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THE BEHRENS-FISHER DISTRIBUTIONS

Venkutai H. Patil

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ABSTRACT

The Behrens-Fisher distributions are important in problems of inference about the difference between the means μ_1 and μ_2 of two normal populations of unknown means and variances, or any linear combinations of them. The purpose of this study is to facilitate the computations of the densities, cumulative probabilities, and percentage points of the Behrens-Fisher distributions with a view of making them available for everyday practical use, much more than they are today.

The Behrens-Fisher variables are the linear combinations of two independent random variables t_1 and t_2, where t_1 and t_2 are distributed according to Student's distribution with f_1 and f_2 degrees of freedom.

An extension of the two-t problem to any linear combination of an arbitrary number of t-like random variables leads to generalizations of the Behrens-Fisher distributions. A recognition of the importance of the many-t problem has influenced the direction of this thesis. For one respect in which any method of computing the Behrens-Fisher distributions must be considered is its adaptibility to the many-t problem as well as the two-t problem.

The following approximations of the Behrens-Fisher distributions were undertaken with a view to putting the computation of the densities, cumulative probabilities, and percentage points of these distributions within the range of a user who has access to a desk-calculator and widely available tables such as those of the normal distributions and its derivatives, and of Student's distribution:

1. The Hermite polynomial methods.
2. Application of the delta method to integral forms due to Ruben.
3. Choosing a dilated Student's t with width h^{-1} and degrees of freedom f.

These approximations were checked against the numerical values published by others, and certain numerical densities which were computed specially for this purpose on the basis of recurrence formulae.

Of the three methods, the third proved the most practical. f and h were so chosen that hd has the same second and fourth moments as that of Student's t with f degrees of freedom. This does a good over-all job that makes it practical to calculate the Behrens-Fisher densities, cumulative probabilities, and percentage points.

The main conclusion is that the tables of ordinary Student's t distribution provide an excellent basis for most applications of the more general Behrens-Fisher distributions.

INTRODUCTION

Scope

In certain applications of statistics, it is important to calculate densities, cumulative probabilities, and percentage points of Behrens-Fisher distributions and of certain natural generalizations. The justifications that have been proposed for these applications are interesting and controversial, but they are secondary here; for the purpose of this thesis is simply to facilitate Behrens-Fisher calculations for those who wish to apply them.

For the problem of inference about the difference between the means of two normal populations of unknown variances, Fisher (1935) proposed solutions based on the Behrens-Fisher distributions. His argument is based on his concept of fiducial probability. Some other frequentists have rejected these solutions because they do not lead to confidence intervals in the Neyman-Pearson sense (Bartlett, 1936; Welch, 1938; James, 1959). Jeffreys (1940) arrives at the same result, and Bayesians borrow his derivation in the personalistic Bayesian theory of statistics.

The Behrens-Fisher distributions, in a slightly generalized sense, are the distributions of linear combinations of pairs of independent random variables t_1 and t_2; that is, the distributions of random variables of the form $bt_2 - at_1$, where a and b are real and not both zero, and t_1 and t_2 are independently distributed according to Student's distribution with f_1 and f_2 degrees of freedom, respectively,

Writing,

$$bt_2 - at_1 = (a^2+b^2)^{1/2}(t_2\cos\theta - t_1\sin\theta)$$

where

$$\sin\theta = a(a^2+b^2)^{-1/2},$$
$$\cos\theta = b(a^2+b^2)^{-1/2},$$

it suffices, of course, to study the distribution of $d = t_2\cos\theta - t_1\sin\theta$. In view of the symmetry of the t-distribution, there is no loss in supposing $0 \leq \theta \leq \pi/2$.

The Behrens-Fisher distributions will, therefore, often be understood in a narrower sense as the distributions of variables of the form $t_2\cos\theta - t_1\sin\theta$, and this is the usual sense.

Heretofore, relatively little has been done to make the Behrens-Fisher distributions available for practical applications. Their direct computation from first principles is prohibitively difficult, especially for daily use, except possibly when both degrees of freedom are very small. Some percentage points and cumulative probabilities have been published. (Refer to Chapter 4 for details.) But these are inadequate even for applications that involve only the percentage points and cumulative probabilities, and do not help at all with applications that involve the densities of the Behrens-Fisher distributions. The latter are important in connection with Bayesian statistics.

A straightforward table of the Behrens-Fisher densities, cumulative probabilities, and percentage points is not a promising solution for an applied statistician. A table of densities or cumulative probabilities yielding even moderate accuracy with the effort of considerable interpolation would seem to require more than one hundred thousand entries. Nonetheless, this solution ought not be abandoned, because with ingenuity, the scheme might be brought within practical bounds. In fact, as more ingenuity is applied to it, the straightforward table would become less and less straightforward and more and more like other devices to be mentioned next.

It might be possible to construct some special tables of practical size which would not be tables of the Behrens-Fisher distributions themselves, but would facilitate the computation of these distributions. Finally, it may be possible to reduce the calculation of Behrens-Fisher distributions to a practical application of tables that are already widely available. I have pretty much confined myself to this last line of investigation. My main conclusion is that the tables of the ordinary Student's t distribution provide an excellent basis for most applications of the more general Behrens-Fisher distributions.

I report here on my exploration of several possible approximations of the Behrens-Fisher distributions undertaken with a view to putting the computation of the densities, cumulative probabilities, and percentage points of these distributions within the range of a user who has access to a desk-calculator and widely available tables such as those of the normal distribution and its derivatives, and of Student's distribution.

What values of d, θ, f_1, and f_2 must a practical method be able to deal with? Values of $|d| > 5$ seem unimportant. The entire natural range of θ, 0 to $\pi/2$, is necessary. It would be desirable to cover all values of f_1 and f_2, but a method confined to values of f_1 and $f_2 > 5$ would meet almost all practical needs.

An extension of the two-t problem to any linear combination of an arbitrary number of t-like random variables leads to generalizations of Behrens-Fisher distributions. Not all methods of calculating the Behrens-Fisher distributions themselves are well adapted to calculating these generalizations. A recognition of the importance of the many-t problem has influenced the direction of this thesis. For one respect in which every particular method of computing the Behrens-Fisher distributions must be considered is its adaptability to the many-t problem as well as the two-t problem.

Conclusion

The following approximations were explored:

1. The Hermite polynomial method.
2. Application of the delta method to integral forms due to Ruben.
3. Choosing a dilated Student's t with width h^{-1} and degrees of freedom f.

The comparison of densities in Table 2 shows that the approximations due to the first two methods are not good for large values of d. Of the three methods, the third is the most practical.

For the third approximation, f and h were so chosen that hd has the same second and fourth moments as a Student's t with f degrees of freedom. This does a good over-all job that makes it practical to calculate densities, cumulative probabilities, and percentage points of d.

An empirical table of corrections Δf to be added to f is given in Table 4, so that certain percentage points can be calculated with higher accuracy.

An attempt was made to justify the use of the third approximation for the three-t problem by calculating the higher cumulants of hd and Student's t with f degrees of freedom, where h and f were calculated on the same principle as for the two-t problem. The comparison is given in Table 7.

Organization

The thesis is so divided into nine chapters that separate chapters are accessible to the reader, depending upon his interest.

To begin with, the Behrens-Fisher problem and the solution based on Behrens-Fisher distributions are stated. The criticisms and justifications of this solution by holders of different views of probability are explained briefly (Chapter 1).

Next, the basic mathematical formulas (in the integral form) of the Behrens-Fisher densities and cumulative probabilities are covered (Chapter 2).

One of the several generating functions that were tried to obtain closed forms of the Behrens-Fisher densities is given. This generating function enabled the writer to calculate certain closed forms and nominally exact numerical values (except for rounding errors in computation) by means of recurrence relations (Chapter 3).

Reference to earlier numerical work is given and that part which is published in widely available tables is reviewed (Chapter 4).

Certain relatively exact numerical values of the Behrens-Fisher densities were computed to serve as the standards against which to check the approximations in which I am ultimately interested. The most time-consuming work for this thesis has been obtaining the recurrence relations and certain numerical values of the Behrens-Fisher densities. For the most part, the study was confined to odd degrees of freedom, since this was the easiest. Just a few numerical values were calculated for even degrees of freedom by numerical integration. This enabled the writer to try harmonic interpolation in f_1 and f_2, suggested by Fisher, which is the same as direct interpolation in f_1^{-1} and f_2^{-1}. The comparison of the interpolated values for the even degrees of freedom from those of odd degrees of freedom with the exact values is given in Table 1 (Chapter 5).

The moments and cumulants of Student's t and d are used in some of the approximations that were explored in this thesis. The expressions for the moments and cumulants of d are calculable from those of Student's t, because of the independence of t_1 and t_2 (Chapter 6).

The different approximations that were explored are subdivided into two categories:

1. Asymptotic approximations.

2. Approximation by one dilated Student's distribution.

The comparison with the exact values is given in Tables 2–7 (Chapters 7, 8).

Finally, the implications of the thesis are exploited to provide a set of practical instructions for computing values of density, cumulative probability, and percentage points of Behrens-Fisher distributions for everyday applications (Chapter 9).

CHAPTER 1

THE BEHRENS-FISHER PROBLEM AND ITS BAYESIAN SOLUTION

1.1. The Behrens-Fisher Problem

The Behrens-Fisher problem, or more accurately, a group of problems, of making inferences about the difference between the means of two normal populations, π_1 and π_2, of unknown and not necessarily equal variances on the basis of independently drawn samples of sizes n_1 and n_2 from π_1 and π_2, is of some practical importance.

Let $(x_{a1}, \ldots, x_{an_a})$ be a sample of size n_a from a normal population π_a with μ_a and variance σ_a^2 $(a = 1,2)$. Then,

$$\bar{x}_a = \sum_{j=1}^{n_a} x_{aj} / n_a \; ; \quad s_a^2 = \sum_{j=1}^{n_a} (x_{aj} - \bar{x}_a)^2 / (n_a - 1)$$

are the sample means and variances.

Fisher (1935) proposed a solution to the problems of estimating and testing the difference between the means μ_1 and μ_2 based on the Behrens-Fisher distributions, sometimes referred to as distributions of d-variables in the following discussion. A d-variable is a linear combination of two independent random variables, each distributed according to Student's distribution with not necessarily equal numbers of degrees of freedom. A test equivalent to Fisher's was given earlier for a special case by Behrens (1929). Many frequentists have criticized and rejected Fisher's solutions because these do not lead to confidence intervals or significance levels (in the Neyman-Pearson sense) (Welch, 1938; Bartlett, 1936; James, 1959).

Scheffé (1943, 1944) and Welch (1947) have offered solutions to the Behrens-Fisher problem, and these, too, have been subjected to severe criticism, as will now be explained.

Scheffé generalized an unpublished solution of Bartlett. The solution is explained here for the simplest case $n_1 = n_2$. Let $x_j = x_{1j} - x_{2j}$. A confidence interval for $\mu_1 - \mu_2$ is obtained by the usual Student technique in terms of x_j. Some statisticians object to this solution, because it depends upon the ordering of the observations in the two samples, which amounts to making a random analysis of the data and ignores aspects of the data that might be relevant.

The Neyman-Pearson school has also tried to offer solutions free of the objection to Scheffé's. Their nearest approach, it seems, is that of Welch (1947) in terms of infinite series which define asymptotic confidence intervals.

Fisher's argument is based on his concept of fiducial probability and depends on the assumption that nothing whatsoever is known in advance about μ_1, μ_2, σ_1, and σ_2. If μ_1 and μ_2 are not mutually irrelevant, as they would not seem to be if there is a serious question of their equality, then the required condition, which is at best somewhat mysterious, is not satisfied.

For estimation, Bayesians believe that there is a cogent argument for the "fiducial intervals" of Fisher based on the Behrens-Fisher distribution. For testing, they distinguish a variety of cases. In one of these, the two-sided version of the test proposed by Fisher is appropriate; in another, a test based on density, not on cumulative distribution, of the Behrens-Fisher distribution is appropriate.

For the problem of estimating the difference of μ_1 and μ_2, Jeffreys (1940) has arrived at the same conclusion as Fisher, but along a more comprehensible path. In fact, Jeffreys' argument is easily modified into the approximate conclusion of the personalistic Bayesian theory of statistics.

A generalization of Jeffreys' argument, which is based on uniform prior distributions of the means and the logarithm of variances, is inherent in the theory of conjugate families of prior distributions of Raiffa and Schlaifer (1961, pages 54–56). This generalization consists of replacing the uniform prior distributions by other distributions that are analytically convenient and that promise to be somewhat more realistic for some applications.

Another important direction of generalization is this: One might be interested not only in the difference between μ_1 and μ_2 but also in their sum or in any linear combination of them. The approach to this generalized Behrens-Fisher estimation problem is easy, not only for any linear combination of two unknown means, but of any number of interest. This generalization is easy in principle, but it leads to generalizations of the Behrens-Fisher distributions, which pose additional computational problems.

1.2. Bayes's Theorem and Personal Probability

The personalistic Bayesian theory of statistics is called "Bayesian" for the rather secondary reason that it finds more occasions to apply Bayes's Theorem than the frequentistic theory of statistics does. The deeper characterization of a personalistic Bayesian is systematic use of a concept of probability called "personal" or "subjective" probability, on which Bayesians believe that statistics can be founded.

The personal probability of an event to a particular person is a certain kind of numerical measure of the confidence of that person in that event. Thus, the personalistic concept of probability is related in an operational sense to the prior opinion about an event. The concept of personal probability is a frankly subjective one. Among the criticisms that have been brought against it, one is, of course, its lack of objectivity. Another is that personal probability can often be determined only crudely, which leads some critics to say that it must be regarded as a nonnumerical concept.

For one introduction to personal probability and for some discussion of other views of probability, see Savage (1954). For a discussion of the application of personal probability to statistics, see Savage *et al.* (1962).

The personal probability of an event for an individual can be thought of roughly in terms of the odds one is prepared to offer in favor of the event and is calculated by the formula,

$$\text{probability} = \text{odds}/(1 + \text{odds}).$$

Personal probability can also be expressed in terms of contingent payments. For example, an individual's personal probability for the event that it will rain today is the price he is prepared to pay for a unit payment to him in case it really does rain today.

The frequency concept of probability defines probability in terms of repetitions of a certain kind of event under certain conditions. This frequency concept is held to be a correct foundation for the theory of statistics by a large number of statisticians. In this concept, the probability of an event is to be determined according to the frequency definition and in no other way. This cuts frequentists off from the application of Bayes's Theorem to problems where the prior probability of the event in question cannot be determined according to their definition. Therefore, frequentists can not speak of the probability of an uncertain hypothesis or the probability distribution of an unknown parameter. They must, on that account, seek to express statistical inference in some other terms.

The third main view of probability is, in the nomenclature of Savage (1954), the necessary view. The holders of this view regard probability as a generalization of implication. For them, probability is a logical relationship between one proposition and another; that is, one proposition partially necessitates the truth of the other proposition. Probability here is much like personal probability except for the assumption that one and only one opinion is justified by any body of evidence. Again, though any such description is at best approximate, Harold Jeffreys (1948) can fairly be described as a holder of this view. Some personalistic Bayesians find his works a source of invaluable material, after making some easy changes. For example, the derivation of the Behrens-Fisher distribution can be modified from Jeffreys (1940) by a Bayesian into a personalistic Bayesian argument for adaption of this solution as an approximation.

1.3. Bayesian Inference

The prior (or initial) probabilities of events are available in the personalistic theory of probability, and Bayes's Theorem is applied to generate the posterior probabilities (new opinions).

Bayes's Theorem is the truism

$$\Pr(H|D) = \frac{\Pr(D|H)\Pr(H)}{\Pr(D)},$$

where H and D are any events, but these letters are chosen to suggest hypothesis and datum, corresponding to the most important application of the theorem.

The concept of inference as the change in opinion induced by the evidence according to Bayes's Theorem, together with the principle of stable estimation which will be explained in the next section, illuminates many questions that have been raised about interval estimation, of which the Behrens-Fisher problem is one.

1.4. Stable Estimation

This section reviews an approximation of great practical value in the personalistic Bayesian theory of statistics and does much to account for the agreement induced in diverse opinions by a common evidence.

This approximation, called "the principle of stable estimation" (Savage et al., 1962), leads to conclusions that are often in harmony with the classical, or frequentistic, theories of statistics.

The principle of stable estimation concerns inference about a continuous parameter, say μ, which is not necessarily one-dimensional, on the basis of the datum D. According to Bayes's Theorem,

$$\rho(\mu | D) = K \Pr(D|\mu) \rho(\mu) ,$$

where K is a normalizing constant, and $\Pr(D|\mu)$ may be a density function.

The idea of stable estimation is that if $\rho(\mu)$ is sufficiently diffuse or gentle relative to $\Pr(D|\mu)$, then for many practical purposes $\rho(\mu|D)$ will be well approximated by $\Pr(D|\mu)$.

In an important extension of the principle, it is assumed not that $\rho(\mu)$ is gentle but that $\rho(\mu) = f(\mu)g(\mu)$, where $g(\mu)$ is a specified function, not necessarily gentle, and $f(\mu)$ is gentle. This extension plays some part in the derivation of the Behrens-Fisher distribution in the personalistic Bayesian theory of statistics.

1.5. Stable Estimation and Behrens-Fisher Distributions

Let $x = (x_1, x_2, \ldots, x_n)$ be a sample of size n from a normal population with mean μ and variance σ^2. Let $\rho(\mu, \sigma^2)$ be the prior density of (μ, σ^2) and let $\rho(x|\mu, \sigma^2)$ be the density of the datum x given μ and σ^2.

According to Bayes's Theorem,

$$\rho(\mu, \sigma^2 | x) = K \rho(x|\mu, \sigma^2) \rho(\mu, \sigma^2) ,$$

where K is a normalizing constant,

$$K = \left[\int_0^{\infty} \int_{-\infty}^{+\infty} \rho(x|\mu, \sigma^2) \rho(\mu, \sigma^2) \, d\mu \, d\sigma^2 \right]^{-1} .$$

We have

$$\rho(x|\mu, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\right)\left[\sum_{i=1}^{n} (x_i - \mu)^2\right]$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\right)\left[(n-1)s^2 + n(\bar{x} - \mu)^2\right] ,$$

where

$$\bar{x} = \Sigma x_i/n \quad , \quad s^2 = \Sigma (x_i - \bar{x})^2/(n-1).$$

Adapting the calculations of Jeffreys (1940) to the principle of stable estimation, Savage has pointed out in an unpublished manuscript that, under suitable circumstances of diffuse prior opinion, the posterior marginal distribution of μ is approximately like that of $\bar{x} + s't$, where $\bar{x}$, s' are constants, $s' = n^{-1/2}s$, and t is a random variable distributed according to Student's distribution with $(n - 1)$ degrees of freedom. The result agrees with that obtained by formal operation with the improper prior density $\rho(\mu, \sigma^2) = \sigma^{-2}$, as defined earlier by Jeffreys.

Instead of taking the improper density $\rho(\mu, \sigma^2) = \sigma^{-2}$, one could take a generalized improper density as a power of σ^{-2}, σ^{-2k}, which would lead to the slightly different marginal posterior distribution that μ is distributed like $\bar{x} + [(n - 1)/(n - 2 + k)]s't$, where t is distributed according to Student's distribution with $(n - 2 + k)$ degrees of freedom. This yields the above result when $k = 1$. Raiffa and Schlaifer (1961, section 3.2.5, pages 54–56) carry this generalization further.

The case $k = 1$ is in harmony with the result in the classical theory of statistics that $t = (\bar{x} - \mu)/s'$ is distributed like Student's t with (n - 1) degrees of freedom. The result stated above, though harmonious with the classical result, says something very different. In the classical theory, the exact distribution of $(\bar{x} - \mu)/x'$ is a Student's distribution for fixed μ. According to the above result in the case k = 1, the posterior distribution of $(\mu - \bar{x})/x'$ is approximately a Student's distribution for specified $\bar{x}$ and s'. This result is also harmonious with the theories of Fisher and Jeffreys.

The result can be extended to two normal distributions with unknown means and variances. Under the assumption that the joint prior distribution of the four parameters μ, $\log \sigma_1$, μ_2, $\log \sigma_2$ is gentle, and that the two samples are drawn independently, (μ_1, σ_1) will be practically independent of (μ_2, σ_2) given the datum $(\bar{x}_1, s'_1; \bar{x}_2, s'_2)$. Each μ_a is distributed approximately like $\bar{x} + s'_a t_a$ $(a = 1, 2)$.

Thus, $\mu_2 - \mu_1$ is distributed approximately like a constant plus a certain linear combination of a pair of independent Student's t like variables; more specifically, $\mu_2 - \mu_1$ is distributed like $\bar{x}_2 - \bar{x}_1 + t_2 s'_2 - t_1 s'_1$. For some purposes it is useful to re-express this by saying that $(\mu_2 - \mu_1 + \bar{x}_1 - \bar{x}_2)/s^*$ is distributed like the d-variable, $d = t_2 \cos\theta - t_1 \sin\theta$, where

$$s^* = (s_1'^2 + s_2'^2)^{1/2}, \quad \sin\theta = s'_1/s^*, \quad \cos\theta = s'_2/s^*,$$

and t_1 and t_2 are random variables distributed according to Student's distribution with $f_1 = (n_1 - 1)$ and $f_2 = (n_2 - 1)$ degrees of freedom, respectively.

The posterior density $\rho(\mu_2 - \mu_1 | \bar{x}_1, s_1'; \bar{x}_2, s_2')$ of $(\mu_2 - \mu_1)$ given the datum $(\bar{x}_1, s_1', \bar{x}_2, s_2')$, evaluated at $\mu_2 - \mu_1 = 0$, is approximately equal to the density $(s^*)^{-1}\rho(d|f_1, f_2; \theta)$ evaluated at $d = (\bar{x}_1 - \bar{x}_2)/s^*$, where $\rho(d|f_1, f_2; \theta)$ is the density of d given f_1, f_2, θ.

The preceding paragraph illustrates the convention of using the same letter ρ to symbolize various densities, so that the function that is meant is determined in part by the letter which is used as its argument. This system is known to have disadvantages, but in this context, all other systems seem to have greater ones.

For the problem of inference about the difference in the means $\mu_2 - \mu_1$, it is, therefore, important to be able to calculate the densities, cumulative probabilities, and percentage points of d.

It is useful, for some purposes, to specify an interval of $\mu_2 - \mu_1$ that has a definite high posterior probability. Such an interval is called a credible interval. Credible intervals are a counterpart of the confidence intervals of the Neyman-Pearson theory, but it is easier and some think more useful to construct credible intervals than to construct confidence intervals. A credible interval of $\mu_2 - \mu_1$ is any interval of $\mu_2 - \mu_1$ that has the required posterior probability according to $\rho(\mu_2 - \mu_1 | \bar{x}_1, s'_1; \bar{x}_2, s'_2)$. In stable estimation, we approximate posterior distributions, and thus, only approximate credible intervals are used.

All known methods of derivation of the Behrens-Fisher distributions lead to the generalized Behrens-Fisher distributions for the problem of inference about any linear combination of an arbitrary number of μ's; Fisher's as well as Jefferys' derivations do; the argument of this section can be extended to the generalized problem; $\Sigma c_a \mu_a$ given the datum $\left\{ \bar{x}_a, s'_a \right\}$ is distributed like a linear combination of t_a, where c_a are constants and t_a are random variables distributed according to Student's distribution with f_a degrees of freedom.

1.6. Bayesian Counterparts of Hypothesis Testing

A fairly general theory is presented here first, and then its application to the Behrens-Fisher testing problem is considered. The paper of Lindley (1961) presents many of the ideas that are sketched below.

Let x be the datum whose density involves two unknown parameters α and β (not necessarily one-dimensional), and suppose that we want to test the null hypothesis H_0 against the alternative hypothesis H_1, where H_0 is a hypothesis that implies $\alpha = 0$ and yields the conditional (upon H_0 being true) prior density $\rho(\beta | H_0)$ for β and H_1 is an alternative hypothesis under which one's prior opinion about α, β is given by $\rho(\alpha, \beta | H_1)$.

Let Ω be the prior odds in favor of H_0,

$$\Omega = \frac{Pr(H_0)}{Pr(H_1)} .$$

The posterior odds $\Omega(x)$, after observing x, are

$$\Omega(x) = \frac{Pr(H_0|x)}{Pr(H_1|x)} .$$

According to Bayes's Theorem,

$$\Omega(x) = \frac{\rho(x|H_0)\, Pr(H_0)}{\rho(x|H_1)\, Pr(H_1)} = L(x)\, \Omega ,$$

where L(x) is the likelihood-ratio,

$$L(x) = \frac{\rho(x | H_0)}{\rho(x | H_1)} = \frac{Num.}{Den.}$$

$$Num. = \rho(x|H_0) = \int \rho(x|\beta, H_0)\, \rho(\beta|H_0)\, d\beta ,$$

$$\text{Den.} = \rho(x|H_1) = \iint \rho(x|\alpha', \beta, H_1)\,\rho(\alpha', \beta|H_1)\,d\alpha'\,d\beta.$$

We assume that under H_1 the conditional distribution of β given that $\alpha = 0$ is the same as the distribution of β given H_0,

$$\rho(\beta|H_0) = \rho(\beta|\alpha = 0,\ H_1),$$

and we also assume

$$\rho(x|\beta, H_0) = \rho(x|\beta, \alpha = 0,\ H_1).$$

In many problems this will be exactly or approximately correct. Under the assumption that $\rho(\beta|H_1,\ \alpha = 0)$ and $\rho(\alpha,\ \beta|H_1)$ are gentle, the principle of stable estimation is applied to Num. and Den. separately to obtain

$$L(x) = \text{Num.} \doteq \frac{\rho(\hat{\beta}_0|\alpha = 0, H_1)\int \rho(x|\beta, \alpha = 0, H_1)\,d\beta}{\rho(\hat{\alpha}, \hat{\beta}\,|H_1)\iint \rho(x|\alpha', \beta, H_1)\,d\alpha'\,d\beta},$$

$$\doteq \frac{\rho(\hat{\beta}_0|\alpha = 0, H_1)}{\rho(\hat{\alpha}, \hat{\beta}\,|H_1)}\ \rho(\alpha = 0|x, H_1),$$

where $\rho(\alpha = 0|x, H_1)$ is the approximate marginal posterior density of α given x and H_1; $\hat{\alpha}$, $\hat{\beta}$ may be taken at the posterior expectations of α, β under H_1 and $\hat{\beta}_0$ may be taken at the posterior expectation of β under $\alpha = 0$ and H_1; $\doteq$ is "approximately equal to."

Under the assumption that $\rho(\alpha, \beta | H_1)$ is gentle, α and β given H_1 are approximately independent, and we could write

$$\rho(\alpha, \beta | H_1) \doteq \rho(\alpha | H_1) \rho(\beta | H_1) \;,$$

and we get

$$L(x) \doteq \frac{\rho(\hat{\beta}_0 | \alpha = 0, H_1)}{\rho(\hat{\alpha} | H_1)\rho(\hat{\beta} | H_1)} \rho(\alpha = 0 | x, H_1) .$$

Further we assume that $\hat{\beta}_0$, $\hat{\beta}$ are very close and

$$\rho(\hat{\beta}_0 | \alpha = 0, H_1) \doteq \rho(\hat{\beta} | H_1) .$$

Then,

$$L(x) \doteq \frac{\rho(\alpha = 0 | x, H_1)}{\rho(\hat{\alpha} | H_1)} .$$

1.7. Application to the Behrens-Fisher Problem

In the Behrens-Fisher problem, $\rho(\alpha = 0 | x, H_1)$ is the approximate posterior density of $(\mu_2 - \mu_1) = D$ evaluated at $D = 0$, which is equal to $(s^*)^{-1} \rho(d | f_1, f_2; \theta)$ evaluated at $d = (\bar{x}_1 - \bar{x}_2)/s^*$, as defined in section 1.5,

$$\Omega(x) \doteq \Omega \frac{(s^*)^{-1}}{\rho(\hat{D} | H_1)} \rho(d = (\bar{x}_1 - \bar{x}_2)/s^* | f_1, f_2; \theta) .$$

If $\hat{D}$ is very close to zero, as it will be in many applications, we have another approximation

$$\Omega(x) \doteqdot \frac{\Omega}{p(D=0|H_1)} (s^*)^{-1} \; p\left(d = (\bar{x}_1 - \bar{x}_2)/s^* \,\middle|\, f_1, f_2 ; \theta\right).$$

In both the approximations, the densities of the d-variable are needed. The only difference in the two approximations is that in one the density is evaluated at $D = \hat{D}$, and in the other, at $D = 0$. If $\hat{D}$ is taken as the approximate posterior expectation $(\bar{x}_1 - \bar{x}_2)$, and if this is close to zero, as it will be in typical applications, both the approximations are almost the same.

CHAPTER 2

BASIC FORMULAS

2.1. Exact Integral Formulas of the Behrens-Fisher Distributions

Let

$$d = t_2 \cos\theta - t_1 \sin\theta ,$$

where t_1 and t_2 are random variables distributed according to Student's distribution with f_1 and f_2 degrees of freedom. Let:

$F(d|f_1, f_2; \theta) = F(d)$ = be the cumulative probability of d,

$\phi(d|f_1, f_2; \theta) = \phi(d)$ = be the density of d,

$T(t|f)$ be the cumulative probability of Student's t with f degrees of freedom,

$\tau(t|f)$ be the density of Student's t with f degrees of freedom.

Then,

(2.1.1)

$$\phi(d) = \int_{-\infty}^{+\infty} \tau(w\cos\theta - d\sin\theta|f_1)\,\tau(w\sin\theta + d\cos\theta|f_2)\,dw ,$$

$$F(d) = \int_{-\infty}^{d} \phi(z)\,dz \quad .$$

The density of a Student's t with f degrees of freedom is well known to be

$$K(f)\left(1+\frac{t^2}{f}\right)^{-(f+1)/2} ,$$

where

$$K(f) = \frac{\Gamma\left(\frac{f+1}{2}\right)}{(f\pi)^{1/2}\,\Gamma\left(\frac{f}{2}\right)} .$$

$\phi(d)$ can be written

(2.1.2)

$$\phi(d) = K(f_1)K(f_2)\int_{-\infty}^{+\infty}\left[1+\alpha^2(w-d\tan\theta)^2\right]^{-u}\left[1+\beta^2(w+d\cot\theta)^2\right]^{-v}dw ,$$

where

$$\alpha^2 = \cos^2\theta/f_1 \quad , \quad \beta^2 = \sin^2\theta/f_2 \quad ,$$

$$u = (f_1+1)/2 \quad , \quad v = (f_2+1)/2$$

In still other terms,

(2.1.3) $$\phi(d) = K(f_1)K(f_2)\pi \; C_{u,v}(d; \alpha, \beta) ,$$

where

$$C_{u,v}(d; \alpha, \beta) = \frac{1}{\pi}\int_{-\infty}^{+\infty}[1+\alpha^2(\omega - d\tan\theta)^2]^{-u}[1+\beta^2(\omega + d\cot\theta)^2]^{-v} d\omega ,$$

where

$$\tan\theta = (f_2/f_1)^{1/2}(\beta/\alpha).$$

(The factor $\frac{1}{\pi}$, though artificial here, will save us from writing π repeatedly later.)

Thus, the problem of computing $\phi(d)$ is basically the problem of computing $C_{u,v}(d; \alpha, \beta)$ with respect to d.

The computation of $C_{u,v}$ is technically an evaluation of elementary integrals or elliptic integrals, depending upon the parity of f_1 and f_2. Specifically, when both f_1 and f_2 are odd, the integrand is rational in w and the integral is an elementary one. When just one of the indices f_1 and f_2 is odd, the integrand is rational in w except for the square root of a quadratic function in w, so it can be evaluated in terms of inverse trigonometric functions. When both indices are even, the integrand is a rational function in w except for the square root of a quartic function in w, so the integration reduces to the evaluation of the complete elliptic integrals.

The integration is comparatively easy when both f_1 and f_2 are odd, but even then it is actually quite complicated, unless f_1 and f_2 are small. Devices to bring order into the calculations are much to be desired. In particular, various generating functions were tried, and are reported in section 3.1, although only one of them seemed to be helpful.

The extension of the Behrens-Fisher problem to several means leads to a linear combination of an arbitrary number of Student's t variables, $d = \Sigma c_a t_a$, where c_a are constants and t_a are random variables distributed according to Student's distribution with f_a degrees of freedom. The study of the exact distribution of this generalized d involves the evaluation of multiple integrals; and except for the odd degrees of freedom, there seems little hope of evaluating them in closed forms suitable for our computing purposes.

2.2. Ruben's Integral Forms

Ruben (1960) has obtained integral forms expressing a d-variable as the ratio of two independent random variables, the numerator of which is a Student's variable and the denominator a function of a beta variable,

$$d(f_1, f_2; \theta) = d = t_3 / \Phi(x) \quad ,$$

where t_3 is a random variable distributed according to Student's distribution with $(f_1 + f_2)$ degrees of freedom, and x is an independent beta variable with parameters $f_1/2$, $f_2/2$, and

$$\Phi(x) = \left[\frac{(f_1 + f_2)\, x\, (1-x)}{x f_2 \cos^2\theta + (1-x) f_1 \sin^2\theta} \right]^{1/2} .$$

Thus,

(2.2.1)

$$\phi(d \mid f_1, f_2; \theta) = \int_0^1 \frac{\tau(\Phi(x) d \mid (f_1+f_2))\, \Phi(x)\, x^{\frac{f_1}{2}-1}}{B\left(\frac{f_1}{2}, \frac{f_2}{2}\right)} \, (1-x)^{\frac{f_2}{2}-1}\, dx ,$$

where

$$B(p, q) = \Gamma(p)\, \Gamma(q) / \Gamma(p+q) .$$

We have

(2.2.2)
$$F(d) = \int_0^1 \frac{T(\Phi(x) d \mid (f_1+f_2))}{B\left(\frac{f_1}{2}, \frac{f_2}{2}\right)}\, x^{\frac{f_1}{2}-1} (1-x)^{\frac{f_2}{2}-1}\, dx .$$

The form (2.2.1) is a relatively simple tautological reformulation of (2.1.1).

The main reason for studying the formula (2.2.1) in addition to the formula (2.1.1) was the hope that the former would be adapted to the approximation method, sometimes called the "delta method," as will be reviewed in section 7.4, and the former might be better adapted to numerical integration than the latter. Neither of these hopes was actually particularly justified. An apparent advantage of (2.2.1) is that the range of integraion is only from 0 to 1, but this may be more than offset by rather violent behavior of the integrand in that interval.

CHAPTER 3

CLOSED FORMS

3.1. Generating Functions, Recurrence Relations

Both f_1 and f_2 are odd; u,v are integers

The idea of obtaining the coefficients $C_{u,v}$ in a closed and compact form by means of a generating function was tried. Several generating functions were considered, and the one given below was found to be the simplest and the only one that seemed to be of use.

(3.1.1) $$\psi(x,y) = \sum_{v=1}^{\infty} \sum_{u=1}^{\infty} C_{u,v}\, x^u y^v \quad ,$$

where $$0 < x, y < 1 .$$

$$\psi(x,y) = \sum_{v=1}^{\infty} \sum_{u=1}^{\infty} \int_{-\infty}^{+\infty} \frac{1}{\pi} \left(\frac{x}{1+\alpha^2(w - d\tan\theta)^2} \right)^u \left(\frac{y}{1+\beta^2(w + d\cot\theta)^2} \right)^v dw .$$

Since, for each u, v, and w, the integrand is at most $x^u y^v$, we can interchange the order of summation and integration to obtain

$$\psi(x,y) = \frac{xy}{\pi}\int_{-\infty}^{+\infty} \frac{dw}{[1-x+\alpha^2(w-d\tan\theta)^2][1-y+\beta^2(w+d\cot\theta)^2]} .$$

This integral is quite elementary and has been thoroughly checked by deriving it by a number of methods. From a statistical point of view, it is interesting to look at it as the convolution of two Cauchy distributions, which is again a Cauchy distribution.

The result is

(3.1.2) $$\psi(x,y) = \frac{xy\left[\alpha(1-x)^{-1/2} + \beta(1-y)^{-1/2}\right]}{\alpha^2\beta^2\left[\left(\alpha^{-1}(1-x)^{1/2} + \beta^{-1}(1-y)^{1/2}\right)^2 + e^2\right]} ,$$

where

$$e = d(\tan\theta + \cot\theta) = d/\sin\theta\cos\theta .$$

Therefore,

(3.1.3) $$\sum_{v=1}^{\infty}\sum_{u=1}^{\infty} C_{u,v}x^u y^v \left[\left(\alpha^{-1}(1-x)^{1/2} + \beta^{-1}(1-y)^{1/2}\right)^2 + e^2\right]$$

$$= xy(\alpha\beta)^{-2}\left[\alpha(1-x)^{-1/2} + \beta(1-y)^{-1/2}\right] .$$

By equating the coefficients of $x^u y^v$, the following recursion formulas have been obtained in terms of M, a_r, b_r, where

$$M = \left[(\alpha+\beta)^2 + \alpha^2 \beta^2 e^2\right]^{-1},$$

$$a_r = \binom{1/2}{r}(-1)^r = -\frac{1 \cdot 1 \cdot 3 \cdot 5 \cdots (2r-5)(2r-3)}{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2r-2)\, 2r},$$

$$b_r = -(2r-1)\, a_r .$$

Case (1), v = 1

(3.1.4) $$C_{u,1} = \left[\alpha b_u + \beta(\alpha+\beta) C_{u-1,1} - 2\alpha\beta \sum_{r=1}^{u-2} C_{r,1} a_{u-r} a_0\right] M.$$

Case (2), u = 1

(3.1.5) $$C_{1,v} = \left[\beta b_v + \alpha(\alpha+\beta) C_{1,v-1} - 2\alpha\beta \sum_{s=1}^{v-2} C_{1,s} a_{v-s} a_0\right] M.$$

Case (3), u,v > 1

(3.1.6) $$C_{u,v} = \Big[(\alpha+\beta)\left(\alpha C_{u,v-1} + \beta C_{u-1,v}\right) - 2\alpha\beta \sum_{s=1}^{v-1} \sum_{r=1}^{u-1} C_{r,s}\, a_{u-r}\, a_{v-s}$$

$$- 2\alpha\beta \sum_{r=1}^{u-2} C_{r,v}\, a_{u-r}\, a_0 \;-\; 2\alpha\beta \sum_{s=1}^{v-2} C_{u,s}\, a_{v-s}\, a_0 \Big] M .$$

These recursion formulas are straightforward and they have been thoroughly tested by their numerical implication.

Some thought was given to the development of $\psi(x,y)$ as a power series in x and y. $\psi(x,y)$ can be re-expressed as:

$$\psi(x,y) = \frac{xy}{\alpha\beta(1-x)^{1/2}(1-y)^{1/2}} \frac{\left[\alpha^{-1}(1-x)^{1/2} + \beta^{-1}(1-y)^{1/2}\right]}{\left[\left(\alpha^{-1}(1-x)^{1/2} + \beta^{-1}(1-y)^{1/2}\right)^2 + e^2\right]}$$

$$= \frac{xy\, e^{-1}}{\alpha\beta(1-x)^{1/2}(1-y)^{1/2}} \frac{\left[A(1-x)^{1/2} + B(1-y)^{1/2}\right]}{\left[\left(A(1-x)^{1/2} + B(1-y)^{1/2}\right)^2 + 1\right]} ,$$

where

$$A = (\alpha e)^{-1} , \qquad B = (\beta e)^{-1} .$$

The basic problem is to express $\psi_1(x,y)$ as a power series in x and y, where

(3.1.7) $$\psi_1(x,y) = \frac{A(1-x)^{1/2} + B(1-y)^{1/2}}{\left[A(1-x)^{1/2} + B(1-y)^{1/2}\right]^2 + 1} .$$

Some ideas that occurred to me are these:

(1) $$\frac{1}{1 - iV} = \frac{1 + iV}{1 + V^2} .$$

$$\therefore \quad \psi_1(x, y) = I_m\left[\frac{1}{1 - i\left(A(1-x)^{1/2} + B(1-y)^{1/2}\right)}\right],$$

where $I_m(z)$ is the imaginary part of z. This largely reduces the problem to developing

$$\left[1 - i\left(A(1-x)^{1/2} + B(1-y)^{1/2}\right)\right]^{-1}$$

in a power series in x and y.

A different direction is this:

$$\psi_1(x, y) = \frac{A(1-x)^{1/2} + B(1-y)^{1/2}}{1 + A^2 + B^2 - A^2x - B^2y + 2AB(1-x)^{1/2}(1-y)^{1/2}}$$

$$= \frac{A(1-x)^{1/2} + B(1-y)^{1/2}}{D^2 - A^2x - B^2y + 2AB(1-x)^{1/2}(1-y)^{1/2}}$$

$$= \frac{\left[A(1-x)^{1/2} + B(1-y)^{1/2}\right]\left[D^2 - A^2x - B^2y - 2AB(1-x)^{1/2}(1-y)^{1/2}\right]}{\left[D^2 - A^2x - B^2y\right]^2 - 4A^2B^2(1-x)(1-y)}$$

$$= \text{Num.}/\text{Den.} ,$$

where

$$D^2 = 1 + A^2 + B^2 .$$

It is very easy to express Num. as a power series in x and y, but it is not so easy to expand $\frac{1}{\text{Den.}}$ as a power series in x and y.

Some thought was given to the function

$$\sum_{v=1}^{\infty} \sum_{u=1}^{\infty} C_{u,v} \, x^u y^v / (u-1)!(v-1)! \; .$$

This leads to the evaluation of the integral

$$\frac{xy}{\pi} \int_{-\infty}^{+\infty} \frac{\exp\left[\frac{x}{1+\alpha^2(\omega - d\tan\theta)^2} + \frac{y}{1+\beta^2(\omega + d\cot\theta)^2}\right]}{\left[1+\alpha^2(\omega - d\tan\theta)^2\right]\left[1+\beta^2(\omega + d\cot\theta)^2\right]} \, d\omega .$$

Another slightly different generating function is

$$\sum_{v=1}^{\infty} \sum_{u=1}^{\infty} C_{u,v} \, x^u y^v / u! \, v! \; .$$

This leads to the evaluation of the integral

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\left[\exp\left(\frac{x}{1+\alpha^2(w-d\tan\theta)^2}\right)-1\right]\left[\exp\left(\frac{y}{1+\beta^2(w+d\cot\theta)^2}\right)-1\right]dw.$$

Both look hopeless.

3.2. Closed Forms

Closed forms of $C_{u,v}(d;\ \alpha,\ \beta)$ have been obtained for u,v = 1(1)7 by starting with $C_{1,1} = (\alpha + \beta)M$ and using the recurrence relations (3.1.4), (3.1.5), (3.1.6). These expressions were checked by calculating the zeroth and second moments of d by using $\phi(d) = K(f_1)K(f_2)\pi\ C_{u,v}$ as the density of d, and comparing them with the known zeroth and second moments of d:

$$\mu_0 = 0 \quad , \quad \mu_2 = \cos^2\theta\,\frac{f_2}{(f_2-2)} + \sin^2\theta\,\frac{f_1}{(f_1-2)} \ .$$

These moments are easily obtained since t_1 and t_2 are independent and the second moment of a Student's t with f degrees of freedom is $f/(f - 2)$ for $f > 2$.

The closed forms are also strongly guaranteed because numerical results derived from them agree with results obtained on a very different basis.

One interesting way to express $C_{u,v}$ obtained from the recurrence relations is for $u > v$,

(3.2.1) $$C_{u,v}(d;\alpha,\beta) = \sum_{j=v}^{u+v-1} L_{j;u,v} M^j,$$

where $L_{j;u,v}$ is a polynomial in α and β. $C_{u,v}$ is a linear combination of u dilated Student's densities with different widths and degrees of freedom ranging from $(2v - 1)$ to $2(u + v - 1) - 1$. For simplicity, writing $L_{j;u,v} = L_j$,

$$C_{u,v} = \sum_{j=v}^{u+v-1} \left[(\alpha+\beta)^{-2j} L_j \,\tau(e_j d\,|\,2j-1) \Big/ K(2j-1)\right],$$

where

$\tau(t|f)$ is, as before, the Student's density with

f degrees of freedom, and

$e_j = (2j - 1)^{1/2}/(\alpha + \beta)f_1f_2$.

From this, and (2.1.3),

(3.2.2) $$p(d|f_1,f_2;\theta) = \pi K(f_1)K(f_2)\sum_{j=v}^{u+v-1}\left[\frac{(\alpha+\beta)^{-2j} L_j\,\tau(e_j d\,|\,2j-1)}{K(2j-1)}\right],$$

and

(3.2.3) $$F(d) = \int_{-\infty}^{d} \phi(z)\,dz = \pi K(f_1) K(f_2) \sum_{j=v}^{u+v-1} \left[\frac{(\alpha+\beta)^{-2j} L_j T(e_j d \,|\, 2j-1)}{K(2j-1)\, e_j} \right],$$

where

T(t|f) is, as before, the cumulative probability of Student's t with f degrees of freedom.

The coefficients $L_{j;u,v}$, polynomials in α and β, do not seem sufficiently regular to be written in a closed and compact form covering all values of the indices; at least any such form was not discovered while studying the closed forms for u,v = 1(1)7, given in the Appendix.

The density and cumulative probability of Behrens-Fisher distributions are linear combinations of dilated Student's densities and cumulative probabilities. If the polynomials $L_{j;u,v}$ were not complicated and the number of terms (= max(u,v)) not potentially numerous, then this would be a very satisfactory way to compute Behrens-Fisher densities and cumulative probabilities when f_1 and f_2 are both odd, though not percentage points. The idea is usable for special purposes, especially for small values of u and v, though it is not promising for everyday practical use.

Since, if f_1 and f_2 are odd, the Behrens-Fisher density is a linear combination of dilated Student's densities with odd degrees of freedom, the same will be true of the generalized Behrens-Fisher density, the density of any linear combination of an arbitrary number of independently distributed t-like random variables with odd degrees of freedom. Thus, the task of computing the density of such a generalized Behrens-Fisher distribution is elementary in the technical sense and, in fact, it can be done with sufficient time, energy, and patience.

The generating function (3.1.1) does not touch the case of even values of f_1 and f_2. Analogous generating functions were derived, but they are much more cumbersome because they involve inverse trigonometric functions and in the worst case, elliptic integrals, and seemed of very little help for our purpose. Their derivation is sketched in the Appendix.

Finally, interest in several terms of in a linear combination of several independent Student's t variables is almost as great as in two. The formula for odd degrees of freedom does generalize in principle, but threatens to become even more cumbersome, and the limitation to odd values becomes even more severe. So, formula (3.2.1), and its possible generalizations, is far from providing the man in the laboratory easy access to densities or cumulative probabilities of the Behrens-Fisher distributions and their generalizations.

Computation of percentage points would still remain a problem even where the formula (3.2.1) is usable (whether because the indices are small or because the computing facilities are large).

So, for all these reasons, it is important to find simple, easily implemented approximations.

CHAPTER 4

EARLIER NUMERICAL WORK

Abstracts of the tables of the Behrens-Fisher variable d are given by Greenwood and Hartley (1962, pages 232–36). Only those that are published in widely available tables are reported here in detail; others are just mentioned.

Let:

F(d), as defined before, be the cumulative probability of d,

d_p be the loop-percent point of d (two tailed).

Behrens (1929): Behrens gives F(d) for some values of d and f, where $f_1 = f_2 = f$, and s_1/s_2, where $\tan \theta = s_1/s_2$.

Sukhatme (1938): The values of the 5-percent and 1-percent points of the distribution of d for values of θ differing by 15 degrees and for all twenty-five combinations of f_1, f_2 in the harmonic series 6, 8, 12, 24, ∞ were tabulated by Sukhatme, at Fisher's suggestion. Sukhatme determined these percentage points to three decimal places, by direct numerical integration. d_p are tabulated for

$f_1, f_2 = 6, 8, 12, 24, \infty;$

$\theta = 0° (15°) 90°;$

$p = 0.05, 0.01.$

Errors in the third place were discovered by Fisher (1941), and these values have been revised by Sukhatme et al. (1951) to correct this error. These revised values have been printed in Fisher and Yates tables (1948, Table V_1; 1957, Table VI).

Fisher (1941): Fisher gave asymptotic expansions in terms of the powers of f_1^{-1} and f_2^{-1} for calculating the cumulative probability and the percentage point in any particular case, and a further range of tables when either f_1 or f_2 is large.

He calculated d_p to three decimal places for

$f_1 = \infty, f_2 = 10, 12, 15, 20, 30, 60, \infty;$

$\theta = 0° (10°) 90°;$

$p = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002.$

These are reprinted in Fisher and Yates (1943, 1948, 1953) as Table V_2 and in 1957 as Table VI_2.

Chapman (1950): Chapman gives F(d) for certain values d, f, where $f_1 = f_2 = f$, and $\theta = 45°$; he also gives certain percentage points d_p, and for $f_1 = f_2 = 11$, a normal approximation.

Fisher and Yates (1957, Table VI_1): Based on the formulas of Fisher and Healy (1956), Fisher and Yates calculated dp to 5 decimal places for

$f_1 = 1(2)7,\ f_2 = f_1(2)7;$

$\theta = 0° \ (15°) \ 90°;$

$p = 0.1,\ 0.05,\ 0.02,\ 0.01.$

CHAPTER 5

NUMERICAL VALUES OF DENSITIES

5.1. Need for Numerical Values

This chapter is interesting not for the user of the Behrens-Fisher distributions as such, but only for someone who wants to compute them accurately on high-speed computing equipment, especially with a view to doing further research.

It seemed impractical to evaluate the errors of various methods of approximations which came up for consideration analytically, so a supply of nominally exact values was found necessary; that is, values exact up to certain significant figures, to check proposed approximations against. For the most part, the study was confined to odd values of f_1 and f_2, since these were easier to compute and seemed adequately representative except, possibly, for very small numbers of degrees of freedom.

5.2. Case when Both f_1 and f_2 Are Odd

Several methods of obtaining numerical values presented themselves; the ordinary numerical integration, the closed forms, are available, as well as recurrence relations. It was thought possible to proceed reliably and with less computing time by the recursion formulas than by numerical integration. Closed forms may be convenient sometimes for computation for small odd values of f_1, f_2; but they become very inconvenient for machine computation with substantial values of f_1 and f_2, because it does not seem easy to have the machine compute the polynomial coefficients $L_{j;u,v}$ nor are they, by any means, easy to compute by hand.

Certain numerical values were computed by writing a program (in the MAD language) for the IBM 709 computer on the basis of the recurrence relations (3.1.4), (3.1.5), (3.1.6), and (2.1.3).

The densities $\phi(d|f_1, f_2; \theta)$ were calculated to seven significant figures, not to be taken seriously after six significant figures, for

$$f_2 = 1(2)9, \; f_1 = f_2(2)19;$$

$$\theta = 0° \; (7.5°) \; 90°;$$

$$d = 0(0.2)7.$$

It would not be practical or useful to present a table of 18720 values here. The numerical values are punched on cards, and copies of these cards could be provided at cost. Any potential user would find the program on the basis of which they were computed far more useful and economical; this is given in the Appendix.

The following checks were made for the machine (IBM 709) calculated values.

Certain numerical values were computed on a desk calculator from the closed forms given in Tables 8–14, and formula (3.2.2), and also for the special case

$$f_1 = f_2 \quad , \qquad \theta = 45^\circ \qquad d = 0$$

(5.2.1) $$\phi(d = 0 \mid f_1, f_1; 45^\circ) = \frac{(K(f_1))^2}{K(2f_1 + 1)} \left(\frac{2f_1}{2f_1 + 1}\right)^{1/2}$$

$K(f_1)$ are available up to six decimal places. $\left(\frac{2f_1}{2f_1 + 1}\right)^{1/2}$ was computed up to nine decimal places, and $\phi(d = 0|f_1, f_1; 45^\circ)$ were computed for $f_1 = 1(2)9$.

For the closed forms, each operation was carried to nine decimal places, and $\phi(d|f_1, f_2; \theta)$ were computed for $f_2 = 1(2)5$, $f_1 = f_2(2)5$; $\theta = 30^\circ, 60^\circ$; $d = 0(1)5$.

In both the checks, the machine values and the desk-computed values agreed to six significant figures.

The number of operations in the program for calculating $\phi(d|f_1, f_2; \theta)$ do not change as d and θ change. According to the three recurrence formulas (3.1.4), (3.1.5), (3.1.6), and (2.1.3), $\phi(d|f_1, f_2; \theta)$ is the sum of uv - 1 terms, where $u = (f_1 + 1)/2$, $v = (f_2 + 1)/2$, and not both u,v = 1, in which case $\phi(d|1, 1, \theta)$ is just one term. The agreement of the decimal places of the machine-calculated and desk-calculated values did not change when the number of terms increased from 1 to 24 as f_1 and f_2 increased from $f_1 = f_2 = 1$ to $f_1 = f_2 = 9$. This is quite encouraging and it appears that the same might be true when the number of terms increased from 24 to 49 as f_1 and f_2 increase from $f_1 = f_2 = 9$ to $f_1 = 19$, $f_2 = 9$.

This is all that can be said about the accuracy of the numerical values that were computed on the IBM 709 machine on the basis of the recurrence relations.

5.3. Case when Either f_1 or f_2 Is Even

Certain numerical values of the densities were calculated for this case by using Ruben's integral form (2.2.1) and the numerical integration method in the book by Hildebrand (1956, formula (3.6.2), pages 71–78). These were obtained by writing a program (in the MAD language) for the IBM 709 computer.

The program was so written that the value of the integral was recomputed with twice as many points in the net until two successive values were the same to four significant figures.

The formula and the program were checked in two ways. A few values were calculated by this method for $f_1 = f_2 = 3$, and checked against the hand-computed values. Secondly, a few values were calculated by using the formula (5.2.1) for the special case $f_1 = f_2$, $\theta = 45°$, and $d = 0$. Both these checks indicate that the values on the high-speed computer appear to be correct to four significant figures.

These numerical values $\phi(d|f_1, f_2; \theta)$ were computed for the following cases:

$(f_1, f_2) = (2,2)$; $\theta = 15°$; $d = 0(1)3$;

$(f_1, f_2) = (4,4)$; $\theta = 15° \ (15°) \ 45°$; $d = 0(2)6$;

$(f_1, f_2) = (6,6), (8,8)$; $\theta = 15°, 45°$; $d = 0(2)6$;

$(f_1, f_2) = (7,6), (8,6)$; $\theta = 15° \ (30°) \ 75°$; $d = 0(2)6$.

These values are given in Table 1.

5.4. Harmonic Interpolation in f_1, f_2

The numerical values obtained in section (5.3) were compared with the values calculated by harmonic interpolation in f_1 and f_2, as suggested by Fisher. The harmonic interpolation was found better than the direct interpolation. Harmonic interpolation in f_1, f_2 is the same as direct interpolation in f_1^{-1}, f_2^{-1}. This was studied to find out whether the numerical densities for even degrees of freedom can be calculated from those of odd degrees of freedom by harmonic interpolation in f_1, f_2. The interpolated value $\phi^*_{f_1, f_2}$ is calculated according to the formula

(5.4.1)
$$\phi^*_{f_1,f_2} = \Big[\frac{(f_1+1)(f_2+1)}{4f_1f_2}\phi_{f_1+1,f_2+1} + \frac{(f_1+1)(f_2-1)}{4f_1f_2}\phi_{f_1+1,f_2-1} + \frac{(f_1-1)(f_2+1)}{4f_1f_2}\phi_{f_1-1,f_2+1} + \frac{(f_1-1)(f_2-1)}{4f_1f_2}\phi_{f_1-1,f_2-1}\Big],$$

where
$$\phi_{f_1,f_2} = \phi(d|f_1,f_2;\theta).$$

The comparison of the interpolated values $\phi^*(d|f_1, f_2; \theta)$ with $\phi(d|f_1, f_2; \theta)$ is given in Table 1, in terms of the percentage error, which is calculated as

$$\%\text{ error} = \frac{100[\phi^*(d|f_1,f_2;\theta) - \phi(d|f_1,f_2;\theta)]}{\phi(d|f_1,f_2;\theta)}.$$

TABLE 1

COMPARISON OF THE DENSITIES $\phi^*(d|f_1,f_2;\theta)$, AS IN (5·4·1), OBTAINED BY HORMONIC INTERPOLATION IN f_1,f_2 WITH THE EXACT DENSITIES $\phi(d|f_1,f_2;\theta)$ IN TERMS OF PERCENTAGE ERRORS IN $\phi(d|f_1,f_2;\theta)$

θ	d	ϕ(d)	Pct. Error	θ	d	ϕ(d)	Pct. Error
			$f_1 = f_2 = 2$				
0°	0	0.3536	+ 0.48	15°	0	0.3251	+ 0.31
	1	0.1925	+ 1.25		1	0.1952	+ 0.51
	2	0.06804	- 2.19		2	0.07423	- 2.96
	3	0.02741	- 8.11		3	0.03004	- 7.33
			$f_1 = f_2 = 4$				
0°	0	0.3750	+ 0.023	15°	0	0.3627	- 0.030
	2	0.06629	- 0.44		2	0.06936	- 0.28
	4	0.006708	- 1.02		4	0.006625	+ 0.60
	6	0.001186	+ 5.06		6	0.001112	+ 6.31
30°	0	0.3474	+ 0.031	45°	0	0.3417	+ 0.032
	2	0.07436	- 0.54		2	0.07631	- 0.66
	4	0.006627	+ 0.75		4	0.006675	+ 0.90
	6	0.0009874	+11.81		6	0.0009334	+13.61
			$f_1 = f_2 = 6$				
0°	0	0.3827	+ 0.0039	15°	0	0.3752	+ 0.0021
	2	0.06404	- 0.13		2	0.06593	- 0.10
	4	0.004055	+ 0.17		4	0.003879	+ 0.41
	6	0.0004220	+ 5.21		6	0.0003769	+15.12

TABLE 1 (Continued)

θ	d	ϕ(d)	Pct. Error	θ	d	ϕ(d)	Pct. Error
			$f_1 = f_2 = 6$				
45°	0	0.3596	+ 0.0011	45°	4	0.003460	+ 1.45
	2	0.07118	- 0.16		6	0.0002357	+11.58
			$f_1 = 7, f_2 = 6$				
0°	0	0.3827	+ 0.0039	15°	0	0.3761	+ 0.0050
	2	0.06404	- 0.13		2	0.06559	- 0.12
	4	0.004055	+ 0.17		4	0.003846	+ 0.26
	6	0.0004220	+ 5.21		6	0.0003747	+ 6.11
45°	0.03623		+ 0.0031	75°	0	0.3777	- 0.0011
	2	0.07025	- 0.071		2	0.06505	+ 0.025
	4	0.003064	+ 0.85		4	0.003165	+ 0.19
	6	0.0001826	+ 7.33		6	0.000239	+ 0.29
			$f_1 = 8, f_2 = 6$				
0°	0	0.3827	+ 0.0039	15°	0	0.3768	+ 0.0011
	2	0.06404	- 0.13		2	0.06536	- 0.12
	4	0.004055	+ 0.17		4	0.003826	+ 0.29
	6	0.0004220	+ 5.21		6	0.0003735	+ 6.10
45°	0	0.3643	- 0.0051	75°	0	0.3796	- 0.0080
	2	0.06951	- 0.12		2	0.06431	- 0.025
	4	0.002796	+ 1.29		4	0.002648	+ 0.57
	6	0.0001541	+10.77		6	0.0001585	+ 5.05

TABLE 1 (Continued)

θ	d	φ(d)	Pct. Error	θ	d	φ(d)	Pct. Error
			$f_1 = f_2 = 8$				
0°	0	0.3867	+ 0.0010	15°	0	0.3813	+ 0.0012
	2	0.06237	- 0.053		2	0.06371	- 0.016
	4	0.002756	+ 0.33		4	0.002597	+ 0.50
	6	0.0001800	+ 4.44		6	0.0001561	+ 5.06
45°	0	0.3690	+ 0.0013	45°	4	0.002133	+ 0.80
	2	0.06771	- 0.015		6	0.00007675	+ 8.46

Conclusion

The comparison in Table 1 shows that harmonic interpolation works very well for $0 \leq |d| \leq 4$. The maximum percentage error in this range, for the cases considered, is 1.45 percent for $f_1, f_2 \geq 4$. the interpolation gets worse as the densities get smaller and the maximum percentage error for $d = 6$, for the cases considered, is 15.12 percent.

CHAPTER 6

MOMENTS AND CUMULANTS OF d

6.1. Moments and Cumulants of Student's Distributions

Owing to the symmetry of Student's distribution, its odd moments are zero. The $2\nu^{th}$ moment of a Student's distribution with f degrees of freedom is

$$\mu_{2\nu} = \frac{1 \cdot 3 \cdot 5 \cdots (2\nu - 1) f^{\nu}}{(f-2)(f-4)(f-6)\cdots(f-2\nu)}, \quad \text{for } 2\nu < f$$

(Cramér, 1945, page 239.) .

In particular, the second, fourth, and sixth moments are

$$\mu_2 = \frac{f}{(f-2)} = \frac{1}{(1 - \frac{2}{f})},$$

$$\mu_4 = \frac{3f^2}{(f-2)(f-4)} = \frac{3}{(1-\frac{2}{f})(1-\frac{4}{f})};$$

$$\mu_6 = \frac{15f^3}{(f-2)(f-4)(f-6)} = \frac{15}{(1-\frac{2}{f})(1-\frac{4}{f})(1-\frac{6}{f})} .$$

Using the relation between moments and cumulants given in Cramér (1945, page 187), the second, fourth, and sixth cumulants are

(6.1.1) $$\kappa_2(f) = \frac{f}{(f-2)} = \frac{1}{(1-\frac{2}{f})} ,$$

$$\kappa_4(f) = \frac{6f^2}{(f-2)^2(f-4)} = \frac{6}{f(1-\frac{2}{f})^2(1-\frac{4}{f})} ,$$

$$\kappa_6(f) = \frac{240f^3}{(f-2)^3(f-4)(f-6)} = \frac{240}{f^2(1-\frac{2}{f})^3(1-\frac{4}{f})(1-\frac{6}{f})} ,$$

and

(6.1.2) $$l_{2\nu}(f) = \kappa_{2\nu}(f)/[\kappa_2(f)]^\nu .$$

6.2. Moments and Cumulants of d

As defined before,

$$d = t_2 c - t_1 s ,$$

where

$$c = \cos\theta, \qquad s = \sin\theta .$$

Since t_1 and t_2 are independently distributed according to Student's distribution with f_1 and f_2 degrees of freedom, the $2\nu^{th}$ cumulant of d is

(6.2.1) $$\kappa_{2\nu}(f_1, f_2, c, s) = c^{2\nu}\kappa_{2\nu}(f_2) + s^{2\nu}\kappa_{2\nu}(f_1) ,$$

and the $2\nu^{th}$ cumulant of $d/\kappa_2^{1/2}(f_1, f_2, c, s)$ is

(6.2.2) $$\ell_{2\nu}(f_1, f_2, c, s) = \ell_{2\nu} = \frac{\kappa_{2\nu}(f_1, f_2, c, s)}{[\kappa_2(f_1, f_2, c, s)]^{\nu}} .$$

The $\ell_{2\nu}$ are calculated by substituting the expressions for $\kappa_{2\nu}(f_1)$, $\kappa_{2\nu}(f_2)$. Expand ℓ_4, ℓ_6 in ascending powers of f_1^{-1}, f_2^{-1}, and $(f_1f_2)^{-1}$. By retaining the terms only up to the order of f_1^{-2}, f_2^{-2}, $(f_1f_2)^{-1}$, the ℓ_4 and ℓ_6 are

(6.2.3) $$\ell_4 \doteq 6\left(\frac{s^4}{f_1} + \frac{c^4}{f_2}\right) + \frac{24 s^4(1+c^2)}{f_1^2} + \frac{24 c^4(1+s^2)}{f_2^2} - \frac{24 c^2 s^2}{f_1 f_2} ,$$

(6.2.4) $$\ell_6 \doteq 240\left(\frac{c^6}{f_2^2} + \frac{s^6}{f_1^2}\right) .$$

Once again, using the relation between cumulants and moments, or proceeding directly, the second, fourth, and sixth moments of d are

$$\mu_2(f_1, f_2, c, s) = \mu_2 = \frac{c^2 f_2}{(f_2 - 2)} + \frac{s^2 f_1}{(f_1 - 2)} ,$$

$$\mu_4(f_1, f_2, c, s) = \mu_4 = \frac{3c^4 f_2^2}{(f_2 - 2)(f_2 - 4)} + \frac{3s^4 f_1^2}{(f_1 - 2)(f_1 - 4)} + \frac{6 s^2 c^2}{(f_1 - 2)(f_2 - 2)} ,$$

$$\mu_6(f_1, f_2, c, s) = \mu_6 = \frac{15 c^6 f_2^3}{(f_2 - 2)(f_2 - 4)(f_2 - 6)} + \frac{15 s^6 f_1^3}{(f_1 - 2)(f_1 - 4)(f_1 - 6)}$$

$$+ \frac{45 c^4 s^2 f_2^2 f_1}{(f_2 - 2)(f_2 - 4)(f_1 - 2)} + \frac{45 c^2 s^4 f_2 f_1^2}{(f_2 - 2)(f_1 - 2)(f_1 - 4)}$$

CHAPTER 7

APPROXIMATIONS

7.1. Aim

Several approximations were explored in a search for methods that would enable a person with a modest kit of equipment, such as standard statistical tables (like Fisher and Yates), a slide rule or a desk-calculator, and perhaps some special table, to compute Behrens-Fisher densities, cumulative probabilities, and percentage points. Different applications require different levels of accuracy and, of course, higher levels of accuracy might be expected to require a more elaborate kit of equipment as well as more computing time. The ideal is to cover all degrees of freedom beginning with (1,1), when the distribution of d is a Cauchy distribution.

In practical applications, very rough accuracy might be adequate. Someone who has an occasion to say that all but 1% of his posterior probability is in a certain interval will not ordinarily be much harmed if only all but 1-1/2% or 2% is in it, nor would he, perhaps, be much harmed if the ostensible 95% interval is a 95-1/2%

interval. Similarly, where densities are to be applied in testing hypotheses, a factor of even 2 or 3 in either direction might be quite unimportant, and errors of as little as 10% or 15% would be negligible.

A major part of the practical problem will be solved if a good approximation is obtained for the cases in which the two degrees of freedom f_1, f_2 are greater than 10. Then, one could look for an approximation which is, perhaps, slightly more complex for the cases in which f_1 and f_2 are smaller.

Several ideas were tried to achieve this aim, and are divided into two categories:

1. Asymptotic approximations.
2. Approximation by one dilated Student's distribution.

7.2. Comparison with the Nominal Exact Values

These approximations were explored by checking them against some of the nominal exact values of the densities which were computed for this purpose. Only approximation (2) was explored further by checking it against some of the nominal exact values of the cumulative probabilities and percentage points that are available, for the simple reason that this was found to be most satisfactory and practical for the densities and therefore, might work for the cumulative probabilities and the percentage points as well.

The merit of any approximation depends upon how well it works for the densities, cumulative probabilities, and percentage points. If it works satisfactorily for the densities and percentage points, we can expect it to work for the cumulative probabilities, too.

In our exploration, the comparison of the densities (cumulative probabilities) is given in terms of the percentage error in density (cumulative probability) calculated according to the formula,

$$(7.2.1) \quad \%\ \text{error} = \frac{(\text{approximate density} - \text{exact density})100}{\text{exact density}} .$$

The comparison of an approximate loop-percent point, d'_p, with the exact loop-percent point, d_p, is given in terms of the percentage error in the level p. This is calculated by considering the probability Δp contained in the interval $[d'_p, d_p]$ and according to the formula

$$(7.2.2) \quad \%\ \text{error in } p = \frac{2\Delta p(100)}{p} .$$

It is not possible to obtain exact nominal Δp, and hence, we can compute only approximate percentage error in p.

Asymptotic approximations are considered in this chapter and the approximation by one dilated Student's distribution is considered in Chapter 8.

The following asymptotic approximations are explored in this chapter:

1. The Hermite polynomial method.
2. Application of the delta method to integral forms due to Ruben.

7.3. The Hermite Polynomial Method

Fisher (1925) gave an asymptotic expansion in terms of the powers of f^{-1} for the Student's density with f degrees of freedom. Then, Fisher (1935) considered the convolution of the two component densities to give an asymptotic expansion in terms of the powers of f_1^{-1}, f_2^{-1} for the cumulative probability and percentage points of d in any particular case. His method can be very easily extended to calculate the density of d and also to the generalized Behrens-Fisher distributions. His method yields the same result as obtained by the application of the Hermite polynomial method directly to d to approximate the Behrens-Fisher distribution.

The density of a random variable x can formally be expressed in terms of the normal density, Hermite polynomials, and cumulants of x.

The density of a random variable x with odd cumulants zero is represented formally by

$$(7.3.1) \qquad \rho(x) = \exp\left[\frac{\kappa_2 - \sigma^2}{2!} D^2 + \frac{\kappa_4 D^4}{4!} + \frac{\kappa_6}{6!} D^6 + \cdots \right] \frac{\nu(x/\sigma)}{\sigma}$$

where

$$\nu(x/\sigma) = \frac{e^{-x^2/2\sigma^2}}{(2\pi)^{1/2}} , \qquad D = \frac{d}{dx} .$$

This development is called "the Edgeworth form of the type A expansion." Refer to Kendall (1943, Vol. 1, section 6.24, pages 156–57).

The use of the series (7.3.1) has been severely criticized by several authors who have objected to it on mathematical as well as statistical grounds. Refer to Kendall (1943, pages 152–53).

In practical applications, however, the important question is not whether an infinite series represents a density, but whether a finite number of terms does so to a satisfactory approximation. Even when the infinite series diverges, its first few terms may, nonetheless, give a satisfactory approximation. This subject has not been fully explored for the present problem (Wallace, 1958).

Suppose $\kappa_{2r}\sigma^{-2r}$, for $r > 1$, is of order n^{1-r}, and $\frac{\kappa_2 - \sigma^2}{\sigma^2}$ is of order of n^{-1}, where n is some parameter.

Put

$$\ell_2' = \frac{(k_2 - \sigma^2)}{\sigma^2} \,, \quad \ell_{2r}' = k_{2r}\sigma^{-2r} \quad \text{for } r > 1 .$$

Then (7.3.1) is

$$\rho(x) = \exp\left[\frac{1}{2}\ell_2'\sigma^2 D^2 + \frac{1}{4!}\ell_4'\sigma^4 D^4 + \frac{1}{6!}\ell_6'\sigma^6 D^6 + \cdots\right]\frac{\nu(x/\sigma)}{\sigma} .$$

Expand the operator and retain terms up to and including $0(n^{-2})$ in ℓ's. The result of this operation is obtained by replacing the operator $\sigma^r\ell^r$ by $(-1)^r H_r(x/\sigma)$ and multiplying by $\frac{\nu(x/\sigma)}{\sigma}$. Then the density of x is

(7.3.2)

$$\rho(x) \doteq \frac{\nu(x/\sigma)}{\sigma}\Big[1 + \frac{1}{2}\ell_2' H_2(x/\sigma) + \frac{1}{8}\ell_2'^2 H_4(x/\sigma)$$
$$+ \frac{1}{48}\ell_2'\ell_4' H_6(x/\sigma) + \frac{1}{24}\ell_4' H_4(x/\sigma) + \frac{\ell_4'^2}{1152} H_8(x/\sigma) + \frac{\ell_6' H_6(x/\sigma)}{720}\Big] ,$$

where

$$H_r(x)\nu(x) = (-D)^r \nu(x) \; ; \quad \text{and}$$

$$\int_{-\infty}^{+\infty} H_r(x) H_s(x) \nu(x) dx = n! \, \delta_{rs},$$

specifically,

$$H_2(x) = x^2 - 1$$
$$H_4(x) = x^4 - 6x^2 + 3$$
$$H_6(x) = x^6 - 15x^4 + 45x^2 - 15$$
$$H_8(x) = x^8 - 28x^6 + 210x^4 - 420x^2 + 105.$$

This is an approximation of Edgeworth's series (7.3.1) in the sense that only few terms of the series are considered. The terms $\nu(z)H_r(z)$ have been tabulated in Aiken <u>et al.</u> (1952).

We obtain two different approximations $H^{(1)}$ and $H^{(2)}$ by application of (7.3.2) to the Behrens-Fisher density due to two choices of σ: $\sigma = 1$ and $\sigma = \kappa_2^{1/2}$.

<u>Approximation $H^{(1)}$</u>

We take $x = d$, $\sigma = 1$. Then $\ell'_2 = \kappa_2 - 1$, $\ell'_{2r} = \kappa_{2r}$ for $r > 1$, and these are calculated from the formulas (6.2.1). From (7.3.2) we obtain,

$$\phi(d) \doteqdot \nu(d)\Big[1 + (k_2 - 1)H_2(d) + \frac{1}{8}k_2^2 H_4(d) + \frac{1}{48}k_2 k_4 H_6(d) + \frac{1}{24}k_4 H_4(d) + \frac{1}{1152}k_4^2 H_8(d) + \frac{1}{720}k_6 H_6(d)\Big]. \tag{7.3.3}$$

This approximation $H^{(1)}$ is the same as that of Fisher's method extended to densities.

Approximation $H^{(2)}$

We take $x = d$, $\sigma = k_2^{1/2}$. Then $\ell'_2 = 0$, $\ell'_{2r} = \ell_{2r}$, for $r > 1$, and these are calculated according to the formulas (6.2.2). From (7.3.2), we have

$$\phi(d) \doteqdot \frac{\nu(y)}{k_2^{1/2}}\Big[1 + \frac{1}{24}\ell_4 H_4(y) + \frac{1}{1152}\ell_4^2 H_8(y) + \frac{1}{720}\ell_6 H_6(y)\Big], \tag{7.3.4}$$

where

$$y = d/k_2^{1/2}.$$

Conclusion

Certain numerical values of $\phi(d)$ were computed according to the approximations $H^{(1)}$ and $H^{(2)}$, and the comparison with the nominally exact values is given in Table 2, in terms of the percentage errors

$I_H(1)$ and $I_H(2)$. The comparison in Table 2 shows that both the approximations can be safely used for the range [0, 3] of $|d|$ for sufficiently many degrees of freedom, say $f_1, f_2 \geq 9$. Both the approximations are equally good or bad. From 3 onward, they are not good. For this reason, more numerical values were not computed.

The Gram-Charlier series of Type A

The density of a random variable x can be expressed formally as

$$\rho(x) = \sum_{j=0}^{\infty} c_j H_j(x)\, \nu(x) .$$

(Refer to Kendall, 1943, Vol. 1, section 6.23, pages 147–50.) Multiplying by $H_r(x)$ and integrating from $-\infty$ to $+\infty$, we have, in view of the orthogonal relationship,

$$c_r = \frac{1}{r!} \int_{-\infty}^{+\infty} \rho(x) H_r(x)\, dx .$$

Application of this formula to $y = d/k_2^{1/2}$ gives, considering only first three terms,

$$\rho(y) \doteq \left[1 + \frac{l_4 H_4(y)}{24} + \frac{l_6 H_6(y)}{720}\right] \nu(y) ,$$

and

$$\phi(d) \doteq K_2^{-1/2} \rho(y) \quad . \tag{7.3.5}$$

The two formulas (7.3.4) and (7.3.5) are the same except for the term $\frac{\ell_4^2 H_8(y)}{1152}$.

No numerical values were computed using the formula (7.3.5), because it was thought of quite late.

7.4. Application of the Delta Method to Integral Forms Due to Ruben

The delta method is usually applied and works satisfactorily when the integral of the product of two functions, one of which is sharp and the other gentle, is to be evaluated. In Ruben's integral form (2.2.1) for the density of d, neither of the two functions is gentle. Hence, we do not altogether expect the delta method to work. However, some numerical values of $\phi(d)$ were calculated by the delta method and empirically, the approximation by the delta method seems to be quite good in the interval [0, 3] of $|d|$ if f_1 and f_2 are fairly large.

Ruben's integral form (2.2.1) for the density of d can be rewritten as

(7.4.1) $$\phi(d \mid f_1, f_2; \theta) = \int_0^1 g(x)\, \beta\left(x; \frac{f_1}{2}, \frac{f_2}{2}\right) dx,$$

where

$$g(x) = \Upsilon(\Phi(x)d)\, \Phi(x),$$

$$\beta\left(x; \frac{f_1}{2}, \frac{f_2}{2}\right) = \frac{x^{\frac{f_1}{2}-1}(1-x)^{\frac{f_2}{2}-1}}{B\left(\frac{f_1}{2}, \frac{f_2}{2}\right)}$$ is the density

of a beta variable with parameters $f_1/2$, $f_2/2$.

We expand g(x) in a power series around the mean x_0 of the beta variable with parameters $f_1/2$, $f_2/2$ where

$$x_0 = \frac{f_1}{(f_1+f_2)}.$$

Consider the first few terms (usually two or three) of this series. The delta method is to integrate these few terms times the beta density and to expect this to approximate the exact integral. Then,

$$\phi(d \mid f_1, f_2; \theta) \doteq g(x_0) + \frac{g''(x_0)}{2} u_2,$$

where

$$g(x_0) = \tau(d \mid (f_1+f_2)) \quad ,$$

$$\mu_2 = \text{variance of } x = \frac{2f_1f_2}{(f_1+f_2)^2(f_1+f_2+2)} \quad ,$$

$$\frac{1}{2}g''(x_0)\mu_2 = \frac{(f_1+f_2)^2 A \tau(d \mid (f_1+f_2))}{4f_1f_2(f_1+f_2+2)(d^2+f_1+f_2)^2} \quad ,$$

$$A = (-m^2 - 4q - 2mp + 3p^2)(1+\frac{d^2}{n})(1-d^2) + (m-p)^2(\frac{n+1}{n})d^2(d^2-3),$$

where

$$m = (f_2 - f_1), \quad n = f_1 + f_2, \quad p = f_2\cos^2\theta - f_1\sin^2\theta, \quad q = f_1f_2 .$$

Thus,

(7.4.2) $$\phi(d \mid f_1, f_2 ; \theta) \doteq \tau(d \mid (f_1+f_2))\left[1 + \frac{n^2 A}{4q(n+2)(d^2+n)^2}\right].$$

Special case: $f_1 = f_2$, $\theta = 45°$

(7.4.3) $$\phi(d \mid f_1, f_2 ; 45°) \doteq \tau(d \mid 2f_1)\left[1 + \frac{2f_1(d^2-1)}{(2f_1+2)(d^2+2f_1)}\right].$$

Conclusion

Some numerical values of $\phi(d|f_1, f_2; \theta)$ were calculated by using the formula (7.4.2) and compared with the nominal exact densities. Table 2 shows the comparison of the approximation with the exact densities of d in terms of percentage errors Π_D in density. It is seen that the approximation works quite well in the range [0, 3] of $|d|$ for $f_1, f_2 > 7$. The approximation is of little use for $|d| > 3$.

CHAPTER 8

APPROXIMATION BY ONE DILATED STUDENT'S DISTRIBUTION

8.1. Choosing a Dilated Student's t for Good Over-all Fit

The idea of picking the degrees of freedom f so that a Student's t with f degrees of freedom would do the job of approximating densities, cumulative probabilities, and percentage points of d, was tried first. This idea did not work. Then an extension of the idea was explored; a dilated Student's t was tried. This calls for finding two numbers f and h so that the second and fourth cumulants of Student's t and f degrees of freedom are the same as those of hd.

Solve the equation

(8.1.1) $$l_4(hd) = l_4(t|f)$$

for f, and then solve the equation

(8.1.2) $$\kappa_2(hd) = \kappa_2(t|f)$$

for h, where $\kappa_{2\nu}(x)$ and $\ell_{2\nu}(x)$ are the $2\nu^{th}$ cumulant and relative cumulants of the random variable x. These are calculated according to the formulas (6.1.1), (6.1.2), and (6.2.1), (6.2.2). We get

$$f = 4 + R(f_1, f_2, c, s) , \tag{8.1.3}$$

where

$$R(f_1, f_2, c, s) = \frac{\left[\frac{c^2 f_2}{(f_2-2)} + \frac{s^2 f_1}{(f_1-2)}\right]^2}{\left[\frac{c^4 f_2^2}{(f_2-2)^2(f_2-4)} + \frac{s^4 f_1^2}{(f_1-2)^2(f_1-4)}\right]} ,$$

and then

$$h^2 = \frac{f}{(f-2)}\left[\frac{c^2 f_2}{(f_2-2)} + \frac{s^2 f_1}{(f_1-2)}\right]^{-1} . \tag{8.1.4}$$

This idea of a dilated Student's t is very simple, and since it is not easily extended to a sequence of better and better approximations, might be, justifiably, regarded as crude. But, as we shall see, it works rather well for the present purpose. A similar idea, but less flexible, has been used by Satterthwaite (1946) for approximating the distribution of a linear combination of random variables, each distributed according to chi-square distribution, by a chi-square distribution.

The densities, cumulative probabilities, and percentage points of a Student's distribution with nonintegral degrees of freedom are obtained by harmonic interpolation in degrees of freedom.

Welch (1947) has derived that, in the classical theory, $d = (\bar{x}_1 - \bar{x}_2)/s^*$ is distributed approximately like a Student's t with f^1 degrees of freedom, where

$$f^1 = \left[\frac{c^4}{(f_2+2)} + \frac{s^4}{(f_1+2)} \right] - 2 \quad .$$

Approximation by one dilated Student's t is better than that of Student's t with f^1 degrees of freedom.

8.2. Alternate Expressions for f and h

Put

$$A_1 = \frac{f_1}{(f_1-2)} \quad , \qquad A_2 = \frac{f_1}{(f_1-4)} \quad ,$$

$$B_1 = \frac{f_2}{(f_2-2)} \quad , \qquad B_2 = \frac{f_2}{(f_2-4)} \quad .$$

Then, calculate

$$Q = (c^2 B_1 + s^2 A_1) \quad ,$$

$$R(f_1, f_2, c, s) = \left[c^4 B_1^2 B_2 f_2^{-1} + s^4 A_1^2 A_2 f_1^{-1} \right]^{-1} Q^2 .$$

f and h are calculated from the expressions

(8.2.1) $$f = 4 + R(f_1, f_2, c, s) ,$$

(8.2.2) $$h^2 = \frac{f}{(f-2)} Q^{-1} .$$

The forms for Q and $R(f_1, f_2, c, s)$ can be rewritten in a slightly different way. From the relation

$$c^2 = \cos^2 \theta = (1 + \cos \theta)/2 ,$$

$$s^2 = \sin^2 \theta = (1 - \cos \theta)/2 ,$$

we get

(8.2.3) $$Q = \frac{1}{2} \left[(A_1 + B_1) + \cos 2\theta \, (B_1 - A_1) \right] ,$$

(8.2.4) $$R(f_1, f_2, c, s) = \left[(1 + \cos 2\theta)^2 B_1^2 B_2 f_2^{-1} + (1 - \cos 2\theta)^2 A_1^2 A_2 f_1^{-1} \right]^{-1} 4 Q^2 .$$

In the Behrens-Fisher problem, the values c^2 and s^2 are calculated before θ, in which case the form given in (8.2.1), (8.2.2) is more convenient to use. Whereas, if we are studying

Behrens-Fisher distribution theoretically, then we may start with the angle θ in which case the forms (8.2.3), (8.2.4) of Q and R are, perhaps, more convenient to use.

8.3. Densities

Some numerical values of the densities $\phi(d)$ were calculated according to the approximation that hd is distributed according to a Student's t with f degrees of freedom.

(8.3.1) $$\phi(d \mid f_1, f_2; \theta) \doteq h\,\tau(hd \mid f) .$$

Table 2 gives the comparison of the approximation with the nominal exact values for four combinations of f_1 and f_2. The approximation works quite well for the range [0, 5] of $|d|$ even for f_1 and f_2 as small as 7. In this range [0, 5] of $|d|$, for the values of d considered, the maximum percentage error in density is 7.42%. For $7 > |d| > 5$, sometimes the percentage error is as high as 47.5% which is not good. At the same time, the densities $\phi(d)$ for $|d| > 5$ get smaller and smaller, and in many practical applications, values of $|d| > 5$ might be unimportant.

This approximation is exact for $\theta = 0°, 90°$.

8.4. Percentage Points of d

Some percentage points were calculated by using the technique of the section (8.1). Two numbers f and h were so chosen that hd is approximately distributed like Student's t with f degrees of freedom. The percentage points were explored for the cases for which the exact values have been tabulated so that the comparison was possible.

The approximate loop-percent values were calculated according to the formula

(8.4.1) $$d'_p \doteq h^{-1} t_p$$

where t_p is the corresponding percentage point of Student's t with f degrees of freedom, for

$$f_1 , f_2 = 6 , 8 , 12 , 24 , \infty ;$$

$$\theta = 15^\circ (15^\circ) 75^\circ ;$$

$$p = 0.05 , 0.01 ;$$

and for

$$f_1 = \infty , \quad f_2 = 10 ;$$

$\Theta = 10^\circ, 40^\circ, 50^\circ, 80^\circ;$

$p = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002.$

For $\theta = 0^\circ, 90^\circ$, the approximation is exact.

The comparison of the approximate percentage points d'_p with the exact percentage points d_p is given in Tables 3 and 5 in terms of the difference $\Delta d_p = d'_p - d_p$, and the percentage error in the level p. We are interested not so much in Δd_p but in how the level p is approximated. This is measured in terms of the percentage error in p.

The percentage error in p is to be calculated according to the formula,

(8.4.2) $$\Pi(p) = \text{percentage error in } p = 2\Delta p(100)/p \; ,$$

where Δp is the probability contained in the interval $[d'_p, d_p]$.

It is not possible to calculate the exact nominal value of Δp from the available tables including those which have been computed. Hence, Δp is computed approximately by considering the probability contained in the interval $[hd'_p, hd_p]$ according to a Student's distribution with f degrees of freedom.

Since the exact percentage points d_p are available only up to three decimal places, certain allowance has to be made in the percentage error in p, $\Pi_{(p)}$. The real difference could be anywhere between $\Delta d_p \pm 0.001$. Then, $\Pi_{(p)}$ would be anywhere in the range

$$\left(\Pi_{(p)} \pm \frac{\Pi_{(p)}(0.001)}{\Delta d_p} \right) .$$

Conclusion

The comparison in Tables 3 and 5 shows that the maximum percentage error in p, for the cases considered, is

-2.20% for $f_1, f_2 > 12$; $p = 0.05, 0.01$;

-3.80% for $f_1 = \infty$, $f_2 = 10$; $p = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002$;

-7.00% for $f_1\ f_2 > 8$; $p = 0.05, 0.01$;

-16.66% for $f_1\ f_2 > 6$; $p = 0.05, 0.01$.

The result that the maximum percentage error in p is -3.80% for $p = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002$ when $f_1 = \infty$, $f_2 = 10$ is extended to other cases of $f_1, f_2 > 10$ on the basis of the following remark. The maximum percentage error in p ($p = 0.05, 0.01$) for $f_2 = 12$, $f_1 > 12$ is greater than the maximum percentage error in p for $f_2 = 24$, $f_1 > 24$. From this, it may be conjectured that the maximum percentage error in p for $f_2 = 10$, $f_1 > 10$ is greater than the maximum percentage error when $f_2 = 12, 24$; $f_1 > f_2$ for p = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002. The maximum percentage error in p for $f_2 = 12, 24$; $f_1 > f_2$ occurs when $f_1 = \infty$. Therefore, it is likely to be maximum for $f_2 = 10$ when $f_1 = \infty$, which is -3.80%. From this, it is concluded that the maximum percentage error in p for p = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002 is -3.80% when $f_1, f_2 > 10$.

8.5. Improvement over the Approximation for the Percentage Points

In order to improve the approximation of section (8.1) as applied to calculating percentage points, an empirical table of corrections Δf to be added to f, given in Table 4, was explored. The percentage points d''_p were calculated by considering hd to be distributed like a Student's variable with $f' = (f + \Delta f)$ degrees of freedom, $\Delta d'_p$, $\Pi'_{(p)}$ are the corresponding differences and percentage errors in p, as defined in section (8.4).

Table 4 gives the corrections Δf for $f_2 = 6, 8$; $f_1 = 6, 8, 12, 24, \infty$; $\theta = 0° (15°) 90°$. The correction for $f_1, f_2 > 8$ is zero, and for $f_1, f_2 > 6$, the corrections are obtained by harmonic interpolation.

Table 5 gives the comparison of the approximate p-percent points d'_p, d''_p with the exact p-percent point d_p in terms of the differences Δd_p, $\Delta d'_p$; and percentage errors $\Pi_{(p)}$ and $\Pi'_{(p)}$. The values are explored for the following cases:

(1) $f_2 = 6, 8;\ f_1 = 6, 8, 12, 24, \infty;$

$\theta = 15^\circ\ (15^\circ)\ 75^\circ;$

$p = 0.05, 0.01;$

(2) $f_1 = f_2 = 7;$

$\theta = 15^\circ\ (15^\circ)\ 45^\circ;$

$p = 0.1, 0.05, 0.02, 0.01;$

Conclusion

The maximum percentage error in p, for the cases considered, has been reduced from 16.66% to 4.42% for $f_1, f_2 > 6$, $p = 0.05, 0.01$, $\theta = 15^\circ\ (15^\circ)\ 75^\circ$.

Table 5 shows that for $f_1 = f_2 = 7$, the approximation gets worse when $p = 10$ due to the correction Δf.

From the above remark, Table 4 of corrections Δf improves the approximation for $f_1, f_2 > 6$ and for $0.01 \leq p \leq 0.05$.

8.6. Cumulative Probabilities

The approximation of the distribution of d by dilated Student's distribution works very well for densities at d = 0, 1, 2, 3 and for the percentage points d_p. This justifies the use of the approximation to compute the cumulative probabilities as well; each percentage point amounts to such a calculation.

A few numerical values of cumulative probabilities have been calculated according to the dilated Student's approximation, and the comparison with the exact values is given in Table 6. The maximum percentage error is 0.32%, for the cases considered and this leads to the conclusion that the approximation can be used to compute cumulative probabilities of d in practical applications.

8.7. Choosing a Dilated Student's t to Approximate Any Linear Combination of Three Independently Distributed Student's t

An attempt to justify the use of dilated Student's t to approximate any linear combination of an arbitrary number of independently distributed Student's t's was made by considering a linear combination of three Student's t-like variables, $d = \Sigma_{a=1}^{3} c_a t_a$, where not all c_a are zero. When one c_a is zero, this reduces to the study of Behrens-Fisher distributions which has been already covered. When all $c_a \neq 0$, nominally exact numerical values of these distributions are not available. These will have to be computed for a thorough investigation of this approximation for a good over-all fit. This has not been done in this thesis. However, the dilated Student's t approximation is studied by comparing the higher cumulants of hd and Student's t and f degrees of freedom, where f and h are calculated on the same principle as that of section (8.1), by equating the second and fourth cumulants of $h\Sigma c_a t_a$ and Student's t with f degrees of freedom.

The comparison of the sixth and eighth cumulants, for certain cases, is given in Table 7. The agreement is not good. The sixth and eighth cumulants were also calculated for certain cases of Behrens-Fisher distributions. The agreement is not good in the latter case either. Still, a thorough investigation of the numerical values of densities, percentage points, and cumulative probabilities of the Behrens-Fisher distributions has proved that the dilated Student's t approximation is satisfactory for many practical purposes. The same might be true for the generalized Behrens-Fisher distributions, and this hope is strengthened by the central limit theorem.

8.8. Description of Tables 2–6

Table 2

In this table $I_{H(1)}$, $I_{H(2)}$, II_D, II_{fh} are the percentage errors in the density $\phi(d)$ due to the approximations by Hermite polynomial method $H^{(1)}$, $H^{(2)}$, the delta method, and one dilated Student's t, according to the formulas (7.3.3), (7.3.4), (7.4.2), and (8.3.1); and the percentage errors are calculated according to the formula (7.2.1).

The densities $\phi(d|f_1, f_2; \theta)$ are explored for the following cases:

$f_1 = f_2 = 7$, $\theta = 15°\ (15°)\ 45°$, $d = 0(1)7$;

$f_1 = f_2 = 9$, $\theta = 15°\ (15°)\ 45°$, $d = 0(1)7$;

$f_1 = 19$, $f_2 = 9$, $\theta = 15°\ (15°)\ 75°$, $d = 0(1)7$;

$f_1 = 9$, $f_2 = 11$, $\theta = 15°\ (15°)\ 75°$, $d = 0(1)7$;

Table 3

In this table, d_p is the loop-percent point of d in Fisher and Yates (1957, Tables VI, VI_1, VI_2)

$$\Delta d_p = d'_p - d_p$$

where d'_p is the approximate loop-percent point calculated according to the formula (8.4.1). The percentage error in p is calculated approximately according to the formula (8.4.2).

Tables 4–5

Table 4 gives the corrections Δf to be added to f calculated from the formula (8.1.3).

d''_p, $\Delta d'_p$, $\Pi'_{(p)}$ are calculated in the same way as in Table 3, by considering $(f + \Delta f)$ degrees of freedom, instead of f.

Table 5 shows the result of adding Δf to f in terms of Δd_p, $\Delta d'_p$, $\Pi_{(p)}$, $\Pi'_{(p)}$.

The values are explored for the following cases:

$f_2 = 6$, $f_1 = 6, 8, 12, 24, \infty$; $\theta = 15° \ (15°) \ 75°$; $p = 0.05, 0.01$;

$f_2 = 8$, $f_1 = 8, 12, 24, \infty$; $\theta = 15° \ (15°) \ 75°$; $p = 0.05, 0.01$;

$f_1 = f_2 = 7$, $\theta = 15° \ (15°) \ 45°$; $p = 0.1, 0.05, 0.02, 0.01$.

Table 6

F(d) is the cumulative probability of d. The comparison of the approximate cumulative probabilities due to one dilated Student's t with F(d) is given in terms of percent of error in F(d) calculated according to the formula (7.2.1).

The values are explored for the following cases:

$f_1 = f_2 = 7$, $f_1 = f_2 = 9$; $\theta = 45°$; $\sqrt{2}\, d = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 4, 5.$

TABLE 2

COMPARISON OF THE APPROXIMATE DENSITIES WITH THE EXACT DENSITIES $\phi(d)$ IN TERMS OF PERCENTAGE ERRORS IN $\phi(d)$[a]

θ	d	$\phi(d)$	II_{fh}	$I_H(2)$	$I_H(1)$	II_D
			$f_1 = f_2 = 7$			
15°	0	0.3787	+ 0.53			
	1	0.2276	- 0.44			
	2	0.06471	- 0.62			
	3	0.01403	+ 1.43			
	4	0.003133	+ 2.88			
	5	0.0008008	+ 3.63			
	6	0.0002376	- 9.17			
	7	0.00008061	+ 2.11			
30°	0	0.3690	+ 1.08			
	1	0.2299	- 0.43			
	2	0.06783	- 1.18			
	3	0.01408	+ 1.42			
	4	0.002813	+ 6.05			
	5	0.0006326	+ 7.42			
	6	0.0001676	+ 1.79			
	7	0.00005204	- 0.77			
45°	0	0.3650	+ 0.82	- 0.54	- 0.052	+ 0.66
	1	0.2306	- 1.30	+ 0.0014	- 0.012	+ 1.30
	2	0.06931	- 2.45	+ 0.58	+ 1.41	- 1.59
	3	0.01418	- 1.41	+ 2.82	+ 6.34	-13.38
	4	0.002669	+ 1.12	+ 6.74	+15.35	-28.84
	5	0.0005471	+ 1.83	-16.27	-98.68	-48.08
	6	0.0001308	- 1.53	-71.45	-98.47	-61.54
	7	0.00003692	- 1.90	-96.36	-100.00	-75.61

[a]$I_H(1)$, $I_H(2)$, II_D, and II_{fh} are the percentage errors in $\phi(d)$ due to the approximations by Hermite polynomial method $H^{(1)}$, $H^{(2)}$, the delta method, and one dilated student's t with width h^{-1} and degrees of freedom f.

TABLE 2 (Continued)

θ	d	φ(d)	II_{fh}	$I_H(2)$	$I_H(1)$	II_D
			$f_1 = f_2 = 9$			
15°	0	0.3833	+ 0.26	- 0.26		+ 0.52
	1	0.2307	- 0.43	+ 0.44		+ 0.0012
	2	0.06288	- 0.16	- 3.89		+ 0.48
	3	0.01198	+ 0.83	- 1.65		- 5.83
	4	0.002200	+ 1.82	+14.89		-25.45
	5	0.0004494	+ 1.34	+ 5.16		-51.67
	6	0.0001062	- 0.0011	-67.66		-71.03
	7	0.0000289	- 3.11	-97.13		-59.86
30°	0	0.3755	+ 0.27	- 0.53		+ 0.53
	1	0.2328	- 0.43	+ 0.20		+ 0.43
	2	0.06530	- 0.46	- 1.38		+ 1.53
	3	0.01181	+ 1.69	+ 0.85		- 8.47
	4	0.001903	+ 3.68	+10.53		-25.26
	5	0.0003316	+ 1.81	-15.15		-47.89
	6	0.00006776	- 4.86	-72.57		-66.22
	7	0.00001644	-15.85	-97.56		-49.75
45°	0	0.3722	+ 0.27	- 0.27	+ 0.051	+ 0.40
	1	0.2335	- 0.43	+ 0.09	- 0.012	+ 0.73
	2	0.06650	+ 0.75	- 0.60	+ 0.75	- 1.05
	3	0.01178	+ 2.54	+ 1.69	+ 2.54	-10.17
	4	0.001756	+ 2.27	+ 7.95	+20.45	-27.78
	5	0.0002694	+ 2.23	-19.23	-82.97	-40.74
	6	0.00004729	- 3.81	-74.63	-97.46	-60.00
	7	0.000009892	-15.57	-97.98	-100.00	-66.67
			$f_1 = 19, f_2 = 9$			
15°	0	0.3853	+ 0.52	- 0.26		+ 0.26
	1	0.2307	- 0.43	+ 0.13		+ 0.0021
	2	0.06219	- 0.16	+ 0.0012		+ 0.19
	3	0.01177	- 0.85	- 0.85		- 5.08
	4	0.002159	+ 1.85	+16.20		-34.72

TABLE 2 (Continued)

θ	d	ϕ(d)	II_{fh}	$I_H(2)$	$I_H(1)$	II_D
			$f_1 = 19,\ f_2 = 9$			
15°	5	0.0004422	+ 1.36	- 2.27		-68.55
	6	0.0001048	- 0.95	-70.00		-86.48
	7	0.00002862	- 3.15	-97.67		-95.94
30°	0	0.3805	+ 0.78	- 0.26		+ 0.21
	1	0.2336	- 0.85	+ 0.0011		+ 0.39
	2	0.06323	- 0.16	+ 0.16		+ 0.95
	3	0.01085	+ 4.63	+ 1.85		- 5.56
	4	0.001684	+ 7.14	+10.12		-33.93
	5	0.0002923	+ 2.39	-21.92		-65.75
	6	0.00006076	- 8.55	-80.42		-86.41
	7	0.00001507	-21.85	-98.47		-95.76
45°	0	0.3790	+ 0.53	- 0.26		+ 0.26
	1	0.2359	- 0.42	+ 0.08		+ 0.42
	2	0.06352	+ 0.47	+ 0.47		- 0.16
	3	0.009716	+ 4.12	+ 2.06		- 7.84
	4	0.001165	+ 6.90	+ 0.0011		-27.16
	5	0.0001447	- 2.07	-32.14		-54.41
	6	0.00002217	-23.42	-83.64		-74.46
	7	0.000004399	-47.50	-98.50		-90.61
60°	0	0.3826	+ 0.0010			
	1	0.2369	- 0.0011			
	2	0.06204	+ 0.97			
	3	0.008733	+ 2.86			
	4	0.0008686	+ 3.34			
	5	0.00007706	+ 0.00011			
	6	0.000007286	-10.56			
	7	0.0000008405	-29.60			
75°	0	0.3897	+ 0.051			
	1	0.2365	- 0.042			
	2	0.05953	- 0.021			

TABLE 2 (Continued)

θ	d	ϕ(d)	II_{fh}	$I_H(2)$	$I_H(1)$	II_D
			$f_1 = 19, f_2 = 9$			
75°	3	0.008231	+ 0.33			
	4	0.0008416	+ 0.18			
	5	0.00007946	- 1.71			
	6	0.000007901	- 4.87			
	7	0.0000008818	- 9.72			
			$f_1 = 9, f_2 = 11$			
15°	0	0.3855	+ 0.23			
	1	0.2327	- 0.44			
	2	0.06185	+ 0.49			
	3	0.1072	+ 1.87			
	4	0.001676	+ 1.79			
	5	0.0002814	+ 0.0011			
	6	0.00005403	- 4.26			
	7	0.00001199	- 8.40			
30°	0	0.3780	+ 0.26			
	1	0.2342	- 0.0022			
	2	0.06427	+ 0.78			
	3	0.01075	+ 3.75			
	4	0.001495	+ 3.33			
	5	0.0002133	+ 0.47			
	6	0.00003470	- 7.20			
	7	0.000006679	-17.51			
45°	0	0.3746	+ 0.27			
	1	0.2344	0.00			
	2	0.06552	+ 0.76			
	3	0.01104	+ 2.73			
	4	0.001514	+ 3.31			
	5	0.0002101	+ 0.95			
	6	0.00003341	- 6.29			
	7	0.000006431	-20.53			

TABLE 2 (Continued)

θ	d	$\phi(d)$	II_{fh}	$I_H(2)$	$I_H(1)$	II_D
			$f_1 = 9$, $f_2 = 11$			
60°	0	0.3772	+ 0.53			
	1	0.2331	- 0.43			
	2	0.06458	+ 0.15			
	3	0.01149	+ 3.48			
	4	0.001812	+ 6.08			
	5	0.0003137	+ 2.87			
	6	0.00006431	- 6.84			
	7	0.00001573	-18.47			
75°	0	0.3840	+ 0.42			
	1	0.2307	- 1.08			
	2	0.06263	- 0.37			
	3	0.01190	+ 1.01			
	4	0.002184	+ 1.87			
	5	0.0004465	+ 1.25			
	6	0.0001056	- 1.14			
	7	0.00002881	- 2.78			

TABLE 3

COMPARISON OF THE APPROXIMATE PERCENTAGE POINTS DUE TO ONE DILATED STUDENT'S t WITH THE EXACT PERCENTAGE POINTS d_p IN TERMS OF THE DIFFERENCES Δd_p AND PERCENTAGE ERRORS $\Pi_{(p)}$ IN p

θ	f_1	$d_{0.05}$	$\Delta d_{0.05}$	$\Pi_{(0.05)}$	$d_{0.01}$	$\Delta d_{0.01}$	$\Pi_{(0.01)}$
				$f_2 = 12$			
15°	12	2.175	+0.001	-0.32	3.029	+0.004	-0.80
	24	2.168	+0.002	-0.40	3.020	+0.005	-1.00
	∞	2.163	+0.001	-0.28	3.014	+0.004	-0.80
30°	12	2.169	+0.001	-0.36	2.978	+0.005	-1.20
	24	2.146	+0.004	-0.76	2.938	+0.009	-2.00
	∞	2.120	+0.005	-1.00	2.909	+0.010	-2.20
45°	12	2.167	0.000		2.954	+0.002	-0.60
	24	2.112	+0.002	-0.36	2.853	+0.005	-1.20
	∞	2.064	+0.005	-1.04	2.775	+0.009	-2.20
60°	12	2.169	+0.001	-0.36	2.978	+0.005	-1.20
	24	2.085	+0.0004	-0.08	2.803	+0.001	-0.20
	∞	2.011	+0.002	-0.32	2.661	+0.002	-0.60
75°	12	2.175	+0.001	-0.32	3.029	+0.004	-0.80
	24	2.069	0.000		2.793	+0.002	-0.40
	∞	1.973	0.000		2.595	0.000	
				$f_2 = 24$			
15°	24	2.062	0.000		2.785	0.000	
	∞	2.056	+0.001	-0.12	2.777	+0.001	-0.20
30°	24	2.058	0.000		2.759	+0.001	-0.40
	∞	2.035	+0.001	-0.20	2.726	+0.001	-0.20

TABLE 3 (Continued)

θ	f_1	$d_{0.05}$	$\Delta d_{0.05}$	$II_{(0.05)}$	$d_{0.01}$	$\Delta d_{0.01}$	$II_{(0.01)}$
				$f_2 = 24$			
45°	24	2.056	0.000		2.747	0.000	
	∞	2.009	0.000		2.664	+0.001	-0.20
60°	24	2.058	0.000		2.759	+0.001	-0.40
	∞	1.983	+0.001	-0.12	2.613	0.000	
75°	24	2.062	0.000		2.785	0.000	
	∞	1.966	0.000		2.585	0.000	

θ	p	d_p	Δd_p	$II_{(p)}$
		$f_1 = \infty$, $f_2 = 10$		
10°	0.10	1.808	0.000	
	0.05	2.219	+0.002	-0.24
	0.02	2.748	+0.003	-0.48
	0.01	3.148	+0.002	-0.52
	0.005	3.553	+0.003	-0.64
	0.002	4.106	+0.001	-0.50
40°	0.10	1.749	+0.002	-0.38
	0.05	2.112	+2.112	+0.008
	0.02	2.559	+0.014	-2.67
	0.01	2.883	+0.017	-3.80
	0.005	3.203	+0.016	-3.52
	0.002	3.630	+0.002	-0.50
50°	0.10	1.721	+0.002	-0.31
	0.05	2.066	+0.006	-1.24
	0.02	2.481	+0.010	-2.37
	0.01	2.775	+0.012	-2.88
	0.005	3.058	+0.010	-2.72
	0.002	3.425	+0.002	-0.30

TABLE 3 (Continued)

θ	p	d_p	Δd_p	$\Pi_{(p)}$
80°	0.10	1.651	0.000	
	0.05	1.967	0.000	
	0.02	2.335	0.000	
	0.01	2.586	0.000	
	0.005	2.818	0.000	
	0.002	3.103	0.000	

TABLE 4

AN EMPIRICAL TABLE OF CORRECTIONS Δf TO BE ADDED TO THE DEGREES OF FREEDOM f OF A DILATED STUDENT'S t

f_1	$f_2 = 6, 8$ θ						
	0°	15°	30°	45°	60°	75°	90°
6	0.0	0.1	0.2	0.4	0.2	0.1	0.0
8	0.0	0.1	0.2	0.4	0.2	0.1	0.0
12	0.0	0.1	0.3	0.6	0.3	0.1	0.0
24	0.0	0.1	0.4	0.8	1.2	0.1	0.0
∞	0.0	0.1	0.5	1.0	4.8	0.1	0.0

TABLE 5

COMPARISON OF THE APPROXIMATE PERCENTAGE POINTS DUE TO ONE DILATED STUDENT'S t WITH f AND (f+Δf) DEGREES OF FREEDOM WITH THE EXACT PERCENTAGE POINTS dp IN TERMS OF THE DIFFERENCES Δd_p, $\Delta d'_p$ AND THE PERCENTAGE ERRORS $\Pi_{(p)}$, $\Pi'_{(p)}$ IN p

θ	Differences & Pct. Errors	f_1				
		6	8	12	24	∞
				$f_2 = 6$		
15°	$d_{0.05}$	+2.440	+2.430	+2.423	+2.418	+2.413
	$\Delta d_{0.05}$	+0.004	+0.009	+0.008	+0.008	+0.007
	$\Delta d'_{0.05}$	0.000	0.000	0.000	-0.001	-0.001
	$\Pi_{(0.05)}$	-1.16	-1.08	-1.12	-1.00	-0.98
	$\Pi'_{(0.05)}$				+0.12	+0.11
	$d_{0.01}$	+3.654	+3.643	+3.636	+3.631	+3.626
	$\Delta d_{0.01}$	+0.020	+0.027	+0.026	+0.024	+0.021
	$\Delta d'_{0.01}$	+0.007	+0.006	+0.004	+0.002	0.000
	$\Pi_{(0.01)}$	-3.80	-3.40	-3.40	-3.00	-2.84
	$\Pi'_{(0.01)}$	-1.33	-0.76	-0.52	-0.25	

TABLE 5 (Continued)

θ	Differ- ences & Pct. Errors	f_1				
		6	8	12	24	∞
				$f_2 = 6$		
30°	$d_{0.05}$	+2.435	+2.398	+2.367	+2.342	+2.322
	$\Delta d_{0.05}$	+0.012	+0.015	+0.019	+0.021	+0.022
	$\Delta d'_{0.05}$	0.000	+0.002	0.000	-0.004	-0.009
	$\Pi_{(0.05)}$	-1.72	-1.90	-2.74	-3.00	-2.93
	$\Pi'_{(0.05)}$		-0.25		+0.57	+1.20
	$d_{0.01}$	+3.557	+3.495	+3.453	+3.424	+3.402
	$\Delta d_{0.01}$	+0.051	+0.061	+0.067	+0.068	+0.066
	$\Delta d'_{0.01}$	+0.020	+0.030	+0.022	+0.007	-0.009
	$\Pi_{(0.01)}$	-6.80	-8.80	-9.42	-9.74	-9.26
	$\Pi'_{(0.01)}$	-2.84	-4.42	-3.12	-1.00	+1.26
45°	$d_{0.05}$	+2.435	+2.364	+2.301	+2.247	+2.201
	$\Delta d_{0.05}$	+0.010	+0.010	+0.016	+0.022	+0.028
	$\Delta d'_{0.05}$	-0.009	-0.005	-0.004	-0.002	-0.001
	$\Pi_{(0.05)}$	-1.56	-1.60	-1.88	-3.16	-4.64
	$\Pi'_{(0.05)}$	+1.40	+0.80	+0.47	+0.29	+0.17
	$d_{0.01}$	+3.514	+3.363	+3.246	+3.158	+3.093
	$\Delta d_{0.01}$	+0.044	+0.044	+0.062	+0.079	+0.091
	$\Delta d'_{0.01}$	-0.004	+0.008	+0.017	+0.023	+0.021
	$\Pi_{(0.01)}$	-6.40	-6.60	-10.56	-13.18	-16.66
	$\Pi'_{(0.01)}$	+0.58	-1.20	-2.80	-4.01	-3.82

TABLE 5 (Continued)

θ	Differ-ences & Pct. Errors	f_1				
		6	8	12	24	∞
				$f_2 = 6$		
60°	$d_{0.05}$	+2.435	+2.331	+2.239	+2.156	+2.082
	$\Delta d_{0.05}$	+0.012	+0.006	+0.003	+0.008	+0.015
	$\Delta d'_{0.05}$	0.000	-0.001	-0.001	-0.002	-0.006
	$\Pi_{(0.05)}$	-1.72	-0.99	-0.48	-1.44	-3.05
	$\Pi'_{(0.05)}$		+0.17	+0.16	+0.36	+1.22
	$d_{0.01}$	+3.557	+3.307	+3.104	+2.938	+2.804
	$\Delta d_{0.01}$	+0.051	+0.020	+0.016	+0.028	+0.046
	$\Delta d'_{0.01}$	+0.021	+0.005	+0.005	+0.006	0.000
	$\Pi_{(0.01)}$	-6.80	-2.86	-2.80	-6.14	-10.96
	$\Pi'_{(0.01)}$	-2.84	-0.72	-0.88	+1.32	
75°	$d_{0.05}$	+2.440	+2.310	+2.193	+2.088	+1.993
	$\Delta d_{0.05}$	+0.004	+0.004	+0.001	+0.001	+0.001
	$\Delta d'_{0.05}$	0.000	0.000	0.000	0.000	+0.001
	$\Pi_{(0.05)}$	-1.16	-0.68	-0.28	-0.12	-0.24
	$\Pi'_{(0.05)}$					-0.24
	$d_{0.01}$	+3.654	+3.328	+3.053	+2.822	+2.627
	$\Delta d_{0.01}$	+0.020	+0.012	+0.004	+0.002	+0.002
	$\Delta d'_{0.01}$	+0.007	+0.002	0.000	0.000	+0.002
	$\Pi_{(0.01)}$	-3.80	-1.98	-0.84	-0.32	-0.20
	$\Pi'_{(0.01)}$	-1.33	-0.33			-0.20

TABLE 5 (Continued)

θ	Differences & Pct. Errors	f_1 = 8	12	24	∞
			f_2 = 8		
15°	$d_{0.05}$	+2.300	+2.292	+2.286	+2.281
	$\Delta d_{0.05}$	+0.005	+0.005	+0.004	+0.005
	$\Delta d'_{0.05}$	+0.001	+0.001	0.000	0.000
	$\Pi_{(0.05)}$	-0.75	-0.29	-0.56	-0.68
	$\Pi'_{(0.05)}$	-0.15	-0.05		
	$d_{0.01}$	+3.316	+3.307	+3.301	+3.295
	$\Delta d_{0.01}$	+0.009	+0.012	+0.010	+0.011
	$\Delta d'_{0.01}$	+0.002	+0.002	0.000	+0.001
	$\Pi_{(0.01)}$	-2.00	-2.04	-1.70	-1.74
	$\Pi'_{(0.01)}$	-0.44	-0.33		-0.16
30°	$d_{0.05}$	+2.294	+2.262	+2.236	+2.215
	$\Delta d_{0.05}$	+0.006	+0.008	+0.011	+0.013
	$\Delta d'_{0.05}$	+0.001	-0.001	+0.001	-0.001
	$\Pi_{(0.05)}$	-1.05	-1.15	-1.94	-2.20
	$\Pi'_{(0.05)}$	-0.16	-0.14	-0.19	-0.17
	$d_{0.01}$	+3.239	+3.192	+3.158	+3.132
	$\Delta d_{0.01}$	+0.022	+0.013	+0.070	+0.030
	$\Delta d'_{0.01}$	+0.009	-0.006	+0.003	-0.002
	$\Pi_{(0.01)}$	-3.80	-2.26	-5.00	-5.30
	$\Pi'_{(0.01)}$	-1.55	+1.04	-0.21	+0.35

TABLE 5 (Continued)

θ	Differ-ences & Pct. Errors	f_1			
		8	12	24	∞
			$f_2 = 8$		
45°	$d_{0.05}$	+2.292	+2.229	+2.175	+2.128
	$\Delta d_{0.05}$	+0.002	+0.005	+0.009	+0.014
	$\Delta d'_{0.05}$	-0.003	-0.003	0.000	+0.004
	$\Pi_{(0.05)}$	-0.68	-0.80	-1.60	-2.58
	$\Pi'_{(0.05)}$	+0.45	+0.48		-0.74
	$d_{0.01}$	+3.206	+3.083	+2.988	+2.916
	$\Delta d_{0.01}$	+0.009	+0.014	+0.024	+0.032
	$\Delta d'_{0.01}$	-0.005	-0.004	+0.004	+0.010
	$\Pi_{(0.01)}$	-2.40	-3.00	-4.00	-7.00
	$\Pi'_{(0.01)}$	-1.33	-0.86	-0.67	-2.22
60°	$d_{0.05}$	+2.294	+2.201	+2.118	+2.044
	$\Delta d_{0.05}$	+0.006	+0.002	+0.002	+0.005
	$\Delta d'_{0.05}$	+0.001	-0.002	-0.002	0.000
	$\Pi_{(0.05)}$	-1.05	-0.29	-0.30	-1.08
	$\Pi'_{(0.05)}$	-0.18	+0.29	+0.30	
	$d_{0.01}$	+3.239	+3.032	+2.862	+2.723
	$\Delta d_{0.01}$	+0.022	+0.005	+0.004	+0.011
	$\Delta d'_{0.01}$	+0.009	-0.003	-0.004	+0.001
	$\Pi_{(0.01)}$	-3.80	-0.98	-0.98	-3.12
	$\Pi'_{(0.01)}$	-1.55	+0.59	+0.98	-0.28

TABLE 5 (Continued)

θ	Differ-ences & Pct. Errors.	f_1			
		8	12	24	∞
		$f_2 = 8$			
75°	$d_{0.05}$	+2.300	+2.183	+2.077	+1.982
	$\Delta d_{0.05}$	+0.005	+0.001	+0.001	+0.001
	$\Delta d'_{0.05}$	+0.001	0.000	0.000	+0.001
	$\Pi_{(0.05)}$	-0.75	-0.30	-0.08	-0.12
	$\Pi'_{(0.05)}$	-0.15			-0.12
	$d_{0.01}$	+3.316	+3.039	+2.805	+2.608
	$\Delta d_{0.01}$	+0.009	+0.004	+0.001	+0.001
	$\Delta d'_{0.01}$	+0.002	0.000	0.000	+0.001
	$\Pi_{(0.01)}$	-2.00	-0.78	-0.24	-0.26
	$\Pi'_{(0.01)}$	-0.44			-0.26

θ	Differ-ences & Pct. Errors	p			
		0.10	0.05	0.02	0.01
		$f_1 = f_2 = 7$			
15°	d_p	+1.89902	+2.35807	+2.97119	+3.45397
	Δd_p	+0.00104	+0.00635	+0.01385	+0.01839
	$\Delta d'_p$	-0.00251	+0.00042	+0.00352	+0.00381

TABLE 5 (Continued)

θ	Differ- ences & Pct. Errors	p			
		0.10	0.05	0.02	0.01
		$f_1 = f_2 = 7$			
15°	$\Pi_{(p)}$	-0.15	-0.98	-2.20	-2.57
	$\Pi'_{(p)}$	+0.39	-0.06	-0.50	-0.54
30°	d_p	+1.91113	+2.35215	+2.92662	+3.36875
	Δd_p	+0.00058	+0.00916	+0.02235	+0.03158
	$\Delta d'_p$	-0.00443	+0.00088	+0.00816	+0.01160
	$\Pi_{(p)}$	-0.090	-1.47	-3.52	-4.78
	$\Pi'_{(p)}$	+0.69	-0.14	-1.30	-1.79
45°	d_p	+1.91788	+2.35161	+2.90869	+3.33071
	Δd_p	+0.00031	+0.00642	+0.01628	+0.02340
	$\Delta d'_p$	-0.00712	-0.00572	-0.00419	-0.00504
	$\Pi_{(p)}$	-0.049	-1.05	-2.69	-3.78
	$\Pi'_{(p)}$	+1.12	+0.94	+0.70	+0.84

TABLE 6

COMPARISON OF THE APPROXIMATE CUMULATIVE PROBABILITIES WITH THE EXACT CUMULATIVE PROBABILITIES F(d) IN TERMS OF PERCENTAGE ERRORS[a]

$\sqrt{2}$ d	F(d)	Pct. Error	$\sqrt{2}$ d	F(d)	Pct. Error
	$f_1 = f_2 = 9$			$f_1 = f_2 = 7$	
0.5	0.6290	+0.061	0.5	0.6265	+0.14
1	0.7438	+0.32	1	0.7392	+0.16
1.5	0.8349	+0.03	1.5	0.8290	+0.072
2	0.9002	-0.10	2	0.8939	+0.061
2.5	0.9415	+0.10	2.5	0.9364	+0.063
3	0.9687	-0.021	3	0.9637	-0.041
4	0.9914	-0.022	4	0.9884	-0.012
5	0.9977	-0.000010	5	0.9964	-0.011

[a]Pct. error = (approx. value - F(d))100/F(d).

TABLE 7

COMPARISON OF THE SIXTH AND EIGHTH CUMULANTS OF hd AND STUDENT'S t WITH f DEGREES OF FREEDOM[a]

c_1^2	c_2^2	c_3^2	Cu-mu-lant	(hd)	$(t\|f)$	Diff.
			Case $f_1 = f_2 = f_3 = 12$			
0.93302	0.06699	0.00000	κ_6	6.6655	6.0352	+ 0.6303
			κ_8	182.8561	139.0572	+43.7989
0.75000	0.25000	0.00000	κ_6	3.1997	2.5393	+ 0.6604
			κ_8	66.3120	41.9825	+24.3295
0.5000	0.5000	0.00000	κ_6	1.7152	1.4701	+ 0.2451
			κ_8	20.3670	16.2587	+ 4.1083
0.81818	0.09091	0.09091	κ_6	4.1375	3.2104	+ 0.9271
			κ_8	96.5436	48.7193	+47.8243
0.47369	0.47369	0.05263	κ_6	1.4212	1.1511	+ 0.2701
			κ_8	18.4787	9.4831	+ 8.9956
0.33333	0.33333	0.33333	κ_6	0.6939	0.5677	+ 0.1262
			κ_8	6.2116	3.1561	+ 3.0555

[a]Let $d = c_1t_1 + c_2t_2 + c_3t_3$, where $\Sigma c_i^2 = 1$, and when $c_3 = 0$, d is the Behrens-Fisher variable, $\kappa_{2\nu}(x)$ is the $2\nu^{th}$ cumulant of a random variable x.

TABLE 7 (Continued)

c_1^2	c_2^2	c_3^2	Cu-mu-lant	(hd)	(t|f)	Diff.
			Case $f_1 = f_2 = f_3 = 24$			
0.93302	0.06699	0.00000	κ_6	0.6833	0.6348	+ 0.0485
			κ_8	4.6019	3.7517	+ 0.8502
0.75000	0.25000	0.00000	κ_6	0.3463	0.2967	+ 0.0496
			κ_8	1.7263	1.1635	+ 0.5628
0.50000	0.50000	0.0000	κ_6	0.1917	0.1816	+ 0.0101
			κ_8	0.6458	0.5493	+ 0.0965
0.81818	0.09091	0.09091	κ_6	0.4413	0.3652	+ 0.0761
			κ_8	2.4649	1.5704	+ 0.8945
0.47369	0.47369	0.05263	κ_6	0.1610	0.1455	+ 0.0155
			κ_8	0.5114	0.3920	+ 1.1194
0.33333	0.33333	0.33333	κ_6	0.0815	0.0758	+ 0.0067
			κ_8	0.1803	0.1458	+ 0.0345

CHAPTER 9

APPLICATIONS OF ONE DILATED STUDENT'S t APPROXIMATION

This chapter is for the person who wishes to compute Behrens-Fisher distributions in practical applications.

9.1. Use of Tabulated Percentage Points

Percentage points have been tabulated in Fisher and Yates (1957, Tables VI, VI_1, VI_2) for the following cases:

(1) $f_1, f_2 = 6, 8, 12, 24, \infty$; $\theta = 0°$ (15°) 90°; p = 0.05, 0.01.

(2) $f_1 = \infty$, $f_2 = 10, 12, 15, 20, 30, 60, \infty$; $\theta = 0°$ (10°) 90°;

p = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002;

(3) $f_1 = 1(2)7$, $f_2 = f_1(2)7$; $\theta = 0°$ (15°) 90°; p = 0.1, 0.05, 0.02, 0.01.

Percentage points for any case falling in this range of f_1, f_2; p can be most conveniently computed from these tables by interpolation as indicated in Fisher and Yates (1957, page 3).

9.2. Computation of Densities, Cumulative Probabilities, and Percentage Points

For $f_1, f_2 > 6$, the approximation by one dilated Student's t of Chapter 8 can be used in practical applications of Behrens-Fisher distributions.

Two numbers f and h are computed by the formulas (8.1.3), (8.1.4). Densities, cumulative probabilities, and percentage points are calculated by considering hd to be distributed approximately according to Student's distribution with f degrees of freedom. Tables of Student's distribution are available in Smirnov (1961).

Table 4 gives corrections Δf to be added to f so that percentage points can be calculated with higher accuracy. Percentage points are calculated with $f' = (f + \Delta f)$ degrees of freedom instead of f.

Case $f_1, f_2 < 6$

When $(f_1, f_2) = (1, 1)$, $(\sin\theta + \cos\theta)^{-1}d$ is a Cauchy variable, a Student's t with one degree of freedom for which densities, cumulative probabilities, and percentage points have been tabulated.

The percentage points for odd values of $f_1, f_2 = 1(2)5$ are available in Fisher and Yates (1957, Table VI_1).

From the closed forms given in Tables 8–10, and formulas (3.2.2), (3.2.3), the densities and cumulative probabilities for odd degrees of freedom are calculated, which is the sum of three terms at most.

The densities, cumulative probabilities, and percentage points for even degrees of freedom can be obtained from the corresponding values for odd degrees of freedom by harmonic interpolation in f_1 and f_2, which has been studied for densities in section (5.4).

Thus, we have bridged the gap between $(f_1, f_2) = (6, 6)$ and (1, 1).

9.3. Generalized Behrens-Fisher Distributions

Generalized Behrens-Fisher distributions are important, and results of section (8.7) indicate that one dilated Student's t approximation might work and this has to be proved with further research by computing few nominally exact numerical values.

APPENDIX

1. Closed Forms of $C_{u,v}(d; \alpha,\beta)$ for u,v = 1(1)7

$C_{u,v}(d; \alpha, \beta)$ have been obtained by starting with

$$C_{1,1}(d; \alpha, \beta) = (\alpha + \beta) M$$

and using the recurrence relations (3.1.4), (3.1.5), and (3.1.6). Since interchanging u,v amounts to interchanging α,β, there is no loss in supposing $u > v$. The general pattern of $C_{u,v}$ is (3.2.1),

$$C_{u,v}(d; \alpha, \beta) = \sum_{j=v}^{u+v-1} L_{j;u,v} M^j .$$

Looking at this form, it is enough to give the coefficients $L_{j;u,v} = L_j$ in order to know $C_{u,v}$.

For each (u,v), the coefficients $L_j (j = v, v + 1, \ldots u + v - 1)$ can be written as

$$L_j = Q_1 Q_2 Q_3 ,$$

where

$Q_1 = 2^{-r}$

$Q_2 = (\alpha + \beta)^s$

Q_3 = the remaining part of L_j.

Thus, it suffices to know r,s and Q_3, and these have been tabulated in Tables 8–14, in this manner:

(u,v)	L	r	s	Q_3
	v			
	v+1			
	⋮			
	u+v-1			

Tables 8–12 give the exact expressions for v = 1(1)5, u = v(1)7, each table covering one value of v and u = v(1)7. Tables 13–14 cover cases (u,v) = (6,6); (7,6); (7,7).

By looking at the expressions, it is seen that the behavior of s is regular. It starts with the value (u + v - 1) for L_{u+v-1} and decreases in steps of 2 until it is > 0, and is equal to zero when it is ≤ 0;

$$L_{j;u,v} = Q_1 Q_2 Q_3 ,$$

$$Q_2 = (\alpha + \beta)^s ,$$

and s corresponding to $L_{j;u,v}$ is 2(u + v - 1) - 2(u + v - 1 - j) = 2j - (u + v - 1), whenever this is greater than zero, and is equal to zero whenever 2j - (u + v - 1) $\leq$ 0.

TABLE 8

CLOSED FORMS OF $C_{u,v}$ FOR v = 1, u = 1(1)7[a]

(u,v)	L	r	s	Q_3
(1,1)	1	0	0	γ_1
(2,1)	1	0	0	$b_1\beta$
	2	0	2	α
(3,1)	1	0	0	$b_2\beta$
	2	2	1	$3\alpha\beta$
	3	0	3	α^2
(4,1)	1	0	0	$b_3\beta$
	2	3	0	$\alpha\beta(4\alpha+5\beta)$
	3	0	2	$\alpha^2\beta$
	4	0	4	α^3
(5,1)	1	0	0	$b_4\beta$
	2	6	0	$5\alpha\beta(5\alpha+7\beta)$

[a] $C_{u,v}(d;\alpha,\beta) = L_v M^v + L_{v+1} M^{v+1} + \ldots + L_{u+v-1} M^{u+v-1}$,

$L_j = Q_1 Q_2 Q_3$; $Q_1 = 2^{-r}$, $Q_2 = (\alpha+\beta)^s$, Q_3 is the remaining part of L_j. $\gamma_i = (\alpha^i+\beta^i)$, $b_i = \binom{-1/2}{i}(-1)^i = \dfrac{1\cdot3\cdot5\ \ldots\ (2i-1)}{2\cdot4\cdot6\ \ldots\ 2i}$.

TABLE 8 (Continued)

(u,v)	L	r	s	Q_3
(5,1)	3	4	1	$5\alpha^2\beta(2\alpha+3\beta)$
	4	2	3	$5\alpha^3\beta$
	5	0	5	α^4
(6,1)	1	0	0	$b_5\beta$
	2	7	0	$21\alpha\beta(2\alpha+3\beta)$
	3	5	0	$\alpha^2\beta(15\alpha^2+42\alpha\beta+28\beta^2)$
	4	4	2	$3\alpha^3\beta(4\alpha+7\beta)$
	5	1	4	$3\alpha^4\beta$
	6	0	6	α^5
(7,1)	1	0	0	$b_6\beta$
	2	9	0	$21\alpha\beta(7\alpha+11\beta)$
	3	7	0	$7\alpha^2\beta(7\alpha^2+21\alpha\beta+15\beta^2)$
	4	6	1	$7\alpha^3\beta(5\alpha^2+16\alpha\beta+12\beta^2)$
	5	3	3	$7\alpha^4\beta(\alpha+2\beta)$
	6	2	5	$7\alpha^5\beta$
	7	0	7	α^6

TABLE 9

CLOSED FORMS OF $C_{u,v}$ FOR v = 2, u = 2(1)7[a]

(u,v)	L	r	s	Q_3
(2,2)	2	0	0	$b_1\gamma_3$
	3	0	3	$2\alpha\beta$
(3,2)	2	0	0	$b_2\beta^3$
	3	1	2	$\alpha(\alpha^2-2\alpha\beta+3\beta^2)$
	4	0	4	$3\alpha^2\beta$
(4,2)	2	0	0	$b_3\beta^3$
	3	2	1	$5\alpha\beta^3$
	4	1	3	$\alpha^2(\alpha^2-3\alpha\beta+6\beta^2)$
	5	0	5	$4\alpha^3\beta$
(5,2)	2	0	0	$b_4\beta^3$
	3	5	0	$5\alpha\beta^3(6\alpha+7\beta)$
	4	4	2	$45\alpha^2\beta^3$

[a] $C_{u,v}(d;\alpha,\beta) = L_vM^v + L_{v+1}M^{v+1} + \ldots + L_{u+v-1}M^{u+v-1}$, $L_j = Q_1Q_2Q_3$; $Q_1 = 2^{-r}$, $Q_2 = (\alpha+\beta)^s$, Q_3 is the remaining part of L_j. $\gamma_i = (\alpha^i+\beta^i)$, $b_i = \binom{-1/2}{i}(-1)^i = \frac{1\cdot3\cdot5\ \ldots\ (2i-1)}{2\cdot4\cdot6\ \ldots\ 2i}$.

TABLE 9 (Continued)

(u,v)	L	r	s	Q_3
(5,2)	5	1	4	$\alpha^3(\alpha^2-4\alpha\beta+10\beta^2)$
	6	0	6	$5\alpha^4\beta$
(6,2)	2	0	0	$b_5\beta^3$
	3	6	0	$7\alpha\beta^3(7\alpha+9\beta)$
	4	5	1	$21\alpha^2\beta^3(3\alpha+4\beta)$
	5	2	3	$21\alpha^3\beta^3$
	6	1	5	$\alpha^4(\alpha^2-5\alpha\beta+15\beta^2)$
	7	0	7	$6\alpha^5\beta$
(7,2)	2	0	0	$b_6\beta^3$
	3	8	0	$21\alpha\beta^3(8\alpha+11\beta)$
	4	7	0	$7\alpha^2\beta^3(28\alpha^2+72\alpha\beta+45\beta^2)$
	5	2	2	$7\alpha^3\beta^3(2\alpha+3\beta)$
	6	3	4	$70\alpha^4\beta^3$
	7	1	6	$\alpha^5(\alpha^2-6\alpha\beta+21\beta^2)$
	8	0	8	$7\alpha^6\beta$

TABLE 10

CLOSED FORMS OF $C_{u,v}$ FOR v = 3, u = 3(1)7[a]

(u,v)	L	r	s	Q_3
(3,3)	3	0	0	$b_2\gamma_5$
	4	2	3	$3\alpha\beta(3\alpha^2-4\alpha\beta+3\beta^2)$
	5	0	5	$6\alpha^2\beta^2$
(4,3)	3	0	0	$b_3\beta^5$
	4	3	2	$3\alpha(\alpha^4-2\alpha^3\beta+3\alpha^2\beta^2-4\alpha\beta^3+5\beta^4)$
	5	0	4	$3\alpha^2\beta(\alpha^2-2\alpha\beta+2\beta^2)$
	6	0	6	$10\alpha^3\beta^2$
(5,3)	3	0	0	$b_4\beta^5$
	4	6	1	$105\alpha\beta^5$
	5	3	3	$3\alpha^2(\alpha^4-3\alpha^3\beta+6\alpha^2\beta^2-10\alpha\beta^3+15\beta^4)$
	6	2	5	$5\alpha^3\beta(3\alpha^2-8\alpha\beta+10\beta^2)$
	7	0	7	$15\alpha^4\beta^2$

[a] $C_{u,v}(d;\alpha,\beta) = L_vM^v + L_{v+1}M^{v+1} + \ldots + L_{u+v-1}M^{u+v-1}$, $L_j = Q_1Q_2Q_3$; $Q_1 = 2^{-r}$, $Q_2 = (\alpha+\beta)^s$, Q_3 is the remaining part of L_j. $\gamma_i = (\alpha^i+\beta^i)$, $b_i = \binom{-1/2}{i}(-1)^i = \frac{1\cdot3\cdot5\ \ldots\ (2i-1)}{2\cdot4\cdot6\ \ldots\ 2i}$.

TABLE 10 (Continued)

(u,v)	L	r	s	Q_3
(6,3)	3	0	0	$b_5\beta^5$
	4	7	0	$21\alpha\beta^5(8\alpha+9\beta)$
	5	2	2	$21\alpha^2\beta^5$
	6	3	4	$3\alpha^3(\alpha^4-4\alpha^3\beta+10\alpha^2\beta^2-20\alpha\beta^3+35\beta^4)$
	7	1	6	$3\alpha^4\beta(3\alpha^2-10\alpha\beta+15\beta^2)$
	8	0	8	$21\alpha^5\beta^2$
(7,3)	3	0	0	$b_6\beta^5$
	4	9	0	$63\alpha\beta^5(9\alpha+11\beta)$
	5	6	1	$63\alpha^2\beta^5(4\alpha+5\beta)$
	6	3	3	$105\alpha^3\beta^5$
	7	3	5	$3\alpha^4(\alpha^4-5\alpha^3\beta+15\alpha^2\beta^2-35\alpha\beta^3+70\beta^4)$
	8	2	7	$21\alpha^5\beta_1(\alpha^2-4\alpha\beta+7\beta^2)$
	9	0	9	$28\alpha^6\beta^2$

TABLE 11

CLOSED FORMS OF $C_{u,v}$ FOR $v = 4$, $u = 4(1)7$[a]

(u,v)	L	r	s	Q_3
(4,4)	4	0	0	$b_3\gamma_7$
	5	1	3	$\alpha\beta(5\alpha^4-8\alpha^3\beta+9\alpha^2\beta^2-8\alpha\beta^3+5\beta^4)$
	6	0	5	$5\alpha^2\beta^2(2\alpha^2-3\alpha\beta+2\beta^2)$
	7	0	7	$20\alpha^3\beta^3$
(5,4)	4	0	0	$b_4\beta^7$
	5	4	2	$5\alpha(\alpha^6-2\alpha^5\beta+3\alpha^4\beta^2-4\alpha^3\beta^3+5\alpha^2\beta^4-6\alpha\beta^5+7\beta^6)$
	6	3	4	$5\alpha^2\beta(5\alpha^4-12\alpha^3\beta+18\alpha^2\beta^2-20\alpha\beta^3+15\beta^4)$
	7	0	6	$5\alpha^3\beta^2(3\alpha^2-6\alpha\beta+5\beta^2)$
	8	0	8	$35\alpha^4\beta^3$
(6,4)	4	0	0	$b_5\beta^7$
	5	5	1	$63\alpha\beta^7$
	6	4	3	$5\alpha^2(\alpha^6-3\alpha^5\beta+6\alpha^4\beta^2-10\alpha^3\beta^3+15\alpha^2\beta^4-21\alpha\beta^5+28\beta^6)$
	7	2	5	$3\alpha^3\beta(5\alpha^4-16\alpha^3\beta+30\alpha^2\beta^2-40\alpha\beta^3+35\beta^4)$

[a] $C_{u,v}(d;\alpha,\beta) = L_vM^v + L_{v+1}M^{v+1} + \dots + L_{u+v-1}M^{u+v-1}$, $L_j = Q_1Q_2Q_3$; $Q_1 = 2^{-r}$, $Q_2 = (\alpha+\beta)^s$, Q_3 is the remaining part of L_j. $\gamma_i = (\alpha^i+\beta^i)$, $b_i = \binom{-1/2}{i}(-1)^i = \frac{1\cdot3\cdot5\ \dots\ (2i-1)}{2\cdot4\cdot6\ \dots\ 2i}$.

TABLE 11 (Continued)

(u,v)	L	r	s	Q_3
(6,4)	8	1	7	$21\alpha^4\beta^2(2\alpha^2-5\alpha\beta+5\beta^2)$
	9	0	9	$56\alpha^5\beta^3$
(7,4)	4	0	0	$b_6\beta^7$
	5	7	0	$21\alpha\beta^7(10\alpha+11\beta)$
	6	6	2	$525\alpha^2\beta^7$
	7	4	4	$5\alpha^3(\alpha^6-4\alpha^5\beta+10\alpha^4\beta^2-20\alpha^3\beta^3+35\alpha^2\beta^4-56\alpha\beta^5+84\beta^6)$
	8	3	6	$35\alpha^4\beta(\alpha^4-4\alpha^3\beta+9\alpha^2\beta^2-14\alpha\beta^3+14\beta^4)$
	9	0	8	$14\alpha^5\beta^2(2\alpha^2-6\alpha\beta+7\beta^2)$
	10	0	10	$84\alpha^6\beta^3$

TABLE 12

CLOSED FORMS OF $C_{u,v}$ FOR v = 5, u = 5(1)7[a]

(u,v)	L	r	s	Q_3
(5,5)	5	0	0	$b_4\gamma_9$
	6	6	3	$25\alpha\beta(7\alpha^6-12\alpha^5\beta+15\alpha^4\beta^2-16\alpha^3\beta^3+15\alpha^2\beta^4-12\alpha\beta^5+7\beta^6)$
	7	4	5	$45\alpha^2\beta^2(5\alpha^4-10\alpha^3\beta+12\alpha^2\beta^2-10\alpha\beta^3+5\beta^4)$
	8	2	7	$35\alpha^3\beta^3(5\alpha^2-8\alpha\beta+5\beta^2)$
	9	0	9	$70\alpha^4\beta^4$
(6,5)	5	0	0	$b_5\beta^9$
	6	7	2	$35\alpha(\alpha^8-2\alpha^7\beta+3\alpha^6\beta^2-4\alpha^5\beta^3+5\alpha^4\beta^4-6\alpha^3\beta^5+7\alpha^2\beta^6-8\alpha\beta^7+9\beta^8)$
	7	5	4	$5\alpha^2\beta(21\alpha^6-54\alpha^5\beta+90\alpha^4\beta^2-120\alpha^3\beta^3+135\alpha^2\beta^4-126\alpha\beta^5+84\beta^6)$
	8	4	6	$105\alpha^3\beta^2(3\alpha^4-8\alpha^3\beta+12\alpha^2\beta^2-12\alpha\beta^3+7\beta^4)$

[a] $C_{u,v}(d;\alpha,\beta) = L_vM^v + L_{v+1}M^{v+1} + \ldots + L_{u+v-1}M^{u+v-1}$, $L_j = Q_1Q_2Q_3$; $Q_1 = 2^{-4}$, $Q_2 = (\alpha+\beta)^s$, Q_3 is the remaining part of L_j. $\gamma_i = (\alpha^i+\beta^i)$, $b_i = \binom{-1/2}{i}(-1)^i = \frac{1\cdot3\cdot5\ \ldots\ (2i-1)}{2\cdot4\cdot6\ \ldots\ 2i}$.

TABLE 12 (Continued)

(u,v)	L	r	s	Q_3
(6,5)	9	0	8	$35\alpha^4\beta^3(2\alpha^2-4\alpha\beta+3\beta^2)$
	10	0	10	$126\alpha^5\beta^4$
(7,5)	5	0	0	$b_6\beta^9$
	6	9	1	$1155\alpha\beta^9$
	7	7	3	$35\alpha^2(\alpha^8-3\alpha^7\beta+6\alpha^6\beta^2-10\alpha^5\beta^3+15\alpha^4\beta^4-$ $21\alpha^3\beta^5+28\alpha^2\beta^6-36\alpha\beta^7+45\beta^8)$
	8	6	5	$35\alpha^3\beta(7\alpha^6-24\alpha^5\beta+50\alpha^4\beta^2-80\alpha^3\beta^3+$ $105\alpha^2\beta^4-112\alpha\beta^5+84\beta^6)$
	9	2	7	$35\alpha^4\beta^2(3\alpha^4-10\alpha^3\beta+18\alpha^2\beta^2-21\alpha\beta^3+$ $14\beta^4)$
	10	1	9	$21\alpha^5\beta^3(10\alpha^2-24\alpha\beta+21\beta^2)$
	11	0	11	$210\alpha^6\beta^4$

TABLE 13

CLOSED FORMS OF $C_{u,v}$ FOR v = 6, u = 6(1)7[a]

(u,v)	L	r	s	Q_3
(6,6)	6	0	0	$b_5\gamma_{11}$
	7	6	3	$21\alpha\beta(9\alpha^8-16\alpha^7\beta+21\alpha^6\beta^2-24\alpha^5\beta+25\alpha^4\beta^4-24\alpha^3\beta^5+21\alpha^2\beta^6-16\alpha\beta^7+9\beta^8)$
	8	5	5	$21\alpha^2\beta^2(28\alpha^6-63\alpha^5\beta+90\alpha^4\beta^2-100\alpha^3\beta^3+90\alpha^2\beta^4-60\alpha\beta^5+28\beta^6)$
	9	1	7	$21\alpha^3\beta^3(7\alpha^4-16\alpha^3\beta+20\alpha^2\beta^2-16\alpha\beta^3+7\beta^4)$
	10	0	9	$63\alpha^4\beta^4(3\alpha^2-5\alpha\beta+3\beta^2)$
	11	0	11	$252\alpha^5\beta^5$
(7,6)	6	0	0	$b_6\beta^{11}$
	7	8	2	$63\alpha(\alpha^{10}-2\alpha^9\beta+3\alpha^8\beta^2-4\alpha^7\beta^3+5\alpha^6\beta^4-6\alpha^5\beta^5+7\alpha^4\beta^6-8\alpha^3\beta^7+9\alpha^2\beta^8-10\alpha\beta^9+11\beta^{10})$
	8	7	4	$21\alpha^2\beta(21\alpha^8-56\alpha^7\beta+98\alpha^6\beta^2-140\alpha^5\beta^3+175\alpha^4\beta^4-196\alpha^3\beta^5+196\alpha^2\beta^6-168\alpha\beta^7+105\beta^8)$

[a] $C_{u,v}(d;\alpha,\beta) = L_v M^v + L_{v+1} M^{v+1} + \ldots + L_{u+v-1} M^{u+v-1}$, $L_j = Q_1 Q_2 Q_3$; $Q_1 = 2^{-r}$, $Q_2 = (\alpha+\beta)^s$, Q_3 is the remaining part of L_j. $\gamma_i = (\alpha^i+\beta^i)$, $b_i = \binom{-1/2}{i}(-1)^i = \frac{1\cdot3\cdot5\ \ldots\ (2i-1)}{2\cdot4\cdot6\ \ldots\ 2i}$.

TABLE 13 (Continued)

(u,v)	L	r	s	Q_3
(7,6)	9	2	6	$7\alpha^3\beta^2(14\alpha^6-42\alpha^5\beta+75\alpha^4\beta^2-100\alpha^3\beta^3+105\alpha^2\beta^4-84\alpha\beta^5+42\beta^6)$
	10	2	8	$63\alpha^4\beta^3(7\alpha^4-20\alpha^3\beta+30\alpha^2\beta^2-28\alpha\beta^3+14\beta^4)$
	11	0	10	$63\alpha^5\beta^4(5\alpha^2-10\alpha\beta+7\beta^2)$
	12	0	12	$462\alpha^6\beta^5$

TABLE 14

CLOSED FORM OF $C_{u,v}$ FOR u = v = 7[a]

(u,v)	L	r	s	Q_3
(7,7)	7	0	0	$b_6\gamma_{13}$
	8	9	3	$147\alpha\beta(11\alpha^{10}-20\alpha^9\beta+27\alpha^8\beta^2-32\alpha^7\beta^3+35\alpha^6\beta^4-36\alpha^5\beta^5+35\alpha^4\beta^6-32\alpha^3\beta^7+27\alpha^2\beta^8-20\alpha\beta^9+11\beta^{10})$
	9	5	5	$49\alpha^2\beta^2(15\alpha^8-36\alpha^7\beta+56\alpha^6\beta^2-70\alpha^5\beta^3+75\alpha^4\beta^4-70\alpha^3\beta^5+56\alpha^2\beta^6-36\alpha\beta^7+15\beta^8)$
	10	3	7	$21\alpha^3\beta^3(42\alpha^6-112\alpha^5\beta+175\alpha^4\beta^2-200\alpha^3\beta^3+175\alpha^2\beta^4-112\alpha\beta^5+42\beta^6)$
	11	2	9	$105\alpha^4b^4(14\alpha^4-35\alpha^3\beta+45\alpha^2\beta^2-35\alpha\beta^3+14\beta^4)$
	12	1	11	$231\alpha^5\beta^5(7\alpha^2-12\alpha\beta+7\beta^2)$
	13	0	13	$924\alpha^6\beta^6$

[a] $C_{u,v}(d;\alpha,\beta) = L_v M^v + L_{v+1}M^{v+1} + \ldots + L_{u=v-1}M^{u+v-1}$, $L_j = Q_1Q_2Q_3$; $Q_1 = 2^{-r}$, $Q_2 + (\alpha+\beta)^s$, Q_3 is the remaining part of L_j. $\gamma_i = (\alpha^i+\beta^i)$, $b_i = \binom{-1/2}{i}(-1)^i = \frac{1\cdot3\cdot5\ \ldots\ (2i-1)}{2\cdot4\cdot6\ \ldots\ 2i}$.

2. To Obtain a Generating Function when Either of the Degrees of Freedom Is Even

Case 1 $f_1 = 2p - 1$, $f_2 = 2q$,

$u = p$, $v = q + 1/2$.

$$\Psi_1(x, y) = \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} C_{p, q+\frac{1}{2}} \, x^p y^q ,$$

where

$$C_{p, q+\frac{1}{2}}(d\, ;\, \alpha, \beta) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dw}{\left[1 + \alpha^2 (w - d\tan\Theta)^2\right]^p \left[1 + \beta^2 (w + d\cot\Theta)^2\right]^{q+\frac{1}{2}}} ,$$

and

$$0 < x, y < 1 .$$

Then,

$$\Psi_1(x, y) = \frac{1}{\pi} \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} \int_{-\infty}^{+\infty} \left[\left(\frac{x}{1 + \alpha^2 (w - d\tan\Theta)^2} \right)^p \left(\frac{y}{1 + \beta^2 (w + d\cot\Theta)^2} \right)^q \text{into} \right.$$

$$\left. \left(\frac{1}{1 + \beta^2 (w + d\cot\Theta)^2} \right)^{1/2} \right] dw .$$

Since, for each p, q, and w, the integrand is at most $x^p y^q$, we obtain, after interchanging summation and integral signs, (except for $\frac{1}{\pi}$),

$$\psi_1(x,y) = \int_{-\infty}^{+\infty} \frac{dw}{\left[1+\beta^2(w+d\cot\theta)^2\right]^{1/2}\left[1-x+\alpha^2(w-d\tan\theta)^2\right]\left[1-y+\beta^2(w+d\cot\theta)^2\right]},$$

which can be transformed into an elementary integral, by suitable change of variable. First express

$$\frac{1}{\left[1-x+\alpha^2(w-d\tan\theta)^2\right]\left[1-y+\beta^2(w+d\cot\theta)^2\right]} = \frac{1}{\alpha^2\beta^2}\sum_{j=1}^{4}\frac{K_j}{(w-w_j)},$$

where

$$\alpha, \beta \neq 0,$$

$$w_1 = d\tan\theta - \frac{i(1-x)^{1/2}}{\alpha},$$

$$w_2 = d\tan\theta + \frac{i(1-x)^{1/2}}{\alpha},$$

$$w_3 = -d\cot\theta - \frac{i(1-y)^{1/2}}{\beta},$$

$$w_4 = -d\cot\theta + \frac{i(1-y)^{1/2}}{\beta},$$

$$K_1 = \frac{1}{(w_1-w_2)(w_1-w_3)(w_1-w_4)},$$

$$K_2 = \frac{1}{(w_2 - w_1)(w_2 - w_3)(w_2 - w_4)} ,$$

$$K_3 = \frac{1}{(w_3 - w_1)(w_3 - w_2)(w_3 - w_4)} ,$$

$$K_4 = \frac{1}{(w_4 - w_1)(w_4 - w_2)(w_4 - w_3)} ,$$

(1)

$$\psi_1(x, y) = \frac{1}{\alpha^2 \beta^2} \sum_{j=1}^{4} \int_{-\infty}^{+\infty} \frac{K_j \, dw}{[1 + \beta^2 (w + d \cot \theta)^2]^{1/2} (w - w_j)} .$$

Let

(2)
$$I_j = \int_{-\infty}^{+\infty} \frac{K_j \, dw}{[1 + \beta^2 (w + d \cot \theta)^2]^{1/2} (w - w_j)} .$$

Put

$$z = \beta (w + d \cot \theta) ,$$

$$dz = \beta \, dw .$$

$$I_j = \int_{-\infty}^{+\infty} \frac{K_j \, dz}{[1 + z^2]^{1/2} (z - z_j)} ,$$

where

$$z_j = \beta(d\cot\theta + w_j)$$

Then,

put $$z^2 + 1 = (z+t)^2$$

$$z = \frac{1-t^2}{2t}$$

$$dz = -\frac{1}{2t^2}(1+t^2)\,dt\ .$$

This reduces I_j to

$$I_j = \int_0^{\infty} \frac{(-2K_j)\,dt}{(t^2 + 2z_j t - 1)} = \int_0^{+\infty} \frac{(-2K_j)\,dt}{(t+z_j)^2 - (1+z_j^2)}$$

$$= (-2K_j)\left[\frac{1}{2(1+z_j^2)^{1/2}} \log \frac{t + z_j - (1+z_j^2)^{1/2}}{t + z_j + (1+z_j^2)^{1/2}}\right]_0^{\infty} .$$

(3) $$I_j = \frac{K_j}{(1+z_j^2)^{1/2}}\left[\log \frac{z_j - (1+z_j^2)^{1/2}}{z_j + (1+z_j^2)^{1/2}}\right] .$$

From (1) and (2), we get

(4) $$\psi_1(x,y) = \frac{1}{\alpha^2\beta^2} \sum_{j=1}^{4} I_j\ .$$

Substitute for I_j according to (3), and thus, $\psi_1(x, y)$ is obtained. $\psi_1(x, y)$, even after simplification, was found very cumbersome and is, therefore, not reported here.

Case 2 $\quad f_1 = 2p,\ f_2 = 2q$

$$C_{p+\frac{1}{2},\, q+\frac{1}{2}}(d;\alpha,\beta) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{dw}{\left[1+\alpha^2(w-d\tan\theta)^2\right]^{p+\frac{1}{2}}\left[1+\beta^2(w+d\cot\theta)^2\right]^{q+\frac{1}{2}}},$$

$$C_{p+\frac{1}{2},\, q+\frac{1}{2}} = A\left[\frac{d^p}{dx^p}\,\frac{d^q}{dy^q}\int_{-\infty}^{+\infty}\frac{}{\left(x+\alpha^2(w-d\tan\theta)^2\right)^{1/2}\left(y+\beta^2(w+d\cot\theta)^2\right)^{1/2}}\right]_{\substack{x=1\\ y=1}},$$

where A is a function of p and q, α, $\beta \neq 0$. Now, according to Byrd and Friedman (1954, page 293, formula 808.01),

$$\int_{-\infty}^{+\infty}\frac{dw}{\alpha\beta\left[(w-d\tan\theta)^2+\frac{x}{\alpha^2}\right]\left[(w+d\cot\theta)^2+\frac{y}{\beta^2}\right]^{1/2}} = \frac{2K(k)}{\alpha\beta\left[\left(\frac{x}{\alpha^2}\right)^{1/2}\left(\frac{y}{\beta^2}\right)^{1/2}g\right]^{1/2}},$$

where

$$g_1 = \frac{d^2\tan^2\theta + d^2\cot\theta^2 + \frac{x}{\alpha^2}+\frac{y}{\beta^2}}{2\left(\frac{x}{\alpha^2}\right)^{1/2}\left(\frac{y}{\beta^2}\right)^{1/2}} = \frac{d^2(\tan^2\theta+\cot^2\theta)\,\alpha^2\beta^2 + x\beta^2 + y\alpha^2}{2\alpha\beta\, x^{1/2}\, y^{1/2}},$$

$$g = g_1 + (g_1^2-1)^{1/2},$$

$$\kappa = \frac{g_1^2 - 1}{g^2} ,$$

$$K(\kappa) = \frac{\pi}{2}\left(1 + \frac{1^2}{2^2}\kappa^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\kappa^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}\kappa^6 + \cdots + \frac{1^2 \cdot 3^2 \cdots (2r-1)^2 \kappa^{2r}}{2^2 \cdot 4^2 \cdots (2r)^2} + \cdots\right);$$

$$\frac{\delta K(\kappa)}{\delta \kappa} = \frac{1}{\kappa}\left(\frac{E(\kappa)}{\kappa'^2} - K(\kappa)\right), \quad \frac{\delta E(\kappa)}{\delta \kappa} = \frac{1}{\kappa}\left(E(\kappa) - K(\kappa)\right), \quad \kappa'^2 = (1-\kappa^2),$$

$$\frac{\delta K(\kappa)}{\delta x} = \frac{\delta K(\kappa)}{\delta \kappa}\frac{\delta \kappa}{\delta x} = \frac{1}{\kappa}\left(\frac{E(\kappa)}{\kappa'^2} - K(\kappa)\right)\frac{\delta \kappa}{\delta x} ,$$

$$\frac{\delta K(\kappa)}{\delta y} = \frac{\delta K(\kappa)}{\delta y}\frac{\delta \kappa}{\delta y} = \frac{1}{\kappa}\left(E(\kappa) - K(\kappa)\right)\frac{\delta \kappa}{\delta y} .$$

$K(\kappa)$, $E(\kappa)$ are the complete elliptic integrals of the first and second kind, and are tabulated.

3. Program in the "MAD" Language for the IBM 709 Computer To Calculate the Densities $\phi(d|f_1,f_2;\theta)$ for Odd Integral Values of f_1 and f_2

In the program, f_1, f_2, and ϕ are changed to N_1, N_2 and ρ.

```
          R  CONVOLUTION OF 2 INDEPENDENT T-DISTRIBU-
          R     TIONS FOR VENKUTAI PATIL, 1-62
          R        INITIALIZATION.
START         READ FORMAT DATA,NSTART,NSTOP,TSTEP,
          1     DSTART,DSTOP,DSTEP
                  VECTOR VALUES DATA=$2I3,4F6.0*$
          R  MAKE GOMMA TABLE.
              SQRPI=1./SQRT.(3.14159265)
              GOMMA (NSTART)=SQRPI
              THROUGH LOOP1A, FOR N=3,2,N.G.N START
LOOP1A          GOMMA(NSTART)=GOMMA(NSTART)*(N-1.0)/
          1       (N-2.0)
              THROUGH LOOP1, FOR N=NSTART+2,2,N.G.NSTOP
LOOP1           GOMMA(N)=GOMMA(N-2)*(N-1.0)/(N-2.0)
          R        MAKE SQUARE ROOT TABLE.
              THROUGH LOOP2, FOR F=NSTART,2.0,F.G.NSTOP
LOOP2           SQR(F)=SQRT.(F)
          R        MAKE K AND K1 TABLES.
              K(1)=-0.5
              K1(1)=1.0
              THROUGH LOOP3, FOR N=2,1,N.G.(NSTOP+1)/2
                X=2.*N
                Y=X-3.
                K(N)=K(N-1)*Y/X
LOOP3           K1(N)=K1(N-1)*Y/(X-2.)
          R        MAKE SEC, SEC2, AND TAN TABLES.
              CONV=0.174532925E-1
              P=0
              THROUGH LOOP4, FOR ARG=TSTEP,TSTEP,
          1       ARG.G.89.9
                P=P+1
                THETA(P)=ARG
                X = ARG*CONV
                Y=COS.(X)
                SEC(P)=1./Y
                SEC2(P)=1./(Y*Y)
LOOP4           TAN(P)=SIN.(X)/Y
              TSTOP=P+1
              THETA(0)=0.0
              THETA(TSTOP)=90.0
          R        CALCULATE RHO
              THROUGH ITER,FOR N2= NSTART,2,N2.G.NSTOP
```

```
                 Q=(N2+1)/2
                 THROUGH ITER,FOR N1=N2,2, N1.G. NSTOP
                 P=(N1+1)/2
                 S=SQR(N1)/SQR(N2)
                 GO = GOMMA (N2)* GOMMA(N1)/SQR(N2)
                 DN=1
                 THROUGH IT, FOR D=DSTART,DSTEP,D.G.DSTOP
                    GD2=D*D/N2
                    GD1=D*D/N1
             R      ***************
                    THROUGH ITT, FOR T=1,1,T.E.TSOP
                       ALPHA=S*TAN(T)
                       ALPH=2*ALPHA
                     ALPH1=ALPHA+1.0
                       ALPH2=ALPHA*ALPH1
                     PHI=1./(ALPH1.P.2+GD2*SEC2(T))
                    C(  1,1)=ALPHI*PHI
                    THROUGH ITA,FOR I=2,1,I.G.P
                       X=0.0
                       THROUGH IT AA, FOR IA=1,1,IA.G.I-2

IT AA                  X=C(  IA,1)*K(I-IA)+X
IT A                C(  I,1)=PHI*(K1(I)+ALPH2*C(  I-1,1)-ALPH *X)
                 THROUGH IT B, FOR J=2,1,J.G.Q
             X=0.0
             THROUGH ITO,FOR JA=1,1,JA.G.J-2
                X=C(1,JA)*K(J-JA)+X
             C(1,J)=PHI*(ALPHA*K1(J)+ALPH1*C(1,J-1)-ALPH*X)
                    THROUGH IT B, FOR I=2,1,I.G.P.
                       X=0.0
                       THROUGH IT B1, FOR IA=1,1,IA.G.I-1
                          THROUGH ITB1   , FOR JA=1,1,JA.G.J-1
IT B1                        X=C(  IA,JA)*K(I-IA)*K(J-JA)+X
                       Y=0.0
                       THROUGH IT B2, FOR IA=1,1,IA.G.I-2
IT B2                     Y=C(  IA,J)*K(I-IA)+Y
                       Z=0.0
                       THROUGH IT B3, FOR JA=1,1,JA.G.J-2
IT B3                     Z=C(  I,JA)*K(J-JA)+Z
IT B                   C(  I,J)=(ALPH1*C(  I,J-1)+ALPH2*C(  I-1,J)
             1         -ALPH*(X+Y+Z))*PHI
ITT                 RHO(DN,T)=C(  P,Q)*GO*SEC(T)
             R *****************
```

```
                     RHO(DN,0)=(GOMMA(N2)*SQRPI/SQR(N2))/
             1          (1.+GD2).P.Q
                     RHO(DN,TSOP)= GOMMA(N1)*SQRPI/SQR(N1)/
             1          (1.+GD1).P.P
IT                   DN=DN+1
             R
             R    PRINT ALL PAGES FOR N1,N2
                  THROUGH ITER, FOR COL=0,6,COL.G.TSOP
                     DN=1
                  WHENEVER COL+5.G.TSTOP
                         L=TSTOP
                       OTHERWISE
                         L=COL+5
                END OF CO   NDITIONAL
                       PRINT FORMAT HEAD,N1,N2,THETA(COL)...
             1             THETA(L)
                       THROUGH ITER, FOR D=DSTART,DSTEP,
             1             D.G.DSTOP
                         PRINT FORMAT LINE,  D, RHO(DN,COL)
             1             ...RHO(DN,L)
ITER                     DN=DN+1
                TRANSFER TO START
             R    DECLARATIONS
                VECTOR VALUES HEAD=$26H1CONVOLUTED T-
             1       DISTRIBUTION.,S10,5HN1 =
             1 I3,7H, N2 = I3/3H- D S50,5HTHETA/S17,F4.1,5(S13,
             1    F4.1)///*$
                VECTOR VALUES LINE=$1H F4.1,S1,6(1PE17.8)*$
                DIMENSION K(50),K1(50),GOMMA(100),SQR(100),
             1    THETA(90),SEC(90),SEC2(90),TAN(90)
             2    C( 400,DMC),RHO(9000,DMR)
                VECTOR VALUES DMC=2,1,20
                VECTOR VALUES DMR=2,2,91
                EQUIVALENCE(GOMMA(2),SQR(1))
                  INTEGER NSTART,NSTOP,N,P,Q,N1,N2,DN,T,I,
             1         IA,J,JA,L,FIXD,
             1       TSTOP,COL
                END OF PROGRAM
```

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