still not widely known. A detailed history and many results concerning this may be found in a recent paper.[6]

The identity is due to Rothe and is as follows:

$$\sum_{k=0}^{n} \frac{x}{x+bk}\binom{x+bk}{k}\frac{y}{y+b(n-k)}\binom{y+b(n-k)}{n-k} = \frac{x+y}{x+y+bn}\binom{x+y+bn}{n} \qquad (7)$$

and, with suitable attention, is valid for all real or complex $x$, $y$, and $b$, and all nonnegative integers $n$. The novel point about (7) is the parameter $b$ which allows this formula to include not only the Vandermonde relation (4) but perhaps ninety percent of the common binomial identities. Relation (5) is particularly useful in manipulating binomial summations. It together with the symmetry $\binom{n}{k} = \binom{n}{n-k}$ and changes of summation variable suffices to reduce most all the known identities to some form of the Vandermonde or other theorem.

[6] H. W. Gould and J. Kaucký, J. Combinatorial Theory 1, 233 (1966).

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 10, NUMBER 1 JANUARY 1969

# Partially Alternate Derivation of a Result of Nelson

PAUL FEDERBUSH
*Department of Mathematics, University of Michigan, Ann Arbor, Michigan*

(Received 17 April 1968)

The result of Nelson that the total Hamiltonian is semibounded for a self-interacting Boson field in two dimensions in a periodic box is derived by an alternate method. It is more elementary in so far as functional integration is not used.

In Ref. 1, Nelson has proved the semiboundedness of the Hamiltonian for a class of two-dimensional self-interacting Boson-field theories in a periodic spatial box. In Ref. 2, Glimm has detailed and extended the result of Ref. 1. We will give an alternate derivation of the results of Nelson avoiding the use of functional integration, central in Ref. 1. As will be seen, the idea of the proof, however, is not essentially different from that of Nelson and we draw on results of his paper. It is hoped that a new method of proof may lead to some new results or insights.

We consider a Hamiltonian of the form

$$H = H_0 + V, \qquad (1)$$

where $H_0$ is the free Hamiltonian of a particle of mass $\mu_0 \neq 0$ expressed in terms of the neutral scalar field $\phi$ and its momentum conjugate $\pi$:

$$H_0 = \frac{1}{2}\int_0^1 dx:[(\nabla\phi)^2 + \mu_0^2\phi^2 + \pi^2]. \qquad (2)$$

As is evident we are working in a periodic box of length 1. $V$ is a polynomial function of the $\phi(x)$. We denote by $H_0$ and ${}^NV$ the parts of $H_0$ and $V$ depending only on the creation and annihilation operators of the $N$ lowest-energy modes of the free Hamiltonian. We always imagine we are working with ${}^NH_0$ and ${}^NV$, but derive inequalities independent of $N$.

*Theorem:* Assume for each $\alpha > 0$ that there is an $M_\alpha$ such that

$$\langle 0|\exp^{-\alpha^N V}|0\rangle \leq M_\alpha, \quad \text{all } N.$$

$|0\rangle$ denotes the vacuum of the free field. Then there is a $B$ such that

$${}^NH_0 + {}^NV \geq B, \quad \text{for all } N.$$

Actually as will be seen it is not necessary to satisfy the condition above for all $\alpha$, but only for some sufficiently large $\alpha$ that one can calculate. We refer to Refs. 1 or 2 for the result that the conditions of the theorem are satisfied for a large class of self-interactions. This much of Nelson's proof requires no functional integration.

We follow the notation of Ref. 2:

$$\phi(x) = \sum_k (2\omega_k)^{-\frac{1}{2}}(a_k + a_{-k}^*)e^{ikx} \qquad (3)$$

[1] E. Nelson, "A Quarticinteraction in Two Dimensions" in *Mathematical Theory of Elementary Particles*, R. Goodman and I. Segal, Eds. (M.I.T. Press, Cambridge, Mass., 1965), pp. 69–73.

[2] J. Glimm, Commun. Math. Phys. 8, 12 (1968).

and define

$$q_0 = \left(\frac{1}{2\omega_0}\right)^{\frac{1}{2}}(a_0 + a_0^*),$$

$$p_0 = -i\left(\frac{\omega_0}{2}\right)^{\frac{1}{2}}(a_0 - a_0^*),$$

$$q_{|k|} = \left(\frac{1}{4\omega_k}\right)^{\frac{1}{2}}(a_k + a_k^* + a_{-k} + a_{-k}^*),$$

$$q_{-|k|} = -i\left(\frac{1}{4\omega_k}\right)^{\frac{1}{2}}(-a_{|k|} + a_{|k|}^* + a_{-|k|} - a_{-|k|}^*), \quad (4)$$

$$p_{|k|} = -i\left(\frac{\omega_k}{4}\right)^{\frac{1}{2}}(a_k - a_k^* + a_{-k} - a_{-k}^*),$$

$$p_{-|k|} = \left(\frac{\omega_k}{4}\right)^{\frac{1}{2}}(a_{|k|} + a_{|k|}^* - a_{-|k|} - a_{-|k|}^*).$$

In terms of these variables,

$$H_0 = \sum_k \tfrac{1}{2}(p_k^2 + \omega_k^2 q_k^2 - \omega_k) = \sum_k H_k. \quad (5)$$

We represent these operators on the $L^2$ space of $E^N$ with measure the product of the measures $\mu_k$,

$$d\mu_k = (\omega_k/\pi)^{\frac{1}{2}} e^{-\omega_k q_k^2}\, dq_k \quad (6)$$

with $q_k$ a multiplicative operator and

$$p_k = i(\partial/\partial q_k) - \omega_k q_k. \quad (7)$$

A complete set of eigenfunctions for $H_k$ is given by

$$\phi_{kn}(q_k) = (2^n n!)^{-\frac{1}{2}} A_n[q_k(\omega_k)^{\frac{1}{2}}], \quad n = 0, 1, 2, \cdots, \quad (8)$$

with corresponding eigenvalues

$$E_{kn} = n\omega_k, \quad (9)$$

$A_n(x)$ is the $n$th Hermite polynomial.

The chief inequality we will exploit is the following numerical inequality for $x$, $y$ real, $y \geq 0$:

$$xy \leq e^x + y \ln y. \quad (10)$$

The expectation value of the interaction $V$ in a state with function $F$ is given by

$$\langle F|\, V\, |F\rangle = \int d\mu\, |F|^2\, V. \quad (11)$$

We apply (10) with $x = rV$ and $y = r^{-1}|F|^2$ to derive the result

$$-\langle F|\, V\, |F\rangle \leq \int d\mu\, e^{-rV} + \frac{1}{r}\int d\mu\, |F|^2 \ln |F|^2 - \frac{1}{r}\ln r. \quad (12)$$

$r$ is a numerical factor to be fixed later. Note that

$$\int d\mu\, e^{-rV} = \langle 0|\, e^{-rV}\, |0\rangle. \quad (13)$$

We intend to bound the second term on the right side of (12) by the expectation value of $H_0$ in the state $F$. We consider the following equation:

$$\int |F|^2 \ln |F|^2\, d\mu = \frac{2}{\lambda}\int F^* H_0 F\, d\mu + \frac{1}{\lambda}\frac{d}{dt}\int [(e^{-H_0 t}F)^*(e^{-H_0 t}F)]^{1+\lambda t}\, d\mu\Big|_{t=0}, \quad (14)$$

which easily follows for functions $F$ nice enough so that all the integrals exist and the differentiation may be moved inside the integral, a dense subspace in $L^2$. We do not discuss domain questions.

We rewrite (12) using (14):

$$-\langle F|\, V\, |F\rangle \leq \int d\mu\, e^{-rV} + \frac{2}{\lambda r}\langle F|\, H_0\, |F\rangle - \frac{1}{r}\ln r + \frac{1}{\lambda r}\frac{d}{dt}\int [(e^{-H_0 t}F)^*(e^{-H_0 t}F)]^{1+\lambda t}\, d\mu\Big|_{t=0}. \quad (15)$$

The theorem we are after is established provided $\lambda r \geq 2$ and we can bound the last term in (15).

The remainder of the paper is devoted to a study of

$$\int [(e^{-H_0 t}F)^*(e^{-H_0 t}F)]^{1+\lambda t}\, d\mu = \int |e^{-H_0 t}F|^{2+2\lambda t}\, d\mu. \quad (16)$$

We consider, corresponding to any $g$ in $L^2(\mu)$, its expression as a sum of products of the functions in (8):

$$g(q) = \sum_{i_1, i_2, \cdots, i_N} C_{i_1, i_2, \cdots, i_N} \prod_s [2^{i_s}(i_s!)]^{-1} e^{i_s} A_{i_s}[q_s(\omega_s)^{\frac{1}{2}}]. \quad (17)$$

(The $q_s$ are merely the $q_k$ in some order.) The $C_{i_1, i_2, \cdots, i_N}$ are now considered as functions on the discrete space whose points are the indices of the $C$'s. To the point $(i_1, i_2, \cdots, i_N)$ is associated the point mass $\Pi_s e^{2i_s}$. With this measure, the transformation $T$ that carries a set of $C$'s into the corresponding function $g$ as in (17) is norm preserving as a map from $l^2$ to $L^2$. We will later show that $T$ is norm decreasing as a map from $l^1$ to $L^4$. Assuming this for a moment, we complete the proof of the theorem.

We apply the Riesz–Thorin convexity theorem to the transformation $T$ obtaining

$$\int |e^{-H_0 t}F|^{2+2\lambda t}\, d\mu \leq \left(\sum_{i_1, i_2, \cdots, i_N} \prod_s e^{2i_s} |\exp[-\omega_{i_1, i_2, \cdots, i_N} t] \times C_{i_1, i_2, \cdots, i_N}|^{[2(1+\lambda t)/(1+3\lambda t)]}\right)^{[(1+3\lambda t)/2(1+\lambda t)]} \quad (18)$$

with

$$\omega_{i_1,i_2,\cdots,i_N} = \sum_s i_s\omega_s. \tag{19}$$

In the right-hand side of (18) we apply the Holder inequality to obtain an expression involving the weighted sum of the squares of the absolute values of the $C$'s which is equal to one:

$$\int |e^{-H_0 t}F|^{2+2\lambda t}\, d\mu \le \left[\sum_{i_1,i_2,\cdots,i_N} \prod_s e^{2i_s} \exp\left(-\omega_{i_1,i_2,\cdots,i_N}\frac{2(1+\lambda t)}{2\lambda}\right)\right]^{2\lambda t}. \tag{20}$$

It follows that

$$\frac{d}{dt}\int |e^{-H_0 t}F|^{2+2\lambda t}\, d\mu\Big|_{t=0} \le 2\lambda \ln\left[\sum_{i_1,\cdots,i_N} \prod_s e^{2i_s} \exp\left(-\frac{1}{\lambda}\,\omega_{i_1,\cdots,i_N}\right)\right]. \tag{21}$$

If $\mu_0/\lambda > 2$, this gives an inequality with finite right-hand side in the limit $N \to \infty$. It is clear that the theorem is now reduced to establishing that $T$ is norm decreasing from $l^1$ to $L^4$.

*Lemma.* Let $S$ be the space of sequences $\{C_\gamma\}$ $\gamma = 0, 1, \cdots, N$ with measure at $\gamma$, $e^{2\gamma}$; and $Y$ the space of functions on $R^1$ with measure

$$(1/\pi)^{\frac{1}{2}} e^{-x^2}\, dx; \tag{22}$$

and $T$ the operator from $S$ to $Y$ given by

$$T(\{C_\gamma\}) = \sum_\gamma C_\gamma \frac{1}{[2^\gamma(\gamma!)]^{\frac{1}{2}}}\, e^\gamma A_\gamma(x) \tag{23}$$

with $A_\gamma$ the $\gamma$th Hermite polynomial; then, $T$ is norm decreasing from $l^1$ to $L^4$.

It is easy to see that this lemma would follow from establishing that

$$\left|\left(\frac{1}{\pi}\right)^{\frac{1}{2}} e^{-a-b-c-d}\int_{-\infty}^{\infty} [2^{a+b+c+d}(a!)(b!)(c!)(d!)]^{-\frac{1}{2}} A_a(x)A_b(x) \times A_c(x)A_d(x)e^{-x^2}\, dx\right| \le 1 \tag{24}$$

for all integers $a$, $b$, $c$, and $d \geq 0$.[3] We use the generating function

$$e^{-t^2+2tZ} = \sum \frac{t^N}{N!} A_N(Z) \tag{25}$$

to obtain

$$\frac{1}{(\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} dx\, e^{-x^2} A_a(x)A_b(x)A_c(x)A_d(x) = \frac{a!\,b!\,c!\,d!}{\frac{1}{2}(a+b+c+d)!}\cdot 2^{\frac{1}{2}(a+b+c+d)} \times (rs+rt+ru+st+su+tu)^{\frac{1}{2}(a+b+c+d)}_{\text{pick-a-power}}, \tag{26}$$

where pick-a-power means to find the coefficient of the monomial $r^a s^b t^c u^d$ in the expansion of the expression. Note that $a + b + c + d$ is even or the integral vanishes.

We make the crude estimate

$$(rs+rt+ru+st+su+tu)^{\frac{1}{2}(a+b+c+d)}_{\text{pick-a-power}} \le 2^{\frac{1}{2}(-a-b-c-d)}\cdot (r+s+t+u)^{a+b+c+d}_{\text{pick-a-power}}. \tag{27}$$

Now,

$$(r+s+t+u)^{a+b+c+d}_{\text{pick-a-power}} = \frac{(a+b+c+d)!}{a!\,b!\,c!\,d!}. \tag{28}$$

Denoting the left-hand side of (24) by LHS and using (27) we obtain

$$\text{LHS} \le e^{-a-b-c-d} \times \frac{(a+b+c+d)!}{(a!\,b!\,c!\,d!)^{\frac{1}{2}}\cdot[\frac{1}{2}(a+b+c+d)]!\,2^{\frac{1}{2}(a+b+c+d)}}. \tag{29}$$

That the right-hand side of (29) is $\le 1$ we leave as any easy exercise.

[3] Actually, it is sufficient to let $a = b = c = d$.