Technique for Measurement of Cross-Spectral Density of Two Random Functions

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The cross-spectral density of two functions may be determined by using two selective filters which have identical impulse responses except for a relative phase difference which should be 0° and 90° for the measurement of cosine and sine components, respectively. A technique is developed which is quite accurate and requires a minimum of special equipment. The operation of the system is checked by measuring the cross-spectral density of two functions whose statistical properties are known.

INTRODUCTION

LET us consider two randomly varying functions of time, $e_1(t)$ and $e_2(t)$, for which the cross-correlation

$$R_{12}(\tau)=\langle e_1(t)e_2(t+\tau)\rangle_{\text{Av}}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e_1(t)e_2(t+\tau)dt$$

exists.[1] The cross-spectral density is defined to be

$$E_{12}(\omega)+iF_{12}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_{12}(\tau)e^{i\omega\tau}d\tau.$$

The functions $E_{12}(\omega)$ and $F_{12}(\omega)$, the cosine and sine components of the cross-spectral density,[2] and $R_{12}(\tau)$ contain essentially equivalent information, being the Fourier transforms of one another. In the past, attention has been given to measurement of $R_{12}(\tau)$.[3,4] However, in many statistical problems E_{12} and F_{12} lend themselves more easily to physical interpretation and their direct measurement is of some interest.

If $e_1(t)$ and $e_2(t)$ are passed through linear filters, the respective outputs $f_1(t)$ and $f_2(t)$ are given by

$$f_s(t)=\int_{-\infty}^{\infty}e_s(x_s)K_s(t-x_s)dx_s,\quad s=1,\,2, \tag{1}$$

where K_1 and K_2 are the impulse responses of the filters and $K_s(x_s)=0$ for $x_s<0$. The average product of f_1 and f_2 is

$$\langle f_1(t)f_2(t)\rangle_{\text{Av}}=\int_{-\infty}^{\infty}R_{12}(\eta)\psi(\eta)d\eta, \tag{2}$$

where

$$\psi(\eta)=\int_{-\infty}^{\infty}K_1(\zeta)K_2(\zeta+\eta)d\zeta. \tag{3}$$

This result is easily proved by substituting expressions (1) in $\langle f_1(t)f_2(t)\rangle_{\text{Av}}$, defining $\zeta=t-x_1$ and $\eta=x_1-x_2$, and reversing the order of the operations integration and $\langle\ \rangle_{\text{Av}}$.[5]

By choosing K_1 and K_2 properly, $\psi(\eta)$ can be made approximately proportional to either $\sin\eta$ or $\cos\eta$, with the result that $\langle f_1(t)f_2(t)\rangle_{\text{Av}}$ yields E_{12} or F_{12}. For example, if the filters are second-order linear systems, the K_s become

$$K_s=A_se^{-\alpha t}\cos(\omega t+\phi_s),\quad s=1,\,2. \tag{4}$$

Evaluation of (3) then gives

$$\begin{aligned}\psi(\eta)&=(A_1A_2/4\alpha)e^{-\alpha|\eta|}\cos(\omega\eta+\phi_1-\phi_2)+0(\alpha/\omega),\\&\cong(A_1A_2/4\alpha)e^{-\alpha|\eta|}\cos(\omega\eta+\phi_1-\phi_2),\end{aligned} \tag{5}$$

for $\alpha/\omega\ll1$. This condition is satisfied if the second-order filters have narrow band width relative to the resonant frequency ω, i.e., they have little damping. Expressions similar to Eq. (5) are obtained when narrow-band filters of order higher than two are analyzed.

Substituting the expression for ψ in Eq. (2) yields

$$\begin{aligned}\langle f_1(t)f_2(t)\rangle_{\text{Av}}&=\frac{A_1A_2}{4\alpha}\int_{-\infty}^{\infty}R_{12}(\eta)e^{-\alpha|\eta|}\cos(\omega\eta+\phi)d\eta,\\&=\frac{A_1A_2}{4\alpha}\Big[\cos\phi\int_{-\infty}^{\infty}R_{12}(\eta)e^{-\alpha|\eta|}\cos\omega\eta d\eta\\&\qquad-\sin\phi\int_{-\infty}^{\infty}R_{12}(\eta)e^{-\alpha|\eta|}\sin\omega\eta d\eta\Big],\end{aligned} \tag{6}$$

where $\phi=\phi_1-\phi_2$.

If α is taken sufficiently small, the damping factor $e^{-\alpha|\eta|}$ will have little effect on the value of $\langle f_1(t)f_2(t)\rangle_{\text{Av}}$, and the following approximate relations result:

$$\int_{-\infty}^{\infty}R_{12}(\eta)e^{-\alpha|\eta|}\cos\omega\eta d\eta\cong E_{12}(\omega), \tag{7}$$

$$\int_{-\infty}^{\infty}R_{12}(\eta)e^{-\alpha|\eta|}\sin\omega\eta d\eta\cong F_{12}(\omega), \tag{8}$$

and

$$\langle f_1(t)f_2(t)\rangle_{\text{Av}}\cong(A_1A_2/4\alpha)[\cos\phi E_{12}(\omega)-\sin\phi F_{12}(\omega)]. \tag{9}$$

[1] The random functions are restricted to those whose average properties are independent of a shift in the origin of time.
[2] The cross-spectral density is assumed to be continuous. In order to include functions with periodic or almost periodic components (line spectrum) it is necessary to use Fourier-Stieltjes integrals, which adds nothing new. The almost periodic components can be handled in a manner similar to that given above.
[3] A. E. Hastings and J. E. Meade, Rev. Sci. Instr. **23**, 347 (1952).
[4] M. J. Levin and J. F. Reintjes, Proc. Natl. Electronics Conf. **8**, 647 (1952).
[5] M. S. Uberoi and L. S. G. Kovasznay, Quart. Appl. Math. **10**, 375 (1953).

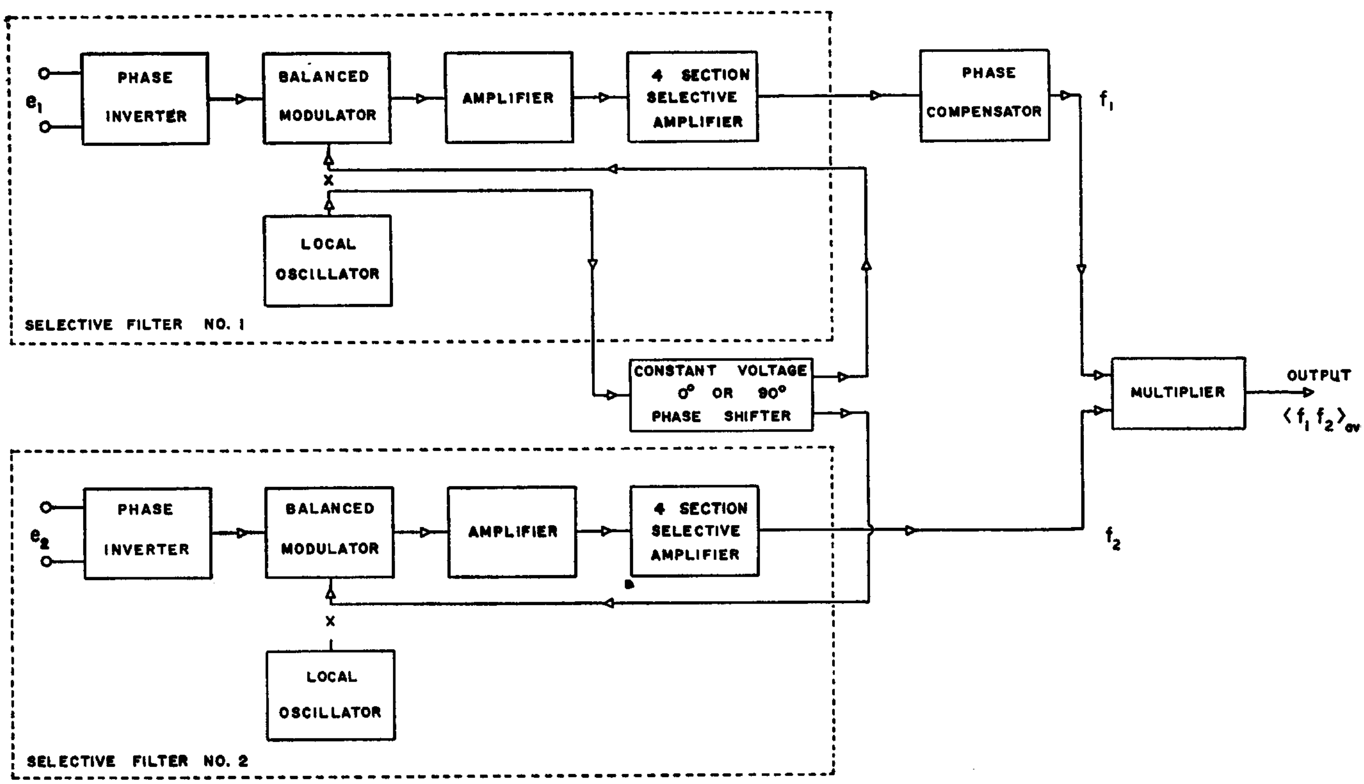

FIG. 1. Block diagram of the arrangement for measuring the cross-spectral density of $e_1(t)$ and $e_2(t)$.

The cosine and sine components of the spectral density can be measured by making ϕ equal to zero and $-90°$, respectively. The fact that α must be small means that the absolute band width of the filter must be taken small enough to assure resolution of all details of $E_{12}(\omega)$ and $F_{12}(\omega)$. Developments similar to the one above can be carried out for higher order linear filters. The basic requirements are: (1) that band widths be small, and (2) that K_1 and K_2 are essentially the same except for a phase difference of ϕ.

EXPERIMENTAL ARRANGEMENT

The filters used in the present investigation are Hewlett-Packard Model 300 A Wave Analyzers. In these filters the input signal produces a balanced modulation of a sinusoidal signal (carrier) of 20 to 36 kc, thereby producing sum and difference frequencies of the input and carrier. The modulated signal is passed through a selective amplifier of variable pass-band, and fixed center frequency (20 kc). The over-all result is a filter with a center frequency variable from zero to 16 kc, and an output at 20 kc. Output at 20 kc causes no difficulty, because subsequent multiplication and averaging of the filter outputs produces an answer which is invariant with frequency translation. The block diagram of the arrangement used is shown in Fig. 1. It is necessary that both filters have: (1) identical transmission characteristics, and (2) maintain a definite phase difference (0° or $-90°$).

The first requirement is met in the following way. Each filter has a four-section selective amplifier nominally tuned at 20 kc. The selective amplifiers are disconnected from preceding equipment, and the same sinusoidal signal of frequency near 20 kc is applied to them. They are tuned stage by stage in such a way that their gains are identical and the phase difference between their outputs is constant (preferably zero) throughout the pass-band. This constant phase shift can be corrected by the phase compensation as indicated in Fig. 1.

The second requirement is met by driving both filters with the same local oscillator. The local oscillators of both units are disconnected. The output of one local oscillator is passed through a constant-voltage phase shifter (see Appendix) whose two outputs are either in phase or $-90°$ out of phase, depending on the position of the selector switch. The two outputs of the phase shifter are used to drive the balance modulators of the two units.

The following procedure is employed to set the phase compensator. A sinusoidal signal (zero to 16 kc) is applied to the two units with the phase shifter at its 0° position,

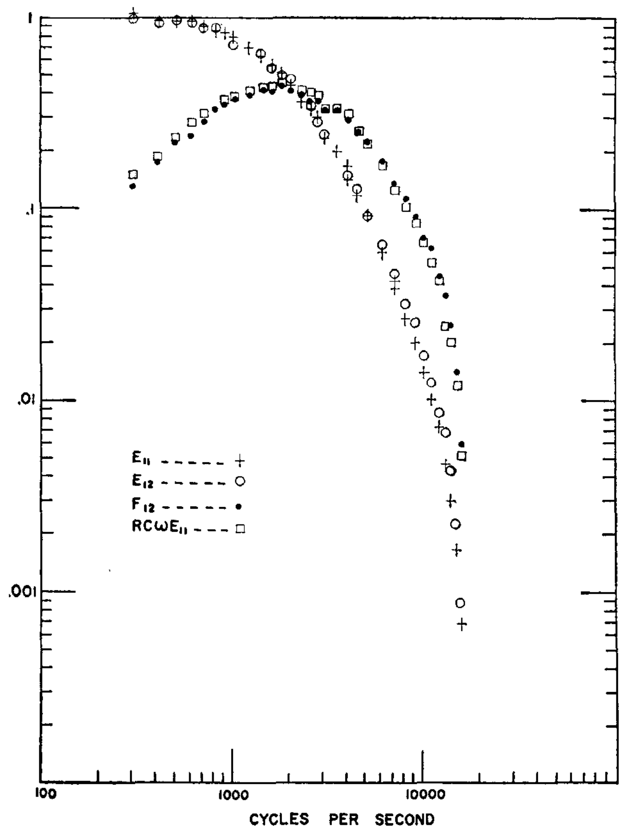

FIG. 2. Data showing agreement between E_{11} and E_{12} and between F_{12} and $RC\omega E_{11}$.

and the outputs of the two units connected to x and y inputs of an oscilloscope. The phase compensator is adjusted until the resulting figure is a straight line and remains so throughout the pass-band of the filter. If now the phase shifter is turned to its $-90°$ position the resulting figure should become a circle (providing the x and y inputs of the oscilloscope have equal gains).

The product of the output of the filters is obtained by a square law multiplier which is described in the Appendix. Short term averaging of the product is provided by the thermocouples utilized in the multiplier.

With the above phase adjustment, the system is ready for use. The signals e_1 and e_2 are supplied to the filters and the phase shifter is set either to its 0° or to its $-90°$ position, so that the average product of the outputs will be either the cosine or sine component, respectively, of the cross-spectral density. The frequency at which the cross-spectral density is measured can be changed from 0 to 16 kc, by changing the local oscillator frequency from 20 to 36 kc.

AN EXAMPLE

The operation of the equipment was checked in the following way. The signal from a Gaussian white noise generator was applied to a resistor R and capacitor C connected in series. If $e_1(t)$ is the voltage across the resistor, and $e_2(t)$ is the voltage at the terminals of the generator, then the latter is

$$e_2(t)=e_1(t)+RC\dot{e}_1(t), \tag{10}$$

and the cross correlation is

$$\langle e_1(t)e_2(t+\tau)\rangle_{\mathrm{Av}}=\langle e_1(t)e_1(t+\tau)\rangle_{\mathrm{Av}}+RC\langle e_1(t)\dot{e}_1(t+\tau)\rangle_{\mathrm{Av}},$$

which is

$$R_{12}(\tau)=R_{11}(\tau)+RC\dot{R}_{11}(\tau). \tag{11}$$

The Fourier transform of Eq. (11) is

$$E_{12}(\omega)+iF_{12}(\omega)=E_{11}(\omega)+iRC\omega E_{11}(\omega). \tag{12}$$

The functions E_{12}, F_{12} and E_{11} were measured, and they satisfy Eq. (12) as shown in Fig. 2.

ACCURACY

The filters have a variable nominal pass-band of 60 to 300 cps, which is equivalent to 15 to 50 cps for a rectangular band-pass filter with the same mean square output. The best tuning was obtained at the widest pass-band. It is necessary to use well-matched wave analyzers to obtain good accuracy at lower pass-bands. The nominal range of the filter is from zero to 16 kc, but the useful range is from 100 cps to 16 kc, unless extreme care is taken to balance the modulator in each unit.

The absolute accuracy of the system is about 3%. However, in many applications the cosine and sine components of the spectral density differ greatly in magnitude. In such cases the rejection of the larger component in measurement of the smaller component is more important than the absolute accuracy. This depends on the phase shifter accuracy, which is better than 1°, and can be further improved by adjusting the phase compensator at every value of ω. With adjustment of the phase compensator, the rejection of the unwanted component is considerably better than 1% of that component. The accuracy also depends on the multiplier, which uses operational amplifiers accurate to better than 0.1%. For very accurate work it is necessary to use an integrator to further average the output of the multiplier.

OTHER TECHNIQUES

There are a number of other straightforward ways of measuring cross-spectral density of two functions. The above method has high accuracy, convenience of operation, and involves minimum design and construction of

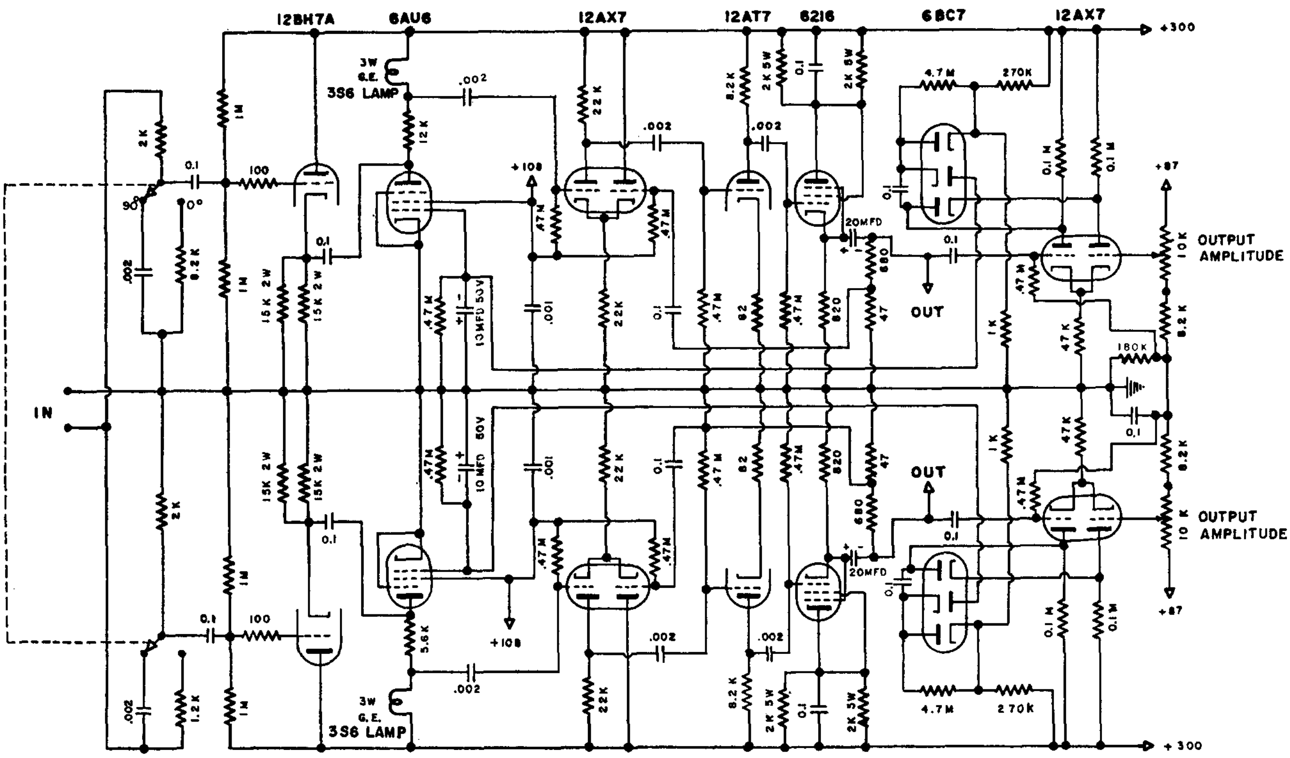

FIG. 3. The constant voltage phase shifter circuit.

special equipment. Huberstrich and Hama[6] have independently developed a similar technique with filters, involving balanced modulators and selective amplifiers, which work at zero cps instead of 20 kc. They have specially designed most of the equipment for this purpose.

For the case of $e_1=e_2$, $R_{11}(\tau)$ is an even function and $F_{11}(\omega)=0$ and $E_{11}(\omega)$ can be measured by passing the signal through a single narrow pass-band filter of variable center frequency and measuring the mean squared output. This technique[7] may be applied to measure the cosine component of the cross-spectral density by consecutively passing e_1+e_2 and e_1-e_2 through the filter. The difference between the two mean squared outputs then gives $4E_{12}$. The sine component can be measured by differentiating one signal, say e_1, and measuring the difference between the spectral densities of $\dot{e}_1+e_2$ and $\dot{e}_1-e_2$, which is $4\omega F_{12}$. The factor ω arise due to the differentiation of one of the signals. Differentiation is a weakness of the technique as it is usually difficult to obtain accurate differentiation over the entire operating frequency range.

APPENDIX

The phase shifter is a pair of voltage amplifiers which produces two 1.8-volt sinusoidal signals of the same magnitude and frequency (20–36 kc) as the input voltage. The output voltages can be in phase or 90° out of phase with an accuracy of 1° for 90° operation and an accuracy of considerably better than 1° for in-phase operation. The circuit diagram is shown in Fig. 3.

For a 90° shift, the circuits shown have the following functions. The *RC* circuits give the input voltage a relative shift of 90° at the grids of the 12BH7 cathode followers. Amplitude control of the voltages at the outputs of the cathode followers is achieved by adjusting the resistance of the G.E. lamps by means of the 6AU6 control tubes. The 12AX7, 12AT7, and 6216 tubes form a feedback amplifier to generate the desired outputs from the voltages across the lamps. Tubes 6BC7 and 12AX7 at the right of the diagram sense the magnitude of the output voltage, and provide control signals for the grids of the 6AU6's. Operation in the 0° mode is the same except that no phase shift is provided at the grids of the 12BH7's.

The multiplier circuit together with the phase compensator is shown in Fig. 4. It is of the square law type,[8] where the squaring elements are thermocouples. The lower diagram gives the circuit details and the upper diagram the basic operations. The operational amplifiers[8] are used

[6] A. Huberstrich and F. R. Hama, AFOSR TN 58-338 (1958).
[7] S. Corrsin and M. S. Uberoi, NACA TN 1050 (1951).
[8] G. A. Korn and T. M. Korn, *Electronic Analog Computers* (McGraw-Hill Book Company, Inc., New York, 1956), second edition.

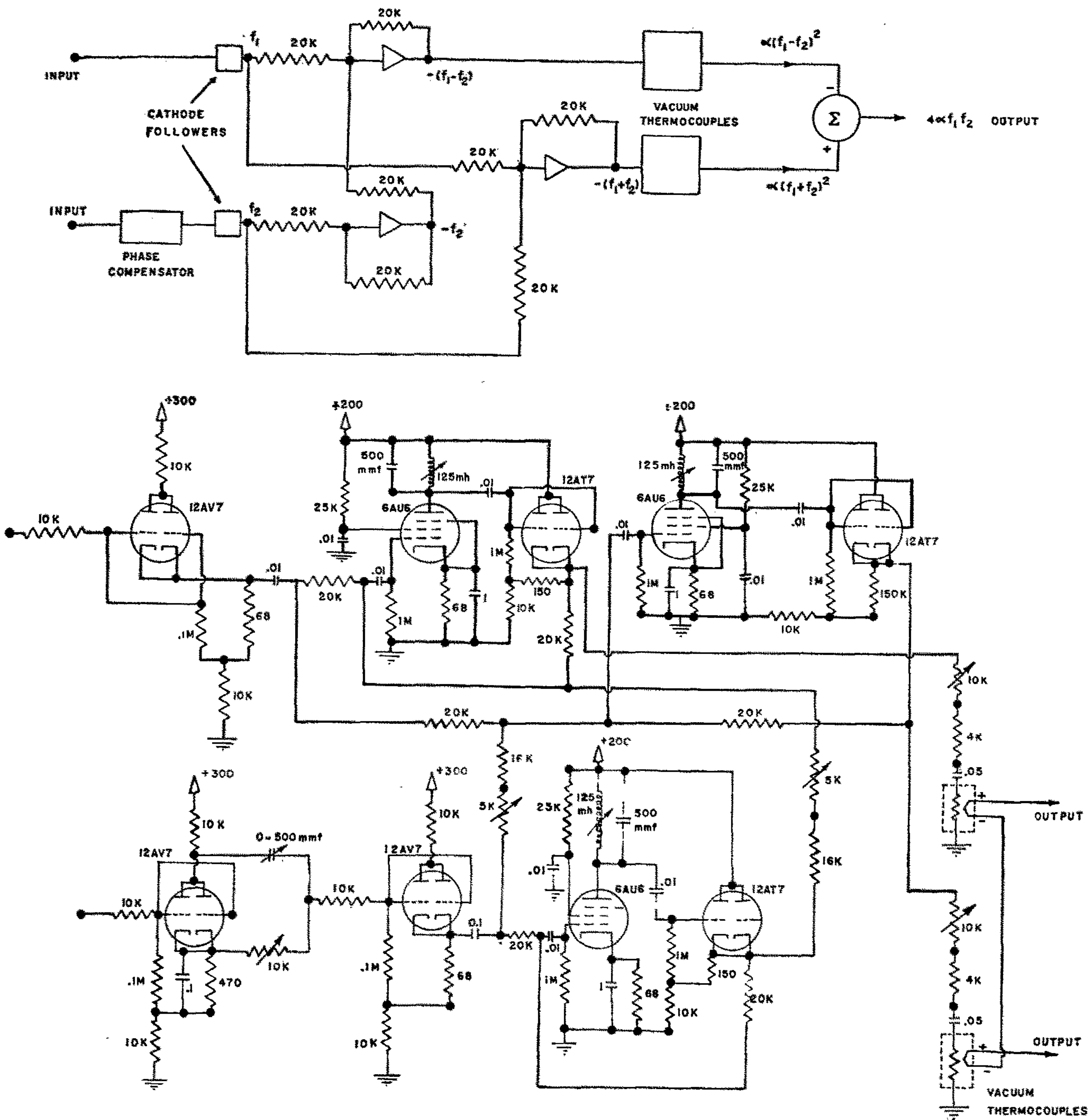

FIG. 4. Circuit diagram and block diagram of phase compensator and multiplier.

to form the necessary voltages for operation of the thermocouples. Open loop gain of the amplifiers is better than 10^4 at 20 kc, and with feedback resistors properly calibrated, the operational amplifiers have an accuracy better than 0.1%.

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