THE UNIVERSITY OF MICHIGAN
OFFICE OF RESEARCH ADMINISTRATION
ANN ARBOR

ON CERTAIN STRATEGIES OF SIGNAL DETECTION USING CLIPPER CROSSCORRELATOR

(Single Signal Size)

Technical Report No. 128
3674-1-T

Cooley Electronics Laboratory
Department of Electrical Engineering

By: G. P. Patil
P. Cota

Approved by: T. G. Birdsall

ORA Project 03674

Contract No. Nonr-1224(36)
Office of Naval Research
Department of the Navy
Washington 25, D. C.

October 1962

TABLE OF CONTENTS

		Page
LIST OF ILLUSTRATIONS		iv
FOREWORD AND BACKGROUND		v
ABSTRACT		vi
ACKNOWLEDGMENT		vi
1.	INTRODUCTION	1
2.	BINOMIAL STRATEGY OF SIGNAL DETECTION (BS)	3
3.	INVERSE BINOMIAL STRATEGY OF SIGNAL DETECTION (IBS)	6
4.	EQUIVALENCE OF BS AND IBS	8
5.	EFFICIENCY OF THE CLIPPER CROSSCORRELATOR	9
6.	MODIFIED BINOMIAL STRATEGY OF SIGNAL DETECTION (MBS)	11
7.	MODIFIED INVERSE BINOMIAL STRATEGY OF SIGNAL DETECTION (MIBS)	13
8.	EQUIVALENCE OF MBS AND MIBS	14
9.	TABLES	15
10.	A SEQUENTIAL STRATEGY	27
	10.1 Sequential Probability Ratio Test	27
	10.2 SPRT when γ, δ are Determined by Wald's Approximation	29
	10.3 ASN for the SPRT	32
	10.4 Comparison of SPRT with MBS	33
11.	CONCLUDING COMMENTS AND REMARKS ABOUT FURTHER WORK	34
REFERENCES		35
DISTRIBUTION LIST		36

LIST OF ILLUSTRATIONS

Figure	Title	Page
1	Values of efficiency for C. C. C. (Based on Table I, using $\bar{n}$.)	21
2(a)	Efficiency of C. C. C. (BS) using actual α, β, n.	25
2(b)	Efficiency of C. C. C. (MBS) using actual α, β, n, under signal plus noise.	25
3	Termination boundaries for clipper crosscorrelator.	30
4	Range of α and β when $\gamma = \frac{\beta_o}{1-\alpha_o}$ and $\delta = \frac{1-\beta_o}{\alpha_o}$.	30
5	Termination boundaries for p = 0.67, α = 0.00098, and β = 0.09871.	33

FOREWORD AND BACKGROUND

The problem of evaluating the performance and the efficiency of the type of detection receiver known as a clipper crosscorrelator has been studied by a number of people in the acoustics and engineering fields. The majority of these studies determine the signal-to-noise ratio at the output of the receiver. By the very nature of the problems considered in these studies a great many approximations are usually made.

In this report Dr. Patil studies a specific detection situation and treats the performance of the receiver in detail, rigorously, and without approximations (beyond the assumption that the input samples are independent). The correlator studied crosscorrelates against a local reference signal.

Four versions of operation of the clipper crosscorrelator are considered. The standard operation of a detection device, integrating the receiver input over a fixed time, is studied under the title of "binomial strategy." A variation of this called the "inverse binomial strategy" operates the receiver accumulator until a fixed threshold is exceeded. Since both of these operations involve observing the output of an accumulator which has a nonnegative input, modifications are in order which quicken the time of decision when the decision is a foregone conclusion. Such quickening, or decreasing the time necessary to reach a decision, is done without effect on the primary measures, the error probabilities, and leads to an increased measure of efficiency.

The results of these studies are presented in both tabular form and graphs. These were calculated by Mr. Cota, who has also added a final section for comparison, which treats the output of the clipper crosscorrelator with a double threshold comparator, following the techniques of Wald's sequential analysis. This section has been added to show the following comparison: in the binomial strategy, inverse binomial strategy, and their modifications the error limits were considered as primary objectives, and the time necessary to reach a satisfactory decision was considered a secondary objective; that is, time was minimized only if its minimization did not affect the error probabilities anticipated on an observation-by-observation basis. In sequential analysis the three variables are considered as primary variables, though not of equal weight, and the average time is minimized subject to the over-all or average error probabilities.

T. G. Birdsall

ABSTRACT

We consider in this report the problem of signal detection using clipper crosscorrelator when the signal of single size and the Gaussian noise are known exactly. We develop strategies in order to meet the requirements dictated by the gravity of the "false alarm" and of the "miss." Four such strategies are suggested which arise in a very natural way, and their interrelations are studied. Efficiency of the clipper crosscorrelator in relation to the usual crosscorrelator is defined and investigated in the set-up as described. Associated tables and charts are given.

ACKNOWLEDGMENT

The authors wish to take this opportunity to express their sincere thanks to Mr. T. G. Birdsall, at whose insistence these investigations in the area of signal detection were started and with whom they have had several instructive discussions. The authors' thanks are due to him also for the fitting foreword that he has very kindly and willingly written for the investigations and results contained in this report.

1. INTRODUCTION

As mentioned in the foreword and background, we consider in this report the problem of signal detection using clipper crosscorrelator when the signal of single size is known exactly. Without loss of generality let the signal size s be positive. The general problem of signal detection is to decide the absence or presence of a possible signal on the basis of a certain number of observations made with, possibly, some noise in the background. Let the random sample of size n of independent observations be $X_1, X_2, \ldots X_i, \ldots X_n$. Under usual assumptions, and under noise alone, let $X_i \sim \eta(0, 1)$; i.e., X_i is normal with zero mean and unit standard deviation when noise alone is operating. Further let it be assumed that under signal plus noise $X_i \sim \eta(s, 1)$; i.e., X_i is normal with mean value s and standard deviation one when signal of positive size s is present in addition to noise.[1]

Now, the clipper crosscorrelator is a device which, instead of recording the magnitude of each observation X_i, records for purposes of simplicity only the count $c(X_i)$ of the observation X_i. To be specific,

$$\begin{aligned} c(X_i) &= 1 \qquad \text{if } X_i > 0 \\ &= 0 \qquad \text{if } X_i \leq 0 \end{aligned}$$

Using these unit and zero counts as basic sample data, strategies can be developed for signal detection purposes. As is well known, the solution to a dichotomous statistical decision problem traditionally involves the recognition and reconciliation to the two types of errors known as α-error and β-error. In the problem under consideration α-error takes the form of "false alarm" and the β-error means "miss." The sizes α and β of the α-error and β-error in making decisions based on a strategy are measured by the chances of committing such errors under such a strategy. Different strategies can imply different sizes of the α-error and the β-error, thus bringing out more effective or less effective roles of the different strategies.

---

[1]The physical interpretation is that s^2 is the signal-to-noise ratio at the input to the clipper.

Depending on the gravity of the situation for the problem at hand one may specify the sizes of "false alarm" and "mistaken miss." On the basis of the observable counts $c(X_1)$, $c(X_2) \ldots$ $c(X_i) \ldots$ recorded by the clipper crosscorrelator we suggest in this report certain strategies of signal detection which can meet the "false alarm" and "mistaken miss" requirements, and further consider the interrelations between such strategies.

2. BINOMIAL STRATEGY OF SIGNAL DETECTION (BS)

As an example, one can think of the following strategy, which makes use of the total count $C = \sum_{i=1}^{n} c(X_i)$ obtained from a sample of size n. The strategy requires specification of a "detection count" d to make the following decisions:

(i) if the total count $C \geq d$, conclude that the signal is present;

(ii) if the total count $C < d$, conclude that the signal is absent.

We propose to call such a strategy a binomial strategy of signal detection (BS) for reasons which will be apparent in the course of the following discussion.

The main problem involved in the BS is the problem of choosing suitable sample size n and the corresponding detection count d. As has been stated before, n and d are chosen so as to meet the α and β requirements.

Note that under noise alone for whatever n may be

$$\text{Prob}\left\{c(X_i) = 1\right\} = \text{Prob}\left\{X_i > 0 \text{ under noise alone}\right\}$$
$$= \frac{1}{2} \qquad i = 1, 2, \ldots, n.$$

Also $X_1, X_2, \ldots X_n$ are independent. Therefore under noise alone

$$C \sim B(n, \tfrac{1}{2})$$

i. e., the total count C based on a random sample of size n is a binomial random variable with parameters n and $p = \frac{1}{2}$. To be specific,

$$\text{Prob}\left\{C = r\right\} = \binom{n}{r}\left(\tfrac{1}{2}\right)^r\left(\tfrac{1}{2}\right)^{n-r} \qquad r = 0, 1, 2 \ldots, n.$$

Further, under signal plus noise with a positive signal of size s,

$$\text{Prob}\left\{c(X_i) = 1\right\} = p\left\{X_i > 0 \text{ under signal plus noise}\right\}$$
$$= \Phi(s) = p \tag{1}$$

where: $\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\frac{t^2}{2}} dt.$

Also, $X_1, X_2 \ldots X_n$ are independent as before. Therefore under signal plus noise

$$C \sim B(n,p)$$

i.e., the total count C is now a binomial random variable with parameters n and p. To be specific,

$$\text{Prob}\left\{C = r\right\} = \binom{n}{r} p^r(1-p)^{n-r} \qquad r = 0, 1, 2, \ldots, n.$$

Note that $p > \frac{1}{2}$ under signal plus noise whereas $p = \frac{1}{2}$ under noise alone.

Further let

$$\text{Prob}\left\{C \geq r\right\} = \sum_{k=r}^{n} \binom{n}{k} p^k(1-p)^{n-k} \qquad r = 0, 1, 2, \ldots, n$$

$$= B(n, r, p) \tag{2}$$

As a consequence of BS one can see that the size of false alarm

$$\alpha = \text{Prob}\left\{C \geq d \text{ under noise alone}\right\}$$
$$= B(n, d, \tfrac{1}{2}). \tag{3}$$

Similarly the size of a miss is obtainable as

$$\beta = \text{Prob}\left\{C < d \text{ under signal plus noise}\right\}$$
$$= 1 - B(n, d, p) \tag{4}$$

where p is given by (1).

Thus, one has the following two equations:

$$B(n, d, \tfrac{1}{2}) = \alpha \tag{5}$$

$$B(n, d, p) = 1 - \beta \tag{6}$$

to be solved for sample size n and detection count d for specified false alarm of size α and miss of size β. The solution of (5) and (6) is the pair of n and d required for BS corresponding to specified α and β. Table I for n and d for different sets of triples of α, β and $p = \Phi(s)$ extends over the range of detection interest; $.01 \leq \beta \leq .90$, $10^{-7} \leq \alpha \leq 10^{-2}$.

3. INVERSE BINOMIAL STRATEGY OF SIGNAL DETECTION (IBS)

It is possible to think of some "tolerance count δ," and to go on recording the individual counts $c(X_i)$ on the clipper crosscorrelator until the observed over-all count $\sum_i c(X_i)$ accumulates to δ. Let the number of individual counts (sample size) required to accumulate the tolerance count δ be denoted by R. As might be expected, one can think of the following strategy for signal detection. Specifying some "tolerance sample size" by η, one makes the following decisions:

(i) if the required sample size $R \leq \eta$, conclude that the signal is present;

(ii) if the required sample size $R > \eta$, conclude that the signal is absent.

We propose to call such a strategy an Inverse Binomial Strategy of Signal Detection (IBS) for reasons which will be apparent in the course of the following discussion.

The main problem involved in the IBS is the problem of choosing a suitable tolerance count δ and the corresponding tolerance sample size η. As has been stated in the introduction, δ and η are chosen so as to meet the α and β requirements.

Note that as before the probability of a unit count is $p = p(s)$ under signal plus noise and $p = \frac{1}{2}$ under noise alone. Individual counts are independent. Therefore under signal plus noise

$$R \sim d\,B(\delta, p)$$

i. e., the sample size required to accumulate the over-all count to δ is an inverse binomial random variable with parameters δ and p. To be specific,

$$\text{Prob}\left\{R = r\right\} = \binom{r-1}{\delta-1} p^{\delta}(1-p)^{r-\delta} \qquad r = \delta, \delta+1, \ldots$$

where: $p = \Phi(s)$.

Similarly, one has under noise alone

$$R \sim d\,B(\delta, \tfrac{1}{2})$$

Further let

$$\text{Prob}\left\{ R \leq r \right\} = \sum_{x=\delta}^{r} \binom{x-1}{\delta-1} p^{\delta}(1-p)^{x-\delta}$$

$$= I(\delta, r, p) \tag{7}$$

As a consequence of IBS one can see that the size of false alarm

$$\alpha = \text{Prob}\left\{ R \leq \eta \text{ under noise alone} \right\}$$
$$= I(\delta, \eta, \tfrac{1}{2})$$

and the size of mistaken miss

$$\beta = \text{Prob}\left\{ R > \eta \text{ under signal plus noise} \right\}$$
$$= 1 - I(\delta, \eta, p)$$

where: $p = \Phi(s)$.

Thus, one has the following two equations

$$I(\delta, \eta, \tfrac{1}{2}) = \alpha \tag{8}$$

$$I(\delta, \eta, p) = 1 - \beta \tag{9}$$

to be solved for δ and η for specified false alarm of size α and miss of size β. The solution of (8) and (9) is the pair of δ and η required for IBS corresponding to specified α and β.

4. EQUIVALENCE OF BS AND IBS

As is apparent, BS and IBS are quite different in approach and character. The following observation, however, brings out the connection between the parameters n and d of BS and the parameters δ and η of IBS and, later, the equivalence of the two strategies. Note that the event E_1 of requiring more than η counts to accumulate the over-all count of δ is equivalent with the event E_2 of obtaining a total count of less than δ in η counts. Therefore

$$\text{Prob}(E_1) = \text{Prob}(E_2)$$

i. e.,

$$1 - I(\delta, \eta, p) = 1 - B(\eta, \delta, p)$$

therefore

$$I(\delta, \eta, p) \equiv B(\eta, \delta, p) \qquad (10)$$

where: I and B are defined by (7) and (2), respectively.

Equation 10 is a very interesting identity and was established by Patil (1960) in a slightly different form. Using this identity it can very easily be seen that

$$\delta = d \qquad \text{and} \qquad \eta = n \qquad (11)$$

where BS and IBS are derived to meet the same α and β requirements. This follows immediately on comparing the pair of equations (5) and (6) with the pair (8) and (9). This shows that tabulation for the parameters of either BS or IBS is enough; there is no need of two separate tables for BS and IBS.

In fact a much stronger relation can be established between BS and IBS. We can show that they are equivalent; i. e., for every possible sample data on the clipper crosscorrelator both BS and IBS come to an identical conclusion regarding presence or absence of the signal. It is very curious to observe that, though totally different in outlook and character, BS and IBS turn out to be equivalent strategies. That they are equivalent easily follows from the equivalence of the events E_1 and E_2 mentioned in this section, together with the established result that δ=d and η = n.

Although equivalent, the BS and IBS differ from one another in one major aspect in practice. BS requires that a fixed number n of counts be taken, whereas the number of counts to be made in order to apply IBS is a random quantity.

5. EFFICIENCY OF THE CLIPPER CROSSCORRELATOR

It is evident that the clipper crosscorrelator does not utilize the entire information on an individual observation. It makes either a zero or a unit count on it, depending on whether the observation is positive or nonpositive, and ignores the magnitude of the observation. Naturally we expect that it be less "efficient" than the crosscorrelator which does take into account the magnitude of each observation. In other words, in order to meet the same α and β requirements, the sample size n required for the clipper crosscorrelator is expected to be larger than the sample size N required for the crosscorrelator. We define the efficiency of the clipper crosscorrelator by the ratio of N to n.

From the discussion in Section 2, we know how to obtain the value of n for some α and β requirement. But how do we obtain the value of N? Following the usual analysis with the crosscorrelator we derive here an expression for N. Table I lists N, and the efficiency of the clipper crosscorrelator for different sets of values of the triple α, β and p.

Now, on the basis of the sample $X_1, X_2, \ldots, X_N$ of size N, the crosscorrelator uses the sample mean $\overline{X}_N$ as the statistic. Choosing a "detection point" D, the crosscorrelator has the following strategy for signal detection:

(i) if the sample mean $\overline{X}_N \geq D$, conclude that the signal is present;

(ii) if the sample mean $\overline{X}_N < D$, conclude that the signal is absent.

The sample size N and the detection point D are chosen so as to meet the α and β requirement.

Note that $\overline{X}_N \sim \eta(0, \frac{1}{\sqrt{N}})$ under noise alone

whereas $\overline{X}_N \sim \eta(s, \frac{1}{\sqrt{N}})$ under signal plus noise

where: s is taken to be positive without loss of generality.

As a consequence, the size of the false alarm

$$\alpha = \text{Prob}\left\{ \overline{X}_N \geq D \text{ under noise alone} \right\}$$

$$= 1 - \Phi(D\sqrt{N})$$

and the size of the miss

$$\beta = \text{Prob}\left\{ \overline{X}_N < D \text{ under signal plus noise} \right\}$$

$$= \Phi\left[(D - s)\sqrt{N}\right]$$

where: $$\Phi(\infty) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt.$$

Thus one has the following two equations

$$\Phi(D\sqrt{N}) = 1 - \alpha$$

$$\Phi\left[(D - s)\sqrt{N}\right] = \beta$$

These are the parametric equations of the normal ROC curve. They can be solved for sample size N and detection point D for specified α and β. Writing F_ϵ for the solution of $\Phi(x) = \epsilon$, the equations become

$$D\sqrt{N} = F_{1-\alpha} \quad (12)$$

and

$$(D - s)\sqrt{N} = F_\beta \quad (13)$$

Subtracting (13) from (12), we have

$$S\sqrt{N} = F_{1-\alpha} - F_\beta$$

therefore

$$N = \left(\frac{F_{1-\alpha} - F_\beta}{s}\right)^2 = \left(\frac{F_\alpha + F_\beta}{s}\right)^2 \quad (14)$$

because $F_{1-\epsilon} = -F_\epsilon$. One may note that F_ϵ is nothing but the ϵ-point of the standard normal distribution for which extensive tables are available.

Incidentally, D can be obtained as

$$D = \frac{F_{1-\alpha}}{\sqrt{N}} \quad (15)$$

6. MODIFIED BINOMIAL STRATEGY OF SIGNAL DETECTION (MBS)

The following discussion will bring out that there is scope for improvement in the BS by taking in practice only as many counts on the clipper crosscorrelator as are essential for the purpose of making the decision on presence or absence of the signal. For example, if one finds that the first d counts are all unit counts, the conclusion under BS of the presence of the signal is clear and certain, and there is no need to observe any more counts, although such an observation would be demanded by sample size n under BS. For that matter, one can stop taking sample data as soon as one has secured an over-all count of d and conclude the presence of the signal as under BS, even if one has not yet exhausted all the required n counts demanded by BS.

Similarly, if one finds that the first n - d+1 counts are all zero counts, the conclusion under BS of the absence of the signal is clear and inevitable. There is no need to observe any more counts because there is no possibility that the over-all count will become even d, as is required for the contrary conclusion, even if the n counts required under BS were completed. For that matter, one can stop taking sample data as soon as one has secured n - d+1 counts, and conclude the absence of the signal, even if one has not yet exhausted all the n counts.

In view of the above discussion we propose the following strategy, to be called the Modified Binomial Strategy of Signal Detection (MBS).

(i) Choose n and d as required by the BS to meet α and β requirement.

(ii) Conclude that the signal is present as soon as the number of unit counts is d, and stop taking sample data.

(iii) Conclude that the signal is absent as soon as the number of zero counts is n-d+1, and stop taking sample data.

The advantage of MBS over BS lies in the curtailment effected in the total number of counts required to make a conclusive decision regarding presence or absence of the signal. Whereas sample size for BS is a fixed quantity n, sample size required for MBS is a random quantity. The Average Sample Number (ASN) can be seen to be

$$\text{ASN} = \sum_{y=d}^{n} y \cdot \binom{y-1}{d-1} p^d (1-p)^{y-d} + \sum_{y=c}^{n} y \cdot \binom{y-1}{c-1} (1-p)^c p^{y-c} \qquad (16)$$

where: $c = n-d+1$ and p is the probability of a count to be unit.

Note that

$$\sum_{y=d}^{n} y \cdot \binom{y-1}{d-1} p^d (1-p)^{y-d}$$

$$= \frac{d}{p} \sum_{y=d}^{n} \binom{y}{d} p^{d+1} (1-p)^{y-d}$$

$$= \frac{d}{p} I(d+1, n+1, p)$$

$$= \frac{d}{p} B(n+1, d+1, p) \qquad (17)$$

where I and B are defined by (7) and (2) respectively.

Similarly,

$$\sum_{y=c}^{n} y \cdot \binom{y-1}{c-1} (1-p)^c p^{y-c} = \frac{c}{1-p} B(n+1, c+1, 1-p) \qquad (18)$$

Therefore, as a consequence of MBS,

$$\text{ASN} = \frac{d}{p} B(n+1, d+1, p) + \frac{n-d+1}{1-p} B(n+1, n-d+a, 1-p) \qquad (19)$$

Substituting $p = \frac{1}{2}$ and $p = p(s)$ gives the value of ASN under noise alone and under signal plus noise, respectively.

7. MODIFIED INVERSE BINOMIAL STRATEGY OF SIGNAL DETECTION (MIBS)

One may discover that IBS can be improved on the same lines as BS has been improved to MBS. We propose the following strategy and call it the Modified Inverse Binomial Strategy of Signal Detection (MIBS).

(i) Choose η and δ as required by IBS to meet the α and β requirement.

(ii) Conclude that the signal is present as soon as the tolerance count δ accumulates, and stop taking sample data.

(iii) Conclude that the signal is absent as soon as the number of zero counts is $\eta-\delta+1$, and stop taking sample data.

The advantage of MIBS over IBS lies in the curtailment effected in the number of counts required in all to make a conclusive decision regarding presence or absence of the signal. It is easy to see that the sample size for both IBS and MIBS is a random quantity. It can be seen that

$$\text{ASN under IBS} = \sum_{r=\delta}^{\infty} r \cdot \binom{r-1}{\delta-1} p^{\delta} (1-p)^{r-\delta} = \frac{\delta}{p} \tag{20}$$

where: p is the probability that a count will be one.

One can show also that under MIBS

$$\text{ASN} = \sum_{r=\delta}^{n} r \cdot \binom{r-1}{\delta-1} p^{\delta} (1-p)^{r-\delta} + \sum_{r=\xi}^{n} r \cdot \binom{r-1}{\xi-1} (1-p)^{\xi} p^{r-\xi} \tag{21}$$

where: $\xi = \eta - \delta + 1$.

Proceeding on the same lines as in the previous section, we have under MIBS

$$\text{ASN} = \frac{\delta}{p} B(\eta+1,\ \delta+1,\ p) + \frac{\eta-\delta+1}{1-p} B(\eta+1, \eta-\delta+2,\ 1-p) \tag{22}$$

Substituting $p = \frac{1}{2}$ and $p = p(s)$ gives the value of ASN under noise alone and under signal plus noise, respectively.

8. EQUIVALENCE OF MBS AND MIBS

When we recall from Section 4 that

$$\delta = d \qquad \text{and} \qquad \eta = n$$

and compare equations (19) and (22), giving ASN under MBS and MIBS, it becomes clear that both MBS and MIBS have the same ASN.

In fact a much stronger relation between MBS and MIBS comes out immediately when we compare the descriptions of the MBS and MIBS as given in Sections 6 and 7. It is very clear, though very curious, to see that MBS and MIBS are equivalent. It may further be said that unlike BS and IBS, MBS and MIBS are identical in practice. The only place they differ in is the starting viewpoint.

9. TABLES

The following tables are calculated for:

α (false alarm probability) = $10^{-7}, 10^{-6}, 10^{-5}, \ldots, 10^{-2}$;

β (miss probability) = .9, .5, .1, .01;

p = .52, .54, .56, .59, .62, .67, .76, .84, .92, .98.

The triples (α, β, p) are chosen so that $10 < n < 1000$.

Two sets of tables were consulted, both of which yield α and β for given values of p, n, d.

"Tables of the Binomial Probability Distribution" (Ref. 4), is a set of 7-place tables tabulated for

$$n = 2(1)\,49; \qquad d = 1(1)\,n.$$

This set of tables was used for $\alpha = 10^{-5}, 10^{-6}, 10^{-7}$ whenever possible (i. e., whenever $n < 50$). Notice that these tables given only one-place accuracy for $\alpha = 10^{-7}$.

"Tables of the Cumulative Binomial Probability Distribution" (Ref. 5), is a set of five-place tables tabulated for

$$n = 1(1)\,50(2)\,100(10)\,200(20)\,500(50)\,1000, \quad d = 0(1)\,n.$$

This set of tables was used for $\alpha = 10^{-2}, 10^{-3}, 10^{-4}$, and, when $n \geq 50$, for $\alpha = 10^{-5}$. Notice that these tables give only one-digit accuracy for $\alpha = 10^{-5}$.

No values were computed for $\alpha = 10^{-6}, 10^{-7}$ when $n \geq 50$.

Table I. n, d, and η of C. C. C. (B. S.) for given α, β, p.

This table is calculated in the following manner. The desired values of p are all available in the tables. For a given α, β pair there are several values of n for which two values of d yield α, β pairs that "straddle" the given α, β pair. For example, if p = .67, α = .01, β = .5, the consulted table gives:

n	d	α	β
46	31	.01295	.45307
	32	.00568	.57754
47	31	.01999	.37349
	32	.00931	.49415

For each value of n, β and d are calculated (by linear interpolation) for $\alpha = .01$. In this case, for n = 46, $\beta(.01) = \overline{\beta} = .50357$, $d(.01) = \overline{d} = 31.406$. For n = 47, $\beta(.01) = \overline{\beta} = .48635$, $d(.01) = \overline{d} = 31.935$. Since the desired value of β is $\beta = .5$, we conclude that $46 < n < 47$, $d_1 = 31.406$, $d_2 = 31.935$. For the purposes of calculating $\eta = \frac{N}{n}$, we calculate (by linear interpolation) n for $\beta = .5$ and denote it by $\overline{n}$.

Linear interpolation is not used, however, for values where the consulted tables give only one-place accuracy, or where $\Delta n \geq 10$ (Δn being the difference between tabulated values of n). In these cases the values of n are given which yield the closest (straddling) α, β pairs.

Table I. n, d, and η of C. C. C. (B. S.) for given α, β, p.

p	β	α	$\hat{n}$	n or $\overline{n}$	η or η (BS)	d_1	d_2
.52	.9	10^{-2}	684	650	.635	355	356
				700	.636	381	382
.54	.9	10^{-4}	925	900	.630	506	507
				950	.638	532	533
	.9	10^{-3}	510	500	.634	285	
				550	.634	311	312
	.9	10^{-2}	170	170	.630	100	101
				180	.633	106	107
	.5	10^{-2}	841	800	.635	433	434
				850	.635	459	460
.56	.9	10^{-5}	615	(600)	(.618)	(353)	
				(650)	(.623)	(380)	
	.9	10^{-4}	410	400	.632	237	238
				420	.633	248	249
	.9	10^{-3}	226	220		133	134

p	β	α	$\hat{n}$	n or $\bar{n}$	η or η (BS)	d_1	d_2
				240	.629	144	145
	.9	10^{-2}	75	77.0	.621	48.7	49.8
	.5	10^{-4}	950	950	.635	532	533
				1000	.632	559	560
	.5	10^{-3}	659	650	.633	364	365
				700	.633	391	392
	.5	10^{-2}	374	360		202	203
				380	.633	213	214
	.1	10^{-2}	900	900	.634	485	486
				950		511	512
.59	.9	10^{-5}	270	(260)	(.664)	(164)	
				(260)	(.608)	(165)	
	.9	10^{-4}	180	180	.618	115	116
				190		121	
	.9	10^{-3}	99	100	.620	65	66
				110		71	72
	.9	10^{-2}	33	34.6	.608	24.3	24.9
	.5	10^{-5}	551	(550)	(.640)	(325)	
	.5	10^{-4}	418	420	.630	248	249
				440		259	260
	.5	10^{-3}	290	280		166	167
				300	.629	177	178
	.5	10^{-2}	164	160		95	96
				170	.628	24.3	24.9
	.1	10^{-5}	931	(950)	(.619)	(542)	
	.1	10^{-4}	758	750	.627	426	427
				800		453	
	.1	10^{-3}	579	500		311	312
				600	.631	338	339
	.1	10^{-2}	394	380		213	214
				400	.632	223	224
	.01	10^{-3}	888	850		470	471
				900	.633	496	497
	.01	10^{-2}	655	650	.634	355	356
				700		381	382
.62	.9	10^{-5}	150	(140)		(95)	(96)
				(150)	(.642)	(101)	
				(150)	(.517)	(102)	

p	β	α	$\hat{n}$	n or $\bar{n}$	η or η (BS)	d_1	d_2
				(160)		(107)	(108)
	.9	10^{-4}	100	100	.614	68	69
				110		74	75
	.9	10^{-3}	55	57.7	.607	39.9	41.2
	.9	10^{-2}	19	20.2	.578	15.7	16.3
	.5	10^{-5}	306	(320)	(.640)	(198)	
				(320)	(.607)	(199)	
	.5	10^{-4}	232	220		138	
				240	.624	149	150
	.5	10^{-3}	161	160	.624	99	100
				170		105	106
	.5	10^{-2}	91	92.3	.628	57.7	58.8
	.1	10^{-5}	517	(550)	(.616)	(326)	
	.1	10^{-4}	420	420	.628	248	249
				440		259	260
	.1	10^{-3}	322	320	.628	188	189
				340		198	199
	.1	10^{-2}	219	220	.628	127	128
				240		138	139
	.01	10^{-5}	730	(750)	(.626)	(434)	
	.01	10^{-4}	615	600	.629	346	
				650		372	373
	.01	10^{-3}	493	480		274	275
				500	.631	285	
	.01	10^{-2}	364	360	.630	202	203
				380		213	214
.67	.9	10^{-5}	72	(76)		(57)	
				(78)	(.594)	(58)	
	.9	10^{-4}	48	52.4	.586	39.7	40.9
	.9	10^{-3}	27	29.3	.577	23.2	23.9
	.5	10^{-5}	148	(150)	(.630)	(101)	
				(150)	(.579)	(102)	
	.5	10^{-4}	112	110		74	75
				120	.607	80	81
	.5	10^{-3}	78	81.4	.606	54.3	55.5
	.5	10^{-2}	44	46.2	.605	31.4	31.9
	.1	10^{-5}	250	(260)	(.639)	(164)	
				(260)	(.610)	(165)	
	.1	10^{-4}	203	200	.618	126	127
				220		138	

p	β	α	$\hat{n}$	n or $\bar{n}$	η or η (BS)	d_1	d_2
	.1	10^{-3}	155	150		94	95
				160	.620	99	100
	.1	10^{-2}	106	100		62.1	
				110	.589	67.7	
	.01	10^{-5}	352	(360)	(.579)	(221)	
	.01	10^{-4}	296	300	.622	182	183
				320		193	194
	.01	10^{-3}	238	240	.622	144	145
				260		155	156
	.01	10^{-2}	175	170		100	101
				180	.626	106	107
.76	.9	10^{-6}	38	45.4	.532	38.5	39.0
	.9	10^{-5}	28	(33.9)	(.526)	(29)	(29.5)
	.9	10^{-4}	19	23.4	.509	20.7	21.3
	.9	10^{-3}	11	13.3	.493	12.4	13.0
	.5	10^{-5}	57	(63.1)	(.578)	(48)	(49)
	.5	10^{-4}	44	48.8	.568	37.1	37.8
	.5	10^{-3}	30	34.1	.562	26.6	27.0
	.5	10^{-2}	17	18.8	.576	14.5	15.0
	.1	10^{-5}	97	(100)		(72)	
				(110)	(.574)	(78)	
	.1	10^{-4}	79	85.6	.586	59.4	60.7
	.1	10^{-3}	60	64.7	.593	44.8	45.9
	.1	10^{-2}	41	43.9	.595	29.6	30.2
	.01	10^{-5}	137	(150)	(.580)	(102)	
	.01	10^{-4}	116	120	.597	80	81
				130		86	87
	.01	10^{-3}	92	97.9	.601	63.6	64.7
	.01	10^{-2}	68	71.9	.603	45.2	46.4
.84	.9	10^{-7}	25	(32)	(.477)	(30)	
				(33)		(31)	
	.9	10^{-6}	20	27.4	.445	25.7	26.4
	.9	10^{-5}	15	20.4	.442	19.5	20.0
	.5	10^{-6}	36	45.2	.506	38.5	39.0
	.5	10^{-5}	29	36.3	.507	30.9	31.6
	.5	10^{-4}	22	27.5	.509	23.4	24.0
	.5	10^{-3}	16	19.4	.498	16.7	17.3
	.1	10^{-5}	49	(54)		(42)	
				(56)	(.553)	(44)	
	.1	10^{-4}	40	46.3	.547	35.9	36.6

p	β	α	$\hat{n}$	n or $\bar{n}$	η or η (BS)	d_1	d_2
	.1	10^{-3}	31	34.8	.556	26.4	27.0
	.1	10^{-2}	21	23.7	.556	17.6	18.2
	.01	10^{-5}	69	76		57	
				78	.566	58	
	.01	10^{-4}	58	65.6	.563	47.1	48.4
	.01	10^{-3}	47	52.4	.566	37.6	38.8
	.01	10^{-2}	36	37.9	.578	26.0	26.7
.92	.9	10^{-7}	12	(23)		(23)	
				(24)	(.354)	(24)	
	.5	10^{-7}	22	(33)	(.416)	(31)	
	.5	10^{-6}	18	27.9	.411	26.4	25.7
	.5	10^{-5}	15	24.4	.377	28.8	29.6
	.5	10^{-4}	11	17.1	.410	16.3	16.9
	.1	10^{-7}	33	(47)		(41)	
				(48)	(.455)	(42)	
	.1	10^{-6}	29	39.5	.467	34.2	34.9
	.1	10^{-5}	24	33.7	.446	28.8	29.6
	.1	10^{-4}	20	27.1	.467	23.4	24.0
	.1	10^{-3}	15	20.3	.478	17.3	17.9
	.1	10^{-2}	10	13.3	.497	11.1	16.8
	.01	10^{-5}	35	44.4	.496	36.3	36.9
	.01	10^{-4}	29	36.6	.505	29.5	30
	.01	10^{-3}	23	29.0	.512	23.2	23.9
	.01	10^{-2}	17	21.2	.517	16.3	16.9
.98	.5	10^{-7}	10	(22)		(22)	(23)
				(23)		(23)	(24)
				(24)	(.297)	(24)	
	.1	10^{-7}	16	(28)	(.350)	(27)	
				(29)		(28)	
	.1	10^{-6}	14	24.1	.356	23.4	24.0
	.1	10^{-5}	11	20.8	.349	19.5	20.0
	.01	10^{-7}	21	(36)		(33)	
				(37)	(.378)	(34)	

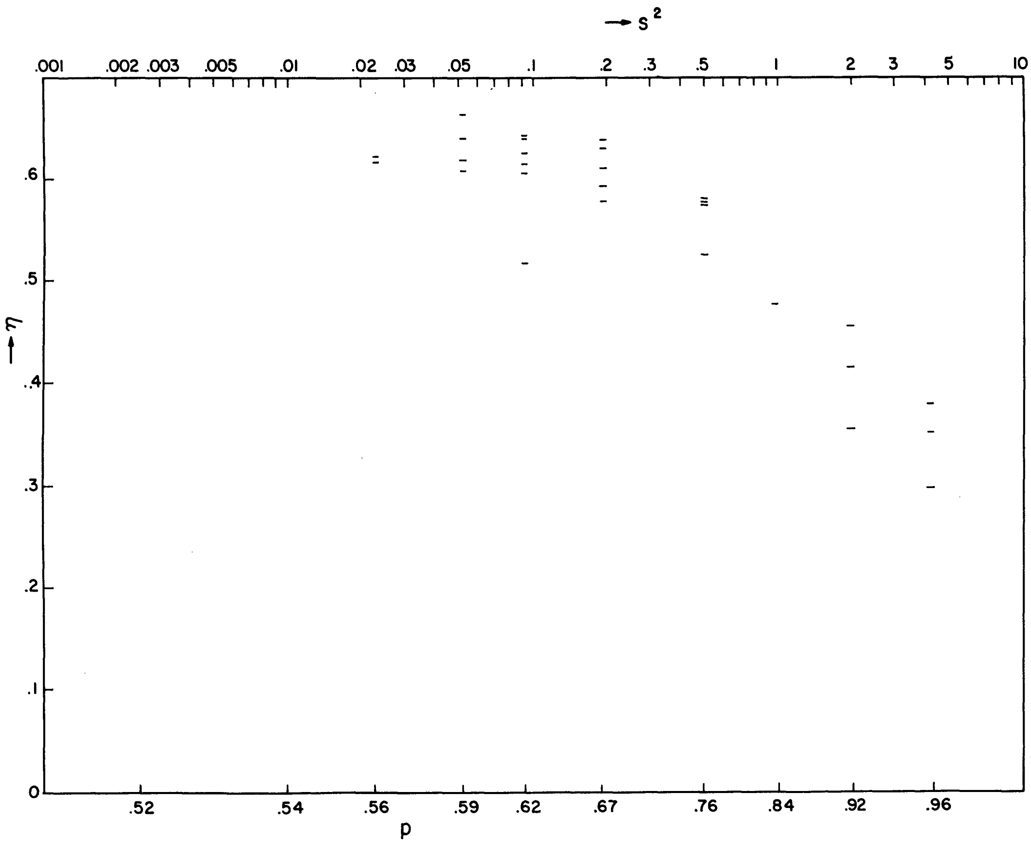

Fig. 1. Values of efficiency for C. C. C.
(Based on Table I, using $\bar{n}$.)

Table II. Efficiency of C. C. C. (BS and MBS) corresponding to α, β, p triples.

In this table, for each α, β, p triple, the value of n is selected which yields the α, β pair closest to the given pair. Then ASN is calculated from the tables by (19). The efficiencies are calculated according to the formulae:

$$\eta(\text{BS}) = \frac{N}{n}, \qquad \eta(\text{MBS}) = \frac{N}{\text{ASN}}.$$

Roughly speaking, η(BS) depends primarily upon β (inversely) and secondarily upon α. No such dependence was observed for η_{SN}(MBS), but for very small n, corresponding to the largest values of α and β, η(MBS) is very high. For η_N(MBS), $\text{ASN}_N = 2(n+1-d)$, to three-place accuracy (except for an occasional error of 1 in the third place).

Table II. Efficiency of C. C. C. (BS and MBS) corresponding to α, β, p triples.

P	β_o	α_o	n	n	d	α	β	ASN_{SN}	η (BS)	η (MBS)	η (MBS)
.52	.9	10^{-2}	684	650	355	.01029	.90249	614	.636	.674	.694
.54	.9	10^{-4}	925	950	533	.00009	.89796	906	.640	.671	.727
		10^{-3}	510	500	285	.00100	.90355	468	.634	.678	.730
		10^{-2}	170	180	106	.01030	.89301	162	.633	.704	.758
	.5	10^{-2}	841	850	459	.01075	.48591	828	.636	.653	.687
.56	.9	10^{-5}	615	600	352	10^{-5}	.89901	564	.653	.695	.787
		10^{-4}	410	420	249	.00008	.90473	389	.635	.686	.775
		10^{-3}	226	220	133	.00117	.89708	199	.630	.697	.785
		10^{-2}	75	78	50	.00843	.90861	65.1	.629	.754	.846
	.5	10^{-4}	950	950	533	.00009	.51251	926	.637	.654	.724
		10^{-3}	659	700	392	.00084	.48421	680	.634	.653	.718
		10^{-2}	374	380	213	.01043	.48681	366	.634	.658	.717
	.1	10^{-2}	900	900	486	.00895	.10718	865	.635	.661	.689
.59	.9	10^{-5}	270	260	164	10^{-5}	.89906	235	.664	.735	.890
		10^{-4}	180	190	121	.00010	.89284	169	.624	.701	.847
		10^{-3}	99	100	66	.00089	.90777	84.5	.625	.739	.893
		10^{-2}	33	35	25	.00834	.90888	26.3	.620	.825	.986
	.5	10^{-5}	551	550	325	10^{-5}	.49896	532	.640	.661	.779
		10^{-4}	418	440	260	.00008	.49497	424	.630	.654	.766
		10^{-3}	290	300	177	.00108	.47521	287	.631	.659	.763
		10^{-2}	164	160	95	.01079	.50448	151	.631	.669	.765
	.1	10^{-5}	931	950	542	10^{-5}	.10523	916	.619	.642	.719
		10^{-4}	758	750	426	.00011	.10367	720	.633	.659	.730
		10^{-3}	579	600	339	.00082	.09938	573	.633	.663	.725
		10^{-2}	394	400	224	.00933	.10221	378	.633	.670	.715
	.01	10^{-3}	888	900	497	.00096	.00990	842	.633	.677	.705
		10^{-2}	655	650	355	.01029	.01062	602	.634	.685	.696
.62	.9	10^{-5}	150	150	101	10^{-5}	.89728	130	.642	.741	.963
		10^{-4}	100	110	75	.00009	.89296	93.6	.611	.718	.933
		10^{-3}	55	56	40	.00092	.90760	44.1	.612	.778	1.008
		10^{-2}	19	20	16	.00591	.92739	12.8	.603	.941	1.206
	.5	10^{-5}	306	320	199	10^{-5}	.50275	307	.607	.633	.796
		10^{-4}	232	240	149	.00011	.48197	229	.625	.655	.815
		10^{-3}	161	160	100	.00098	.51688	151	.624	.662	.818
		10^{-2}	91	94	59	.00860	.47524	87.0	.681	.736	.889

P	β_o	α_o	n	n	d	α	β	ASN_{SN}	η (BS)	η (MBS)	η (MBS)
	.1	10^{-5}	517	550	326	10^{-5}	.08709	524	.616	.647	.753
		10^{-4}	420	420	248	.00012	.09778	398	.629	.664	.763
		10^{-3}	322	320	188	.00103	.10511	302	.629	.666	.756
		10^{-2}	219	220	128	.00904	.10867	205	.630	.677	.741
	.01	10^{-5}	730	750	434	10^{-5}	.00921	700	.626	.671	.740
		10^{-4}	615	600	346	10^{-4}	.01331	558	.629	.677	.739
		10^{-3}	493	500	285	10^{-3}	.00977	460	.631	.686	.727
		10^{-2}	364	380	214	.00790	.01018	345	.632	.696	.718
.67	.9	10^{-5}	72	78	58	10^{-5}	.89822	62.7	.594	.739	1.103
		10^{-4}	48	54	41	.00009	.89647	41.6	.590	.766	1.138
		10^{-3}	27	29	23	.00116	.88996	21.4	.590	.800	1.222
	.5	10^{-5}	148	150	101	10^{-5}	.49606	141	.630	.670	.945
		10^{-4}	112	120	81	.00008	.50333	112	.611	.655	.916
	10	10^{-3}	78	80	54	.00116	.48512	73.6	.614	.667	.910
		10^{-2}	44	47	32	.00931	.49415	42.3	.616	.685	.906
	.1	10^{-5}	250	260	165	10^{-5}	.10114	245	.610	.647	.826
		10^{-4}	203	220	138	.00010	.07896	205	.618	.664	.819
		10^{-3}	155	160	100	.00098	.09871	148	.621	.671	.814
		10^{-2}	106	110	68	.00837	.10538	101	.624	.679	.795
	.01	10^{-5}	352	360	221	10^{-5}	.01085	330	.618	.674	.795
		10^{-4}	296	300	183	.00008	.01235	273	.624	.686	.793
		10^{-3}	238	240	144	.00117	.00954	215	.625	.698	.772
		10^{-2}	175	180	106	.01030	.00918	158	.627	.714	.751
.76	.9	10^{-6}	38	46	39	.0000009	.8927683	32.4	.544	.773	1.564
		10^{-5}	28	35	30	10^{-5}	.87754	24.1	.551	.800	1.607
		10^{-4}	19	24	21	.00014	.86233	15.7	.540	.825	1.620
		10^{-3}	11	14	13	.00092	.88373	7.73	.528	.957	1.849
	.5	10^{-5}	57	64	49	10^{-5}	.47351	58.0	.588	.648	1.176
		10^{-4}	44	48	37	.00011	.49086	42.8	.577	.647	1.154
		10^{-3}	30	35	27	.00094	.47040	39.9	.580	.657	1.015
		10^{-2}	17	19	15	.00961	.49363	16.1	.586	.692	1.117
	.1	10^{-5}	97	110	78	10^{-5}	.08887	102	.574	.619	.957
		10^{-4}	0.79	86	61	.00007	.11160	79.2	.589	.639	.974
		10^{-3}	60	66	46	.00093	.09229	59.8	.598	.661	.940
		10^{-2}	41	44	30	.01131	.08558	38.9	.606	.686	.890
	.01	10^{-5}	137	150	102	10^{-5}	.01005	134	.580	.649	.888
		10^{-4}	116	130	87	.00007	.00720	114	.565	.645	.834
		10^{-3}	92	98	65	.00080	.01116	85.4	.606	.695	.873
		10^{-2}	68	70	45	.01123	.00959	59.2	.612	.724	.825

P	β_o	α_o	n	n	d	α	β	ASN_{SN}	η (BS)	η (MBS)	η (MBS)
.84	.9	10^{-7}	25	32	30	10^{-7}	.9052782	17.9	.477	.853	2.544
		10^{-6}	20	28	26	.0000015	.8479863	17.4	.479	.771	2.235
		10^{-5}	15	21	20	.0000105	.8715202	11.5	.469	.856	2.462
	.5	10^{-6}	36	46	39	.0000009	.4592471	40.6	.523	.592	1.504
		10^{-5}	29	35	30	.0000112	.4969924	30.1	.521	.606	1.520
		10^{-4}	22	28	24	.00009	.47195	24.0	.526	.614	1.473
		10^{-3}	16	20	17	.00129	.40100	17.3	.539	.623	1.348
	.1	10^{-5}	49	56	44	10^{-5}	.10206	51.5	.553	.601	1.191
		10^{-4}	40	46	36	.00008	.10685	42.0	.553	.606	1.156
		10^{-3}	31	35	27	.00094	.09542	31.5	.564	.626	1.097
		10^{-2}	21	25	19	.00732	.09204	22.1	.575	.650	1.029
	.01	10^{-5}	69	78	58	10^{-5}	.00965	69.0	.566	.639	1.051
		10^{-4}	58	64	47	.00011	.00996	55.9	.573	.656	1.019
		10^{-3}	47	54	39	.00075	.00868	46.4	.578	.672	.974
		10^{-2}	36	37	26	.01004	.01022	30.9	.589	.705	.910
.92	.9	10^{-7}	12	24	24	10^{-7}	.8648214	10.8	.354	.787	4.248
	.5	10^{-7}	22	33	31	10^{-7}	.4981990	27.0	.416	.508	2.288
		10^{-6}	18	28	26	.0000015	.3905792	24.0	.443	.517	2.067
		10^{-5}	15	25	23	.0000097	.5911838	22.0	.331	.376	1.379
		10^{-4}	11	18	17	.00007	.42812	14.8	.448	.545	2.017
	.1	10^{-7}	33	48	42	10^{-7}	.0859854	44.8	.455	.487	1.560
		10^{-6}	29	40	35	.0000007	.0967268	37.2	.475	.511	1.583
		10^{-5}	24	32	28	.0000097	.1084891	29.6	.480	.519	1.536
		10^{-4}	20	28	24	.00009	.06861	25.6	.495	.542	1.386
		10^{-3}	15	21	18	.00074	.08193	19.1	.504	.554	1.323
		10^{-2}	10	14	12	.00647	.09583	12.6	.520	.578	1.217
	.01	10^{-5}	35	45	37	.0000077	.0084345	40.2	.507	.568	1.267
		10^{-4}	29	37	30	.00010	.00788	32.6	.515	.584	1.191
		10^{-3}	23	30	24	.00072	.00825	26.0	.527	.608	1.129
		10^{-2}	17	22	17	.00845	.00637	18.5	.548	.652	1.007
.98	.5	10^{-7}	10	24	24	10^{-7}	.3842197	19.2	.297	.371	3.564
	.1	10^{-7}	16	28	27	10^{-7}	.1074660	26.4	.350	.371	2.450
		10^{-6}	14	25	24	.0000008	.0886451	23.6	.357	.378	2.231
		10^{-5}	11	21	20	.0000105	.0653488	19.9	.374	.394	1.964
	.01	10^{-7}	21	37	34	10^{-7}	.0062433	34.6	.378	.404	1.748

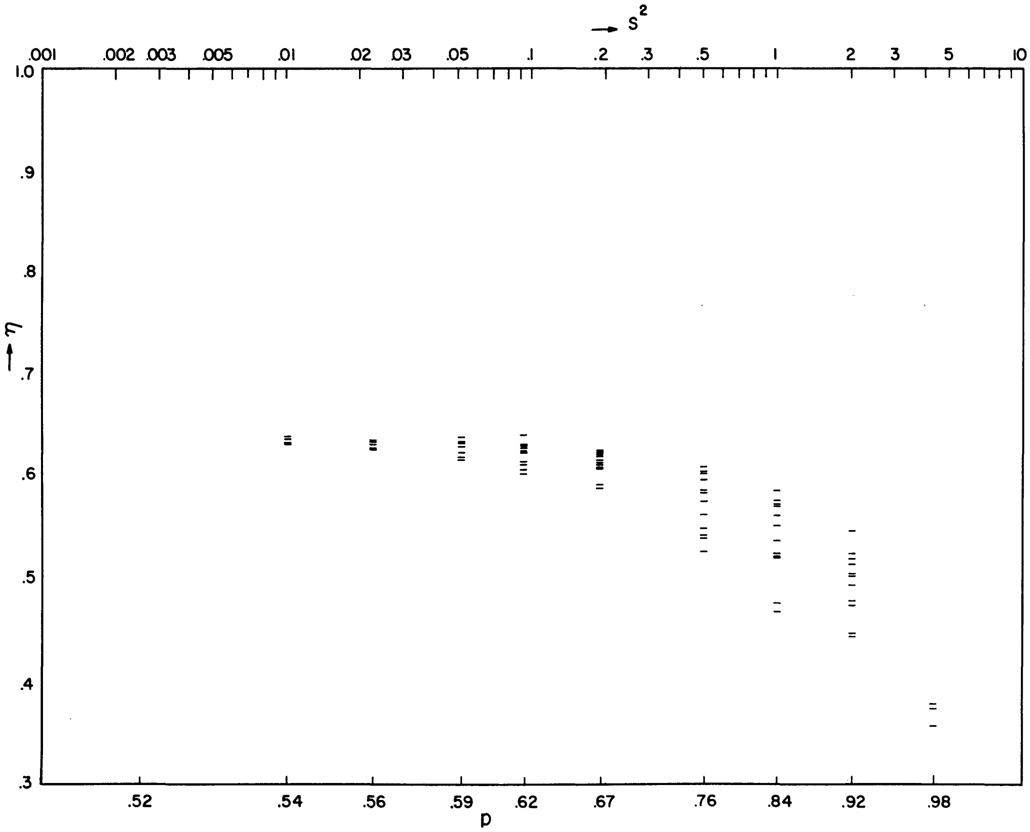

Fig. 2(a). Efficiency of C. C. C. (BS) using actual α, β, n.

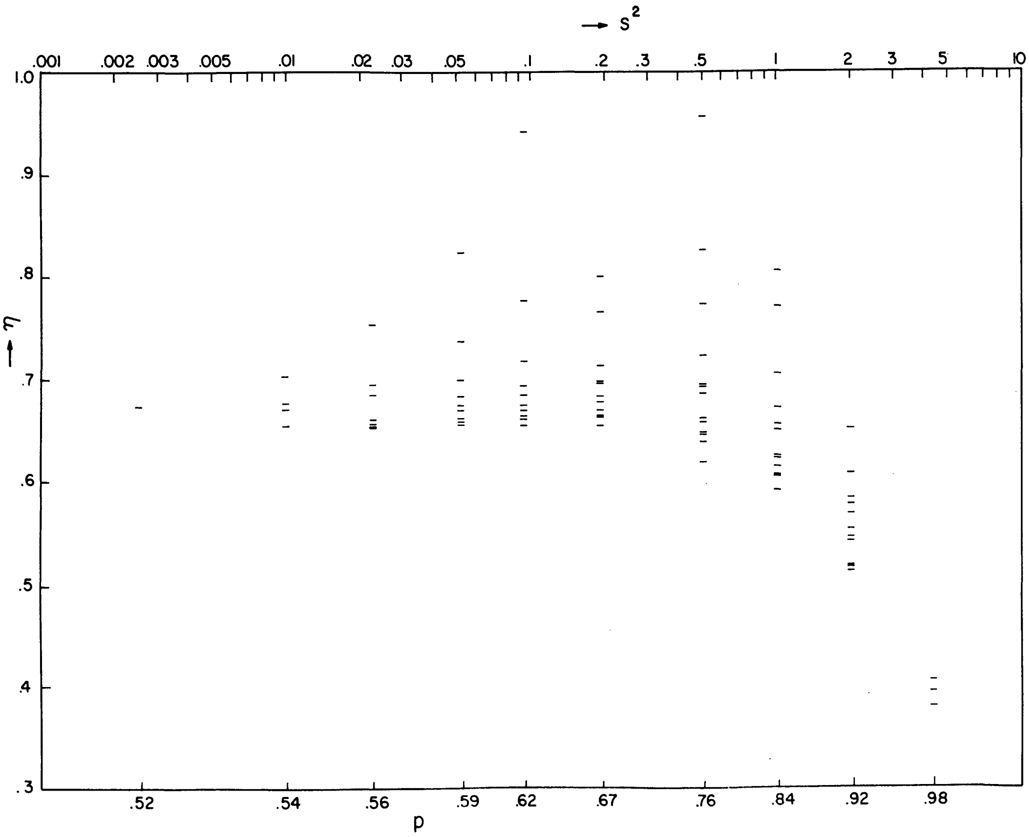

Fig. 2(b). Efficiency of C. C. C. (MBS) using actual α, β, n, under signal plus noise.

In Ref. 6 a simple expression was developed to be used as an approximation for the clipper crosscorrelator efficiency when $\beta = .50$. It was simply $(2p-1/s)^2$. Table III lists this value, and the range $\eta(BS)$ at $\beta_o = .50$ from Table II.

p	s	$(2p-1/s)^2$	$\eta(BS)$ at $\beta_o = .5$ from II
very small		$\frac{2}{\pi} = .6366$	---
.54	.1004	.6349	.636
.56	.1510	.6315	.634-.637
.59	.2275	.6260	.619-.640
.62	.3055	.6172	.602-.681
.67	.4399	.5974	.611-.630
.76	.7063	.5420	.577-.588
.84	.9945	.4675	.521-.539
.92	1.405	.3574	.331-.448
.98	2.054	.2184	.297

Table III. Comparison of Ref. 6 and some values of Table II.

10. A SEQUENTIAL STRATEGY

The preceding strategies all consist of choosing, before taking any observations, a sample size n (and a cut-point d) necessary to obtain a given α, β, p triple.

In the case of MBS and MIBS, however, we notice that it may sometimes be possible to make a decision before n observations have been taken. The MBS (as well as the MIBS) is a form of sequential test. That is, after each observation, we make one of the following three decisions: (1) accept hypothesis A (signal is present), (2) accept hypothesis B (signal is not present), (3) take an additional observation. We use two cut-points to make our decision. In the case of MBS, we have

1) if C (no. of ones) = d, accept A;
2) if z (no. of zeros) = n-d+1, accept B;
3) otherwise, take another observation.

10.1 Sequential Probability Ratio Test

A much more efficient sequential test is the sequential probability ratio test (see Ref. 3). This is an optimum test when the cut-points are chosen correctly. The test is defined as follows. Denote by θ_i the probability that $x_i > 0$. Testing hypothesis SN against hypothesis N is equivalent to testing the hypothesis $\theta = p = \Phi(s)$ against the hypothesis $\theta = \frac{1}{2}$.

The probability of obtaining a sample $[c(x_1), c(x_2), \ldots, c(x_m)]$ with Z zeros and C ones is given by

$$P(C,z) = \theta^C (1-\theta)^z, \qquad \text{where } C + z = m. \tag{23}$$

Under hypothesis SN the above probability is given by

$$P_{SN}(c,z) = p^C (1-p)^z, \tag{24}$$

and under hypothesis N by

$$P_N(c,z) = \frac{1}{2}^C \frac{1}{z}^z = \frac{1}{2}^{C+z} = \frac{1}{2}^m . \tag{25}$$

The likelihood ratio for a sample with the same number of zeros and ones as the given sample is given by

$$\ell(C,z) = \frac{P_{SN}(C,z)}{P_N(C,z)} = 2^{C+z}\, p^C (1-p)^z . \tag{26}$$

The likelihood ratio for the given sample can therefore be computed from (26), and depends on the number of zeros, Z, and ones, C, in the sample.

The test is carried out in the following manner: two positive constants γ and $\delta(\gamma < \delta)$ are chosen. After each trial, $\ell(C,z)$ is computed. If $\ell \geq \delta$, judge SN to be true. This is denoted by "A." If $\ell \leq \gamma$, judge N to be true. This is denoted by "B." If $\gamma < \ell < \delta$, take another observation. The values of α and β are fixed by the values of p, γ, δ.

For practical purposes, it is easier to calculate log $\ell(C,z)$, after the mth observation, than to calculate $\ell(C,z)$. The log-likelihood ratio, log $\ell(C,z)$, is given by

$$\begin{aligned} \log \ell(C,z) &= \log [2^C p^C 2^z (1-p)^z] \\ &= C \log 2p + z \log 2(1-p) . \end{aligned} \tag{27}$$

The test now takes the form:

If $\log \ell(C,z) \geq \log \delta$, A occurs; (28)

if $\log \ell(C,z) \leq \log \gamma$, B occurs; (29)

if $\log \gamma < \log \ell(C,z) < \log \delta$, take another observation. (30)

Equation 27 can be written in the form

$$z = \frac{\log \ell(m)}{\log 2(1-p)} - \frac{\log 2p}{\log 2(1-p)} C . \tag{31}$$

If we define

$$Z_A = \frac{\log \delta}{\log 2(1-p)} - \frac{\log 2p}{\log 2(1-p)} C, \text{ and} \tag{32}$$

$$Z_B = \frac{\log \gamma}{\log 2(1-p)} - \frac{\log 2p}{\log 2(1-p)} C, \tag{33}$$

we can write the sequential test as follows: For a given (C, Z),

$$\text{If } Z \leq Z_A(C), \text{ accept A,} \tag{34}$$

$$\text{if } Z \geq Z_B(C), \text{ accept B,} \tag{35}$$

$$\text{if } Z_A(C) < Z < Z_B(C), \text{ take another observation.} \tag{36}$$

Graphically, BS, IBS, MBS, MIBS, and the sequential probability ratio test (SPRT) can be summed up as shown in Fig. 3 (for given α, β, p).

10.2 SPRT when γ, δ are Determined by Wald's Approximation

Let α_o, β_o be the design values of false alarm and miss probabilities, respectively. Pick the values of γ and δ using Wald's approximation:

$$\delta = \frac{1 - \beta_o}{\alpha_o} \tag{37}$$

and

$$\gamma = \frac{\beta_o}{1 - \alpha_o} \tag{38}$$

The actual values of α, β are not, in general, equal to the values of α_o, β_o, because the test does not necessarily terminate with specific likelihood ratios $\ell = \delta$ or $\ell = \gamma$. The fact that m must take on integral values makes possible a "spill-over" in likelihood ratio at the boundaries (δ and γ). For all observations terminating in an A decision, the expected value of the likelihood ratio is

$$E_A[\ell(C, Z)] = \frac{1-\beta}{\alpha} \tag{39}$$

similarly for B decisions

$$E_B[\ell(C, Z)] = \frac{\beta}{1-\alpha} \tag{40}$$

It is evident that there is also a possible "spill-over" in the values of α, β. We shall now calculate these "spill-overs."

Suppose that after n observations an A decision is reached; i.e.,

$$\ell(C, Z) = 2^{Z+C}(1-p)^Z p^C \geq \delta, \qquad \text{with } C+Z = n. \tag{41}$$

On the (n-1) observation, the likelihood ratio satisfied the inequality

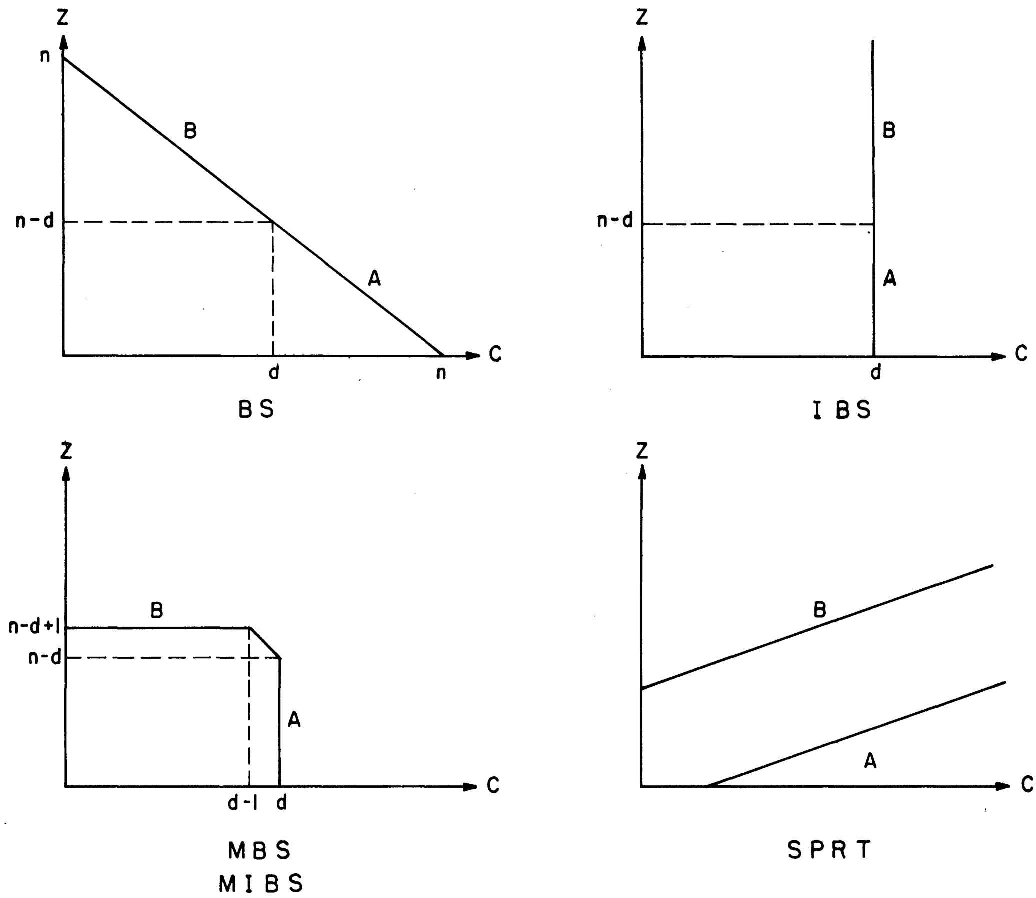

Fig. 3. Termination boundaries for clipper crosscorrelator.

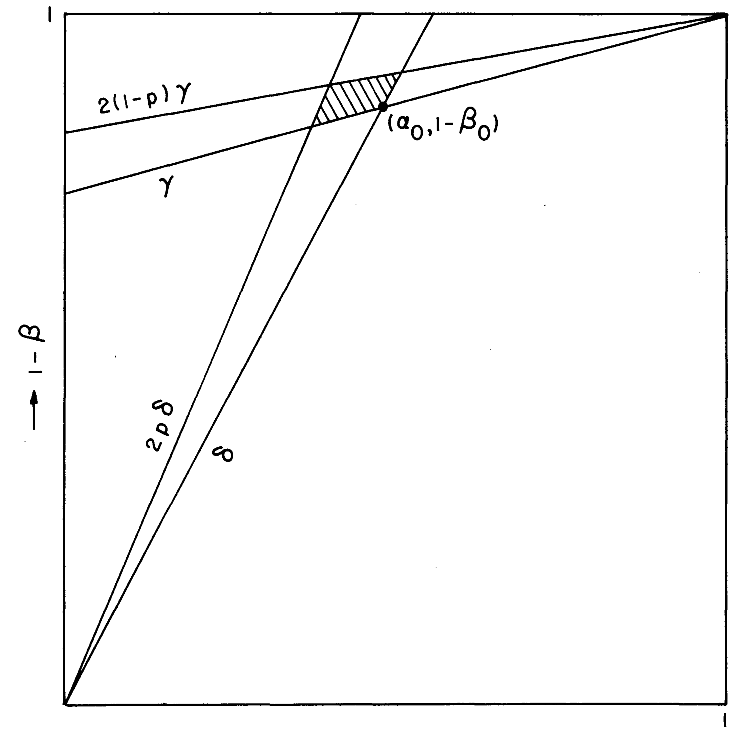

Fig. 4. Range of α and β when $\gamma = \dfrac{\beta_o}{1-\alpha_o}$ and $\delta = \dfrac{1-\beta_o}{\alpha_o}$.

$$\gamma < \ell(C-1, Z) = 2^{Z+C-1} (1-p)^Z p^{C-1} < \delta .$$

There will be a real number x ($C-1 < x \leq C$) such that

$$2^{Z+X} (1-p)^Z p^X = \delta . \tag{42}$$

Dividing (41) by (42) we have

$$\frac{\ell(C, Z)}{\delta} = 2^{C-x} p^{C-x} , \tag{43}$$

and since $p > \frac{1}{2}$,

$$\delta \leq \ell(C, Z) < 2 p \delta . \tag{44}$$

Similarly, if a B decision is reached at $C+Z = n$, we have

$$\ell(C, Z) = 2^{Z+C} (1-p)^Z p^C \leq \gamma$$

and

$$\gamma < \ell(C, Z-1) = 2^{Z-1+C} (1-p)^{Z-1} p^C < \delta ,$$

and so there is an x ($Z-1 < x \leq z$) such that

$$2^{x+c} (1-p)^x p^c = \gamma .$$

Therefore we have

$$\frac{\ell(C, Z)}{\gamma} = 2^{z-x} (1-p)^{z-x}$$

and

$$2(1-p)\gamma < \ell(C, Z) \leq \gamma . \tag{45}$$

Equations 44 and 45 set bounds on the likelihood ratio at termination, and these two equations, with Eqs. 37, 38, 39, and 40, can be used to set bounds on α and β:

$$\frac{1 - \beta_o}{\alpha_o} \leq \frac{1 - \beta}{\alpha} < 2p \frac{1 - \beta_o}{\alpha_o} \tag{46}$$

and

$$2(1-p) \frac{\beta_o}{1 - \alpha_o} < \frac{\beta}{1 - \alpha} \leq \frac{\beta_o}{1 - \alpha_o} \tag{47}$$

These relationships can be seen graphically on the ROC curve (Fig. 4). The range of α, β is indicated by the hatched region.

10.3 ASN for the SPRT

On page 53 of Ref. 3 the following formula is derived:

$$E_\theta(n) = \frac{E_\theta(\log \ell_1 + \ldots + \log \ell_n)}{E_\theta(\log \ell)} \tag{48}$$

where: $E_\theta(y)$ is the expected value of y for a given value of θ,

n is the number of observations necessary for termination,

$$\log \ell_i = \log \frac{P_{SN}[c(X_i)]}{P_N\ [c(X_i)]} \quad ,$$

and

$$\ell = \frac{P_{SN}[c(X)]}{P_N\ [c(X)]} \quad .$$

The numerator is the expected value of log $\ell(m)$ at termination:

$$E_\theta(\log \ell_1 + \ldots + \log \ell_n) = E_\theta[\log \ell(C, Z)], \text{ where } C+Z = n$$

$$\cong P_\theta(A) \log \delta + P_\theta(B) \log \gamma \ . \tag{49}$$

The denominator is given by

$$E_\theta(\log \ell) = \theta \log 2p + (1-\theta) \log 2\,(1-p)$$

The average sample number can thus be written as

$$E_{SN}(n) \cong \frac{(1-\beta) \log \delta + \beta \log \gamma}{p \log 2p + (1-p) \log 2(1-p)} \tag{50}$$

and

$$E_N(n) \cong \frac{\alpha \log \delta + (1-\alpha) \log \gamma}{1/2 \log 2p + 1/2 \log 2(1-p)} \tag{51}$$

Since Eq. 49 is based on Wald's approximation (Eqs. 39 and 40), there is no loss of accuracy in writing (50) and (51) in the form

$$E_{SN}(n) \cong \frac{(1-\beta)\log\frac{1-\beta}{\alpha} + \beta\log\frac{\beta}{1-\alpha}}{p\log 2p + (1-p)\log 2(1-p)} \tag{52}$$

$$E_N(n) \cong \frac{\alpha\log\frac{1-\beta}{\alpha} + (1-\alpha)\log\frac{\beta}{1-\alpha}}{1/2\log 2p + 1/2\log 2(1-p)} \tag{53}$$

10.4 Comparison of SPRT with MBS

In Fig. 5, the termination boundaries of SPRT are compared with those of MBS for the triple p = .67, α = .00098, β = .09871.

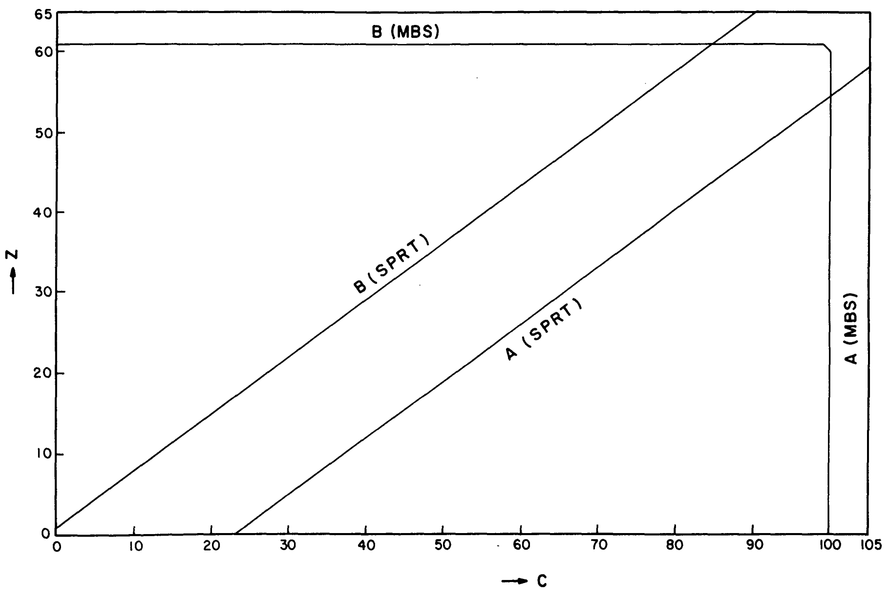

Fig. 5. Termination boundaries for p = 0.67, α = 0.00098, and β = 0.09871.

11. CONCLUDING COMMENTS AND REMARKS ABOUT FURTHER WORK

We have considered in this report the problem of signal detection using a clipper-crosscorrelator when the signal of known single size is possibly present and with Gaussian noise in the background.

As is well known, the solution to a dichotomous statistical decision problem always involves the recognition and reconciliation to the two types of errors known as α-error and β-error. In the signal detection problem the α-error takes the form of "false alarm" and the β-error means "miss."

We have suggested five such strategies which arise in a very natural way and have studied their interrelations. We have also defined and investigated the efficiency of the clipper-crosscorrelator in relation to the usual crosscorrelator.

In this report the investigations and results have been all theoretical in nature and yet have been intuitively very meaningful. Associated tables and charts may be useful for application.

REFERENCES

1. G. P. Patil, Contributions to Estimation in a Class of Discrete Distributions, Doctoral Dissertation at The University of Michigan, Ann Arbor, Michigan, 1959.

2. G. P. Patil, "On the Evaluation of the Negative Binomial Distribution With Examples," Technometrics, Vol. 2: 501-505, 1960.

3. A. Wald, Sequential Analysis, John Wiley and Sons, New York, 1947.

4. National Bureau of Standards, Tables of the Binomial Probability Distribution, Applied Mathematics Series 6, Issued January 1950.

5. Staff of the Computation Laboratory, Harvard University, Tables of the Cumulative Binomial Probability Distribution, Harvard University Press, 1955.

6. T. G. Birdsall, "On the Extension of the Theory of Signal Detectability," U. S. Navy Journal of Underwater Acoustics, Vol. 11, No. 2, p. 183, April 1961.

DISTRIBUTION LIST

Office of Naval Research (Code 468)
Department of the Navy
Washington 25, D. C. (3 copies)

Office of Naval Research (Code 436)
Department of the Navy
Washington 25, D. C. (1 copy)

Office of Naval Research (Code 437)
Department of the Navy
Washington 25, D. C. (1 copy)

Director
U. S. Naval Research Laboratory
Technical Information Division
Washington 25, D. C. (6 copies)

Director
U. S. Naval Research Laboratory
Sound Division
Washington 25, D. C.
Attn: Mr. W. J. Finney (1 copy)

Commanding Officer
Office of Naval Research Branch Office
The John Crerar Library Building
86 East Randolph Street
Chicago 1, Illinois (1 copy)

Commanding Officer
Office of Naval Research Branch Office
Box 39, Navy No. 100
FPO, New York (8 copies)

Armed Services Technical Information Agency
Arlington Hall Station
Arlington 12, Virginia (10 copies)

Commander
U. S. Naval Ordnance Laboratory
Acoustics Division
White Oak
Silver Spring, Maryland (2 copies)
Attn: Mr. Derrill, J. Bordelon,
Dr. Ellingson

Commanding Officer and Director
U. S. Navy Electronics Laboratory
San Diego 52, California (1 copy)

Director
National Bureau of Standards
Connecticut Avenue and Van Ness St. N.W.
Washington 25, D. C.
Attn: Mrs. Edith Corliss (1 copy)

Office of Chief Signal Officer
Department of the Army
Pentagon Building
Washington 25, D. C. (1 copy)

Commanding Officer and Director
David Taylor Model Basin
Washington 7, D. C. (1 copy)

Superintendent
U. S. Navy Postgraduate School
Monterey, California
Attn: Prof. L. E. Kinsler (1 copy)

Commanding Officer
Air Force Cambridge Research Center
230 Albany Street
Cambridge 39, Massachusetts (1 copy)

Chief
Office of Ordnance Research
Box C. M., Duke Station
Durham, N. C. (1 copy)

National Science Foundation
1520 H Street N. W.
Washington D. C. (1 copy)

Commanding General
Wright-Patterson AFB
Dayton, Ohio (1 copy)

Commanding Officer
U. S. Navy Mine Defense Laboratory
Panama City, Florida (1 copy)

U. S. Naval Academy
Annapolis, Maryland
Attn: Library (1 copy)

Chief, Physics Division
Office of Scientific Research
HQ Air Research and Development Command
Andrews AFB
Washington 25, D. C. (1 copy)

University of California
Marine Physical Laboratory of the
Scripps Institution of Oceanography
San Diego 52, California (2 copies)
Attn: Dr. V. C. Anderson
Dr. Philip Rudnick

Director
U. S. Navy Underwater Sound Reference Laboratory
Office of Naval Research
P. O. Box 8337
Orlando, Florida (1 copy)

Commanding Officer and Director
U. S. Navy Underwater Sound Laboratory
Fort Trumbull
New London, Connecticut (2 copies)
Attn: Mr. W. R. Schumacher,
Mr. L. T. Einstein

Commander
U. S. Naval Air Development Center
Johnsville, Pennsylvania (1 copy)

Dr. M. J. Jacobson
Department of Mathematics
Rensselaer Polytechnic Institute
Troy, New York (1 copy)

Director
Columbia University
Hudson Laboratories
145 Palisades Street
Dobbs Ferry, N. Y. (1 copy)

Woods Hole Oceanographic Institution
Woods Hole, Massachusetts
Attn: A. C. Vine (1 copy)

Johns Hopkins University
Institute for Cooperative Research
34th and Charles Street
Baltimore 18, Maryland
Attn: Dr. W. H. Huggins (1 copy)

Edo Corporation
College Point, Long Island, N. Y.
Attn: Mr. Charles J. Loda (1 copy)

Melpar, Inc.
Applied Sciences Division
11 Galen Street
Watertown, Mass.
Attn: Dr. David Van Meter (1 copy)

Director
University of Miami
Marine Laboratory
Miami, Florida
Attn: Dr. J. C. Steinberg (1 copy)

Harvard University
Acoustics Laboratory
Division of Applied Science
Cambridge 38, Massachusetts (1 copy)

Brown University
Department of Physics
Providence 12, R. I. (1 copy)

Western Reserve University
Department of Chemistry
Cleveland, Ohio
Attn: Dr. E. Yeager (1 copy)

University of California
Department of Physics
Los Angeles, California (1 copy)

Institute for Defense Analysis
Communications Research Division
von Neumann Hall
Princeton, New Jersey (1 copy)

Dr. B. F. Barton, Director
Cooley Electronics Laboratory
The University of Michigan
Ann Arbor, Michigan (1 copy)

Project File
Cooley Electronics Laboratory
The University of Michigan
Ann Arbor, Michigan (30 copies)

Project File
The University of Michigan Office of Research Administration
Ann Arbor, Michigan (1 copy)

Director
The University of Michigan Office of Research Administration
Cooley Electronics Laboratory
Ann Arbor, Michigan (1 copy)
Attn: Mr. T. G. Birdsall