Effect of x-y coupling on the beam breakup instability

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In solenoidal beam transport systems, motions in the x and y directions are coupled by the $\mathbf{v}\times\mathbf{B}$ force. A two-dimensional coupled mode description of the beam breakup (BBU) instability is presented; its dispersion relation is derived and compared with the one-dimensional BBU dispersion relation. In the two-dimensional description, instability growth is doubled and two additional wave modes are found in the regime of strong focusing. In the weak focusing regime, the two-dimensional description gives the same dispersion relation as the one-dimensional model.

We examine the effect of coupling between the x and y directions of transverse motion in the beam breakup (BBU) instability. A dispersion relation is derived from a two-dimensional coupled-mode description, yielding four wavemodes. In the strong focusing regime, two of these modes are identical to those found with the existing one-dimensional theory, but with twice the coupling constant. In addition, we find two modes not given by the one-dimensional theory. In the weak focusing regime, our results reduce to the dispersion relation found by one-dimensional theory. This is expected because x-y coupling vanishes in the weak focusing limit.

Existing descriptions of the BBU instability have considered the interaction of an electron (or ion) beam with linearly polarized microwave oscillations in cylindrical pillbox cavities. Calculations in different regimes[1-3] have recently been unified by a one-dimensional coupled-mode theory[4] which reproduces the earlier findings in their regimes of validity. This coupled-mode theory also shows a different scaling in a previously undescribed regime.

The one-dimensional coupled-mode equations are[4]

$$\frac{d}{dt}\gamma\frac{dx}{dt}+\gamma\omega_c^2x=a, \tag{1a}$$

$$\left(\frac{\partial^2}{\partial t^2}+\frac{\omega_0}{Q}\frac{\partial}{\partial t}+\omega_0^2\right)a=2\gamma\omega_0^4\epsilon x, \tag{1b}$$

where $\gamma=(1-\beta^2)^{-1/2}$ is the relativistic mass factor, ω_c is the betatron frequency ($\omega_c=eB/mc\gamma$ is the relativistic electron cyclotron frequency in a solenoidal field), and $d/dt=\partial/\partial t+v\partial/\partial z$ is the convective derivative.

Assuming a disturbance of the form $e^{i\omega t-ikz}$, these equations yield the dispersion relation

$$\Omega^2-(\omega_c^2-\Gamma)=0, \tag{2}$$

where $\Omega=\omega-kv$ and $\Gamma=2\omega_0^4\epsilon/(-\omega^2+\omega_0^2+i\omega\omega_0/Q)$.

In a solenoidal magnetic field, the magnetic force is a velocity-dependent force coupling the x and y motions, so that a one-dimensional description, such as given by Eq. (1), may be inadequate.[5] In particular, the description of the focusing force as a harmonic potential force ($\gamma\omega_c^2x$) may be inaccurate for solenoidal focusing. In order to examine this possibility for the BBU instability, we use a coupled-mode description which describes motion in both the x and y directions, which are transverse to the beam.

We consider a magnetized beam ($\omega_{pe}\ll\omega_{ce}$) in a solenoid, passing through cylindrical cavities whose radius is large compared to the beam radius, and whose length and spacing are small compared to the scale length of the beam disturbance. The TM_{110} frequency of the cavities is denoted ω_0. The cyclotron motion of the magnetized beam couples the transverse motions in the x and y directions. The equation of motion for a beam influenced by the transverse force per unit mass $\mathbf{a}(z,t)$ is

$$\frac{d}{dt}\gamma\frac{dx}{dt}+\gamma\omega_c\frac{dy}{dt}=a_x, \tag{3a}$$

$$\frac{d}{dt}\gamma\frac{dy}{dt}-\gamma\omega_c\frac{dx}{dt}=a_y, \tag{3b}$$

where $\gamma=(1-\beta^2)^{-1/2}$ is the relativistic mass factor, $\omega_c=eB/mc\gamma$ is the relativistic electron cyclotron frequency in the solenoidal field, and $d/dt=\partial/\partial t+v\partial/\partial z$ is the convective derivative.

The TM_{110} modes are excited by e-beam displacements according to

$$\left(\frac{\partial^2}{\partial t^2}+\frac{\omega_0}{Q}\frac{\partial}{\partial t}+\omega_0^2\right)a_x=2\gamma\omega_0^4\epsilon x, \tag{4a}$$

$$\left(\frac{\partial^2}{\partial t^2}+\frac{\omega_0}{Q}\frac{\partial}{\partial t}+\omega_0^2\right)a_y=2\gamma\omega_0^4\epsilon y, \tag{4b}$$

where Q is the quality factor of the microwave cavity. The mode giving rise to force a_x is the linearly polarized mode with magnetic field in the y direction on axis, while a_y arises from the orthogonal linearly polarized mode. Equations (3) and (4) are invariant under rotation in the x-y plane. The quantity ϵ is the dimensionless coupling constant of the e-beam to the TM_{110} mode, given by[6]

$$\epsilon=0.422(l/L)(I/17kA)\beta/\gamma, \tag{5}$$

where l is the length of the microwave cavities, L is their spacing, and I is the beam current. For disturbances with

$\omega_0 < 0$, Q must be replaced by $-Q$. This yields the same equations, henceforth we shall only consider $\omega_0 > 0$.

If we set $a_y = 0$ in Eq. (3b) and disregard Eq. (4b), we obtain the one-dimensional BBU description of Eq. (1), as well as the solution $x = dy/dt = 0$, i.e., $\Omega = 0$. This situation would arise, e.g., if the microwave mode giving rise to a_y were negligible as a result of preferential damping, etc. In cases with cylindrical symmetry, both microwave modes may be excited, and their combination may not yield a linearly polarized wave. The two-dimensional description of transverse beam dynamics may be utilized in this case.

For a disturbance of form $e^{-i(kz-\omega t)}$, Eqs. (3) and (4) yield the dispersion relation

$$\Omega^4 + \Omega^2(-\omega_c^2 + 2\Gamma) + \Gamma^2 = 0, \tag{6}$$

where $\Omega = \omega - vk$, $\Gamma = 2\omega_0^4\epsilon/(-\omega^2 + \omega_0^2 + i\omega\omega_0/Q)$, and $\Gamma|_{\omega_0} = -2i\omega_0^2\epsilon Q$.

The resonant modes ($\omega = \omega_0$) obey

$$k|_{\omega_0} = \frac{1}{v}\left[\omega_0 \pm \left(\frac{\omega_c^2 + 4i\omega_0^2\epsilon Q \pm (\omega_c^4 + 8i\omega_0^2\omega_c^2\epsilon Q)^{1/2}}{2}\right)^{1/2}\right], \tag{7}$$

$$v\frac{\partial k}{\partial\omega}\bigg|_{\omega_0} = 1 \pm \left(\frac{\omega_c^2 + 4i\omega_0^2\epsilon Q \pm (\omega_c^4 + 8i\omega_0^2\omega_c^2\epsilon Q)^{1/2}}{2}\right)^{-1/2} \times [1 \pm \omega_c^2(\omega_c^4 + 8i\omega_0^2\omega_c^2\epsilon Q)^{-1/2}](\epsilon Q^2\omega_0) \times (2 - i/Q). \tag{8}$$

Equations (7) and (8) describe four resonant modes. Two of the modes have $\mathrm{Im}(k) > 0$, and thus can give rise to convective growth. Equation (8) is useful in estimating the group velocity of a disturbance with frequencies centered at ω_0 according to

$$v_g = [\mathrm{Re}(\partial k/\partial\omega)|_{\omega_0}]^{-1}. \tag{9}$$

The expected behavior of a disturbance (for sufficiently large time) is that it will be dominated by the fastest growing mode; i.e., $\omega = \omega_0$. The peak of the disturbance will move at the velocity given by Eq. (9), while its amplitude will evolve as $\mathrm{Im}(k)z$. In order to examine the amplitude at fixed z, the Green's function may be constructed from the dispersion relation.

In the limit of strong focusing ($|\Gamma|_{\omega_0}| \ll \omega_c^2$), the dispersion relation is approximated by

$$[\Omega^2 - (\omega_c^2 - 2\Gamma)][\Omega^2 - \Gamma^2/\omega_c^2] = 0. \tag{10}$$

For comparison, the dispersion relation of the one-dimensional description of the BBU[4] is $\Omega^2 - (\omega_c^2 - \Gamma) = 0$. For strong focusing, the modes obey

$$k|_{\omega_0} = (1/v)[\omega_0 \pm (\omega_c + 2i\omega_0^2\epsilon Q/\omega_c)], \tag{11a}$$

$$k|_{\omega_0} = (1/v)[\omega_0 \pm 2i\omega_0^2\epsilon Q/\omega_c], \tag{11b}$$

$$\mathrm{Re}(\partial k/\partial\omega|_{\omega_0}) = (1/v)(1 \pm 4\epsilon Q^2\omega_0/\omega_c). \tag{12}$$

Equation (12) describes the modes of (11a) and (11b). We note that the e-folding length of the growing modes [$\mathrm{Im}(k) > 0$] is the same for (11a) and (11b); the group velocities are also equal. The e-folding length is proportional to the magnetic field strength.

A growing disturbance described by Eq. (11a) [with the plus sign, so $\mathrm{Im}(k) > 0$] will propagate at the group velocity found with Eq. (12), growing as $\exp[\mathrm{Im}(k)z]$, where $\mathrm{Im}(k)$ is given by (11a). The growth rate is twice that found with the one-dimensional approach[2,7] [Eqs. (5.14) and (5.15) of Ref. 2], while the group velocity is reduced. This is a result of the factor of two multiplying Γ in the first factor of Eq. (10) which does not appear in Eq. (2).

In order to describe the modes of (11a) using the existent one-dimensional theory, it is sufficient to use double the value of coupling constant ϵ given by Eq. (5). This accounts for the existence of twice as many microwave modes as were included in the one-dimensional approach.

Two additional wave modes [Eq. (11b)], not found in the one-dimensional description, result from consideration of the coupling between x and y degrees of freedom. In the limit of low beam current ($\epsilon \to 0$), these modes have $\Omega = 0$, while the modes of Eq. (11a) have $\Omega = \pm(-\omega_c)$. The modes of Eq. (11b) have growth rates equal to those of Eq. (11a). Their contribution to the Green's function $\{G(z,t) \sim \int d\omega \exp[i\omega t - ik(\omega)z]\}$ has the same asymptotic response [multiplied by $\exp(i\omega_c z/v)$]. As a result, they may be expected to play a significant role in BBU growth.

In the case where the wavelengths of the convectively unstable modes [Eqs. (11a) and (11b) with plus signs] do not greatly exceed the lengths of the microwave cavities, finite transit time effects will reduce BBU growth.[8] The convectively unstable mode of Eq. (11b) has a longer wavelength, so that its growth reduction may be less severe. Thus, it may be the dominant unstable mode.

In the limit of weak focusing ($|\Gamma|_{\omega_0}| \gg \omega_c^2$), the dispersion relation reduces to

$$\Omega^4 + 2\Gamma\Omega^2 + \Gamma^2 = (\Omega^2 + \Gamma)^2 = 0, \tag{13}$$

$$k|_{\omega_0} = (1/v)\{\omega_0 \pm (1+i)\omega_0\epsilon^{1/2}Q^{1/2}\}, \tag{14}$$

$$\partial k/\partial\omega|_{\omega_0} = (1/v)[1 \pm (1-i)\epsilon^{1/2}Q^{3/2}(1 - i/2Q)]. \tag{15}$$

These solutions are identical to those obtained in the one-dimensional approach with weak focusing. For weak focusing, motion in the two transverse directions is decoupled, so that the one-dimensional and two-dimensional results are expected to be the same.

Using Eqs. (11), (12), (14), and (15), the BBU instability behavior can be modeled for strong or weak focusing. The wavelength of a resonant unstable mode is $\lambda = 2\pi/\mathrm{Re}(k)$, the e-folding length is $l_e = [\mathrm{Im}(k)]^{-1}$, the convective growth time is $\tau_e = l_e/v_g$. For a disturbance resulting from an impulse at $(z,t) = (0,0)$, the disturbance at position "z" will be maximum at the time $t_m = z/v_g$.

In summary, the effects of x-y coupling on the BBU instability have been examined with a two-dimensional description of the BBU instability. In the strong focusing regime, the x and y directions of motion are coupled, so that the two-dimensional description yields results not found in the one-dimensional description with harmonic focusing force. Two wave modes are found which are identical to those of the one-dimensional description for twice-the coupling constant. Two additional modes are found which do not appear in the one-dimensional approach.

In the weak focusing regime, the two-dimensional description gives the same dispersion relation as the one-dimensional model.

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