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Weber's Mixed Boundary-Value Problem in Electrodynamics

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Electrodynamics problems with mixed boundary values promise to assume increasing practical importance in fields such as plasma physics. A new method of attacking such problems in three dimensions is presented and discussed.

1. STATEMENT OF THE PROBLEM

THE emergence of plasma physics as an important scientific discipline has disclosed many imperfections in the traditional approaches to certain aspects of mathematical physics. One of these is the mixed boundary-value problem in electrodynamics originating when a moderately conducting domain is incompletely bounded by perfect conductors. While it is evident that this problem can be solved in all two-dimensional symmetries, the simplest examples of complete three-dimensional character present great difficulties.

The classical example of such a problem is Nobili's[1] colored rings first considered by Riemann.[2] This is the problem of an infinite slab of conducting material on whose plane sides two circular metallic disks held at opposite potential are placed. The conductivity of the slab is finite, that of the disks infinite. The formulation of the problem is as follows:

A solution of Laplace's equation $\phi(\rho, z)$ is to be found such that for $z = \pm a$,

$$\phi(\rho, \pm a) = \pm\phi_0, \ \rho < c,$$
$$(\partial\phi/\partial z)_{\pm a} = 0, \qquad \rho > c.$$

Here ρ is the two-dimensional distance $(x^2 + y^2)^{\frac{1}{2}}$, the thickness of the slab is $2a$, the radius of each electrode disk is c. A presumed solution of this cylindrically symmetric, mixed boundary-value problem was given by Weber,[3] but it was actually only an approximation for the case where the disk size is small compared with the slab thickness.

The method presented in the following pages reduces the problem to a standard integral equation of the Fredholm type (integral equation of the "second kind"). In another paper[4] the construction of solutions of this equation is presented in detail together with numerical results. Recently, the same problem was attacked by Tranter[5] using the method of integral transforms and an approximate solution was constructed. However, since the method given here is quite different and capable of considerable generalization (see Sec. 6 below), we think it merits a detailed presentation.

2. THE SINGLE CIRCULAR DISK

As a preparation, let the well-known solution for the metallic disk be rederived. We start with

$$\phi(\rho, z) = \frac{2}{\pi}\int_0^\infty d\lambda J_0(\rho\lambda)e^{-|z|\lambda}A(\lambda), \tag{1}$$

where $A(\lambda)$ is to be determined so that $\phi(\rho, 0)$ is equal to the constant potential ϕ_0 for $\rho < c$. Rather than trying to find $A(\lambda)$ directly, we make the following *Ansatz:*

$$A(\lambda) = \int_0^c d\xi \cos\lambda\xi f(\xi) \tag{2}$$

and attempt to find $f(\xi)$ from

$$\frac{2}{\pi}\int_0^\infty d\lambda J_0(\rho\lambda)\int_0^c d\xi \cos\lambda\xi f(\xi) = \phi_0. \tag{3}$$

Replacing ρ by η momentarily, we now operate on this with $(d/d\rho)\int_0^\rho \eta\, d\eta/(\rho^2 - \eta^2)^{\frac{1}{2}}$:

$$\frac{2}{\pi}\frac{d}{d\rho}\int_0^\rho \frac{\eta\, d\eta}{(\rho^2 - \eta^2)^{\frac{1}{2}}}\int_0^\infty d\lambda J_0(\eta\lambda)\int_0^c d\xi \cos\lambda\xi f(\xi) = \phi_0. \tag{4}$$

After drawing the λ integration to the left, the η integral can be performed:

$$\int_0^\rho \frac{\eta\, d\eta}{(\rho^2 - \eta^2)^{\frac{1}{2}}} J_0(\eta\lambda) = \frac{\sin\lambda\rho}{\lambda}$$

[1] L. Nobili, Poggendorf Ann. **9**, 183 (1827); **10**, 393, 410 (1827).
[2] B. Riemann, Poggendorf Ann. **95**, 130 (1855).
[3] H. Weber, Z. Angew. Math. **75**, 75 (1873).
[4] O. Laporte and R. G. Fowler, Phys. Rev. **148**, 170 (1966).
[5] C. J. Tranter, Quart. J. Math. **2**, 60 (1951).

and (3) becomes

$$\frac{2}{\pi}\frac{d}{d\rho}\int_0^\infty d\lambda\frac{\sin\lambda\rho}{\lambda}\int_0^c d\xi\cos\lambda\xi f(\xi)=\phi_0.$$

Again exchanging the order of integration gives

$$\frac{2}{\pi}\frac{d}{d\rho}\int_0^c d\xi f(\xi)\int_0^\infty\frac{d\lambda}{\lambda}\sin\lambda\rho\cos\lambda\xi=\phi_0, \quad (5)$$

in which the λ integral is recognizable as the "Dirichlet Discontinuous Factor." Therefore, we have

$$f(\rho)=\phi_0,$$

and $A(\lambda)$ of (2) can now be substituted into (1).

3. THE SLAB PROBLEM

Exactly the same method is now employed for the slab, with the modification that the vanishing of the normal derivative at the plates is brought about by assuming infinitely many equidistant image plates held at alternate potentials. (See Fig. 1.) Let a potential of the following form be assumed:

$$\phi=\frac{2}{\pi}\int_0^\infty d\lambda(\cdots+e^{-|z+3a|\lambda}-e^{-|z+a|\lambda}$$
$$+e^{-|z-a|\lambda}-e^{-|z-3a|\lambda}\cdots)J_0(\rho\lambda)A(\lambda) \quad (6)$$

with $A(\lambda)$ given by (2). This can be summed to be

$$\phi=\frac{2}{\pi}\int_0^\infty d\lambda\frac{\sinh\lambda z}{\cosh\lambda a}J_0(\rho\lambda)A(\lambda),\ -a\leq z\leq+a. \quad (6a)$$

With solutions of this form we now seek to satisfy the boundary conditions on each of the plates. Let

$$\phi=+\phi_0 \quad \text{for} \quad z=(4m+1)a,$$
$$\phi=-\phi_0 \quad \text{for} \quad z=(4m-1)a.$$

Substituting these conditions into (6) leads to results such as the following: For

$$z=-3a:\quad +\phi_0=\frac{2}{\pi}\int_0^\infty(\cdots+1-e^{-2a\lambda}$$
$$+e^{+4a\lambda}-e^{-6a\lambda}+\cdots)J_0A\,d\lambda,$$
$$z=-a:\quad -\phi_0=\frac{2}{\pi}\int_0^\infty(\cdots+e^{-2a\lambda}-1$$
$$+e^{-2a\lambda}-e^{-4a\lambda}\cdots)J_0A\,d\lambda,$$
$$z=+a:\quad +\phi_0=\frac{2}{\pi}\int_0^\infty(\cdots+e^{-4a\lambda}-e^{-2a\lambda}$$
$$+1-e^{-2a\lambda}\cdots)J_0A\,d\lambda.$$

These and all other boundary conditions are seen to be identical, and give after summing

$$\frac{2}{\pi}\int_0^\infty d\lambda J_0(\lambda\rho)\tanh a\lambda\int_0^c d\xi\cos\lambda\xi f(\xi)=\phi_0. \quad (7)$$

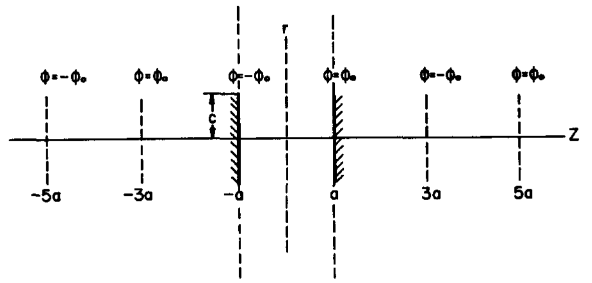

FIG. 1. Geometry of the image system for the two-disk problem.

This equation, which should be compared with (3) of the previous section is now subjected to the same transformations, which in the previous section led from (3) to (5). The result is

$$\frac{d}{d\rho}\int_0^c d\xi f(\xi)\int_0^\infty\frac{d\lambda}{\lambda}\tanh a\lambda\sin\lambda\rho\cos\lambda\xi=\phi_0. \quad (8)$$

In order to perform the differentiation with respect to ρ with complete safety, let the hyperbolic tangent be split by writing

$$\tanh a\lambda=1-2/(e^{2a\lambda}+1).$$

This results in the integral equation

$$f(\xi)-\frac{2}{\pi}\int_{-c}^{+c}d\xi_1K(\xi\xi_1)f(\xi_1)=\phi_0, \quad (8a)$$

with the symmetric kernel

$$K(\xi\xi_1)=\int_0^\infty d\lambda\,\frac{\cos\lambda\xi\cos\lambda\xi_1}{e^{2a\lambda}+1}. \quad (8b)$$

The range of integration with respect to ξ was extended to $-c$ by assuming $f(\xi)$ to be even.

4. EXPANSION IN LEGENDRE POLYNOMIALS

We introduce dimensionless variables into the integral equation by writing

$$\xi=cx,\quad \xi_1=cx_1,\quad \lambda=\mu/2a,\quad f(\xi)/\phi_0=F(x) \quad (9)$$

and later also into the equation for ϕ itself as

$$\rho/c=\sigma,\quad z/a=\zeta. \quad (10)$$

Equations (8a, b) are now

$$F(x)-\frac{2}{\pi}\epsilon\int_{-1}^{+1}dx_1F(x_1)K(x,x_1)=1, \quad (11a)$$

with

$$K(x,x_1)=\int_0^\infty d\mu\,\frac{\cos\epsilon\mu x\cos\epsilon\mu x_1}{e^\mu+1}. \quad (11b)$$

The constant

$$\epsilon=c/2a$$

is the important ratio of the problem. For infinite plate distance (11a) reduces to $F(x)=1$, and the potential $\phi(z, \rho)$ becomes the original Weber expression

$$\phi(z, \rho) = \frac{2}{\pi}\phi_0 \int_0^\infty d\lambda \frac{\sinh z\lambda}{\cosh a\lambda} \frac{\sinh \lambda a}{\lambda} J_0(\rho\lambda).$$

Because of the range of the variables x and x_1, it was found most convenient to expand the unknown function $F(x)$ into a series of even Legendre polynomials

$$f(x) = \sum_0^\infty A_{2m} P_{2m}(x). \tag{12}$$

The integral equation (11a) now becomes an infinite system of linear equations for the expansion coefficients A_{2m}

$$(4n+1)^{-1} A_{2n} - (\epsilon/\pi) \sum_m A_{2m} M_{2m,2n} = \delta_{0,2n},$$

with the matrix elements

$$M_{2m,2n} = \int_{-1}^{+1} dx \int_{-1}^{+1} dx_1 P_{2m}(x) P_{2n}(x_1) K(xx_1).$$

In the paper referred to in Ref. 4, we have reported the calculations necessary to obtain actual solutions and have shown that not only the matrix M decreases satisfactorily with increasing m and n, thereby making early truncation possible, but also the A_{2n} decrease rapidly.

5. THE FIELD FOR $\rho > c$ AS A RAPIDLY CONVERGING FOURIER SERIES

The expression for the complete potential $\phi(\rho, z)$ is using (6a), (2), (9), and (12)

$$\frac{\phi}{\phi_0} = \left(\frac{2\epsilon}{\pi}\right)^{\frac{1}{2}} \sum_m (-1)^m A_{2m} \times \int_0^\infty \frac{d\mu}{\mu^{\frac{1}{2}}} \frac{\sinh \frac{1}{2}\mu\zeta}{\cosh \frac{1}{2}\mu} J_0(\sigma\epsilon\mu) J_{2m+\frac{1}{2}}(\epsilon\mu).$$

The appearance of the half integer Bessel functions is explained below. The occurrence of the hyperbolic cosine in the denominator shows that in the complex μ plane there is a string of first-order poles along the imaginary axis located at

$$\mu = (2n+1)\pi i.$$

This, therefore, invites one to decompose J_0 into two Hankel functions $H_0^{(1)}$ and $H_0^{(2)}$ and draw the integral containing the former into the upper, and the one containing the latter into the lower half-plane. (The integrals along the two large quarter circles in the first and fourth quadrants do not contribute, as can be shown readily.) Each integral of the m series thus becomes the sum of an integral from zero to $i\infty$ containing $H_0^{(1)}$ and a second one from zero to $-i\infty$ with $H_0^{(2)}$. Due to the circulation relation

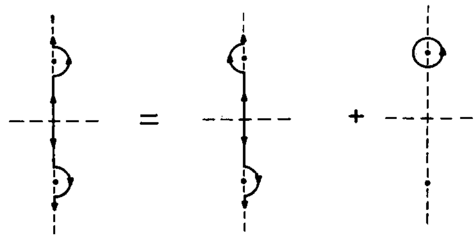

FIG. 2. Decomposition of the path of complex integration for the potential.

$$H_0^{(1)}(iz) = -H_0^{(2)}(-iz),$$

we cancel these integrals, as soon as the paths become symmetrical with respect to the origin. As the "pictorial equation" Fig. 2 shows this can be achieved while, at the same time, the residues at the poles along the positive imaginary axis have to be taken into account. The result,

$$\frac{\phi}{\phi_0} = \left(\frac{2\epsilon}{\pi}\right)^{\frac{1}{2}} 4\pi \sum_m (-1)^m A_{2m} \sum_{n=0}^\infty (-1)^n H_0^{(1)}[i\sigma\epsilon(2n+1)\pi] \times \frac{J_{2m+\frac{1}{2}}[i\epsilon(2n+1)\pi]}{[i(2n+1)\pi]^{\frac{1}{2}}} \sin \tfrac{1}{2}(2n+1)\pi\zeta,$$

although a double series, should be very useful for the numerical calculation in the space outside the cylinder formed by the two plates $\sigma > 1$, where it converges rapidly. The appearance of the imaginary unit is only apparent. No corresponding expression for $\sigma < 1$, i.e., $\rho < a$ seems to exist.

6. THE NORMAL DERIVATIVE FOR $z = a$

A more detailed calculation of the normal derivative nets us the surface charge on the plates and also serves as a check on the fulfillment of the boundary condition for $\rho > c$. We have, from (2) and (6a) and using (9) and (10).

$$\left(\frac{\partial(\phi/\phi_0)}{\partial\zeta}\right)_{\zeta=1} = \frac{\epsilon}{\pi} \int_0^\infty \mu\, d\mu J_0(\epsilon\sigma\mu) \times \int_0^1 dx \cos \epsilon\mu x \sum A_{2m} P_{2m}(x).$$

The x integration can be performed and leads to spherical Bessel functions, so that we have

$$\left(\frac{\partial\phi/\phi_0}{\partial\zeta}\right)_{\zeta=1} = \frac{1}{(2\pi\epsilon)^{\frac{1}{2}}} \sum A_{2m} \times \int_0^\infty \mu_1^{\frac{1}{2}}\, d\mu_1 J_0(\sigma\mu_1) J_{2m+\frac{1}{2}}(\mu_1),$$

where, for the sake of simplicity, $\mu_1 = \epsilon\mu$ is introduced as variable of integration. The integrals which appear here belong to the family of the Sonine–Schafheitlin discontinuous integrals.[6] They all represent different hypergeometric functions of σ, according as this variable is less or greater than one. What is of interest to us here is that for $\sigma > 1$ they all vanish, so that we have the result

$$[\partial(\phi/\phi_0)/\partial\zeta]_{\zeta=1} = 0, \quad \rho > c.$$

It is therefore seen that our form of solution does indeed satisfy the boundary condition outside the disks.

On the disks, i.e., for $\sigma < 1$ the reduction of these integrals to hypergeometric series would not constitute a particular advantage, were it not for the fact that these series can all be summed and reduced to Jacobi polynomials.[7] The result for the normal derivative is therefore, for $\sigma < 1$,

$$\left(\frac{\partial(\phi/\phi_0)}{\partial\zeta}\right)_{\zeta=1} = \frac{1}{2\pi\epsilon}\sum_{m=0}^{\infty} 2^{2m}\frac{(m-1)!}{(2m-1)!}A_{2m} \times \left(\frac{d}{d\tau}\right)^m [\tau^m(1-\tau)^{2m-1/2}], \quad (13)$$

where $\tau = \sigma^2$. For $m = 0$, unity should be substituted for the quotient of the two factorials. The ratio of Eq. (13) to Eq. (14) is plotted in Fig. 3.

To calculate the total charge, or for the current problem, the reciprocal resistance, the normal derivative has to be integrated over the disk surface. Because of the appearance of the m fold derivative with respect to $\tau = \sigma^2$, it is immediately seen that the contributions of all series terms with $m > 0$ vanish. The result is

$$\int_0^1 \sigma\, d\sigma \left(\frac{\partial(\phi/\phi_0)}{\partial\zeta}\right)_{\zeta=1} = \frac{1}{2\pi\epsilon}A_0. \quad (14)$$

7. THE PROBLEM OF IMPERFECT INFINITE ELECTRODES

Case I: A Single Plate

The family of inverse problems to those just solved presents interesting aspects. We consider first the case of a single infinite plate with a circular hole in it in a partially conducting medium, the hole being closed by a nonconductor. Here, the boundary conditions are

$$\phi = 0, \quad \rho > c, \quad z = 0,$$
$$\partial\phi/\partial z = 0, \quad \rho < c, \quad z = 0.$$

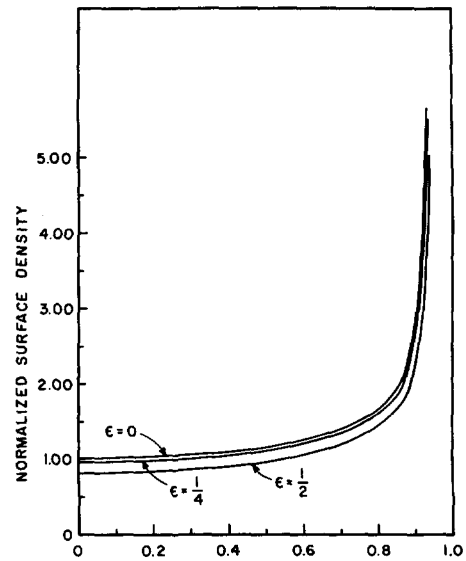

FIG. 3. The surface density of charge on either disk of the two-disk problem.

We introduce

$$\phi = -kz + \omega.$$

Then the new function ω fulfills the boundary conditions

$$\omega = 0, \quad \rho > c, \quad z = 0,$$
$$(\partial\omega/\partial z) = +k, \quad \rho < c, \quad z = 0.$$

This time, we propose to find the function $\partial\omega/\partial z$, and to make it satisfy the condition outside automatically. We assume that

$$\frac{\partial\omega}{\partial z} = \frac{2}{\pi}\int_0^\infty d\lambda e^{-\lambda z}J_0(\rho\lambda)A(\lambda)$$

with the new *Ansatz* that

$$A(\lambda) = \lambda\int_0^c \sin\lambda\xi f(\xi)\, d\xi.$$

Once again we rename ρ as η, and now operate on both sides with a new choice of operator, namely:

$$\frac{1}{\rho}\frac{d}{d\rho}\int_0^\rho \eta\, d\eta(\rho^2 - \eta^2)^{\frac{1}{2}}.$$

Then

$$k\rho = \frac{2}{\pi\rho}\frac{d}{d\rho}\int_0^\rho \eta\, d\eta(\rho^2-\eta^2)^{\frac{1}{2}} \times \int_0^\infty \lambda\, d\lambda J_0(\lambda\eta)\int_0^c \sin\lambda\xi f(\xi)\, d\xi.$$

The integral over η can be performed by use of the discontinuous integrals of Weber and Schafheitlin. It yields

$$k\rho = \frac{2}{\pi\rho}\frac{d}{d\rho}\int_0^\infty \lambda\, d\lambda\left((\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{\rho^{\frac{3}{2}}}{\lambda^{\frac{3}{2}}}J_{\frac{3}{2}}(\rho\lambda)\right)\int_0^c \sin\lambda\xi f(\xi)\, d\xi.$$

[6] N. Nielsen, *Handbuch der Theorie der Cylinderfunktionen* (B. G. Teubner, Leipzig, 1905), formulas (4) and (11) of Sec. 74, p. 191 *et seq.* See also G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1958), Sec. 13.4, Eq. (2), p. 401.

[7] See R. Courant and D. Hilbert, *Methoden der Mathematischen Physik* (Julius Springer, Leipzig, 1924), Vol. I, p. 74.

The λ integral, after interchanging with ξ integration, can be reduced to

$$\frac{\rho^{\frac{1}{2}}}{\xi^{\frac{1}{2}}}\int_0^\infty d\lambda J_{\frac{3}{2}}(\rho\lambda)J_{\frac{1}{2}}(\xi\lambda),$$

and this is one of the generalized Dirichlet factors whose value is 1 or 0 according as $\xi < \rho$ or $\xi > \rho$. Therefore, the equation reduces to

$$k\rho = \frac{1}{\rho}\frac{d}{d\rho}\int_0^\rho \xi f(\xi)\,d\xi,$$

the solution of which is $f(\xi) = k\xi$. When this is returned to the equation for $A(\lambda)$, the ξ integration can be performed, yielding

$$\partial\omega/\partial z = (\tfrac{1}{2}\pi)^{\frac{1}{2}}c^{\frac{3}{2}}k\int_0^\infty e^{-\lambda z}J_0(\rho\lambda)J_{\frac{3}{2}}(\lambda c)\lambda^{\frac{1}{2}}\,d\lambda.$$

Hence

$$\omega = -(\tfrac{1}{2}\pi)^{\frac{1}{2}}c^{\frac{3}{2}}k\int_0^\infty e^{-\lambda z}J_0(\rho\lambda)J_{\frac{3}{2}}(\lambda c)\frac{d\lambda}{\lambda^{\frac{1}{2}}},$$

and it can be shown that this integral does indeed have the desired property that it is zero for $\rho > c$, $z = 0$.

Case II. Two Plates

The second case is that of two electrodes with opposite holes closed with a nonconductor. Here, the generalization employed in moving from the one-disk problem to the two-disk problem can be carried out again. We again imagine an infinite set of image plates so charged that our problem repeats between each pair of plates. As in the previous case, we note the need for a uniform field in the case of infinite plates, and let

$$\phi = -kz + \omega.$$

Then the boundary conditions are

$$z = a,\quad \rho < c,\quad (\partial\phi/\partial z) = 0,\quad (\partial\omega/\partial z) = k, \tag{15a}$$

$$z = -a,\quad \rho < c,\quad (\partial\phi/\partial z) = 0,\quad (\partial\omega/\partial z) = k, \tag{15b}$$

$$z = a,\quad \rho > c,\quad \phi = \phi_0,\quad \omega = \phi_0 + ka, \tag{15c}$$

$$z = -a,\quad \rho > c,\quad \phi = -\phi_0,\quad \omega = -\phi_0 - ka. \tag{15d}$$

If now we choose $\phi_0 = -k/a$, then $\omega = 0$ at $z = \pm a$.

An examination of the possibilities shows that the solution of the Laplace equation which fits the need to have the function be antisymmetrical, and zero at $z = \pm a$, while its derivative is symmetrical over the region $-a \leq z \leq a$ is

$$\frac{\partial\omega}{\partial z} = \frac{2}{\pi}\int_0^\infty d\lambda\,\frac{\cosh\lambda z}{\sinh\lambda a}J_0(\rho\lambda)A(\lambda) \tag{16}$$

and

$$\omega = \frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\frac{\sinh\lambda z}{\sinh\lambda a}J_0(\rho\lambda)A(\lambda).$$

Following the method of the single plate case, we search for an equation for $A(\lambda)$ which will satisfy (15a) and (15b). This leads to the integral equation

$$f(\rho) + \frac{4}{\pi}\int_0^a d\xi f(\xi)K_2(\rho\xi) = -k\rho,$$

where

$$K_2(\rho\xi) = \int_0^\infty d\lambda\,\frac{\sin\lambda\rho\,\sin\lambda\xi}{e^{2a\lambda}-1}$$

is again a symmetric kernel with properties analogous to those of the two-disk kernel.

Once more, with the introduction of the dimensionless variables used before, the integral equation can be solved in terms of Legendre polynomials, this time of odd order. Letting

$$F(x) = \sum_0^\infty B_{2m+1}P_{2m+1}(x),$$

one finally obtains the formal solution

$$\omega(\rho, z) = \left(\frac{2c}{\pi}\right)^{\frac{1}{2}}k\sum_{m=0}^\infty(-1)^m B_{2m+1} \times\int_0^\infty \frac{d\lambda}{\lambda^{\frac{1}{2}}}\frac{\sinh\lambda z}{\sinh\lambda a}J_0(\lambda\rho)J_{2m+\frac{3}{2}}(\lambda c).$$

When $z = \pm a$, the integral over λ is a discontinuous function for all m, which has a zero value for $\rho > c$, so that the boundary conditions (15c) and (15d) are fulfilled identically.

8. CONCLUSION

The application of this method to hydrodynamical problems of interest such as counterflowing liquids that might be present in heat exchangers or reaction cells is evident. Our method has certain points in common with a paper by Sommerfeld[8] in which he employs a series of discontinuous integrals as the starting point for the solution of the problem of an oscillating disk.

[8] A. Sommerfeld, Ann. Physik **42**, 389 (1943).