A deformation of the general zero-curvature equations associated to simple Lie algebras

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We construct deformations and rational reductions for all the general zero-curvature equations associated to simple complex Lie algebras [known as AKNS equations for sl(2, $\mathbb{C}$)].

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I. INTRODUCTION

One of the fundamental properties of the Korteweg–de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx}, \quad u = u(x,t), \tag{1.1}$$

is that it has an infinite number of local conservation laws: that is, there is an infinite sequence of identities

$$\partial_t H_q = \partial J_q, \quad q = 1,2,... \tag{1.2}$$

that follow formally from (1.1). (We write $\partial_t = \partial/\partial t$, $\partial \equiv \partial/\partial x$.) The "conserved densities" H_q and "fluxes" J_q are elements of $\mathbb{C}[u^{(j)}] =$ the set of differential polynomials in u, i.e., polynomials in u and its x-derivatives $u^{(j)} = \partial^j(u)$.

Here is the charming construction, due to Gardner,[1] of these conservation laws. Consider another equation

$$w_t = 6ww_x - w_{xxx} + 6\epsilon^2 w^2 w_x, \quad w = w(x,t). \tag{1.3}$$

It is easy to check that if w satisfies (1.3), then $u(x,t)$, given by the formula

$$u = w + \epsilon^2 w^2 + \epsilon w_x \tag{1.4}$$

satisfies (1.1). Now, rewrite (1.3) in the conservation form

$$\partial_t w = \partial(3w^2 - w_{xx} + 2\epsilon^2 w^3), \tag{1.5}$$

invert (1.4) (understood as an automorphism of differential rings $\mathbb{C}[u^{(j)}]\,[[\epsilon]] \to \mathbb{C}[w^{(j)}]\,[[\epsilon]]$)

$$w = u + \sum_{k=1}^{\infty} \epsilon^k P_k, \quad P_k \in \mathbb{C}[u^j], \tag{1.6}$$

substitute (1.6) into (1.5), and identify ϵ^q-coefficients on both sides of the resulting equality: you get (1.2).

What is the meaning of the Gardner trick? Notice that under the homomorphism (1.4), conservation laws (1.2) for the KdV equation (1.1) become conservation laws for the Gardner equation (1.3) which, therefore, is also an *integrable* system, that is, it has an infinite number of conservation laws. Thus, starting with the KdV equation (1.1), we have a curve (1.3) parametrized by ϵ in the space of evolution equations, and an *integrable curve* at that. Moreover, the map (1.4) tells us that we also have a *reduction* of our curve, that is, a regular map which sends any point on the curve into a base point with parameter $\epsilon = 0$ and is the identity at $\epsilon = 0$ (We do not distinguish between evolution equations and their solutions; see, e.g., Ref. 2 for the spirit of algebraic treatment of evolution equations.) It is, then, irresistible to conjecture that (a) most, if not all, integrable systems currently in circulation (see, e.g., Refs. 3–5) can be included in integrable one-parameter families, which we call *deformations*, and, moreover, (b) these curves carry with them reductions. There is some evidence available to back up this conjecture (see, e.g., Refs. 6 and 7), though it is not at all clear what could be the underlying reasons for the existence of such a general phenomenon.

The main result of this paper establishes the existence of deformations and *rational* reductions for all the general zero-curvature equations associated to simple Lie algebras.[5] Details will be given in the course of the paper via the following route: in Sec. II, we review the general zero-curvature equations associated to simple complex Lie algebras; in Sec. III, we study two different coordinate systems on the tangent bundle of the manifold of Cartan subalgebras of a given simple Lie algebra. We find that these coordinate systems are related by a rational map. In Sec. IV, we interpret constructions of Sec. III as providing deformations and desired reductions.

II. THE GENERAL ZERO-CURVATURE EQUATIONS

In this section, we summarize the Wilson construction of the general zero-curvature equations.[5]

Let $\mathfrak{g}$ be a complex simple Lie algebra, F a regular semisimple element of $\mathfrak{g}$, $\mathfrak{h}$ a unique Cartan subalgebra containing F, so that we have

$$\mathfrak{g} = \mathfrak{h} \oplus [\mathfrak{h}, \mathfrak{g}] = \mathfrak{h} \oplus \operatorname{Im} \operatorname{ad} F. \tag{2.1}$$

Let $l = \dim \mathfrak{h}$ denote the rank of $\mathfrak{g}$ and let $R \subset \mathfrak{h}^*$ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. For every $\alpha \in R$, let E_α be a nonzero element of the corresponding root space, so that

$$[\mathfrak{h}, \mathfrak{g}] = \bigoplus_{\alpha \in R} \mathbb{C} E_\alpha. \tag{2.2}$$

Let u_α, $\alpha \in R$, be differentially independent variables, and let $B = \mathbb{C}\,[u_\alpha^{(j)}]$ be the differential algebra of polynomials in variables $u_\alpha^{(j)}$ with the derivation ∂ acting on B through $\partial(u_\alpha^{(j)}) = u_\alpha^{(j+1)}$. We introduce a grading on B by setting $\deg u_\alpha^{(j)} = j + 1$. We set $\tilde{\mathfrak{g}} = B \otimes_{\mathbb{C}} \mathfrak{g}$ and extend the derivation ∂ and the grading deg to $\tilde{\mathfrak{g}}$ by $\partial(1 \otimes \mathfrak{g}) = 0$, $\deg(1 \otimes \mathfrak{g}) = 0$.

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Define $u = \Sigma\, u_\alpha \otimes E_\alpha$, which, for brevity, we shall write as $\Sigma\, u_\alpha E_\alpha$. Let λ be a formal parameter commuting with everything; set

$$U = -u + \lambda F \in \tilde{\mathfrak{g}}. \qquad (2.3)$$

For $V \in \tilde{\mathfrak{g}}[\lambda]$, our general zero-curvature equations[5]

$$[\partial - U, \partial_t - V] = 0 \qquad (2.4)$$

are equivalent to

$$\partial_t u = -\partial V + [U,V]. \qquad (2.4')$$

These equations make sense if and only if the rhs of (2.4') does not depend upon λ and lies in $\widetilde{[\mathscr{h}, \mathfrak{g}]} = B \underset{\mathbb{C}}{\otimes} [\mathscr{h}, \mathfrak{g}]$. Here is the full description of all possible V's:

Fix a natural number $r \geqslant 1$ and an element $v \in \mathscr{h}$. Let

$$W = \sum_{i=0}^{\infty} v_i \lambda^{-i}, \quad v_i \in \tilde{\mathfrak{g}}, \quad v_0 = v \qquad (2.5)$$

be a unique solution of the equation $\partial W = [U,W]$ such that v_i is homogeneous of degree i. Set

$$V = \sum_{i=0}^{r} v_i \lambda^{r-i}, \quad V_- = \sum_{i=r+1}^{\infty} v_i \lambda^{r-i}. \qquad (2.6)$$

From $\lambda^r W = V + V_-$, we have

$$-\partial V + [U,V] = \partial V_- - [U,V_-], \qquad (2.7)$$

and since the lhs does not involve any negative powers of λ whereas the rhs does not contain any positive powers, (2.7) is λ-independent. Picking out the terms of order zero in λ, we rewrite (2.4') as

$$u_t = -v_{r,x} + [u,v_r] = [F, -v_{r+1}], \qquad (2.8)$$

which shows that the rhs belongs to $\widetilde{[\mathscr{h},\mathfrak{g}]} = \widetilde{\mathrm{Im\,ad}\, F}$.

Now denote by $\partial_t = \partial_t(v,r)$ the evolutionary (i.e., commuting with ∂) derivation of B, which is defined by (2.8) via

$$\partial_t(v,r) u_\alpha = \alpha\text{–component of } [F, -v_{r+1}] \text{ in } \widetilde{[\mathscr{h},\mathfrak{g}]}. \qquad (2.9)$$

These are the equations we are going to deform. The properties of these equations are given in the following proposition:

Proposition 2.1: (i) If $v \neq 0$ then $\partial_t(v,r) \neq 0$. Thus we get l linearly independent derivations for each r. (ii) Let K denote the Killing form on $\mathfrak{g}$ naturally extended to $\tilde{\mathfrak{g}}$, and set $H_s = H_s(v) = s^{-1} K(v_{s+1}, F)$. Then the elements H_s, $s \geqslant 1$ are common nontrivial (i.e., they do not lie in ∂B) conserved densities of all evolution equations (2.9) (that is, $\partial_t H_s \in \partial B$). Thus we obtain l linearly independent conservation laws for each $s \geqslant 1$. (iii) Equations (2.9) can be written in Hamiltonian form

$$\partial_t(v,r) u_\alpha = -\alpha(F) \frac{\delta H_{r+1}}{\delta u_{-\alpha}}, \quad \alpha \in R, \qquad (2.10)$$

where $\delta/\delta u_\alpha$ is the functional derivative with respect to u_α. (See, e.g., Ref. 2 for the differential-algebraic version of calculus.) Thus all derivations $\partial_t(v,r)$ (or corresponding "flows") commute with each other.

The proofs follow from the general theory of Lax[4] equations. We shall not need them: our object of study is just Eqs. (2.4).

III. MANIFOLD OF CARTAN SUBALGEBRAS

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, $\mathscr{H}$ the set of all Cartan subalgebras in $\mathfrak{g}$. $\mathscr{H}$ is a homogeneous space for the Lie group Aut $(\mathfrak{g})$ and we may regard $\mathscr{H}$ as a nonsingular algebraic variety. In this section, we study relations between two coordinate systems on (open piece of) $\mathscr{H}$. (To avoid any confusion with the notations of the preceding section, the reader would do well to ignore temporarily the existence of Sec. II).

Let F be a semisimple element of $\mathfrak{g}$ such that the dimension of the centralizer $\mathfrak{g}^F$ of F in $\mathfrak{g}$ is minimal; in other words, F is regular semisimple. Then $\mathscr{h} := \mathfrak{g}^F$ is a Cartan subalgebra of $\mathfrak{g}$, and the rank of $\mathfrak{g}$ is given by rk $\mathfrak{g} := \dim \mathscr{h}$. We then have

$$\mathfrak{g} = \mathscr{h} \oplus [F,\mathfrak{g}],\ [\mathscr{h}, \mathfrak{g}] = [F,\mathfrak{g}] = \mathrm{Im}(\mathrm{ad}\, F).$$

Let $\mathscr{H}'$ be the open subset of $\mathscr{H}$ consisting of all $\mathscr{h}' \in \mathscr{H}$ satisfying the transversality condition $\mathscr{h}' \cap [F, \mathfrak{g}] = \{0\}$. We define a smooth map $P: \mathscr{H}' \to [F, \mathfrak{g}]$ by requiring $(F - P(\mathscr{h}')) \in \mathscr{h}'$ for all $\mathscr{h}' \in \mathscr{H}'$. Then $P(\mathscr{h}')$ uniquely determines $\mathscr{h}'$ by $\mathscr{h}' = \mathfrak{g}^{F - P(\mathscr{h}')}$ provided $F - P(\mathscr{h}')$ happens to be regular semisimple; we denote by $\mathscr{H}''$ the set of all such $\mathscr{h}' \in \mathscr{H}'$ (for which it does happen). Then $\mathscr{h} \in \mathscr{H}''$ [since $F - P(\mathscr{h}) = F$] and $\mathscr{H}''$ is open in $\mathscr{H}$ (since $\mathscr{H}'' = [\mathscr{H}'$ which is open in $\mathscr{H}]$ $\cap$ [{the set of all regular semisimple elements} which is open open in $\mathfrak{g}$]). Therefore P is smooth and injective on the nonempty open subset $\mathscr{H}''$ of $\mathscr{H}$. Since $\dim \mathscr{H} = \dim [F,\mathfrak{g}]$ ($= \dim \mathrm{Im}\, P$), it follows that the differential dP of the map P is surjective at all points $\mathscr{h}' \in \mathscr{H}''$.

Now let us consider the tangent bundle $\mathrm{T}(\mathscr{H}'')$ of $\mathscr{H}''$. Regarding P as providing a coordinate system for $\mathscr{H}''$, we identify $\mathrm{T}(\mathscr{H}'')$ with $\mathrm{T}(P(\mathscr{H}'')) \cong P(\mathscr{H}'') \times [F, \mathfrak{g}]$, the tangent bundle of $P(\mathscr{H}'')$. Denote the coordinate system thus obtained on $\mathrm{T}(\mathscr{H}'')$ by $(P, P_x) \in P(\mathscr{H}'') \times [F,\mathfrak{g}]$.

Another coordinate system (P,u) on $P(\mathscr{H}'') \times [F,\mathfrak{g}]$ may be defined as follows. Any $w \in \mathfrak{g}$ determines a holomorphic vector field X_w on $\mathscr{H}$ as the generator of a one-parameter family of diffeomorphisms $\exp(\tau\, \mathrm{ad}\, w)$ restricted to $\mathscr{H}$: for any $f \in C^\infty(\mathscr{H})$,

$$(X_w f)(\mathscr{h}') = \lim_{\tau \to 0} \tau^{-1} [f((\exp(\mathrm{ad}\, \tau w)) \cdot \mathscr{h}') - f(\mathscr{h}')].$$

The map $w \mapsto X_w$ is a Lie algebra homomorphism. Denote by $X_w(\mathscr{h}') \in T_{\mathscr{h}'}(\mathscr{H})$ the value of X_w at $\mathscr{h}'$. Then $X_w(\mathscr{h}') = 0$ iff $w \in \mathscr{h}'$. Thus the correspondence

$$(\mathscr{h}', X_{F-u}(\mathscr{h}')) \leftrightarrow (P(\mathscr{h}'), u), \quad u \in [F, \mathfrak{g}]$$

defines our second coordinate system on $\mathrm{T}(\mathscr{H}'')$.

Our goal is to connect these coordinate systems. To obtain a connection between (P, P_x) and (P,u), consider a holomorphic curve $\gamma: \overline{B} \to \mathscr{H}''$ defined on an open ball $\overline{B} \subset \mathbb{C}$ containing zero. Put $P(\tau) = P(\gamma(\tau))$ for $\tau \in \overline{B}$, set $P := P(0)$, $P_x = dP(\tau)/d\tau|_{\tau=0}$, and let (P,u) correspond to (P,P_x). For the Cartan subalgebra $\gamma(\tau)$, when $\tau \to 0$, its general regular semisimple element near $F - P$ [recall that $\mathfrak{g}^{F-P} = \gamma(0)$] can be written in two different forms (according to the two coordinate systems introduced above) as

$$F - P - \tau P_x + O(\tau^2),$$

and

$$F-P+\tau[F-u,F-P]+O(\tau^2)$$
$$=(\exp \operatorname{ad} \tau(F-u))\cdot(F-P)+O(\tau^2).$$

Since $\gamma(\tau)$ is commutative, we have

$$[F-P-\tau P_x, F-P+\tau[F-u,F-P]]=O(\tau^2),$$

or, putting

$$T:=\operatorname{ad}(F-P), \tag{3.1}$$

we get

$$T^2(P-u)=T(P_x). \tag{3.2}$$

We shall now analyze the correspondence $(P,P_x)\leftrightarrow(P,u)$ given by (3.2), and we regard F as variable as well (we shall need this later). We note that the results below require only the assumption that $\mathfrak{g}$ is a simple Lie algebra over a field k of characteristic zero, and in such a context we formulate our statements (one often needs $k=\mathbb{R}$ rather than $k=\mathbb{C}$. We want to cover this case as well).

The following lemma from linear algebra provides the basis for our analysis.

Lemma 3.1: Let L be a vector space over a field k, with fixed basis $w_1,\dots,w_n$. Let r be an integer with $1\leqslant r\leqslant n$, and consider the algebraic variety

$$M=\{(T,T')\in \operatorname{End} L\times \operatorname{End} L \mid \operatorname{rank} T'\leqslant r\}.$$

Define an element $A\in k\,[M]$ ($=$ regular functions on M) by

$$A(T,T')w_1\wedge\cdots\wedge w_n$$
$$=TT'w_1\wedge\cdots\wedge TT'w_r\wedge w_{r+1}\wedge\cdots\wedge w_n. \tag{3.3}$$

(Both sides lie in the one-dimensional space $\Lambda^n(L)$ with the basic vector $w_1\wedge\cdots\wedge w_n$.) Then there exists $T''\in k\,[M]\otimes_k \operatorname{End} L$ such that

$$T'TT''=T'A \tag{3.4}$$

in $k\,[M]\otimes_k \operatorname{End} L$ [in other words, considered as a regular function on M with values in $\operatorname{End} L$, T'' satisfies $T'TT''(T,T')=T'A(T,T')$].

Proof: Write $TT'w_i=\Sigma_{j=1}^n a_{ij}w_j$, $1\leqslant i\leqslant n$, where $a_{ij}\in k\,[M]$. Define the matrix $a=(a_{ij})_{1\leqslant i,j\leqslant r}$, then $A=\det a$. Next we take the matrix $a'=(a'_{ij})_{1\leqslant i,j\leqslant r}$ such that $aa'=a'a=A\mathbb{1}_r$, so that $a'_{ij}\in k\,[M]$. Set, for $1\leqslant i\leqslant r$, $\bar{w}_i=\Sigma_{j=1}^r a'_{ij}w_j$, so $Aw_i=\Sigma_{j=1}^r a_{ij}\bar{w}_j$. Now we can define T'' (for the fixed T and T') by $T''w_j=T'\bar{w}_j$, $1\leqslant j\leqslant r$, $T''w_j=0$, $r<j\leqslant n$. Since, for $1\leqslant i\leqslant r$, $TT'\bar{w}_i=TT'\Sigma_{j=1}^r a'_{ij}w_j=\Sigma_{j=1}^r a'_{ij}TT'w_j=\Sigma_{j=1}^r a'_{ij}\Sigma_{k=1}^n a_{jk}w_k \equiv\Sigma_{j,k=1}^r a'_{ij}a_{jk}w_k \pmod{L_r:=kw_{r+1}+\cdots+kw_n}=Aw_i \pmod{L_r}$, we have, again for $1\leqslant i\leqslant r$, $T''TT'\bar{w}_i=T''(Aw_i \bmod L_r)=T''(Aw_i)=AT''w_i=AT'\bar{w}_i$. Thus $(T''TT'-AT')=0$ on $\bar{L}':=k\bar{w}_1+\cdots+k\bar{w}_r$ and since $Aw_i\in\bar{L}'$, we get $(T''TT'-AT')A=0$ on $L':=kw_1+\cdots+kw_r$. Now $\operatorname{rk} T'\leqslant r$, so $T'(L)=T'(L')$. Thus if $A\neq0$, then $T''TT'-AT'=0$ on L. But M is irreducible, and A is regular and not identically zero on M. Therefore $T''TT'-AT'\equiv0$ always.

Q.E.D.

We now use Lemma 3.1 in the situation: $L=\mathfrak{g}$, $r=\dim\mathfrak{g}-\operatorname{rk}\mathfrak{g}$, and T, T' on M are given by $T=\operatorname{ad}(F-P)$, $T'=\operatorname{ad}F$ where $P\in\mathfrak{g}$ and $F\in\mathfrak{g}$ is regular semisimple. If F is regular semisimple, i.e., $\dim\mathfrak{g}^F=\operatorname{rk}\mathfrak{g}$, and $[F,\mathfrak{g}]\cap\mathfrak{g}^{F-P}=\{0\}$, we choose a basis $w_1,\dots,w_n$ in $\mathfrak{g}$ in the following manner: take $w_1,\dots,w_r$ such that $[F,w_1],\dots,[F,w_r]$ are linearly independent (we can do this since F is regular so $\dim(\operatorname{Im}\operatorname{ad}F)=n-\dim\mathfrak{g}^F=r$), then $TT'w_1=T([F,w_1]),\dots,TT'w_r=T([F,w_r])$ are also linearly independent (since $[F,\mathfrak{g}]\cap\mathfrak{g}^{F-P}=\{0\}$). Now choose complementary vectors $w_{r+1},\dots,w_n$ such that $A\neq0$, see (3.3). Now define, via Lemma 3.1, the element

$$S=A^{-1}T''T. \tag{3.5}$$

Claim: S is the projection of $\mathfrak{g}$ onto $[F,\mathfrak{g}]$ along $\mathfrak{g}^{F-P}$. Indeed, $ST'=A^{-1}T''TT'=A^{-1}AT'=T'$, thus $S=\operatorname{Id}$ on $\operatorname{Im}T'=\operatorname{Im}\operatorname{ad}F$. On the other hand, if $y\in\mathfrak{g}^{F-P}=\operatorname{Ker}\operatorname{ad}(F-P)=\operatorname{Ker}T$, that is, $Ty=0$, then $Sy=A^{-1}T''Ty=0$. To sum up, S is a rational function of $(F,P)\in\mathfrak{g}\times\mathfrak{g}$ defined whenever $\dim\mathfrak{g}^F=\operatorname{rk}\mathfrak{g}$ and $[F,\mathfrak{g}]\cap\mathfrak{g}^{F-P}=\{0\}$.

Applying this claim to our basic equation (3.2), written in the form

$$P_x=S([F-P,P-u]), \tag{3.6}$$

we get P_x as a regular function on the following Zariski-locally-closed subset Z of $\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}$:

$$Z:=\{(F,P,u)|\dim\mathfrak{g}^F=\operatorname{rk}\mathfrak{g};\ P,u\in[F,\mathfrak{g}];$$
$$[F,\mathfrak{g}]\cap\mathfrak{g}^{F-P}=\{0\}\}. \tag{3.7}$$

Denote by Y the analogous set of (F,P,P_x) obtained by replacing u by P_x in (3.7).

Now let Q be the projection of $\mathfrak{g}$ onto $[F-P,\mathfrak{g}]$ along $\mathfrak{g}^{F-P}$. Then, as above for S, Q is a rational function of $(F,P)\in\mathfrak{g}\times\mathfrak{g}$ defined whenever $F-P$ is regular semisimple (perhaps we should stress that in all matters unrelated to our problem of connecting coordinate systems, we treat F and P as free variables).

Turning back to our basic equation (3.2), we rewrite it with the help of Q as

$$T(P-u)=Q(P_x). \tag{3.8}$$

Since $\operatorname{ad}F$ is an isomorphism on $[F,\mathfrak{g}]=\operatorname{Im}(\operatorname{ad}F)$, we can define $P',u'\in[F,\mathfrak{g}]$ by $P=[F,P']$, $u=[F,u']$. Then $P-u=T'(P'-u')$ and we have, by Lemma 3.1,

$$A(P-u)=AT'(P'-u')=T''TT'(P'-u')$$
$$=T''T(P-u)=T''(Q(P_x)).$$

This shows that u is a regular function on the Zariski-locally-closed set

$$Y':=\{(F,P,P_x)\in Y|(F-P)\text{ is regular semisimple}\}$$

(recall that $A\neq0$ on Y). Denoting $Z'=\{(F,P,u)\in Z|(F-P$ is regular semisimple$\}$, we collect the results of this reasoning in the following theorem.

Theorem 3.2: The maps $Z\to Y$, given by (3.6), and $Y'\to Z'$, given by (3.8), are regular. In particular, our basic Eq. (3.2) determines an everywhere-defined birational correspondence $Y'\leftrightarrow Z'$.

Now we can analyze different asymptotics connected with the correspondence $Y'\leftrightarrow Z'$.

Theorem 3.3: (a) Fix a regular semisimple F_0 and $P, P_x \in [F_0, \mathfrak{g}]$. Set $F = \epsilon^{-1}F_0$. Then $u = P + O(\epsilon)$ as $\epsilon \to 0$. (b) Fix a regular semisimple $F \in \mathfrak{g}$ and $P_0, P_{0,x} \in [F, \mathfrak{g}]$. Set $P = \nu P_0$, $P_x = \nu P_{0x}$. Then $[F,u] = \nu([F,P_0] - P_{0,x}) + O(\nu^2)$, as $\nu \to 0$ (equivalently, $u = \nu(P_0 - (\text{ad } F)^{-1} P_{0,x}) + O(\nu^2)$, where $(\text{ad } F)^{-1}$ is an isomorphism on $[F, \mathfrak{g}] \ni P_{0,x}$). (c) Fix a regular semisimple $F \in \mathfrak{g}$ and let $\mathscr{L} = \mathfrak{g}^F$ be the corresponding Cartan subalgebra with root system Δ. Then the determinant of the Fréchet derivative of the linearization of the map $(P, P_x) \mapsto u$, is given by

$$\prod_{\alpha \in \Delta} (1 - \alpha(F)^{-1}\partial). \tag{3.9}$$

Proof: (a) In all three cases we use the fact that by Theorem 3.2, u is rational in the parameters involved: in the present case, as a function of ϵ. Let us write then $u = \epsilon^s(u_0 + O(\epsilon))$ with some $s \leqslant 0$ and require $u_0 \neq 0$ for $s < 0$ (we thus allow $u_0 = 0$ for $s = 0$, taking care of the possibility of u having positive s-asymptotics in ϵ). Rewriting (3.2) in long hand, we have

$$[\text{ad}(\epsilon^{-1}F_0 - P)]^2[P - \epsilon^s(u_0 + O(\epsilon))]$$
$$= [\text{ad}(\epsilon^{-1}F_0 - P)]P_x = O(\epsilon^{-1}) = O(\epsilon^{s-2}).$$

This yields

$$(\text{ad } F_0)^2 u_0 = 0 \quad \text{if} \quad s < 0,$$
$$(\text{ad } F_0)(P - u_0) = 0 \quad \text{if} \quad s = 0.$$

But ad F_0 is nonsingular on $[F_0, \mathfrak{g}]$ which forces $u_0 = 0$ for $s < 0$, a contradiction with the choice $u_0 \neq 0$ made for $s < 0$. Thus $s = 0$ and $P - u_0 = 0$, proving (a). (b) For $\nu \to 0$, write $u = \nu^s(u_0 + O(\nu))$, where $s \leqslant 1$ and $u_0 \neq 0$ for $s < 1$. Then, as above, (3.2) can be rewritten as

$$[\text{ad}(F - \nu P_0)^2](\nu P_0 - u) = \nu[F, P_{0,x}] + O(\nu^2),$$

which yields

$$(\text{ad } F)^2 u_0 = 0, \quad \text{if} \quad s < 1,$$
$$(\text{ad } F)^2(P_0 - u_0) = (\text{ad } F)P_{0,x} \quad \text{if} \quad s = 1,$$

which forces, as above, $s = 1$ and $P_{0,x} = [F, P_0 - u_0]$. (c) If $u = f(P, P_x)$ is a locally smooth map with $0 = f(0,0)$, its linearization $l(f)$ (at zero) is defined by $\bar{u} = l(f)(P, P_x) := (d/d\nu) f(\nu P, \nu P_x)|_{\nu=0}$. By (b), we get $\bar{u} = P_0 - (\text{ad F})^{-1} P_{0,x}$. Choose a nonzero vector E_α in the root space $\mathfrak{g}_\alpha$ for every root $\alpha \in \Delta$. Then we can rewrite the above formula as

$$\bar{u}_\alpha = P_{0\alpha} - \alpha(F)^{-1}P_{0\alpha,x}. \tag{3.10}$$

Now recall that the Fréchet derivative of any map ϕ given by a nonlinear differential operator of the form $u_i = u_i(v_j^{(m)})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant N$, is an $n \times N$ matrix $D(\phi)$ of differential operators, defined by

$$D(\phi)_{ij} = \Sigma_{m>0} \frac{\partial u_i}{\partial v_j^{(m)}} \partial^m, \quad \partial = \frac{d}{dx}.$$

Thus for the case (3.10), we get the matrix-diag$(\ldots, 1 - \alpha(F)^{-1}\partial, \ldots)$, which implies (3.9).

Q.E.D.

IV. EVOLUTION EQUATIONS

In this section we construct deformations of Eqs. (2.4). The idea is to interpret them as defined on $\mathscr{H}''$ rather than on $\mathfrak{g}$.

We begin by recalling some simple notions from the calculus. Suppose N is a manifold (smooth, like everything else in this section), let $D(N)$ denote the set of all vector fields on N [that is, derivations of $C^\infty(N)$.] Consider separately $\mathbb{R}^1$ with the coordinate x and vector field d/dx. If $X \in D(N)$, a (local) trajectory of X is a map $\gamma: I^1 \to N$ such that

$$\frac{d}{dx}\gamma^* = \gamma^* X, \quad I^1 \text{ is an interval in } \mathbb{R}^1, \tag{4.1}$$

understood as the equality of operators on $C^\infty(N)$ [with values in $C^\infty(I^1)$]. If one chooses a (local) coordinate system $(y_1, \ldots, y_n)$ on N and if $X = \Sigma g_i(y)\partial/\partial y_i$, then (4.1) is equivalent to the familiar form of a system of ODE's:

$$\frac{d\gamma^*(y_i)}{dx} = \gamma^*(g_i), \quad 1 \leqslant i \leqslant n. \tag{4.2}$$

If the field X depends upon parameters $\mu_1, \ldots, \mu_k$ (x may be one of them), definition (4.2) works equally well.

What we need is a bit more. Suppose X and Y are two families of vector fields on N, and they both depend upon two parameters which we denote x and t. Consider $I^2 = I^1 \times I^1$ with these coordinates x and t and fix two vector fields $\partial = \partial/\partial x$ and $\partial_t = \partial/\partial t$ on I^2.

Definition 4.1: A trajectory of the pair X, Y is a map $\gamma: I^2 \to N$ such that

$$\frac{\partial}{\partial x}\gamma^* = \gamma^* X, \tag{4.3}$$

$$\frac{\partial}{\partial t}\gamma^* = \gamma^* Y. \tag{4.4}$$

Definition–Proposition: Let $X = X(\mu)$ be a family of vector fields on N depending upon parameters $\mu = (\mu_1, \ldots, \mu_k)$. Then the set of operators $X_{\mu_i} := \dfrac{\partial X}{\partial \mu_i}$, $1 \leqslant i \leqslant k$, defined by

$$X_{\mu_i}(h) = \frac{\partial}{\partial \mu_i} X(h), \quad h \in C^\infty(N),$$

is again a family of vector fields on N with parameters μ.

Proof: Differentiate, with respect to μ_i, the equality

$$X(h_1 h_2) = h_1 X(h_2) + h_2 X(h_1), \quad h_1, h_2 \in C^\infty(N).$$

Proposition 4.2: A trajectory γ for the pair X, Y can be drawn through every point $(x,t,n) \in I^2 \times N$ [i.e., $\gamma(x,t) = n$] if and only if

$$[X,Y] - X_t + Y_x = 0, \tag{4.5}$$

as operator on $C^\infty(N)$.

Proof: The only integrability condition for γ is

$$\frac{\partial}{\partial t}(4.3) = \frac{\partial}{\partial x}(4.4),$$

as follows, e.g., from writing (4.3), (4.4) in local coordinates. We have then

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}\gamma^* = \frac{\partial}{\partial t}\gamma^* X = (\gamma^* Y)X + \gamma^* X_t,$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial t}\gamma^* = \frac{\partial}{\partial x}\gamma^* Y = (\gamma^* X)Y + \gamma^* Y_x,$$

so

$$\gamma^*([X,Y] - X_t + Y_x) = 0.$$

Q.E.D

We remark in passing how to transform (4.5) into a more familiar form. Consider $\overline{N}: = N \times I^2$ and extend naturally on $\overline{N}$ vector fields X, Y, ∂ and ∂_t, continuing to denote them by the same symbols. Then (4.5) can be written as

$$[\partial + X, \partial_t + Y] = 0. \tag{4.6}$$

It is obvious now how to proceed. Rewrite (2.4) as

$$[-U, -V] - (-U_t) + (-V_x) = 0, \tag{4.7}$$

and consider the representation of $\mathfrak{g}$ in $D(\mathscr{H}'')$, as in Sec. III. Applying proposition 4.2, we see that (4.7) is the integrability condition for the system

$$\frac{\partial}{\partial x}\gamma^* = \gamma^*(-X_{\lambda F - u}), \tag{4.8}$$

$$\frac{\partial}{\partial t}\gamma^* = \gamma^*(-X_V), \tag{4.9}$$

for $u = u(x,t)$ fixed. Applying Theorem 3.2 to Eq. (4.8), we find u (or rather $-u$) as a rational function of $P(\gamma)$. This enables us to eliminate u in favor of γ in (4.9) which thus becomes our deformed equation, if we define the deformation parameter ϵ as $-\lambda^{-1}$. Indeed, Theorem 3.3(a) then yields that $-u = P(\gamma) + O(\epsilon)$, thus $-P(\gamma)$ (or $-\gamma$) is the deformed variable [analog of w in (1.3)].

Proposition 4.3: Deformed Eq. (4.9) is not equivalant to undeformed one (2.4) under any change of variables.

Proof: Use existence of the reduction (4.8). If (4.9) were equivalent to (2.4), the map $u \mapsto \gamma$ which inverts (4.8) (understood as a rational map $\gamma \mapsto u$ by Theorem 3.2) will be a (finite) differential operator. In particular, its linearization will invert the linearization of (4.8). Therefore the Fréchet derivative of this linearization will invert the Fréchet derivative of linearized (4.8). Taking determinants, we get a differential operator which inverts expression (3.9), which is a differential operator of positive order [since $\alpha(F) \not\equiv 0$ for $\alpha \in \Delta$], a contradiction.

Q.E.D.

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[1]R. M. Miura, C. S. Gardner, and M. D. Kruskal, "Korteweg–de Vries Equation and Generalizations. II. Existence of Conservation Laws and Constants of Motion," J. Math. Phys. **9**, 1204–1209 (1968).

[2]Yu. I. Manin, "Algebraic Aspects of Nonlinear Differential Equations," J. Sov. Math. **11**, 1–122 (1979).

[3]V. G. Drinfel'd and V. V. Sokolov, "Korteweg–de Vries Type Equations and Simple Lie Algebras," Dokl. Acad. Nauk SSSR **258**, 11–16 (1981) (in Russian).

[4]G. Wilson, "Commuting Flows and Conservation Laws for Lax Equations," Math. Proc. Cambridge Philos. Soc. **86**, 131–143 (1979).

[5]G. Wilson, "The Modified Lax and Two-Dimensional Toda Lattice Equations Associated with Simple Lie Algebras," Ergodic Theory Dynamical Systems **1**, 361–380 (1982).

[6]B. A. Kupershmidt, "On the Nature of the Gardner Transformation," J. Math. Phys. **22**, 449–451 (1981).

[7]B. A. Kupershmidt, "Deformations of Integrable Systems," Proc. R. Ir. Acad. Sect. A (to appear).