Generalized parastatistics and internal symmetry

R. Y. Levine and Y. Tomozawa
Department of Physics, University of Michigan, Ann Arbor, Michigan 48109

(Received 29 June 1982; accepted for publication 2 January 1983)

We exhibit generalized parastatistical double commutation relations which allow a solution less restrictive than Green's ansatz. This leads to local interaction terms in the Lagrangian consistent with an internal symmetry among Green indices. The generalized commutation relations are covariant under this symmetry transformation rather than invariant. The formulation may be applicable to the generation problem as well as supersymmetric model building.

PACS numbers: 05.30. – d

I. INTRODUCTION

Green[1] generalized the usual quantum statistics by postulating double commutation relations among fields as alternative solutions of Heisenberg's equation of motion. Taking paraboson relations in this discussion, define

$$H = \tfrac{1}{2} \sum_{\mathbf{k}} \omega_{\mathbf{k}} \{a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger\}, \tag{1}$$

and require

$$[H, a_{\mathbf{k}}] = -\omega_{\mathbf{k}} a_{\mathbf{k}}. \tag{2}$$

One has generalized conditions[1]

$$\begin{aligned}
&[\{a_{\mathbf{k}}^\dagger, a_{\mathbf{k}'}\}, a_{\mathbf{k}''}] = -2\delta_{\mathbf{k}'',\mathbf{k}} a_{\mathbf{k}'},\\
&[\{a_{\mathbf{k}}, a_{\mathbf{k}'}\}, a_{\mathbf{k}''}] = 0,\\
&[\{a_{\mathbf{k}}, a_{\mathbf{k}'}\}, a_{\mathbf{k}''}^\dagger] = 2\delta_{\mathbf{k}'',\mathbf{k}'} a_{\mathbf{k}} + 2\delta_{\mathbf{k}',\mathbf{k}''} a_{\mathbf{k}},\\
&[\{a_{\mathbf{k}}^\dagger, a_{\mathbf{k}'}^\dagger\}, a_{\mathbf{k}''}^\dagger] = 0.
\end{aligned} \tag{3}$$

Parafermi relations are similar with all commutators on the lhs of Eqs. (3) and (1). We exhibit these later in Eqs. (31)–(34).

A solution of Eqs. (3) is given by Green's ansatz

$$a_{\mathbf{k}} = \sum_{i=1}^{p} a_{\mathbf{k}}^{(i)}, \tag{4}$$

where the commutation relations among operators with the same Green index are normal boson relations, while those of different Green components are "abnormal,"

$$[a_{\mathbf{k}}^{(i)}, a_{\mathbf{k}'}^{(i)\dagger}] = \delta_{\mathbf{k},\mathbf{k}'}, \quad \text{etc.,} \tag{5a}$$

and

$$\{a_{\mathbf{k}}^{(i)}, a_{\mathbf{k}'}^{(j)}\} = \{a_{\mathbf{k}}^{(i)}, a_{\mathbf{k}'}^{(j)\dagger}\} = 0 \quad (i \neq j). \tag{5b}$$

Greenberg and Messiah[2] showed that the ansatz, Eq. (4), is a unique solution in the case of a unique vacuum state, $|0\rangle$, such that

$$a_{\mathbf{k}} |0\rangle = 0, \tag{6a}$$

and derived

$$a_{\mathbf{k}}^{(i)} a_{\mathbf{k}'}^{(j)\dagger} |0\rangle = \delta_{\mathbf{k},\mathbf{k}'} \delta_{ij} |0\rangle, \tag{6b}$$

which can be written

$$a_{\mathbf{k}} a_{\mathbf{k}'}^\dagger |0\rangle = \delta_{\mathbf{k},\mathbf{k}'} p |0\rangle \tag{6c}$$

for integer p.

One may ask the significance of the simple sum in Eq. (4) in light of the fact that Eq. (5) indicates that Green components $a_{\mathbf{k}}^{(i)}$ and $a_{\mathbf{k}}^{(i)\dagger}$ are independent operators. The form of Eq. (4) is due to the postulated double commutators in Eq. (3), and may be unsatisfactory because it results in an undue constraint on the theory. In Sec. II of this article we generalize the double commutator to accommodate mixing of the Green components as independent fields. In Sec. III we consider local interaction terms with the Green index representing an internal symmetry. A discussion follows in Sec. IV.

II. GENERALIZATION OF PARASTATISTICS

Define the Hamiltonian and equations of motion for boson fields $A_{\mathbf{k}}^{(i)}$,

$$H = \tfrac{1}{2} \sum_{\mathbf{k},i} \omega_{\mathbf{k}} \{A_{\mathbf{k}}^{(i)}, A_{\mathbf{k}}^{(i)\dagger}\} \tag{7}$$

and

$$[H, A_{\mathbf{k}}^{(i)}] = -\omega_{\mathbf{k}} A_{\mathbf{k}}^{(i)}, \tag{8}$$

where $i = 1, 2, ..., p$. The solution to Eqs. (7) and (8) is given by

$$[\{A_{\mathbf{k}}^{(i)}, A_{\mathbf{k}'}^{(j)\dagger}\}, A_{\mathbf{k}''}^{(l)}] = -2\delta_{\mathbf{k}',\mathbf{k}''} \sum_n \gamma_n^{ijl} A_{\mathbf{k}}^{(n)}, \tag{9a}$$

$$[\{A_{\mathbf{k}}^{(i)}, A_{\mathbf{k}'}^{(j)}\}, A_{\mathbf{k}''}^{(l)\dagger}] = 2\delta_{\mathbf{k}',\mathbf{k}''} \sum_n \rho_n^{ijl} A_{\mathbf{k}}^{(n)} + 2\delta_{\mathbf{k},\mathbf{k}''} \sum_n \rho_n^{jil} A_{\mathbf{k}'}^{(n)}, \tag{9b}$$

$$[\{A_{\mathbf{k}}^{(i)}, A_{\mathbf{k}'}^{(j)}\}, A_{\mathbf{k}''}^{(l)}] = 0, \tag{9c}$$

with Hermitian conjugation of the above, where γ and ρ are complex numbers. From Eqs. (7) and (8),

$$\sum_i \gamma_n^{iil} = \delta_{ln}. \tag{10}$$

The Jacobi identity

$$[\{A,B\},C] + [\{B,C\},A] + [\{C,A\},B] = 0, \tag{11}$$

applied to Eq. (9), gives

$$\rho_n^{ijl} = \gamma_n^{ilj}. \tag{12}$$

We further impose a symmetry relation

$$\rho_n^{ijl} = \rho_n^{jil}, \tag{13}$$

which implies, from Eq. (12),

$$\gamma_n^{ijl} = \gamma_n^{lji}. \tag{14}$$

Similar to Greenberg and Messiah,[2] we derive further

 0022-2488/83/061603-05$02.50

constraints due to the existance of a unique vacuum. We have, from Eq. (9),

$$[\{A_{\mathbf{k}}^{(i)},A_{\mathbf{k}'}^{(j)\dagger}\},A_{\mathbf{k}''}^{(l)}A_{\mathbf{k}'''}^{(m)\dagger}] = -2\delta_{\mathbf{k}',\mathbf{k}''}\Big(\sum_n \gamma_n^{ijl}A_{\mathbf{k}}^{(n)}\Big)A_{\mathbf{k}'''}^{(m)\dagger} + 2\delta_{\mathbf{k},\mathbf{k}'''}A_{\mathbf{k}''}^{(l)}\Big(\sum_n (\gamma_n^{jim})^*A_{\mathbf{k}'}^{(n)\dagger}\Big). \tag{15}$$

Assume there exists a unique vacuum, $|0\rangle$. Then it is easy to show that

$$A_{\mathbf{k}}^{(j)}|0\rangle = 0 \tag{16}$$

from Heisenberg's equation of motion, provided $\omega_{\mathbf{k}} \neq 0$ (an exception occurs for zero modes, $\omega_{\mathbf{k}} = 0$, which may lead to symmetry breaking). From Eq. (9) this implies there exists numbers $C_{\mathbf{k},\mathbf{k}'}^{ij}$ such that

$$A_{\mathbf{k}}^{(i)}A_{\mathbf{k}'}^{(j)\dagger}|0\rangle = C_{\mathbf{k},\mathbf{k}'}^{ij}|0\rangle. \tag{17}$$

Taking the vacuum expectation value on both sides of Eq. (17) yields

$$C_{\mathbf{k},\mathbf{k}'}^{jl} = C_{\mathbf{k}',\mathbf{k}}^{lj*}. \tag{18}$$

Operating both sides of Eq. (15) on the vacuum yields

$$(C_{\mathbf{k}',\mathbf{k}}^{jl})^*C_{\mathbf{k}'',\mathbf{k}'''}^{lm} - (C_{\mathbf{k}''',\mathbf{k}''}^{ml})^*C_{\mathbf{k},\mathbf{k}'}^{ij} = -2\delta_{\mathbf{k}',\mathbf{k}''}\sum_n \gamma_n^{ijl}C_{\mathbf{k},\mathbf{k}'''}^{nm} + 2\delta_{\mathbf{k},\mathbf{k}'''}\sum_n (\gamma_n^{jim})^*C_{\mathbf{k}'',\mathbf{k}'}^{ln}. \tag{19}$$

The lhs of Eq. (19) vanishes due to Eq. (18), and inspection of the rhs of Eq. (19) gives the condition

$$C_{\mathbf{k},\mathbf{k}'}^{ij} = C^{ij}\delta_{\mathbf{k},\mathbf{k}'}, \tag{20}$$

with

$$C^{ij} = (C^{ji})^*, \tag{21}$$

and implies the equation

$$\sum_n C^{nm}\gamma_n^{ijl} = \sum_n (\gamma_n^{jim})^*C^{ln}. \tag{22}$$

Up to this point all relations are implied by Eqs. (9), (12), and (16). In order to find an explicit solution to Eq. (22), we rewrite Eq. (17) with Eq. (20),

$$A_{\mathbf{k}}^{(i)}A_{\mathbf{k}'}^{(j)\dagger}|0\rangle = \delta_{\mathbf{k},\mathbf{k}'}C^{ij}|0\rangle. \tag{23}$$

From the assumed uniqueness of the physical vacuum, we observe that it is natural to postulate

$$C^{ij} = \delta^{ij}. \tag{24}$$

First, notice that the uniqueness of the physical vacuum would lead to a less restrictive condition $C_{ij} = C_i\delta_{ij}$ ($C_i > 0$). But this condition can be reduced to Eq. (24) by normalization of the fields $A_k^i \rightarrow \sqrt{C_i}A_k^i$. Secondly, we should point out that $C^{ij} \neq \delta^{ij}$ implies that the vacuum is a superposition of states containing arbitrary numbers of particles. This can be seen by repeated application of the operator $A_iA_j^\dagger$ on the vacuum, $|0\rangle$. From Eqs. (22) and (24) we have

$$\gamma_m^{ijl} = (\gamma_l^{jim})^*. \tag{25}$$

In order to find an explicit solution for γ_n^{ijl} we take the special case

$$\gamma_n^{ijl} = \sum_m \alpha_{im}\beta_{mj}\omega_{lm}\xi_{mn}. \tag{26}$$

Conditions Eqs. (10), (12), (13), (14), and (25) then imply

$$\beta = \alpha^\dagger, \quad \omega = \alpha, \quad \xi = \alpha^\dagger, \tag{27}$$

where α is a unitary matrix. We prove this in the Appendix. Hence, the matrix γ is given by

$$\gamma_n^{ijl} = \sum_m \alpha_{im}\alpha_{jm}^*\alpha_{lm}\alpha_{nm}^*. \tag{28}$$

Comparison of Eqs. (5), (9a), and (28) implies $A_{\mathbf{k}}$ can be written

$$A_{\mathbf{k}}^{(i)} = \sum_j \alpha_{ij}a_{\mathbf{k}}^{(j)}, \tag{29}$$

where $a_{\mathbf{k}}^{(j)}$ is a Green component satisfying the commutation relations (5). We have shown that Eq. (29) is a unique solution of the double commutators (9), under the vacuum constraints (16) and (24), the symmetry relation (13), and the ansatz (26).

Note that mathematically a more general solution in the form of Eq. (26) exists without the symmetry relation, Eq. (13). Using an argument similar to the proof in the Appendix, Eqs. (9), (10), and (25) imply the nonsingular solution,

$$\gamma_n^{ijl} = \sum_m \frac{\alpha_{im}\alpha_{mj}^\dagger\xi_{lm}\xi_{mn}^\dagger}{(\alpha^\dagger\alpha)_{mm}}, \tag{30}$$

where ξ is a unitary matrix and α is an arbitrary nonsingular matrix. Equation (30) results from the fact that Eq. (9a) and the equation

$$[\{A_{\mathbf{k}}^i,A_{\mathbf{k}'}^j\},B_{\mathbf{k}''}^l] = -2\delta_{\mathbf{k}',\mathbf{k}''}\sum_n \gamma_n^{ijl}B_{\mathbf{k}}^n,$$

where

$$A_{\mathbf{k}}^i = \sum_j \frac{\alpha_{ij}}{\sqrt{(\alpha^\dagger a)_{jj}}}a_{\mathbf{k}}^j$$

and

$$B_{\mathbf{k}}^i = \sum_j \xi_{ij}a_{\mathbf{k}}^j \quad (\xi\xi^\dagger = 1)$$

are not differentiated in Heisenberg's equation of motion (8). The physical interpretation of this solution is clear, in that the symmetry of the Hamiltonian is reflected in the commutation relations defining the particles.

The entire argument above can be duplicated for parafermions. We exhibit the key relations:

$$[[B_{\mathbf{k}}^{(i)},B_{\mathbf{k}'}^{(j)\dagger}],B_{\mathbf{k}''}^{(l)}] = 2\delta_{\mathbf{k}',\mathbf{k}''}\sum_n \eta_n^{ijl}B_{\mathbf{k}}^{(n)}, \quad \text{etc.}, \tag{31}$$

and

$$\eta_n^{ijl} = \sum_m \beta_{im}\beta_{jm}^*\beta_{lm}\beta_{nm}^*, \tag{32}$$

with β unitary. The operator $B_{\mathbf{k}}^{(i)}$ can be represented by

$$B_{\mathbf{k}}^{(i)} = \sum_j \beta_{ij}b_{\mathbf{k}}^{(j)}, \tag{33}$$

where

$$\{b_{\mathbf{k}}^{(i)},b_{\mathbf{k}'}^{(i)\dagger}\} = \delta_{\mathbf{k},\mathbf{k}'}, \tag{34a}$$

$$[b_{\mathbf{k}}^{(i)},b_{\mathbf{k}'}^{(j)\dagger}] = [b_{\mathbf{k}}^{(i)},b_{\mathbf{k}'}^{(j)}] = 0 \quad (i \neq j). \tag{34b}$$

Equations (28) and (29) for parabosons, as well as Eqs.

(32) and (33) for parafermions, imply the covariance of the commutation relations (9) and (31) under unitary transformation of the Green components of the parafields. That is,

$$A_{\mathbf{k}}^{\prime(i)} = \sum_j A^{ij} A_{\mathbf{k}}^{(j)} \tag{35}$$

corresponds to Eq. (9) with

$$(\gamma')_n^{ijl} = \sum_{i',j',l',n'} A_{ii'} A_{jj'}^* A_{ll'} A_{nn'}^* \gamma_{n'}^{i'j'l'}. \tag{36}$$

A similar equation holds for parafermions. (For this reason, we call Eqs. (9) and (31) *covariant commutation relations*.) It should be noted that Green's parastatistical commutation relations, as well as normal commutation relations, are invariant under unitary transformations.

It should be noted that, without the assumption of Eq. (26), solutions for γ_n^{ijl} exist which are associated with relative single commutations relations more general than Green's ansatz. An example of this is shown in the next section.

III. LOCAL INTERACTIONS

From Eqs. (29) and (33) it is suggested that the Green index can be viewed as an internal symmetry index. One is then led to the question of local interaction terms which satisfy an internal symmetry. We exhibit an example of an interaction obeying an SU(2) symmetry.

Consider paraboson and parafermion fields, ϕ_i and ψ_i ($i = 1,2,3$) (ϕ could be scalar or gauge boson), which are vectors under SU(2) with "i" a Green index as defined in Eqs. (9) and (31). Also, define canonical Green components ϕ_i^g and ψ_i^g ($i = 1,2,3$), which satisfy normal Green commutation relations Eqs. (5) and (34). By an SU(2) symmetry on the Green index we mean that α and β [in Eq. (29) and (33), respectively] are orthogonal matrices. In other words,

$$\psi_i = \sum_j \beta_{ij}\psi_j^g, \quad \phi_i = \sum_j \alpha_{ij}\phi_j^g, \quad i,j = 1,2,3. \tag{37}$$

Assume relative commutation relations between ψ_i^g and ϕ_i^g, ($i = 1,2,3$):

$$[\psi_i^g, \phi_i^g] = 0 \tag{38a}$$

and

$$\{\psi_i^g, \phi_j^g\} = 0 \quad (i \neq j). \tag{38b}$$

Consider the interaction term ($i,j,k = 1,2,3$)

$$\epsilon_{ijk}\bar{\psi}_i\psi_j\phi_k = \epsilon_{ijk}\bar{\psi}_i^g\psi_j^g\phi_k^g = \bar{\psi}_1^g\psi_2^g\phi_3^g - \cdots, \tag{39}$$

where the first equality follows from SU(2) invariance. Locality of the interaction term in Eq. (39) follows due to the fact that the rhs contains Green fields of different indices, and provided relative commutation relations between parafields are of the normal type, Eq. (38). (As a result, ϕ_j^g and ψ_j^g commute with this interaction at spacelike separation.) A Klein transformation,[3]

$$\phi_i^g \to \phi_i'^g = \phi_i^g(-1)^{N_j + N_k} \quad (i,j,k \text{ cyclic}, \psi_j \text{ unchanged}), \tag{40}$$

where N_j is the number operator of the jth quantum number, will change the relative commutation to all commutators:

$$[\psi_i^g, \phi_j'^g] = 0, \quad (\text{all } i,j). \tag{41}$$

This will violate locality of the interaction term in Eq. (39) with ϕ_j^g replaced by $\phi_j^{g\prime}$; e.g., $\bar{\psi}_1^g\psi_2^g\phi_3'^g$ and $\bar{\psi}_1^g\psi_3^g\phi_2'^g$ anticommute at spacelike separation instead of commute. This indicates that this Klein transformation has a physical implication in the sense that ψ and ϕ' cannot form a local O(3) Yukawa interaction. Of course, if ψ_j^g in Eq. (40) is altered by a Klein transformation similar to ϕ_j^g in Eq. (40), all commutation relations remain the same and locality of the interactions is maintained. Equations (40) and (41) reveal the relationship between locality of interactions and commutation relations between parafields.

As is mentioned earlier, we note that the fields ϕ_i of the Yukawa interaction in Eq. (39) can be Lorentz scalar or vector fields. The latter case naturally arises in the case of gauge field theory to which our discussion can be applied. In other words, the symmetry between Green indices can be a local symmetry. Also note that a scalar ϕ^4 interaction, which is essential in spontaneously broken gauge theory, is easily made local; for example,

$$\Big(\sum_i (\phi_i^2)\Big)^2.$$

The above results can be generalized to a lemma: Any group invariant interaction between generalized Green components is local provided suitable relative double commutation relations are chosen among the components. We show this for SO(N) and SU(N) symmetries. In general, two types of identifications of the Green index with an internal symmetry representation are possible; e.g., either spinor or vector representations. If the vector (adjoint) representation ϕ_i^g is chosen for fields satisfying single commutation-anticommutation relations, Eq. (5) [and Eq. (34) for parafermions], the spinor representation ϕ_β^α, where

$$\phi_\beta^\alpha = \frac{1}{\sqrt{2}}\sum_i \sigma_{i\beta}^\alpha \phi_i^g \tag{42}$$

satisfies only double commutation relations. [In Eq. (42) $\{\frac{1}{2}\sigma_{i\beta}^\alpha\}$ are the generators of SU(N).] On the other hand, if the spinor representation ϕ_α^g is assumed to satisfy single commutation–anticommutation relations, one has single commutation–anticommutation relations for $\phi_\beta^{g\alpha}$ by the composite rule:

$$\{\phi_\beta^{g\alpha}, \phi_\gamma^{g\alpha}\} = \{\phi_\beta^{g\alpha}, \phi_\gamma^{g\alpha\dagger}\} = \{\phi_\alpha^{g\beta}, \phi_\alpha^{g\gamma}\} = \{\phi_\alpha^{g\beta}, \phi_\alpha^{g\gamma\dagger}\} = 0 \tag{43}$$

for $\beta \neq \gamma$ and $\beta,\gamma \neq \alpha$; and the rest commute at spacelike separation. In this case, the fields in an adjoint representation

$$\phi_i = \sum_{\alpha,\beta} \sigma_{i\beta}^\alpha \phi_\alpha^{g\beta} \tag{44}$$

satisfy only double commutation relations (9), in which γ_n^{ijl} is more general than the solution in Eq. (26).

The interaction term which is local (the group-invariant term associated with the lemma above), in the case of spinor fields satisfying Eq. (43), is given by

$$\sum_{\alpha\beta} \bar{\psi}^\alpha \psi^\beta \phi_\beta^\alpha. \tag{45}$$

Equation (45) is local because terms with repeated indices commute with all other fields at spacelike separation. Similarly, in a theory in which only vector and tensor fields exist, the commutation relations in Eq. (38) yield group-invariant local interaction terms. In the case of only the vector (adjoint) representations allowed, one has only the SU(2) group, with local interaction equation (39), or its generalization, $SU(2)\times\cdots\times SU(2)$. This is the only case where the group invariant interaction does not have repeated indices on the fields. Incidently, the uniqueness of the SU(2) group, mentioned above, is a property of group invariance and is independent of the statistics.

Taking the viewpoint that locality of interactions is an essential ingredient in constructing a field theory, one notes that general double commutation relations (9) and (31) may break the internal symmetry between Green components by causing invariant terms to be nonlocal. A mechanism based on this idea may replace the Higg's mechanism for spontaneously breaking internal gauge symmetry with mass and mass-splitting generated dynamically. This concept of dynamical symmetry breaking by a locality condition may also be applied to a global internal symmetry or supersymmetry.

IV. DISCUSSION

The liberation of the Green index as described in Sec. II made possible the assignment of an internal symmetry to that index. Note that the $A^{(i)}$'s do not obey bilinear commutation relations in contrast to normal parastatistics in which physical fields satisfy bilinear commutation relations (Green components). We are departing from the usual point of view in parastatistics that says that only the full parastatistics field has independent status, not the individual components. We also noted that the double commutation relations proposed in the text, Eqs. (9) and (31), are covariant under group transformations, while the canonical commutation relations, as well as ordinary parastatistical double commutation rules, are invariant. In fact, this invariance is the reason for the restricted form of the Green ansatz. This implies the object γ_n^{ijl} in the double commutation relations [Eqs. (9) and (31)] forms a basis for the representations of the symmetry group [see Eq. (36)]. The three conditions on γ_n^{ijl}, Eqs. (10), (13), and (25), are invariant conditions which are analogous to the tracelessness condition on tensor representations of the symmetry group. This condition selects irreducible representations.

It was emphasized that locality of the group-invariant interaction terms is associated with a specific subset of possible double commutators [Eqs. (38) and (43) and group transformations on components in these equations]. Also mentioned was the fact that the spinor rules for commutators [Eq. (43)] and the vector rules [Eq. (38)] resulted in a different formulation of the theory. An interesting question is whether the two formulations lead to different physical results. Note that if canonical commutation relations for the fields are assumed, the vector and spinor forms of the field [related by Eq. (44)] both satisfy canonical commutation relations. This follows due to the invariance of the canonical commutation relations under the internal symmetry group.

It may be possible that this type of symmetry could exist between the three generations of quark and leptons. However, it should be mentioned that the representations of the Higgs scalars for symmetry breaking between Green components are constrained. This is due to the fact that anticommutivity between the scalars of different components (or, more generally, double commutators) does not allow a constant shift in the scalar fields, unless anticommuting c-numbers (Grassmann algebra) are introduced. Therefore, the Higgs in standard spontaneously broken gauge theories would have to be either a singlet, or the adjoint representation; or, more generally, a tensor of even rank, in the Green index symmetry. (For example, the spinor representation is not allowed.) For this reason, symmetry breaking of the horizontal gauge symmetry, which explains mass splittings between generations, is subject to the constraints described above, if the generation symmetry is associated with generalized Green indices. In Sec. III we suggested the basis of a possible alternative symmetry breaking mechanism.

Also, this concept may be applicable to a supersymmetric theory with parastatistics. This has been considered by the authors elsewhere.[4]

ACKNOWLEDGMENT

This work is supported in part by the U. S. Department of Energy.

APPENDIX: REPRESENTATION OF MATRIX γ_n^{ijl}

We want to find the representation for γ_n^{ijl} satisfying conditions (10), (12), (14), and (25):

$$\sum_i \gamma_n^{iil} = \delta_{ln}, \tag{A1}$$

$$\gamma_n^{ijl} = \gamma_n^{lji}, \tag{A2}$$

$$\gamma_n^{ijl} = (\gamma_l^{jin})^*. \tag{A3}$$

In this Appendix all summation of indices are explicitly shown and a double index does not imply summation. Assume the form

$$\gamma_n^{ijl} = \sum_m \alpha_{im}\beta_{mj}w_{lm}\xi_{mn}, \tag{A4}$$

where α,β,w,ξ are nonsingular. This corresponds to a decomposition of an ordinary matrix,

$$\gamma^{ij} = \sum \alpha_{im}\beta_{mj}$$

or, in matrix notation, $\gamma = \alpha\beta$. Equation (A4) is probably the most general form compatible with the symmetry requirements; Eqs. (A1)–(A3).

Multiplying Eq. (A2) by $\beta_{jk}^{-1}\xi_{nm}^{-1}$ and summing over j and n, we obtain

$$\alpha_{im}w_{lm} = \alpha_{lm}w_{im}. \tag{A5}$$

This implies, by nonsingularity of w,

$$w = \alpha E, \tag{A6}$$

where E is a diagonal matrix which can be absorbed into the definition of β or ξ. As a result, without loss of generality, we write

$$w = \alpha. \tag{A7}$$

Multiplying Eq. (A1) by $\alpha_{il}^{-1}\xi_{nj}^{-1}$ and summing over l and n yields

$$(\alpha^{-1}\xi^{-1})_{ij} = \delta_{ij}(\beta\alpha)_{ii}. \tag{A8}$$

Define the diagonal matrix, D,

$$\alpha^{-1}\xi^{-1} \equiv D \equiv (\delta_{ij} d_i), \tag{A9}$$

where

$$d_i \equiv (\beta\alpha)_{ii}. \tag{A10}$$

Note $d_i \neq 0$ because of the nonsingularity of $(\xi\alpha)$. Therefore,

$$\xi = D^{-1}\alpha^{-1}. \tag{A11}$$

Multiplying Eq. (A3) by $(\alpha^{-1})_{sl}(\xi^{-1})_{nq}(\beta^{\dagger})_{ti}^{-1}$ and summing over l, q, and i yields

$$\delta_{sq}(\beta^{\dagger -1})_{ts}\beta_{sj} = (\alpha^{\dagger})_{tj}(\alpha^{-1}\xi^{\dagger})_{st}(\alpha^{\dagger}\xi^{-1})_{tq}. \tag{A12}$$

From δ_{sq} on the lhs of Eq. (A12) one has that $(\alpha^{-1}\xi^{\dagger})$ and $(\alpha^{\dagger}\xi^{-1})$ on the rhs are diagonal matrices, which then implies $((\beta^{\dagger})^{-1}\alpha)$ on the lhs is diagonal. Define

$$\alpha^{-1}\xi^{\dagger} \equiv F_1, \quad \alpha^{\dagger}\xi^{-1} \equiv F_2,$$

and (A13)

$$(\beta^{\dagger})^{-1}\alpha \equiv F_3.$$

Substituting Eq. (A13) into Eq. (A12), one obtains

$$\beta = F_1 F_2 F_3^{-1}\alpha^{\dagger}. \tag{A14}$$

Note that the F_i are not independent and are related by

$$F_2^{\dagger}F_1 = 1 \tag{A15}$$

and

$$F_3^{-1}F_3^{\dagger}F_1F_2 = 1. \tag{A16}$$

Equations (A10) and (A14) imply

$$\alpha^{\dagger}\alpha = F_2 D^{-1} = G, \tag{A17}$$

where G is a positive definite diagonal matrix. From Eq. (A14), using Eq. (A17),

$$\beta\alpha = F_1F_2F_3^{-1}F_2D^{-1} \equiv F, \tag{A18}$$

where F is diagonal. Therefore, from Eq. (A10) and the fact that F is diagonal, we have

$$F = D. \tag{A19}$$

In other words,

$$\beta = D\alpha^{-1}. \tag{A20}$$

Recalling Eq. (A8), $\xi = D^{-1}\alpha^{-1}$, we arrive at the expression

$$\gamma_n^{ijl} = \sum_m \alpha_{im}\alpha_{mj}^{-1}\alpha_{lm}\alpha_{mn}^{-1}, \tag{A21}$$

where

$$\alpha^{\dagger}\alpha = G. \tag{A22}$$

Define

$$\alpha' = \alpha X, \tag{A23}$$

which results in

$$\alpha'^{\dagger}\alpha' = X^{\dagger}\alpha^{\dagger}\alpha X = X^{\dagger}GX. \tag{A24}$$

Choose

$$X = \sqrt{G}^{-1}, \tag{A25}$$

using the fact that G is a positive definite diagonal matrix, and note that X is diagonal. We arrive at Eq. (27) in the text by substituting

$$\alpha = \alpha' X^{-1} \tag{A26}$$

into Eq. (A21). Notice that X cancels, yielding Eq. (A21) (removing primes) with

$$\alpha^{\dagger}\alpha = 1. \tag{A27}$$

[1]G. Gentile, Nuovo Cimento **17**, 493 (1940); T. Okayama, Progr. Theoret. Phys. 7, 517 (1952); H. S. Green, Phys. Rev. **90**, 270 (1953); I. E. McCarthy, Proc. Cambridge Philos. Soc. **51**, 131 (1955); S. Kamefuchi and Y. Takahashi, Nucl. Phys. **36**, 177 (1962); L. O'Raifeartaigh and C. Ryan, Proc. Roy. Irish Acad. Sect. A **62**, 93 (1963); S. Kamefuchi, Matscience Rep. **24**, (1963).

[2]O. W. Greenberg and A. Messiah, Phys. Rev. B **138**, 1155 (1965).

[3]O. Klein, J. Phys. Radium **9**, 1 (1938); K. Nishijima, Progr. Theoret. Phys. **5**, 187 (1950); S. Oneda and H. Umezawa, Progr. Theoret. Phys. **9**, 685 (1953); T. Kinoshita, Phys. Rev. **96**, 199 (1954); **110**, 978 (1958); G. Luders, Z. Naturforsch. **139**, 251 (1958); H. Araki, J. Math. Phys. **2**, 267 (1961).

[4]R. Y. Levine and Y. Tomozawa, Preprint UM HE 82-14, University of Michigan, 1982.