Excitation and propagation of lower-hybrid waves in a bounded, inhomogeneous plasma

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The excitation and propagation of lower-hybrid waves in an inhomogeneous, cylindrical plasma is studied theoretically for finite-length electrostatic sources. The boundary-value problem for the electrostatic potential in a cold, inhomogeneous plasma is solved numerically as a superposition of the radial eigenmodes excited by a finite-length source. Radial eigenmodes are found numerically by an algorithm which includes the case where the lower-hybrid resonance layer occurs in the plasma. The eigenmode superposition is carried out for several phased-ring sources. The plasma response is found to be composed of resonance-cone surfaces along which the potential is a maximum. When the resonance layer does not occur in the plasma, the resonance-cone surfaces reflect from the column axis and at the plasma boundary. For the case when the resonance layer does occur, the resonance-cone surfaces become asymptotic to the resonance layer and do not penetrate to the center. The presence of damping causes the resonance-cone singularities to dissolve axially leaving the lowest-order radial mode excited by the source.

I. INTRODUCTION

Radio-frequency plasma heating near the lower-hybrid frequency has become important as a possible means of providing supplemental heating in fusion devices. The possibility exists that plasma waves launched by external sources may couple either linearly[1] or nonlinearly[2] to hot-plasma waves that may give rise to plasma heating through collisionless damping processes. In order to properly describe the excitation and propagation of externally excited lower-hybrid waves, it is necessary to account for both the plasma density inhomogeneity and the finite size of the plasma and the source.

The results of previous theoretical studies[3-6] indicate that electrostatic lower-hybrid waves launched by finite-sized sources give rise to singularities in the wave field in an infinite, cold plasma. In homogeneous plasma the singularity due to a point source exists along the surface of a cone with its axis parallel to the magnetic field lines.[3] Briggs and Parker[4] showed that these so-called resonance cones are curved surfaces in an inhomogeneous plasma and do not penetrate past the resonance layer where the wave frequency equals the lower-hybrid frequency. Simonutti[5] and Bellan and Porkolab[6] extended this work in planar geometry to include first-order thermal effects and the effects of finite-sized sources. They found that linearly converted hot-plasma waves appear as weak singularities at the point at which the resonance-cone surfaces reach the lower-hybrid resonance layer. The particular character of the plasma response depends on the superposition of plane waves excited by the source.

Not all of the waves excited by a given source may necessarily penetrate the plasma due to the density and magnetic field inhomogeneity. An accessibility condition has been derived by several authors which stipulates a minimum allowable value for the parallel wavenumber k_z for propagating waves.[7,8] Troyon and Perkins[9] showed that Landau damping of the lower-hybrid waves imposes a condition on the maximum allowable k_z value for wave penetration. Hence, a window in the

k_z spectrum is formed by Landau damping and accessibility through which a source should inject the bulk of its power. Phased waveguide arrays have been proposed[10] for rf power coupling which concentrate the excited k_z spectrum in the penetration window.

The purpose of this paper is to examine the propagation of lower-hybrid waves in a cylindrical, bounded cold plasma which are excited by an arbitrarily phased array of electrostatic sources. In particular, attention is focused on the properties of the resonance-cone surfaces in cylindrical geometry. The density is allowed to vary radially and the electrostatic plasma response is calculated for the source configurations and parameters of several recent experiments. In Sec. II the boundary-value problem in cylindrical geometry is developed. In Sec. III approximate solutions are discussed which show the relationship of the bounded-plasma waves to resonance cones in an infinite plasma. The effects of damping are also considered. The numerical calculation of the eigenmodes of the bounded system is discussed in Sec. IV, including the case where the lower-hybrid resonance layer occurs in the plasma. Full numerical results are then presented in Sec. V for the total plasma response due to several specific sources. The results are summarized in Sec. VI.

II. CYLINDRICAL BOUNDARY-VALUE PROBLEM

The plasma is taken to be an infinitely long cylinder of radius a with a uniform magnetic field imposed in the z direction and a radially varying density profile. The electrostatic source array is localized at a fixed radius b with an arbitrarily specified axial and azimuthal rf potential variation. A vacuum region may occur between the source and the plasma as shown in Fig. 1.

It can be shown that the lower-hybrid waves are nearly electrostatic in all but a thin layer at the plasma surface.[7] Hence, the electrostatic assumption may be made provided that only those waves that satisfy the accessibility condition are included. It is also assumed that the plasma is moderately collisional in agreement
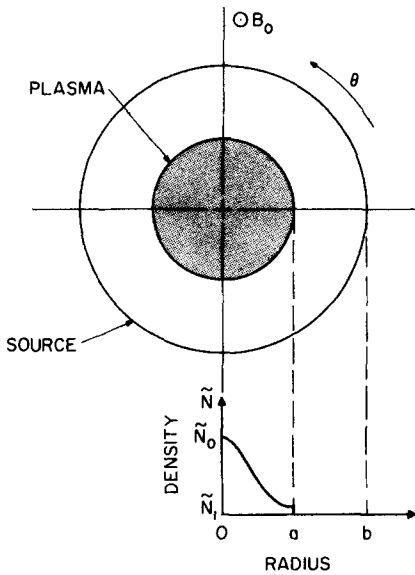
FIG. 1. Sketch of the bounded-plasma system. A constant magnetic field is in the z direction and the plasma density varies radially.

with experimental conditions.[4,11,12] For the collisional case Bellan and Porkolab[6] showed that the hot-plasma waves are not appreciably excited and thermal effects may be neglected.

The resulting electrostatic wave equation in the plasma can be found from Poisson's equation in a source-free region

$$\nabla \cdot (\mathbf{K} \cdot \nabla \phi) = 0, \tag{1}$$

where ϕ is the electrostatic potential and $\mathbf{K}$ is the cold-plasma dielectric tensor. Fourier transforming the axial variation and Fourier analyzing the azimuthal variation as $\exp(im\theta)$ leads to the equation

$$K_1 \frac{d^2\phi}{dr^2} + \left(\frac{dK_1}{dr} + \frac{K_1}{r}\right)\frac{d\phi}{dr} - \left(k_z^2 K_\parallel + \frac{im}{r}\frac{dK_x}{dr} + \frac{m^2}{r^2} K_1\right)\phi = 0, \tag{2}$$

where K_1, K_x, and $K_\parallel$ are the elements[13] of $\mathbf{K}$.

The potential in the vacuum region is described by solutions of the modified Bessel equation which is given by

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \left(k_z^2 + \frac{m^2}{r^2}\right)\phi = 0. \tag{3}$$

To completely specify the solution, boundary conditions at the plasma–vacuum interface and at the source are required. At the plasma–vacuum boundary it is assumed that the tangential components of $\mathbf{E}$ and the nor-

mal components of D are continuous. E_z and E_θ are continuous when

$$\phi_p(a) = \phi_v(a), \tag{4}$$

where the subscripts refer to the values on the plasma and vacuum sides of the boundary, respectively. Continuity of D_r requires

$$K_1(a)\frac{d\phi_p(a)}{dr} - i\frac{K_x(a)}{a}m\phi_p(a) = \frac{d\phi_v(a)}{dr}. \tag{5}$$

The source is Fourier analyzed as

$$\phi_s = \sum_{m=-\infty}^{\infty} \exp(im\theta)\,\hat{\phi}_s(m, k_z), \tag{6}$$

where $\hat{\phi}_s(m, k_z)$ is the Fourier transform-Fourier series coefficient. In order to match the source potential term-by-term in the Fourier series, the following condition is required

$$\phi_v(r = b, m, k_z) = \hat{\phi}_s(m, k_z). \tag{7}$$

The boundary conditions given by Eqs. (4) through (7) and Eqs. (2) and (3) constitute a well-defined boundary-value problem for the potential.

The eigenfunctions of this problem are of the form

$$\phi_m(r)\exp(im\theta - ik_z z), \tag{8}$$

and it is necessary to find the radial function $\phi_m(r)$ from Eqs. (2) and (3). In the vacuum region the solution is a linear combination of modified Bessel functions

$$\phi_v = B_1 I_m(k_z r) + B_2 K_m(k_z r). \tag{9}$$

In the plasma there are also two independent solutions denoted by $\phi_m^{(1)}$ and $\phi_m^{(2)}$ and the general solution is

$$\phi_p = A_1 \phi_m^{(1)} + A_2 \phi_m^{(2)}. \tag{10}$$

The functions $\phi_m^{(1)}$ and $\phi_m^{(2)}$ cannot be given in closed form because of the complexity of Eq. (2) which, in general, has singular points at $r = 0$, $r \to \infty$ and at the lower-hybrid resonance layer defined by $K_1(r) = 0$. Since $r = 0$ is a regular singular point, series solutions for $\phi_m^{(1)}$ and $\phi_m^{(2)}$ can be found by the method[14] of Frobenius. One of the solutions is analytic at $r = 0$ and the other is singular; the singular solution is denoted by $\phi_m^{(2)}$. It is assumed that the potential at this point is finite and hence

$$A_2 = 0. \tag{11}$$

Application of the boundary conditions yields the following system of equations for the coefficients A_1, B_1, and B_2:

$$\begin{bmatrix} \phi_m(a) & -I_m(k_z a) & -K_m(k_z a) \\ K_1(a)\phi_m'(a) - \dfrac{imK_x(a)}{a}\phi_m(a) & -k_z I_m'(k_z a) & -k_z K_m'(k_z a) \\ 0 & I_m(k_z b) & K_m(k_z b) \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hat{\phi}_s \end{bmatrix}. \tag{12}$$

The coefficients are well defined in terms of $\hat{\phi}_s$ provided that the determinant of the coefficients does not vanish. Vanishing of the determinant defines the dispersion relation for the eigenmodes of the system.

In the nonsingular case the eigenfunctions $\phi_m^{(1)}$, K_m, and I_m may be used to find the total plasma response due to

an arbitrarily specified source ϕ_s. The potential in real space is found as a superposition over all possible axial and azimuthal modes and is given by

$$\phi(r, \theta, z) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \exp(im\theta) \int_{-\infty}^{\infty} dk_z \exp(ik_z z)$$

$$\times \begin{cases} A_1 \phi_m^{(1)}(r, k_z), & 0 \leq r \leq a \\ B_1 I_m(k_z r) + B_2 K_m(k_z r), & a \leq r \leq b \end{cases} \quad (13)$$

with the coefficients specified by Eq. (12). In general, the integration in Eq. (13) must be found numerically and the results for several specific sources are presented in Sec. V. It is first useful, however, to examine approximations to Eq. (13) for simplified cases.

III. APPROXIMATE UNIFORM-COLUMN SOLUTIONS

It is found that the superposition of bounded-plasma eigenmodes implied in Eq. (13) gives rise to resonance-cone surfaces similar to those found in infinite plasma. To show this the case is considered where the plasma is homogeneous with no surrounding vacuum region and with $\omega > \omega_{lh}$. The superposition integral of Eq. (13) is given by

$$\phi(r, \theta, z)$$
$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \exp(im\theta) \int_{-\infty}^{\infty} dk_z \exp(ik_z z) \hat{\phi}_s(m, k_z) \frac{J_m(k_x r)}{J_m(k_x a)}, \quad (14)$$

where

$$k_x^2 = -k_z^2 (K_{\parallel}/K_{\perp}) \quad (15)$$

and J_m is the Bessel function of the first kind of order m resulting from the solution of Eq. (2) for the homogeneous plasma. The integral in Eq. (14) can be evaluated as a residue sum over all the poles:

$$\phi(r, \theta, z) = \sum_{m=-\infty}^{\infty} \frac{i}{a} \left(-\frac{K_{\perp}}{K_{\parallel}}\right)^{1/2} \exp(im\theta) \sum_{n=1}^{\infty} \exp(ik_{zn}z)$$

$$\times \hat{\phi}_s(m, k_z) \frac{J_m[p_{mn}(r/a)]}{J_m'(p_{mn})}, \quad (16)$$

where the poles are given by the eigenvalue equation

$$J_m(p_{mn}) = 0, \quad p_{mn} = k_{xn} a \quad (17)$$

obtained by equating the determinant of Eq. (12) to zero and

$$k_{zn} = \frac{p_{mn}}{a} \left(-\frac{K_{\perp}}{K_{\parallel}}\right)^{1/2}. \quad (18)$$

The potential given by Eq. (16) will be recognized as a sum of Trivelpiece–Gould eigenmodes[15] for the homogeneous plasma waveguide. In the case of an inhomogeneous plasma the Bessel functions are replaced by the eigenmode solutions $\phi_m(r)$ of Eq. (2).

To show that resonance cones arise from the eigenfunction superposition in Eq. (13), the homogeneous plasma solution given by Eq. (14) is again used. The integral may be approximated by the substitution of the large argument expansion of the Bessel function which is given by

$$J_m(x) \sim \frac{1}{(2\pi x)^{1/2}} \cos\left[x - \frac{\pi}{4}(2m + 1)\right]. \quad (19)$$

Since $k_x r \gg 1$ except at small values of k_z and r, the use of this expansion does not introduce a large error into the integration. Since the integrand is an even function of k_z, the equivalent cosine transform may be used and the potential may be expressed as

$$\phi(r, \theta, z) \simeq \sum_{m=-\infty}^{\infty} a_m \exp(im\theta) \frac{L}{2\pi} \int_0^{\infty} dk_z \frac{2}{k_z L} \left(\frac{a}{r}\right)^{1/2}$$

$$\times \cos k_z z \, \sin \frac{k_z L}{2} \frac{\cos\left[k_x r - (2m+1)\frac{\pi}{4}\right]}{\cos\left[k_x a - (2m+1)\frac{\pi}{4}\right]}, \quad (20)$$

where the z dependence of the source has been taken to be constant for $|z| < L/2$ and zero for $|z| > L/2$. In this case it is assumed that the azimuthal dependence of the source may be expanded in a Fourier series with expansion coefficient a_m. Expanding $1/\cos x$ in the form

$$\frac{1}{\cos x} = 2\exp(-ix) \sum_{n=0}^{\infty} (-1)^n \exp(-2inx) \quad (21)$$

and expressing the remaining trigonometric functions as complex exponentials puts Eq. (20) into a form that may be evaluated by the saddle-point method. It is found that the integrand is an infinite sum of exponentials each having the form of

$$\exp[ik_z f_n(r, z)], \quad (22)$$

where f_n is a function of spatial variables only. According to the saddle-point method, the integrand will be highly oscillatory and the oscillations will largely cancel except when

$$f_n(r, z) = 0, \quad n = 0, 1, 2, \ldots . \quad (23)$$

Thus, the integral will have a maximum in real space on the infinite set of discrete surfaces defined in Eq. (23). These surfaces are the resonance-cone surfaces in the cylindrical, bounded plasma. Equation (23) may be written explicitly as

$$z \pm \frac{L}{2} \pm r\left(-\frac{K_{\parallel}}{K_{\perp}}\right)^{1/2} \pm (2n+1)a\left(-\frac{K_{\parallel}}{K_{\perp}}\right)^{1/2} = 0, \quad n = 0, 1, 2, \ldots, \quad (24)$$

where any combination of signs is allowed. The azimuthal mode number m does not appear in Eq. (24) but enters in a phase factor multiplying $a_m \exp(im\theta)$ in Eq. (20). This factor is important because it may affect the θ variation. In Fig. 2 the projection of these surfaces is plotted on the $r-z$ plane for $r < a$. It is found that there are two sets of resonance-cone surfaces; one from each end of the source, at the cone angle defined by

$$\tan^2 \xi_c = -K_{\perp}/K_{\parallel}. \quad (25)$$

This angle was discussed previously by Fisher and Gould for the upper-hybrid frequency.[16] Similar results were found for the lower-hybrid frequency in an infinite plasma for a finite-length source.[5,6] It is also seen from Fig. 2 that the resonance-cone surfaces appear to reflect at the column axis and at the plasma boundary. For asymmetric excitation the m-dependent phase factor in Eq. (20) indicates that the resonance-cone surfaces reflect only from the boundary and pass through the axis, retaining the asymmetry of the source,
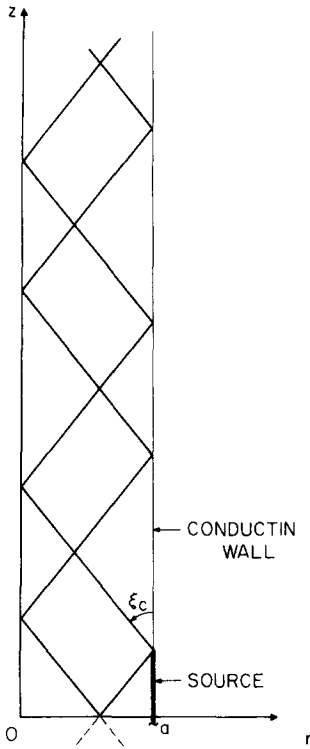
FIG. 2. Projection of the resonance-cone surfaces in the r-z plane corresponding to saddle points in the integrand of the superposition integral as given by Eq. (24). The figure is symmetric about the axes.

IV. EIGENMODES

In order to find the total plasma response for a given source, the $\phi_m^{(1)}$ eigenfunctions must be found in the range $0 \leq r \leq a$ at every k_z value. To expedite the numerical calculations a Lorentzian density profile is chosen as

$$\tilde{N} = \frac{\tilde{N}_0 - \tilde{N}_1}{1 + \gamma^2 r^2} + \tilde{N}_1, \tag{28}$$

where $\tilde{N}_0$ is the peak density, $\tilde{N}_1$ is the minimum density, and γ is the profile parameter. The densities are normalized to unity at the lower-hybrid density. With some manipulation the dielectric tensor components can be written as

$$K_\perp = \frac{1 - \tilde{N}_0 + (1 - \tilde{N}_1)\gamma^2 r^2}{1 + \gamma^2 r^2}, \tag{29}$$

$$K_{\parallel} = \frac{1 - \alpha_c^2 \tilde{N}_0 + (1 - \alpha_c^2 \tilde{N}_1)\gamma^2 r^2}{1 + \gamma^2 r^2}, \tag{30}$$

and

$$K_x = i\eta_c \frac{\tilde{N}_0 + \tilde{N}_1 \gamma^2 r^2}{1 + \gamma^2 r^2}, \tag{31}$$

where

whereas the axisymmetric mode reflects at the axis. The superposition integral may thus be interpreted in terms of a sum of Trivelpiece–Gould eigenmodes and in terms of resonance-cone surfaces. It may be shown, by use of a Green's function technique, that it is entirely equivalent to view the response in a bounded plasma as a superposition of Trivelpiece–Gould modes or as a superposition of resonance cones from a collection of point sources.[17]

It is possible to evaluate the effect of damping on the resonance cones in a homogeneous plasma using the superposition integral of Eq. (14) in the residue series form of Eq. (16). If it is assumed that

$$\omega_{ci}^2 \ll \omega^2 \ll \omega_{ce}\omega_{ci} \tag{26}$$

and $\nu/\omega \ll 1$, then k_{zn} may be written as

$$k_{zn} \approx \frac{p_{mn}}{a} \frac{\omega^2 - \omega_{pi}^2 + i\frac{\nu}{\omega}\omega_{pi}^2}{\omega_{pe}^2}, \quad n = 1, 2, \ldots, \tag{27}$$

where ν is the Krook-model collision frequency for ion–neutral collisions. Since the imaginary part of k_{zn} is positive and increases with the radial mode number n, the higher-order radial modes are more strongly damped than the lower-order modes. This implies that damping reduces the sharpness of the resonance-cone singularity as the distance from the source is increased. After a sufficiently large axial distance, the resonance-cone surfaces can be expected to disappear altogether leaving the lowest-order radial mode excited by the source.

Although the results of this section were derived for a homogeneous plasma, it is found from the numerical results that the same behavior is characteristic of inhomogeneous plasma.
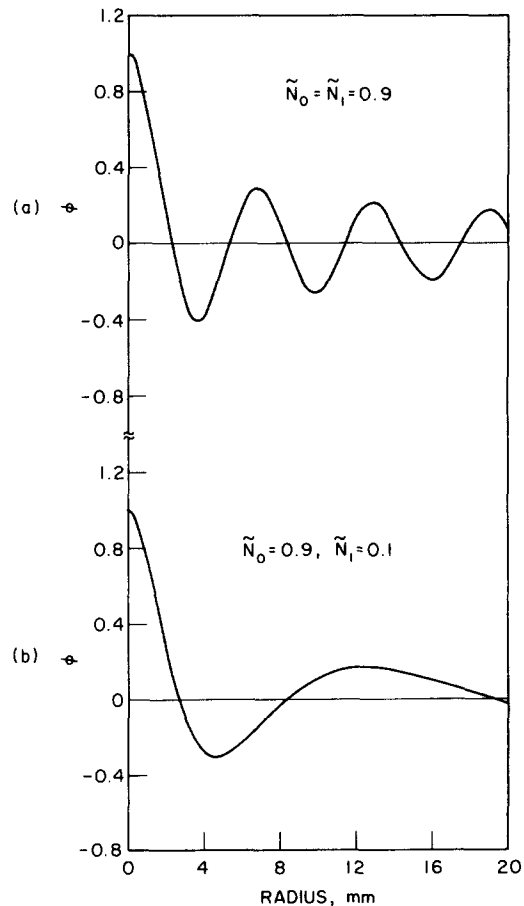

FIG. 3. Radial eigenfunctions for $m = 0$. (a) Homogeneous case (Trivelpiece–Gould mode) and (b) inhomogeneous case. The Trivelpiece–Gould mode is evaluated at the peak density, $\tilde{N}_0$. All values of the potential are given in arbitrary units but are plotted on the same scale unless otherwise noted ($\nu/\omega = 0$ and $k_z = 4$ m^{-1}).

$$\alpha_c^2 = \frac{(\omega^2 - \omega_{ce}^2)(\omega^2 - \omega_{ci}^2)}{\omega^2(\omega^2 - \omega_{ce}\omega_{ci})} \qquad (32)$$

and

$$\eta_c = \frac{\omega(\omega_{ce} - \omega_{ci})}{\omega^2 - \omega_{ce}\omega_{ci}} . \qquad (33)$$

Hence, the eigenmode equation becomes

$$\phi_m'' + p(r, k_z)\phi_m' + q(r, k_z)\phi_m = 0, \qquad (34)$$

where

$$p(r, k_z) = \frac{1}{r} + \frac{2r\gamma^2(\tilde{N}_0 - \tilde{N}_1)}{(1 + \gamma^2 r^2)[1 - \tilde{N}_0 + (1 - \tilde{N}_1)\gamma^2 r^2]} \qquad (35)$$

and

$$q(r, k_z) = -\frac{m^2}{r^2} - k_z^2 \frac{1 - \alpha_c^2 \tilde{N}_0 + (1 - \alpha_c^2 \tilde{N}_1)\gamma^2 r^2}{1 - \tilde{N}_0 + (1 - \tilde{N}_1)\gamma^2 r^2}$$
$$+ \frac{2\eta_c m\gamma^2(\tilde{N}_1 - \tilde{N}_0)}{(1 + \gamma^2 r^2)[1 - \tilde{N}_0 + (1 - \tilde{N}_1)\gamma^2 r^2]} . \qquad (36)$$

It is noted that Eq. (34) has a regular singularity at $r = 0$ and, for the case of $\tilde{N}_0 > 1$ and $\tilde{N}_1 < 1$, at $r_0 = \gamma^{-1}[(\tilde{N}_0 - 1)/(1 - \tilde{N}_1)]^{1/2}$. Equation (34) can be solved numerically for the eigenfunctions as an initial-value problem by integrating from $r = 0$ to $r = a$. The values of ϕ_m and ϕ_m' at $r = 0$ required to begin the integration may be found by a Frobenius expansion for ϕ_m about this singular point. If the resonance layer occurs in
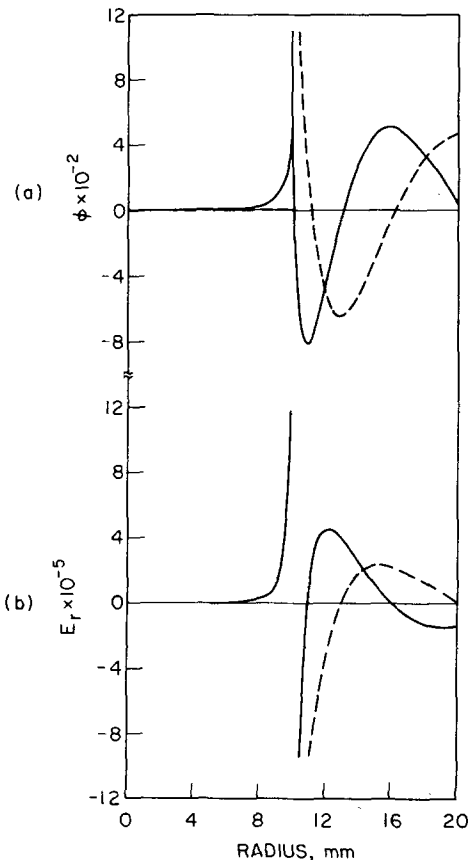

FIG. 4. Inhomogeneous plasma eigenfunctions for $m = 0$. (a) Electrostatic potential and (b) corresponding radial electric field. The resonance layer occurs at $r_0 = 10$ mm. The real parts are the solid lines and the imaginary parts are the dashed lines $(\nu/\omega = 0, \tilde{N}_0 = 1.9, \tilde{N}_1 = 0.1, \text{ and } k_z = 4 \text{ m}^{-1})$.
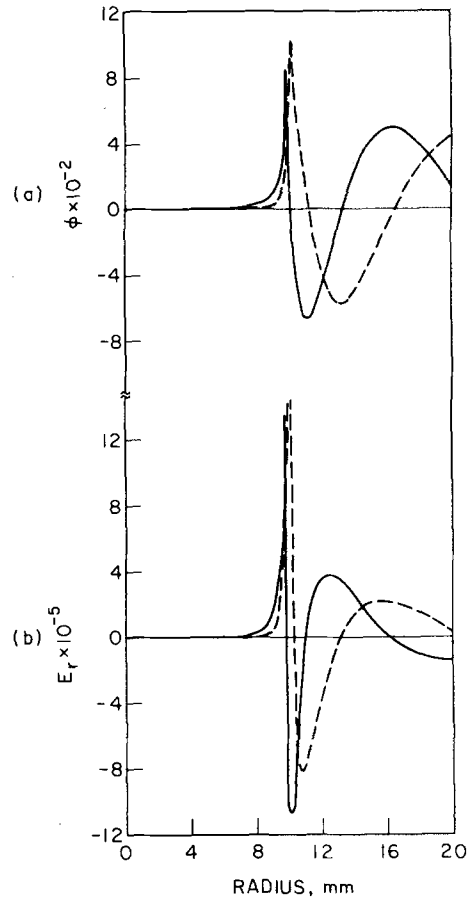

FIG. 5. The effect of collisions on inhomogeneous plasma modes. (a) Potential and (b) radial electric field. The real parts are the solid lines and the imaginary parts are the dashed lines $(\nu/\omega = 0.02, \tilde{N}_0 = 1.9, \tilde{N}_1 = 0.1, m = 0, \text{ and } k_z = 4 \text{ m}^{-1})$.

the plasma, damping may be included to remove the singularity at $r = r_0$. When the damping is zero, a connection formula can be derived from the Frobenius expansions about the singular point at $r = r_0$. The connection formula can be used to continue the solution through the resonance layer.[17]

The numerical results indicate that the eigenfunctions fall into two classes, depending on whether or not the resonance layer occurs in the plasma. Since it is assumed that $\tilde{N}_1 < 1$, the resonance layer will occur in the plasma when $\tilde{N}_0 > 1$. A comparison of an eigenfunction for the homogeneous plasma with one for an inhomogeneous plasma with the same parameters is shown in Fig. 3 for $\tilde{N}_0 < 1$. The inhomogeneous plasma eigenfunctions closely resemble the Trivelpiece-Gould modes. The main difference is that the radial wavelength increases as the density decreases. For the case of $\tilde{N}_0 > 1$, the potential and radial electric field of an eigenfunction are shown in Fig. 4. A logarithmic singularity occurs at the resonance layer at $r = r_0$. The solution is oscillatory on the low density side and is evanescent on the high density side. It is noted that the boundary value of ϕ_m is complex even in the absence of collisions. In Fig. 5 the eigenfunction found by numerical integration of Eq. (34) through the resonance layer is shown for the same parameters but with
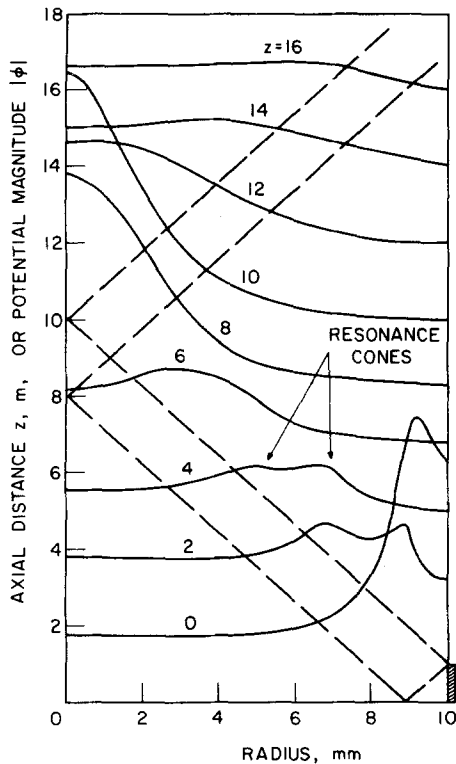
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FIG. 6. Potential response from a single $m=0$ source in a homogeneous plasma for $\tilde{N}_0 < 1$. The characteristic lines from Eq. (24) are shown as dashed lines. Reference positions for each curve are taken at the corresponding axial positions on the ordinate. Values of the potential are given in arbitrary units but all curves are plotted on the same scale unless otherwise noted $(\nu/\omega = 0.3$ and $\tilde{N}_0 = 0.99 = \tilde{N}_1)$.

finite damping. The value of the potential at the resonance layer is decreased while the boundary value increases. In general, it is found that damping effects the higher k_z modes more than the lower k_z modes.

V. NUMERICAL RESULTS FOR PHASED-RING SOURCES

In this section results are presented for the plasma response due to one or more split-cylinder sources that are concentric with the plasma at the radius $r = a$ $= b$ and that have arbitrary relative polarity. When the cylinder halves are driven in phase, the excitation is purely $m = 0$. For out-of-phase drive the azimuthal variation is approximated by the $|m| = 1$ modes. When more than one ring is used, it is assumed that adjacent rings are driven out-of-phase and in the $m = 0$ mode.

The first case considered is that of a single $m = 0$ cylinder of length 2.0 m as considered in Sec. III. The results for a homogeneous plasma with $\tilde{N}_0 < 1$ are shown in Fig. 6. The damping parameter ν/ω is chosen for a weakly ionized low-density plasma. It is seen that weak singularities emanate from the source along the characteristic lines predicted by Eq. (24). A region of enhanced potential occurs at the point where the resonance cones reach the column axis and reflect outward again. The resonance-cone singularities are strongly damped axially and after sufficient distance disappear altogether leaving the lowest-order modes excited by

the source.

The response to $|m| = 1$ excitation is similar except that a potential minimum occurs at the column axis due to the antisymmetric excitation as shown in Fig. 7. The qualitative behavior of the resonance-cone surfaces is in close agreement with previously reported experimental results.[4,11]

The results for a 1 m long, $m = 0$ source in an inhomogeneous plasma with $\tilde{N}_0 < 1$ are shown in Fig. 8. The response is qualitatively similar to the homogeneous plasma response except that the resonance-cone surfaces are curved due to the inhomogeneous density. The damping appears to have a stronger effect on the resonance-cone singularities in this case. The plasma response for the same parameters but with $\tilde{N}_0 > 1$ is shown in Fig. 9. The resonance layer is chosen to occur midway between the axis and the plasma boundary. In this case the resonance-cone singularities become asymptotic to this layer and do not penetrate to the column axis. This behavior is the same as that found for the planar-geometry case by Simonutti[5] and Bellan and Porkolab.[6] For the cases shown in Figs. 5-9, the electric field is mainly radial except near the ends of the source and near the column axis.

The case of an inhomogeneous plasma driven by eight $m = 0$ rings with alternating polarities is shown
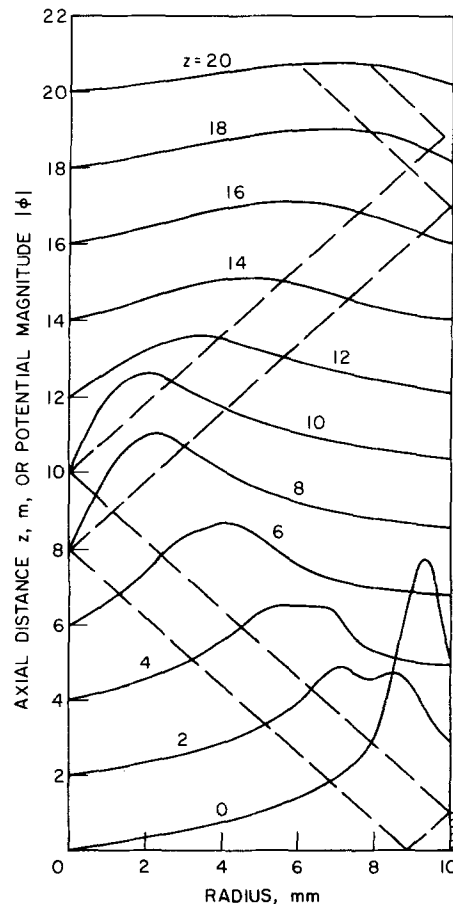

FIG. 7. Potential response from a single $|m| = 1$ split-cylinder source in a homogeneous plasma for $\tilde{N}_0 < 1$. The characteristic lines are the dashed lines $(\nu/\omega = 0.3$ and $\tilde{N}_0 = 0.99 = \tilde{N}_1)$.
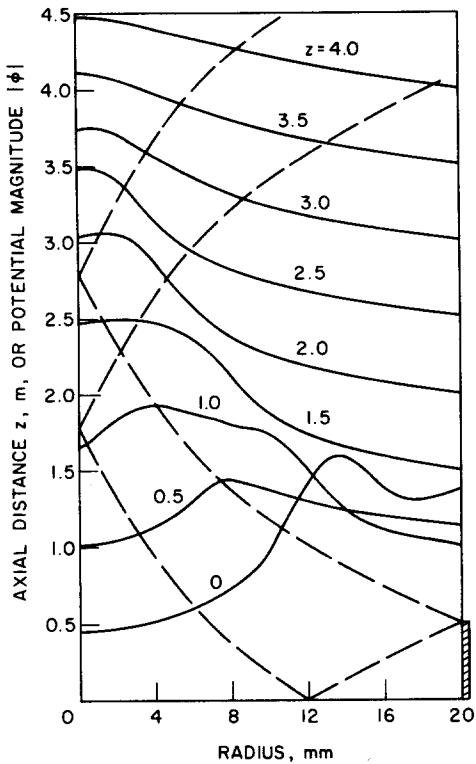
FIG. 8. Potential response from a single $m = 0$ source in an inhomogeneous plasma for $\tilde{N}_0 < 1$. The dashed lines are the characteristic lines $(\nu/\omega = 0.2$, $\tilde{N}_0 = 0.9$, and $\tilde{N}_1 = 0.1)$.

in Fig. 10. A sinusoidal wave with the periodicity of the source is enclosed within the resonance-cone surfaces from the ends of the source in agreement with the experimental results of Bellan and Porkolab.[12] The damping included in this model causes the wave to disappear axially and radially, leaving the lowest-order radial mode excited by the source.

VI. SUMMARY

The boundary-value problem was formulated and solved for the electrostatic potential in a cold, collisional, inhomogeneous, cylindrical plasma. It was shown theoretically that the response in a bounded plasma from a finite-sized source is characterized by resonance-cone singularities composed of a superposition of bounded-plasma eigenmodes. It was found that
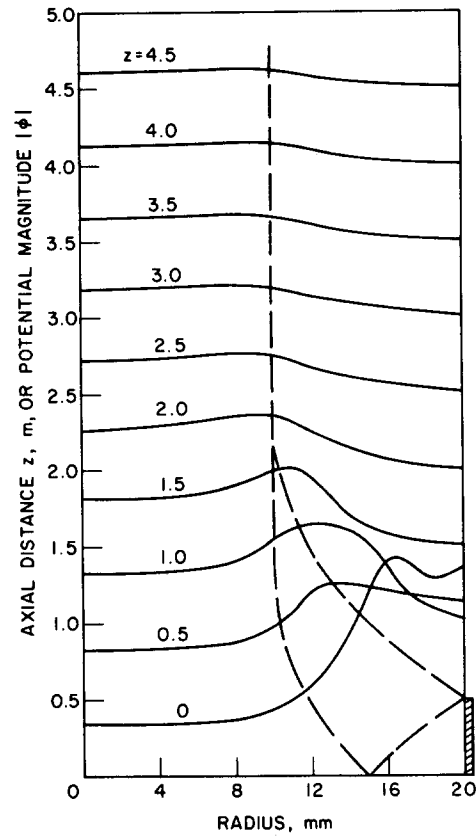

FIG. 9. Potential response from a single $m = 0$ source in an inhomogeneous plasma for $\tilde{N}_0 > 1$. The dashed lines are the characteristic lines $(\nu/\omega = 0.2$, $\tilde{N}_0 = 1.9$, and $\tilde{N}_1 = 0.1)$.

a transition between the well-defined resonance-cone surfaces and the lowest-order eigenmodes takes place over a finite axial distance as the higher-order eigenmodes are damped. Examination of the approximate superposition integral shows that the resonance-cone surfaces reflect from the plasma column axis for axisymmetric sources, but for an azimuthally localized source the resonance-cone surfaces appear to pass through the axis and reflect from the boundary. If the resonance layer is present, the resonance-cone surfaces approach the resonance layer but do not penetrate to the center of the plasma. The results are found to agree qualitatively with those of the planar, infinite plasma theory and with recent experiments in low-den-
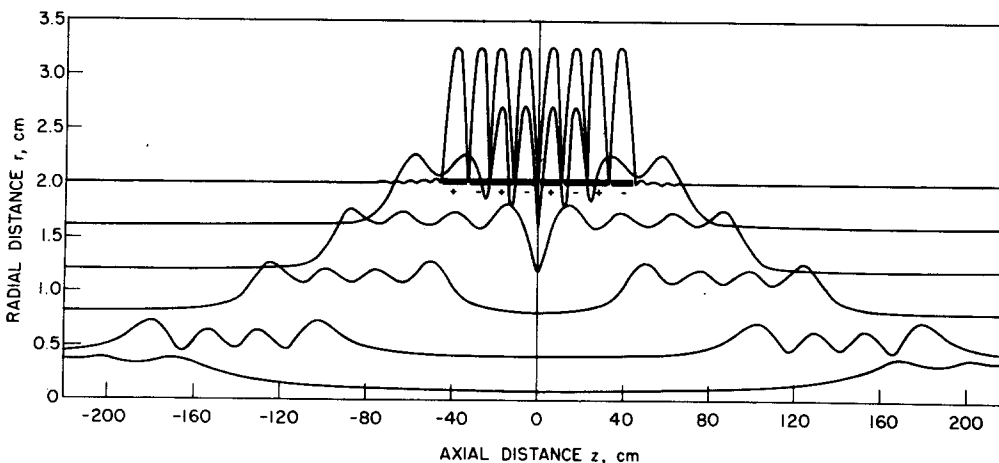

FIG. 10. Potential response from eight $m = 0$ rings with alternating polarity in an inhomogeneous plasma for $\tilde{N}_0 < 1$ $(\nu/\omega = 0.05$, $\tilde{N}_0 = 0.9$, and $\tilde{N}_1 = 0.1)$.

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ACKNOWLEDGMENT

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