoustic-type of wave has been investigated by Hessel and Shmoys,[10] Whale,[11] Chen,[12] Wait[13] and others.

* This work was supported by the Rome Air Development Center.

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(3) Compressible plasma with externally impressed constant magnetic field. The problem encountered with these assumptions is much more complicated due to the fact that the transverse and longitudinal waves are coupled and modified by each other. Seshadri[14] has treated the radiation from a line magnetic-current source.

A general, unified investigation of the excitation problem of third category is presented in this paper. Both two-dimensional and three-dimensional excitation fields due to different types of sources are studied, thus including the Seshadri's treatment.

The inhomogeneous Maxwell–Euler equations are reformulated through linear operator and generalized transform techniques into an equivalent matrix integral equation. Operator methods are a well-known and potent tool in quantum mechanics, and the introduction of the operator method into electromagnetic theory has been explored by Bresler and Marcuvitz,[15,16] Moses,[17] and others. Recently, Diament[18] has studied the formalism of an operator method combined with a generalized transform method in obtaining the formal solution of Maxwell's equations for general linearized media. Because of the systematic approach, compact notation, and convenience for numerical analysis, these formal operator transform techniques are extended in the present work to the system of linearized equation describing the excitation of disturbances in inhomogeneous, anisotropic and compressible plasma.

Equivalence relations between the magnetic-current source, the electric-current source, the mechanical-body source, and the fluid-flux source are easily obtained from the four-vector forcing function of the integral equation.

There are also some other papers by Seshadri[19,20] which are related to the present work.

II. BASIC EQUATIONS

In this section, the basic equations governing weak disturbances produced by various kinds of sources in a compressible plasma are presented.

The following assumptions are used:

(a) The plasma as a whole is stationary.

(b) The plasma is constantly under the action of a dc magnetic field.

(c) The motions of ions and neutral particles can be neglected.

(d) The sources of the disturbances are weak.

(e) Ideal gas law can be applied.

(f) Collisional dissipation effects can be neglected.

In this case, a set of linearized equations is usually considered to be adequate to relate the disturbances to their respective sources. Considering just one Fourier component of the disturbances in the form of $\exp(-i\omega t)$, and employing the rationalized mks system of units, this set of equations is the following linearized inhomogeneous Maxwell and Euler equations (Oster,[3] Tanenbaum and Mintzer,[21] Cohen[9,22,23]

(a) The Maxwell equations:

$$\nabla \times \mathbf{E} - i\mu_0\omega\mathbf{h} = -\mathbf{K}, \qquad (1)$$

$$\nabla \times \mathbf{h} + i\epsilon_0\omega\mathbf{E} + eN_0\mathbf{V} = \mathbf{J}. \qquad (2)$$

(b) The momentum-transport equation:

$$-i\omega N_0 m\mathbf{V} + mU^2\nabla n + eN_0(\mathbf{E} + \mathbf{V}\times\mathbf{B}_0) = \mathbf{F}. \qquad (3)$$

(c) The mass-transport equation

$$N_0\nabla\cdot\mathbf{V} + \mathbf{V}\cdot\nabla N_0 - i\omega n = Q \qquad (4)$$

The following notation has been used in the above equations:

- $\mathbf{h}$: varying component of the magnetic field
- $\mathbf{E}$: varying component of the electric field (constant component is not considered in this investigation)
- ϵ_0: permittivity of free space
- μ_0: permeability of free space
- $\mathbf{B}_0$: externally applied dc magnetic field
- $\mathbf{V}$: fluid velocity of the electron gas
- $\mathbf{K}$: magnetic-current source
- $\mathbf{J}$: electric-current source
- n: varying component of the number density of the electron gas
- e: absolute value of the charge of an electron
- $\mathbf{F}$: mechanical body source for the electron gas
- Q: fluid-flux source for the electron gas
- N_0: the number density of electrons in the undisturbed plasma
- m: the mass of the electron
- U: the acoustic velocity for the electron gas under adiabatic condition.

III. OPERATOR-TRANSFORM METHOD

A formal solution to the set of Eqs. (1)–(4) can be obtained by an operator-transform method. This method is an extension of that used by Diament[18] for the formal solution of Maxwell's equations in general linear media.

First of all, Eqs. (1)–(4) are rearranged to obtain a proper matrix form. Thus, $\mathbf{E}$ and $\mathbf{V}$ are solved in terms of $\mathbf{h}$, n, $\mathbf{J}$ and $\mathbf{F}$ by employing Eqs. (2) and (3).

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Their results are:

$$A_{11}\mathbf{h}+A_{12}n+\mathbf{E}=\mathbf{S}_1, \tag{5}$$

$$A_{21}\mathbf{h}+A_{22}n+\mathbf{V}=\mathbf{S}_2, \tag{6}$$

$$A_{11}=\frac{\omega_p{}^2\omega/i\epsilon_0}{(\omega^2-\omega_p{}^2)^2-\omega_c{}^2\omega^2}\left[\frac{i\omega_c}{\omega}\hat{b}\times(\nabla\times\mathbf{1})+\frac{(\omega^2-\omega_p{}^2)}{\omega^2}\nabla\times\mathbf{1}-\frac{\omega_c{}^2}{(\omega^2-\omega_p{}^2)}\hat{b}\hat{b}\cdot\nabla\times\mathbf{1}\right]-\frac{i\nabla\times\mathbf{1}}{\epsilon_0\omega}, \tag{7}$$

$$A_{12}=\frac{eU^2\omega/i\epsilon_0}{(\omega^2-\omega_p{}^2)^2-\omega_c{}^2\omega^2}\left[\omega_c\hat{b}\times(\nabla\cdot\mathbf{1})+\frac{(\omega^2-\omega_p{}^2)}{i\omega}\nabla\cdot\mathbf{1}+\frac{i\omega\omega_c{}^2}{(\omega^2-\omega_p{}^2)}\hat{b}\hat{b}\cdot(\nabla\cdot\mathbf{1})\right], \tag{8}$$

$$A_{21}=[A_{11}+(i\nabla\times\mathbf{1}/\epsilon_0\omega)](\epsilon_0\omega/ieN_0), \tag{9}$$

$$A_{22}=A_{12}(\epsilon_0\omega/ieN_0), \tag{10}$$

$$\mathbf{S}_1=\frac{\omega/i\epsilon_0}{(\omega^2-\omega_p{}^2)^2-\omega_c{}^2\omega^2}\left\{\frac{e}{m}\left[\omega_c\hat{b}\times\mathbf{F}+\frac{(\omega^2-\omega_p{}^2)}{i\omega}\mathbf{F}+\frac{i\omega\omega_c{}^2}{(\omega^2-\omega_p{}^2)}\hat{b}\hat{b}\cdot\mathbf{F}\right]\right.$$
$$\left.+\omega_p{}^2\left[\frac{i\omega_c}{\omega}\hat{b}\times\mathbf{J}+\frac{(\omega^2-\omega_p{}^2)}{\omega^2}\mathbf{J}-\frac{\omega_c{}^2}{(\omega^2-\omega_p{}^2)}\hat{b}\hat{b}\cdot\mathbf{J}\right]\right\}-i\frac{\mathbf{J}}{\epsilon_0\omega} \tag{11}$$

$$\mathbf{S}_2=[\mathbf{S}_1+i(\mathbf{J}/\epsilon_0\omega)](\epsilon_0\omega/ieN_0). \tag{12}$$

In the expression given by Eqs. (7)–(12), $\hat{b}$ is the unit vector in the direction of the externally applied constant magnetic field, $\omega_c=eB_0/m$ is the conventional electron cyclotron frequency, $\omega_p{}^2=e^2N_0/\epsilon_0 m$ is the electron plasma frequency, and also we have employed some dyadic operations with their associated matrices as given in the Appendix.

Now the original Eqs. (1) and (4), together with the rearranged Eqs. (5) and (6) can be put into the following desirable matrix form:

$$\begin{bmatrix} 1 & 0 & i(\nabla\times\mathbf{1}/\omega\mu_0) & 0 \\ 0 & 1 & 0 & (iN_0/\omega)(\nabla\cdot\mathbf{1})'+(i/\omega)\nabla N_0\cdot\mathbf{1} \\ A_{11} & A_{12} & 1 & 0 \\ A_{21} & A_{22} & 0 & 1 \end{bmatrix}\begin{bmatrix}\mathbf{h}\\ n\\ \mathbf{E}\\ \mathbf{V}\end{bmatrix}=\begin{bmatrix}-i\mathbf{K}/\omega\mu_0\\ iQ/\omega\\ \mathbf{S}_1\\ \mathbf{S}_2\end{bmatrix} \tag{13}$$

where $(\nabla\cdot\mathbf{1})'$ is the transpose of $\nabla\cdot\mathbf{1}$.

This matrix equation can, then, be put into the form of an operator equation

$$\mathcal{W}\psi(r)=\phi(r) \tag{14}$$

Thus, the basic Maxwell–Euler's Eqs. (1) through (4) have been reformulated into a single abstract relation between the sources and the resultant fields. $\psi(r)$ is a ten-vector containing the field quantities, $\phi(r)$ is a ten-vector representing the source quantities, and $\mathcal{W}$ is the system matrix differential operator relating the fields to the sources. Two identity submatrices of $\mathcal{W}$ are significant in deriving an integral equation of the second kind in an inhomogeneous medium.

Here, we introduce the generalized transform techniques, which amounts to choosing some convenient basis of representation for the solution and transforming the operator differential equation in real space to an operator integral equation in transform space. The generic summation symbol S, such as used in Quantum Mechanics,[24] will be used, which requires that the expression following this symbol be integrated or summed over the entire range of the repeated variable.

Let $\Psi(s)$ and $\Phi(s)$ be the transforms of the vectors $\psi(r)$ and $\phi(r)$, respectively, then

$$\begin{cases}\Psi(s)=\mathsf{S}d(s,r)\psi(r)\\ \psi(r)=\mathsf{S}c(r,s)\Psi(s)\end{cases}, \tag{15}$$

and

$$\begin{cases}\Phi(s)=\mathsf{S}d(s,r)\phi(r)\\ \phi(r)=\mathsf{S}c(r,s)\Phi(s)\end{cases}, \tag{16}$$

where the transformation kernel and inverse transformation kernel satisfy the properties

$$\mathsf{S}c(r,s)d(s,p)=\mathbf{1}(r,p), \tag{17}$$

[24] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Co., Inc., New York, 1955), p. 128.

and

$$\mathsf{S}d(u,r)c(r,s)=\mathbf{1}(u,s). \tag{18}$$

The idemfactor $\mathbf{1}(u, s)$ comprises a Dirac delta function or a Kronecker delta and a unit dyadic, as required. Also, we take the transformation law for the matrix operator $\mathcal{W}$ as

$$\mathcal{W}(u,s)=\mathsf{S}d(u,r)\mathcal{W}c(r,s). \tag{19}$$

Now, we proceed to the transformation of the operator Eq. (14). Premultiplying both sides of Eq. (14) by $d(u, r)$, and then substituting the expansion for $\psi(r)$ as given by the transform pair in Eq. (15), and summing or integrating over the complete r space, the operator Eq. (14) in the real space becomes the operator integral equation in the transform space

$$\mathsf{S}\mathcal{W}(u,s)\Psi(s)=\Phi(u). \tag{20}$$

This equation has the character of a generalized integral equation of the first kind, with $\Phi(s)$ as the forcing function, $\Psi(s)$ as the unknown function, and $\mathcal{W}(u, s)$ as the kernel. Also, this equation may be put into the generalized forms of the telegraphist's equations[25,26] by the application of the following partitionings:

$$\Psi(s)=\mathsf{S}d(s,r)\begin{bmatrix}\mathbf{h}\\ n\\ \mathbf{E}\\ \mathbf{V}\end{bmatrix}=\begin{bmatrix}I_t(s)\\ V_e(s)\\ V_t(s)\\ I_e(s)\end{bmatrix}, \tag{21}$$

$$\Phi(s)=\mathsf{S}d(s,r)\begin{bmatrix}-i\mathbf{K}/\omega\mu_0\\ iQ/\omega\\ \mathbf{S}_1\\ \mathbf{S}_2\end{bmatrix}=\begin{bmatrix}J_t(s)\\ W_e(s)\\ W_t(s)\\ J_e(s)\end{bmatrix}, \tag{22}$$

$$\mathcal{W}(u,s)=\begin{bmatrix}\mathbf{1}(u,s) & 0 & -Y_t(u,s) & 0\\ 0 & \mathbf{1}(u,s) & 0 & -Z_e(u,s)\\ -Z_t(u,s) & -T_{te}(u,s) & \mathbf{1}(u,s) & 0\\ -T_{et}(u,s) & -Y_e(u,s) & 0 & \mathbf{1}(u,s)\end{bmatrix}. \tag{23}$$

The orthonomality property of the transformation kernels as given by Eq. (18) is utilized in Eq. (23). Thus, the generalized telegraphist's equations are

$$I_t(u)=J_t(u)+\mathsf{S}Y_t(u,s)V_t(s), \tag{24}$$

$$V_t(u)=W_t(u)+\mathsf{S}Z_t(u,s)I_t(s)+\mathsf{S}T_{te}(u,s)V_e(s), \tag{25}$$

$$I_e(u)=J_e(u)+\mathsf{S}T_{et}(u,s)I_t(s)+\mathsf{S}Y_e(u,s)V_e(s), \tag{26}$$

$$V_e(u)=W_e(u)+\mathsf{S}Z_e(u,s)I_e(s). \tag{27}$$

The general Fredholm integral equation of the first kind, Eq. (20), which is equivalent to the original Maxwell–Euler's equations, will now be reformulated into a general Fredholm integral equation of the second kind which is more amenable to analysis. At the same time we have reduced the order of the matrices to be manipulated from 10×10 to 4×4. In order to effect this reformulation, $\Psi(s)$, $\Phi(s)$ and $\mathcal{W}(u, s)$ will be partitioned in the following way:

$$\Psi(s)=\begin{bmatrix}\Psi_1(s)\\ \Psi_2(s)\end{bmatrix},\qquad \Phi(s)=\begin{bmatrix}\Phi_1(s)\\ \Phi_2(s)\end{bmatrix}, \tag{28}$$

where

$$\Psi_1(s)=\begin{bmatrix}I_t(s)\\ V_e(s)\end{bmatrix},\qquad \Psi_2(s)=\begin{bmatrix}V_t(s)\\ I_e(s)\end{bmatrix},$$
$$\Phi_1(s)=\begin{bmatrix}J_t(s)\\ W_e(s)\end{bmatrix},\qquad \Phi_2(s)=\begin{bmatrix}W_t(s)\\ J_e(s)\end{bmatrix}, \tag{29}$$

and

$$\mathcal{W}(u,s)=\begin{bmatrix}\mathbf{1}(u,s) & -\mathcal{W}_{12}(u,s)\\ -\mathcal{W}_{21}(u,s) & \mathbf{1}(u,s)\end{bmatrix}, \tag{30}$$

where

$$\mathcal{W}_{12}(u,s)=\begin{bmatrix}Y_t(u,s) & 0\\ 0 & Z_e(u,s)\end{bmatrix},$$
$$\mathcal{W}_{21}(u,s)=\begin{bmatrix}Z_t(u,s) & T_{te}(u,s)\\ T_{et}(u,s) & Y_e(u,s)\end{bmatrix}. \tag{31}$$

The introduction of these partitioned matrices into the integral Eq. (20) gives the following coupled

[25] S. A. Schelkunoff, Bell System Tech. J. **34**, 995 (1955).
[26] N. Marcuvitz, *Waveguide Handbook* (McGraw-Hill Book Co., Inc., New York, 1951), Sect. 3.5c.

integral equations:

$$\Psi_1(u)=\Phi_1(u)+\mathcal{S}\mathcal{W}_{12}(u,s)\Psi_2(s), \tag{32}$$

$$\Psi_2(u)=\Phi_2(u)+\mathcal{S}\mathcal{W}_{21}(u,s)\Psi_1(s), \tag{33}$$

and the substitution of Eq. (33) into Eq. (32) gives rise to the desired integral equation of the second kind

$$\Psi_1(u)=F(u)+\mathcal{S}K(u,s)\Psi_1(s), \tag{34}$$

where the compound source is

$$F(u)=\Phi_1(u)+\mathcal{S}\mathcal{W}_{12}(u,s)\Phi_2(s), \tag{35}$$

and the four-dyadic kernel is

$$K(u,s)=\mathcal{S}\mathcal{W}_{12}(u,v)\mathcal{W}_{21}(v,s). \tag{36}$$

IV. SOLUTION IN A HOMOGENEOUS PLASMA

The integral Eq. (34) can be easily solved for a homogeneous plasma because the kernel has the ideal form $K(u,s)=N(s)1(u,s)$.

Choosing a Fourier transform and thus using the transformation kernels

$$d(s,r)=[1/(2\pi)^3]\exp(-ir\cdot s),$$
$$c(r,s)=\exp(ir\cdot s), \tag{37}$$

we can obtain

$$-Y_t(u,s)=-(\mathsf{s}/\omega\mu_0)1(u,s), \tag{38}$$

$$-Z_e(u,s)=-(N_0/\omega)s'1(u,s), \tag{39}$$

$$-Z_t(u,s)=\frac{\omega\omega_p^2/i\epsilon_0}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[\frac{-\omega_c}{\omega}\mathsf{b}\mathsf{s}+i\frac{(\omega^2-\omega_p^2)}{\omega^2}\mathsf{s}-i\frac{\omega_c^2}{(\omega^2-\omega_p^2)}bb'\mathsf{s}\right]1(u,s)+\frac{\mathsf{s}}{\epsilon_0\omega}1(u,s), \tag{40}$$

$$-T_{te}(u,s)=\frac{e\omega U^2/i\epsilon_0}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[i\omega_c\mathsf{b}s+\frac{(\omega^2-\omega_p^2)}{\omega}s-\frac{\omega\omega_c^2}{(\omega^2-\omega_p^2)}bb's\right]1(u,s), \tag{41}$$

$$-T_{et}(u,s)=(\epsilon_0\omega/ieN_0)[-Z_t(u,s)-(\mathsf{s}/\epsilon_0\omega)1(u,s)], \tag{42}$$

$$-Y_e(u,s)=-(\epsilon_0\omega/ieN_0)T_{te}(u,s), \tag{43}$$

where

$$\mathsf{b}\equiv\begin{bmatrix}0 & -b_z & b_y\\ b_z & 0 & -b_x\\ -b_y & b_x & 0\end{bmatrix},\quad b\equiv\begin{bmatrix}b_x\\ b_y\\ b_z\end{bmatrix},\quad b'\equiv[b_x\ \ b_y\ \ b_z], \tag{44}$$

$$s\equiv\begin{bmatrix}s_1\\ s_2\\ s_3\end{bmatrix},\quad s'\equiv[s_1\ \ s_2\ \ s_3],\quad \mathsf{s}\equiv\begin{bmatrix}0 & -s_3 & s_2\\ s_3 & 0 & -s_1\\ -s_2 & s_1 & 0\end{bmatrix}.$$

The kernel of the integral Eq. (34) is

$$K(u,s)=\mathcal{S}\mathcal{W}_{12}(u,v)\mathcal{W}_{21}(v,s)$$
$$=\begin{bmatrix}\mathsf{S}Y_t(u,v)Z_t(v,s) & \mathsf{S}Y_t(u,v)T_{te}(v,s)\\ \mathsf{S}Z_e(u,v)T_{et}(v,s) & \mathsf{S}Z_e(u,v)Y_e(v,s)\end{bmatrix}, \tag{45}$$

with

$$\mathsf{S}Y_t(u,v)Z_t(v,s)=\frac{-c^2\omega_p^2/\omega}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\mathsf{s}\left[i\omega_c\mathsf{b}+\frac{\omega(\omega^2-\omega_p^2-\omega_c^2)}{\omega_p^2}1-\frac{\omega\omega_c^2}{\omega^2-\omega_p^2}bb'\right]\mathsf{s}1(u,s), \tag{46}$$

$$\mathsf{S}Y_t(u,v)T_{te}(v,s)=\frac{iec^2U^2}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\mathsf{s}\left[i\omega_c\mathsf{b}-\frac{\omega\omega_c^2}{(\omega^2-\omega_p^2)}bb'\right]s1(u,s) \tag{47}$$

$$\mathsf{S}Z_e(u,v)T_{et}(v,s)=\frac{i\omega_p^2/e}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}s'\left[i\omega_c\mathsf{b}-\frac{\omega\omega_c^2}{(\omega^2-\omega_p^2)}bb'\right]\mathsf{s}1(u,s), \tag{48}$$

$$\mathsf{S}Z_e(u,v)Y_e(v,s)=\frac{\omega U^2}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}s'\left[i\omega_c\mathsf{b}+\frac{(\omega^2-\omega_p^2)}{\omega}1-\frac{\omega\omega_c^2}{(\omega^2-\omega_p^2)}bb'\right]s1(u,s), \tag{49}$$

where $c=(\mu_0\epsilon_0)^{-1/2}$ is the velocity of light in free space. Thus, the kernel has the ideal form

$$K(u,s)=\mathbf{S}\mathbf{W}_{12}(u,v)\mathbf{W}_{21}(v,s)=N(s)\,1\,(u,s),\quad(50)$$

where $N(s)$ is a 4×4 matrix.

Substitution of this ideal form of the kernel given by Eq. (50) into the integral equation produces the solution of the integral equation directly as

$$\begin{bmatrix}I_t(s)\\ V_e(s)\end{bmatrix}=[1-N(s)]^{-1}F(s),\quad(51)$$

where the Fourier transform of the four-vector general source function

$$F(s)=\begin{bmatrix}J_t(s)+(\mathbf{S}/\omega\mu_0)W_t(s)\\ W_e(s)+(N_0s'/\omega)J_e(s)\end{bmatrix}.\quad(52)$$

In real space the magnetic field and the density fluctuation field are given by

$$\begin{bmatrix}\mathbf{h}(r)\\ n(r)\end{bmatrix}=\mathbf{S}[1-N(s)]^{-1}F(s)\exp(ir\cdot s),\quad(53)$$

and the dispersion relation is given as

$$\det.[1-N(s)]=0.\quad(54)$$

V. EQUIVALENCE RELATIONS

Equivalence relations between different types of sources in real space are obtained from the equivalence relations in transform space, which are derived from the four components of the four-vector general source function. These four components are found from Eq. (52) to be

$$F_1(s)=J_{tx}(s)+\frac{s_3ec^2}{\omega m(\omega^2-\omega_p^2)}f_y(s)+\frac{is_2c^2}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{\frac{e}{m}\left[\omega_cf_x(s)+i\frac{(\omega^2-\omega_p^2)}{\omega}f_z(s)\right]+i\frac{\omega_c\omega_p^2}{\omega}j_x(s)\right\}$$
$$+\frac{is_3c^2}{(\omega^2-\omega_p^2)}j_y(s)-\frac{is_2c^2(\omega^2-\omega_p^2-\omega_c^2)}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}j_z(s),\quad(55)$$

$$F_2(s)=J_{ty}(s)+\frac{ic^2}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left(-\frac{e}{m}\left\{\omega_c[s_3f_z(s)+s_1f_x(s)]+\frac{(\omega^2-\omega_p^2)}{i\omega}[s_3f_x(s)-s_1f_z(s)]\right\}\right.$$
$$\left.-\frac{i\omega_c\omega_p^2}{\omega}[s_3j_z(s)+s_1j_x(s)]-(\omega^2-\omega_p^2-\omega_c^2)[s_3j_x(s)-s_1j_z(s)]\right),\quad(56)$$

$$F_3(s)=J_{tz}(s)-\frac{s_1ec^2}{\omega m(\omega^2-\omega_p^2)}f_y(s)-\frac{is_1c^2}{(\omega^2-\omega_p^2)}j_y(s)$$
$$+\frac{is_2c^2}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{\frac{e}{m}\left[\omega_cf_z(s)+\frac{(\omega^2-\omega_p^2)}{i\omega}f_x(s)\right]+\frac{i\omega_c\omega_p^2}{\omega}j_z(s)+(\omega^2-\omega_p^2-\omega_c^2)j_x(s)\right\},\quad(57)$$

$$F_4(s)=W_e(s)-\frac{s_2}{(\omega^2-\omega_p^2)}\left[-\frac{i}{m}f_y(s)+\frac{\omega_p^2}{e\omega}j_y(s)\right]$$
$$-\frac{\omega/e}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{\frac{es_1}{m}\left[\omega_cf_z(s)+\frac{(\omega^2-\omega_p^2)}{i\omega}f_x(s)\right]+s_1\omega_p^2\left[i\frac{\omega_c}{\omega}j_z(s)+\frac{(\omega^2-\omega_p^2)}{\omega^2}j_x(s)\right]\right.$$
$$\left.+\frac{es_3}{m}\left[-\omega_cf_x(s)+\frac{(\omega^2-\omega_p^2)}{i\omega}f_z(s)\right]+s_3\omega_p^2\left[-i\frac{\omega_c}{\omega}j_x(s)+\frac{(\omega^2-\omega_p^2)}{\omega^2}j_z(s)\right]\right\}.\quad(58)$$

The transform of the electric current source, $j(s)\equiv\mathbf{S}d(s,r)\mathbf{J}$, and the transform of the mechanical body source, $f(s)\equiv\mathbf{S}d(s,r)\mathbf{F}$, have been used in Eqs. (55)–(58), in addition to the transform of the magnetic-current source $J_t(s)$, and the transform of the fluid-flux source $W_e(s)$ as defined by Eq. (22).

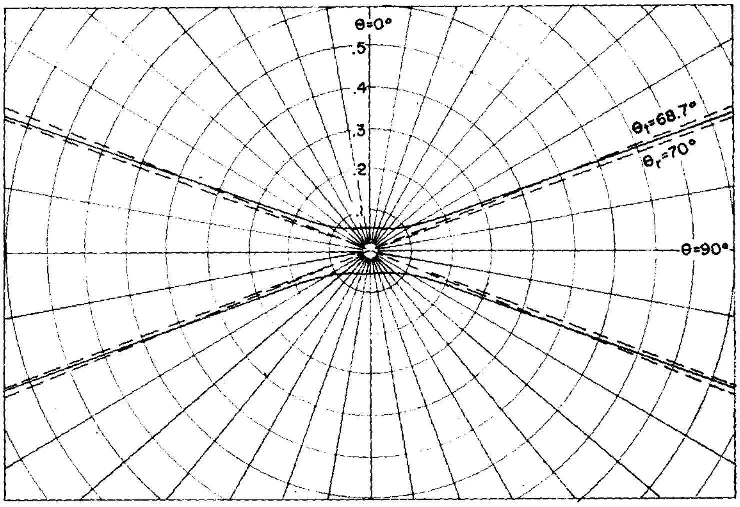

FIG. 1. Dispersion curve at $\omega=3\times10^6$.

Some equivalence relations between different types of sources in the real space, which can be used to obtain the excited field due to one type of source from the solutions obtained for another type of source, are given as follows:

$$-\frac{i}{\mu_0}\mathbf{K}=\frac{ec^2}{m[(\omega^2=\omega_p^2)^2-\omega_c^2\omega^2]}\left\{i(\omega^2-\omega_p^2)\nabla\times\mathbf{F}+\frac{i\omega_c^2\omega^2}{(\omega^2-\omega_p^2)}\hat{y}\times\nabla F_y-\omega\omega_c\hat{y}\nabla\cdot\mathbf{F}+\omega\omega_c\frac{\partial\mathbf{F}}{\partial y}\right\},\tag{59}$$

$$-\frac{i}{\mu_0}\mathbf{K}=\frac{c^2}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{-\omega(\omega^2-\omega_p^2-\omega_c^2)\nabla\times\mathbf{J}-\frac{\omega\omega_c^2\omega_p^2}{(\omega^2-\omega_p^2)}\hat{y}\times\nabla J_y-i\omega_c\omega_p^2\hat{y}\nabla\cdot\mathbf{J}+i\omega_c\omega_p^2\frac{\partial\mathbf{J}}{\partial y}\right\},\tag{60}$$

$$\frac{iQ}{\omega}=\frac{1}{m(\omega^2-\omega_p^2)}\frac{\partial F_y}{\partial y}+\frac{\omega/m}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{i\omega_c\frac{\partial F_z}{\partial x}+\frac{(\omega^2-\omega_p^2)}{\omega}\frac{\partial F_x}{\partial x}-i\omega_c\frac{\partial F_x}{\partial z}+\frac{(\omega^2-\omega_p^2)}{\omega}\frac{\partial F_z}{\partial z}\right\},\tag{61}$$

$$\frac{iQ}{\omega}=\frac{i\omega_p^2}{e\omega(\omega^2-\omega_p^2)}\frac{\partial J_y}{\partial y}-\frac{\omega_p^2/e}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{\omega_c\frac{\partial J_z}{\partial x}-i\frac{(\omega^2-\omega_p^2)}{\omega}\frac{\partial J_x}{\partial x}-\omega_c\frac{\partial J_x}{\partial z}-i\frac{(\omega^2-\omega_p^2)}{\omega}\frac{\partial J_z}{\partial z}\right\}.\tag{62}$$

Equation (59) is the equivalence relation between the magnetic-current source and the mechanical-body source; Eq. (60) is the equivalence relation between the magnetic-current source and the electric-current source; Eq. (61) is the equivalence relation between the fluid-flux source and the mechanical-body source; Eq. (62) is the equivalence relation between the fluid-flux source and the electric-current source.

VI. SAMPLE CALCULATIONS

In order to show the salient features of the present technique, the radiation fields due to a point current source is obtained. The point electric-current source can

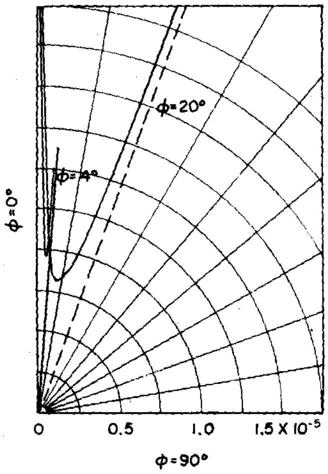

FIG. 2. E_ϕ' vs ϕ at $\omega=3\times10^6$.

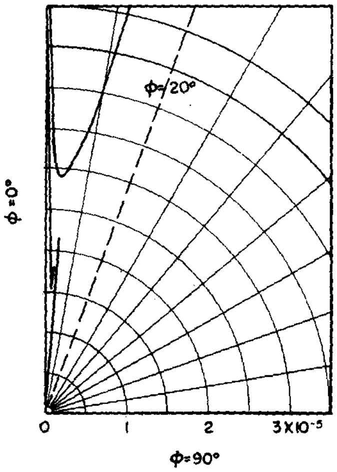

FIG. 3. E_M' vs ϕ at $\omega=3\times10^6$.

be expressed as

$$\mathbf{J}=\hat{y}(2\pi)^3 J_0\delta(x)\delta(y)\delta(z), \qquad (63)$$

and its transform given by

$$j_y(s)\text{-}\mathrm{S}d(s,r)(2\pi)^3 J_0\delta(x)\delta(y)\delta(z)=J_0. \qquad (64)$$

To obtain the solution in real space, it is necessary to evaluate the following integral,

$$\mathcal{G}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{L(s)\exp[i(s_1x+s_2y+s_3z)]}{G(s)}ds_1ds_2ds_3. \qquad (65)$$

The asymptotic solution to the integral Eq. (65) can be obtained by applying the principle of stationary phase. Usually, threefold Fourier integrals with axial symmetry are treated by conversion into Hankel transforms, but such a conversion complicates the asympotic evaluation and also loses sight of the close relation existing between the radiation fields and the dispersion relation. Thus, we will use the asymptotic solution developed by Lighthill.[27] The solution of Eq. (65) satisfying the radiation condition is asymptotically given by

$$\mathcal{G}=\frac{4\pi^2}{r}\sum\frac{CL\exp[i(s_1x+s_2y+s_3z)]}{|\nabla G||K|^{1/2}}+O(r^2)^{-1} \qquad (66)$$

as $r\rightarrow\infty$. The summation is over all points (s_1, s_2, s_3) of the surface $G(s)=\det.[1-N(s)]=0$ where the normal to the surface is parallel to the direction of observation and $(\mathbf{r}\cdot\nabla G)/(\partial G/\partial\omega)<0$. At each of these summation points the Gaussian curvature, K, can not be zero. C is $\pm i$ where $K<0$ and ∇G is in the direction of $\pm r$, and ± 1 where $K>0$ and the surface is convex to the direction of $\pm\nabla G$.

In the present problem the axis of symmetry is in the y direction and so the dispersion relation is a function of s_2^2 and $(s_1^2+s_3^2)$, or just a function of s and the angle θ measured from y axis, i.e.,

$$\det[1-N(s)]=f(s_2^2, s_1^2+s_3^2)=f(s,\theta). \qquad (67)$$

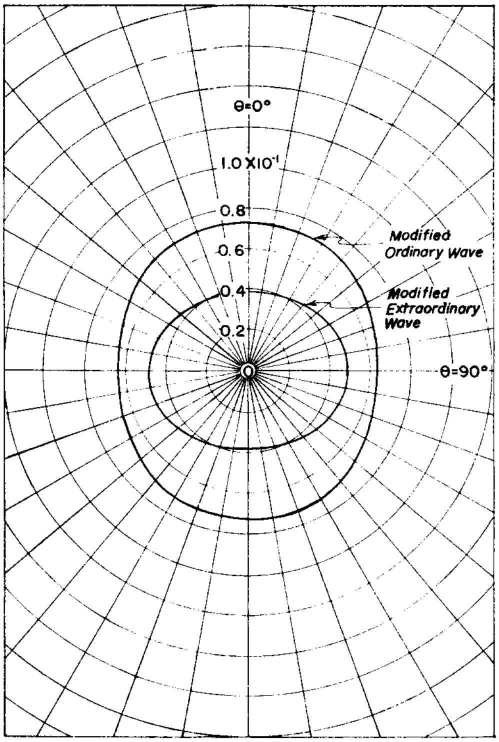

FIG. 4. Dispersion curves for modified ordinary and extra ordinary waves at $\omega=3\times10^7$.

The stationary points are given by the equations

$$\begin{cases} f=0 \\ (\partial f/\partial s_\rho)-|\tan\phi|(\partial f/\partial s_2)=0, \end{cases} \qquad (68)$$

where $s_\rho^2\equiv s_1^2+s_3^2$. It is very complicated to solve Eq. (68) directly. Instead, we find the radiation direction ϕ corresponding to each point on the dispersion curves,[8] which are the plots of $f(s,\theta)=0$. In order to perform this calculation, the following form of Eq. (68) is used;

$$\tan(\theta-\phi)=(1/s)(ds/d\theta). \qquad (69)$$

Also, the Gaussian curvature must be evaluated at each stationary phase point in the following form

$$K=\frac{[1-(\cot\theta/s)(ds/d\theta)][1+(2/s^2)(ds/d\theta)^2-(1/s)(d^2s/d\theta^2)]}{[1+(1/s^2)(ds/d\theta)^2]^2 s^2}, \qquad (70)$$

and, the absolute value of ∇G evaluated in the form

$$|\nabla G|=\{1+[(1/s)(ds/d\theta)]^2\}^{1/2}. \qquad (71)$$

The first derivative and the second derivative of s with respect to θ are obtained from the simplified form of the dispersion relation Eq. (54);

$$\begin{aligned} s^6(\Omega^2\cos^2\theta-1)&+s^4[(1-\omega_0^2)(\beta_e^2+2\beta_0^2)-\Omega^2(\beta_e^2+2\beta_0^2\cos^2\theta-\beta_e^2\omega_0^2\cos^2\theta)] \\ &+s^2\beta_0^2[-(1-\omega_0^2)^2(2\beta_e^2+\beta_0^2)+\Omega^2(2\beta_e^2+\beta_0^2\cos^2\theta-\beta_e^2\omega_0^2\cos^2\theta-\beta_e^2\omega_0^2)] \\ &+\beta_e^2\beta_0^4(1-\omega_0^2)[(1-\omega_0^2)^2-\Omega^2]=0, \end{aligned} \qquad (72)$$

[27] M. J. Lighthill, Phil. Trans. Roy. Soc. London **252**, 397 (1960).

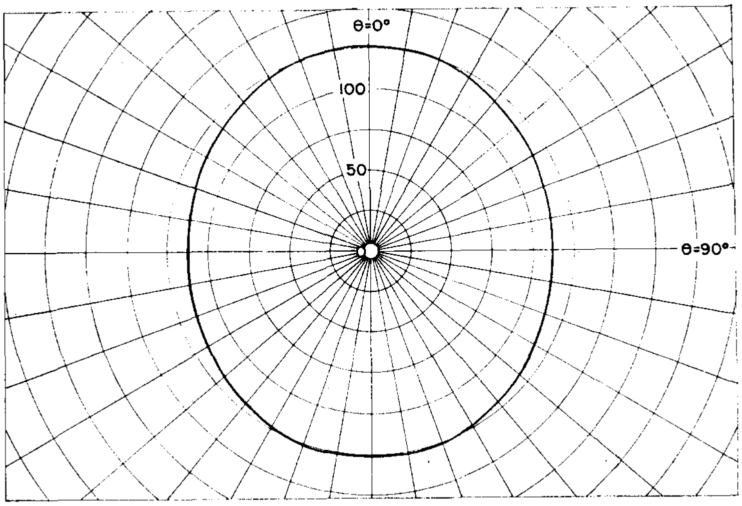

FIG. 5. Dispersion curve for modified plasma wave at $\omega=3\times10^7$.

where $\beta_0\equiv\omega/C$, $\beta_e\equiv\omega/U$, $\Omega\equiv\omega_c/\omega$, $\omega_0\equiv\omega_p/\omega$. If we should be only interested in the amplitude variations, we can easily see from Eqs. (67) and (68) that it is not necessary to check the radiation condition $(\mathbf{r}\cdot\nabla G)/(\partial G/\partial\omega)<0$, which will essentially select one point among two symmetrical points (s_2, s_ρ) and $(-s_2, -s_\rho)$.

The asymptotic solutions for the three components of the electric field are obtained as follows:

$$E_y=\frac{4\pi^2}{r}\sum\frac{C\exp(i\mathbf{s}\cdot\mathbf{r})\omega^4(\omega^2-\omega_c^2)}{|\nabla G|(|K|)^{1/2}U^2c^4}\left(\frac{J_0}{i\omega\epsilon_0}\right)\Bigg\{\frac{s_2^2U^2}{\omega^4(\omega^2-\omega_c^2)}[\omega_p^2(2\omega^2-\omega_p^2-2c^2s^2)-c^2s_\rho^2(c^2s^2-\omega^2)]$$
$$-[1/\omega^2(\omega^2-\omega_c^2)][\omega_p^2(2\omega^2-\omega_p^2-c^2s^2-c^2s_2^2)-U^2s_\rho^2(c^2s^2+\omega_p^2-\omega^2)]$$
$$+[1-(U^2/\omega^2)s_2^2][1-(c^2/\omega^2)(s^2+s_2^2)+(c^4/\omega^4)s_2^2s^2]\Bigg\},\quad(73)$$

$$E_\rho=\frac{4\pi^2}{r}\sum\frac{C\exp(i\mathbf{s}\cdot\mathbf{r})\omega^4(\omega^2-\omega_c^2)}{|\nabla G|(|K|)^{1/2}U^2c^4}\left(\frac{J_0}{i\omega\epsilon_0}\right)s_\rho\Bigg\{\frac{c^2\omega_\rho^2s_2}{\omega^2(\omega^2-\omega_c^2)}$$
$$-[s_2U^2/\omega^4(\omega^2-\omega_c^2)][(c^2s^2+\omega_p^2-\omega^2)(c^2s_\rho^2+\omega_p^2)+c^2\omega_p^2s_2^2]-[1-(U^2/\omega^2)s_2^2][1-(c^2/\omega^2)s^2](c^2/\omega^2)s_2\Bigg\},\quad(74)$$

$$E_\phi=\frac{4\pi^2}{r}\sum\frac{C\exp(i\mathbf{s}\cdot\mathbf{r})J_0\omega_c\omega_p^2}{|\nabla G|(|K|)^{1/2}U^2c^4\epsilon_0}s_2s_\rho(c^2-U^2),\quad(75)$$

where the sum of $\hat{x}E_x$ and $\hat{z}E_z$ are expressed in terms of two vectors in the direction of $\hat{\rho}$ and $\hat{\phi}$ by using the fact that

$$\hat{x}s_1+\hat{z}s_3=\mathbf{s}_\rho=|s_\rho|\hat{\rho}$$

and

$$-\hat{x}s_3+\hat{z}s_1=\mathbf{s}_\rho\times\hat{y}=|s_\rho|\hat{\phi}\quad(76)$$

($\hat{\rho}$ and $\hat{\phi}$ are unit vectors in the cylindrical coordinate system.)

Some calculations of the dispersion curves and E_ϕ', E_M' vs ϕ are given in Figs. 1–8, where

$$E_\phi'\equiv36\pi\epsilon_0r|E_\phi|/J_0,\quad(77)$$

and

$$E_M'\equiv36\pi\epsilon_0r|E_M|/J_0.\quad(78)$$

$|E_M|$ is the magnitude of the projection of the electric field on the meridian plane, which is obtained as

$$|E_M|=(|E_y|^2+|E_\rho|^2)^{1/2}.\quad(79)$$

Due to the symmetrical nature of the physical system and the source involved, E_ϕ' and E_M' vs ϕ are plotted only for the range $\phi=0°$ to $\phi=90°$. The angle $\phi=0°$ and $\theta=0°$ corresponds to the direction of the earth's magnetic field. Ionospheric plasma is considered, and the electron temperature, T, of 1487°K and the electron density, N_0, of 1.7×10^5 electrons/cc are taken as the representative values above the F-peak region of the ionosphere. The earth's magnetic field is assumed to be 0.5 G. Calculations are made for four frequencies, which are $\omega=3\times10^5$, $\omega=3\times10^6$, $\omega=3\times10^7$, and $\omega=3\times10^8$.

The relatively large magnitude of the modified plasma wave[28] compared with the modified electromagnetic waves should be comparable with the results of Hessel and Shmoys,[10] and Wait.[13] Hessel and Shmoys

[28] The choice of this terminology is explained in Refs. 29 and 31 together with a careful analysis of the dispersion relation.

considered the excitation by a point current source without static magnetic field. Wait has studied the radiation from a slotted sphere antenna immersed in a compressible plasma, without static magnetic field, and concluded that the relative power in the acoustic type of wave is increased as the dimension of the antenna is reduced.

When $\omega=3\times10^6$, the dispersion curve is given by Fig. 1, which has a transition angle[29,30] between a modified ordinary wave and a modified plasma wave at $\theta_t=68.7°$, and a resonance angle[29,30] for a modified plasma wave at $\theta_r=70°$. Due to a turning point in Fig. 1, there are three rays existing inside the cone $\phi\leq4°$, as shown in Figs. 2 and 3. This terminology "ray" has been used by Arbel and Felsen[8] for each stationary phase-point contribution. At $\phi=20°$, corresponding to $\theta_r=70°$, the stationary point goes to infinity and the asymptotic solution can not be applied. There are two rays existing in the region $4°\leq\phi\leq20°$. A large modified plasma wave contribution exists near and inside the boundary cone $\phi=20°$.

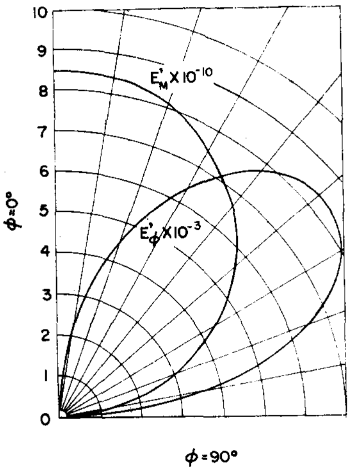

FIG. 6. E_ϕ',E_M' vs ϕ for modified plasma wave at $\omega=3\times10^7$.

When $\omega=3\times10^7$, three separate dispersion curves exist for each type of wave, as shown in Figs. 4 and 5. The field patterns of a modified plasma wave as plotted in Fig. 6 shows that this type of wave is essentially linearly polarized with E_M component, as can be expected from its longitudinal nature. The field patterns for modified ordinary and extraordinary waves as given by Figs. 7 and 8 have the similar features as compared with those patterns for the ordinary and extraordinary modes calculated by Arbel and Felsen.[8]

When $\omega=3\times10^5$, the dispersion curve belongs to the same region in ω_p^2/ω^2 vs ω_c^2/ω^2 plane[30,31] as in the case of $\omega=3\times10^6$, and similar features appear in the field patterns. The transition angle and the resonance angle are, respectively, $\theta_t=\tan^{-1}$ (29.18) and $\theta_r=\tan^{-1}$ (29.3). Due to a turning point of the dispersion curve there are two rays existing inside the cone $\phi\leq17.75°$. A large contribution due to a modified plasma wave is restricted to a very narrow region near the axis $\phi=0°$.

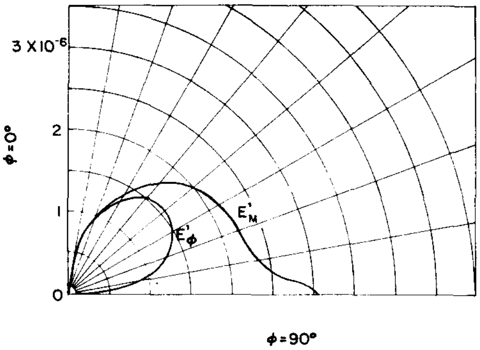

FIG. 7. E_ϕ', E_M' vs ϕ for modified ordinary wave at $\omega=3\times10^7$.

When $\omega=3\times10^8$, the dispersion curve, consisting of three separate curves, belongs to the same region in ω_p^2/ω^2 vs ω_c^2/ω^2 plane as in the case of $\omega=3\times10^7$, and all the features explained with regard to $\omega=3\times10^7$ applies here. In addition, the propagation constants of both modified ordinary and extraordinary waves are nearly equal to the propagation constant of light in free space at $\omega=3\times10^8$.

Similar calculations are also made for the F-peak region, the E region and the D region of the ionosphere, and their graphs are given elsewhere.[29] The dependence of the radiation patterns of the excited fields on the altitude of the ionosphere is found to be not too conspicuous. However, there is quite a variation in the magnitude of the excited fields. Also, the calculation for the D region gives some different types of dispersion curves which belong to some different regions in ω_p^2/ω^2 vs ω_c^2/ω^2 plane.

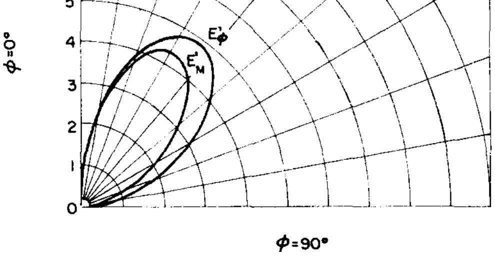

FIG. 8. E_ϕ', E_M' vs ϕ for modified extrordinary wave at $\omega=3\times10^7$.

[29] Y. K. Wu, *Unified Approach to Excitation Problems in Compressible Plasma* (Ph.D. dissertation, The University of Michigan, 1965).

[30] W. P. Allis, S. J. Buchsbaum, and A. Bers, *Waves in Anisotropic Plasmas* (MIT Press, Cambridge, 1963).

[31] Y. K. Wu, Radio Science **2,** 1019 (1967).

VI. CONCLUSION

An over-all picture of the wave excitation in the compressible plasma immersed in the constant magnetic field can be seen most clearly from this compact and systematic operator-transform method. Equivalence relations given above show in one way the powerfulness of this method. The close relationship existing between the excited field and the dispersion relation is most clearly shown by Eqs. (53) and (54), and also by the sample calculation applying a convenient form of the stationary phase method formulated by Lighthill.

Through our general treatment, we can see the more complete picture of many of the results obtained with conventional methods either by neglecting the earth's magnetic field or by neglecting the compressibility of the plasma. Some other results of the calculation are: (i) Very large field contributions due to modified plasma wave are found at low frequencies for certain space directions only, and at high frequencies for all directions. (ii) The radiation fields are composed of many rays, and at higher frequencies each ray corresponds to each different type of wave.

The present treatment should be applicable to the inhomogeneous medium, but its practical application needs further study.

ACKNOWLEDGMENTS

The authors are grateful for the criticism and help given by R. E. Hiatt, G. Hok, and J. J. LaRue of the Radiation Laboratory, The University of Michigan.

APPENDIX

Some dyadic operations with their associated matrices as used in the text are given in the following expressions:

$$\mathbf{1}\equiv\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z}\rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\nabla\times\mathbf{1}\equiv[\hat{x}(\partial/\partial x)+\hat{y}(\partial/\partial y)+\hat{z}(\partial/\partial z)]\times(\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z})\rightarrow\begin{bmatrix}0&-\partial/\partial z&\partial/\partial y\\\partial/\partial z&0&-\partial/\partial x\\-\partial/\partial y&\partial/\partial x&0\end{bmatrix}$$

$$\nabla\cdot\mathbf{1}\equiv[\hat{x}(\partial/\partial x)+\hat{y}(\partial/\partial y)+\hat{z}(\partial/\partial z)]\cdot(\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z})\rightarrow\begin{bmatrix}\partial/\partial x\\\partial/\partial y\\\partial/\partial z\end{bmatrix}$$

$$\hat{b}\hat{b}\equiv(\hat{x}b_x+\hat{y}b_y+\hat{z}b_z)(\hat{x}b_x+\hat{y}b_y+\hat{z}b_z)\rightarrow\begin{bmatrix}b_x\\b_y\\b_z\end{bmatrix}\begin{bmatrix}b_x&b_y&b_z\end{bmatrix}$$

$$\hat{b}\times\mathbf{1}\equiv(\hat{x}b_x+\hat{y}b_y+\hat{z}b_z)\times(\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z})\rightarrow\begin{bmatrix}0&-b_z&b_y\\b_z&0&-b_x\\-b_y&b_x&0\end{bmatrix}.$$