Slow Viscous Shear Flow past a Plate in a Channel

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Flow past a plate midway between two walls is studied analytically using the Stokes approximation. An exact solution is found for the semi-infinite plate using the Wiener–Hopf technique. For the finite plate an approximate technique related to variational principles is discussed which provides both upper and lower bounds on the drag.

THE problem of slow flow past a flat plate in a channel is treated in the framework of the Stokes approximation, i.e., the assumption is made that the Reynolds number based on the channel width is small. For the case of the semi-infinite plate, an exact solution is obtained using a Fourier integral approach and the Wiener–Hopf technique. The answer obtained agrees closely with an earlier approximate solution by Koiter.[1] The exact solution necessitates more arithmetical computation than does Koiter's first approximation, but little additional conceptual complexity is introduced. An improved approximation used by Koiter has, in fact, practically the same degree of complexity and offers little if any advantage over the exact solution.

An exact solution for the finite plate seems to be unobtainable. However, variational theorems are at hand for bounding the drag from above and below, which will lead to approximations with minimum errors in the least mean-square sense.

The present problem was pursued with the hope of clarifying some of the aspects of flow near a sharp edge. The Stokes approximation has, of course, definite drawbacks in this regard beyond the assumption of low Reynolds number; it does not differentiate between a leading edge and a trailing edge, and it is not capable generally of yielding a solution for flow past an edge without the confinement of walls, except in the sense of Proudman and Pearson[2] and of Kaplun.[3] It is felt, however, that the confining effect of the walls is of great advantage in simplifying the physical aspects of the problem, since it eliminates the two-dimensionality of the flow far upstream and downstream from the edge and gives very simple flows there. Another particular advantage of the present solution is the generation of simple asymptotic formulas in the vicinity of the edge which indicate the nature of the singularity to be found there.

FORMULATION OF THE PROBLEM FOR THE SEMI-INFINITE PLATE

The problem considered in this section is the flow between two infinite plates ($-\infty \leq x \leq \infty$, $y = \pm 1$), each moving with constant velocity U, and surrounding a semi-infinite stationary plate ($-\infty \leq x \leq 0$, $y = 0$). The flow is considered slow enough so that the rate of vorticity change by convection is small compared to vorticity transfer by viscosity. (The maximum Reynolds number for which this assumption is valid is not known *a priori*, although it can be presumed to be of order one.) Symmetry about the x axis is assumed. Existence of constant pressure gradients at $x = \pm\infty$ are allowed, the pressure gradients being different at each extremity owing to the difference in boundaries at $y = 0$. A sketch of the flow situation far upstream and downstream from the trailing edge is shown in Fig. 1.

The constant pressure gradient as x approaches minus infinity coupled with the moving boundary at $y = 1$ and the stationary one at $y = 0$ give a velocity profile quadratic in y; thus $u = Uy +$

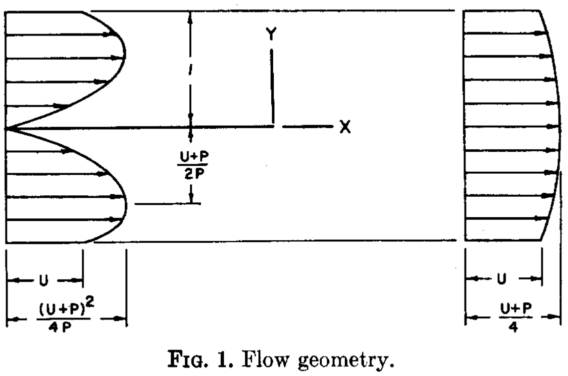

FIG. 1. Flow geometry.

* The work was completed at the Douglas Aircraft Company's Missile and Space Systems Division.
[1] W. T. Koiter, Kgl. Ned. Akad. Wetenschappen **57B**, 558 (1954).
[2] I. Proudman and J. R. A. Pearson, J. Fluid Mech. **2**, 237 (1957).
[3] S. Kaplun, J. Math. Mech. **6**, 595 (1957).

$P(y - y^2)$, where P is six times the average upstream velocity due to the pressure gradient and U is the velocity of the upper plate. For positive x along $y = 0$, by symmetry considerations the shear stress $(\mu\, \partial u/\partial y)$ must vanish. Thus as x approaches plus infinity, $u = \frac{1}{4}(1 + 3y^2)U + \frac{1}{4}(1 - y^2)P$ is the parabolic distribution satisfying the boundary conditions and which possesses the same discharge as the upstream profile.

The mathematics of the problem then is to find a stream function which is a solution of the biharmonic equation in the strip $(-\infty \leq x \leq \infty,\ 0 \leq y \leq 1)$ satisfying the following boundary conditions:

$$\psi(x, 1) = \tfrac{1}{2}U + \tfrac{1}{6}P, \quad (1)$$

$$(\partial\psi/\partial y)(x, 1) = U, \quad (2)$$

$$\psi(-\infty, y) = \tfrac{1}{2}(U + P)y^2 - \tfrac{1}{3}Py^3, \quad (3)$$

$$\psi(\infty, y) = \tfrac{1}{4}(U + P)(y + y^3) - \tfrac{1}{3}Py^3, \quad (4)$$

$$\psi(x, 0) = 0, \quad (5)$$

$$(\partial\psi/\partial y)(x < 0, 0) = 0, \quad (6)$$

$$(\partial^2\psi/\partial y^2)(x > 0, 0) = 0. \quad (7)$$

An inconvenience in the mathematics of the problem is that the flow situation is different at the two extremities of the channel. To handle this feature of the flow, an auxiliary problem is first solved wherein boundary condition (7) is replaced by

$$(\partial\psi/\partial y)(x > 0, 0) = \tfrac{1}{4}(U + P) \quad (8)$$

corresponding to a moving boundary at the right half of the x axis. This replacement has the proper velocity at $x = \pm\infty$ but not in the vicinity of the edge of the strip for $x > 0$.

For the auxiliary problem the velocity is known everywhere on the boundary. The domain being an infinite strip, a Fourier integral solution is suggested. Taking the stream function in the form $G(k, y) \exp(-ikx)$ and substituting in the biharmonic equation, it is found that G must satisfy $G^{IV} - 2k^2G^{II} + k^4G = 0$. Imposing the boundary conditions given by Eqs. (1), (2), and (5), the form of the solution for this problem can thus be written as

$$\psi_0(x, y) = \tfrac{1}{2}(U + P)y^2 - \tfrac{1}{3}Py^3 + \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} B(k)\, G(k, y) \exp(-ikx)\, dk, \quad (9)$$

where

$$G(k,y) = k(y - 1) \sinh ky + y \sinh k \sinh k(1 - y) \quad (10)$$

and $B(k)$ is to be determined by the remainder of the boundary conditions. The ϵ in the limits of the integral is a small positive number to ensure a path of integration which lies above $k = 0$.

Use of the inversion theorem for Fourier integrals gives

$$B(k)G(k, y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} [\psi_0(x, y) - \tfrac{1}{2}(U + P)y^2 + \tfrac{1}{3}Py^3] \exp(ikx)\, dx.$$

This together with boundary conditions (6) and (8) yields

$$B(k) = \frac{i(U + P)}{4(2\pi)^{\frac{1}{2}}k(\sinh^2 k - k^2)}.$$

Thus,

$$\psi_0(x, y) = -\tfrac{1}{3}y^3P + \frac{U + P}{4}\left[2y^2 + \frac{i}{2\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{G(k, y) \exp(-ikx)\, dk}{k(\sinh^2 k - k^2)}\right] \quad (11)$$

with the series expansions found by the residue theorem to be

$$\psi_0(x > 0, y) = \tfrac{1}{4}(U + P)(y + y^3) - \tfrac{1}{3}Py^3 + \frac{U + P}{4} \sum_{j=1}^{\infty} \sum_{\delta=1}^{2} \frac{G(k_{j\delta}^-, y) \exp(-ixk_{j\delta}^-)}{k_{j\delta}^-(\sinh 2k_{j\delta}^- - 2k_{j\delta}^-)},$$

$$\psi_0(x < 0, y) = \tfrac{1}{2}(U + P)y^2 - \tfrac{1}{3}Py^3 - \frac{U + P}{4} \sum_{j=1}^{\infty} \sum_{\delta=1}^{2} \frac{G(k_{j\delta}^+, y) \exp(-ixk_{j\delta}^+)}{k_{j\delta}^+(\sinh 2k_{j\delta}^+ - 2k_{j\delta}^+)}.$$

Here $k_{j\delta}^+$, $k_{j\delta}^-$ are the nonzero roots of $\sinh k = (-1)^\delta k$ in the upper and lower half-planes, respectively. Note that if k is a root, so are $-k$, k^*, $-k^*$, where the asterisk denotes the complex conjugate. The values of the first 20 of these roots are given in Table I; the remainder can be obtained by consideration of the asymptotic values of the roots

TABLE I. The first 10 roots of $\sinh k = (-1)^\delta k$ lying in the first quadrant of the k plane.

$\delta = 1$	$\delta = 2$
2.2507 + 4.2124 i	2.7687 + 7.4977 i
3.1031 + 10.7125 i	3.3522 + 13.9000 i
3.5511 + 17.0734 i	3.7168 + 20.2385 i
3.8588 + 23.3984 i	3.9831 + 26.5545 i
4.0937 + 29.7081 i	4.1933 + 32.8597 i
4.2838 + 36.0099 i	4.3668 + 39.1588 i
4.4434 + 42.3068 i	4.5146 + 45.4541 i
4.5811 + 48.6007 i	4.6434 + 51.7468 i
4.7021 + 54.8924 i	4.7575 + 58.0377 i
4.8100 + 61.1826 i	4.8599 + 64.3272 i

of $\sinh k = (-1)^\delta k$, which in the first quadrant are given by

$$k_{j1} \sim \ln[(4j-1)\pi] + i(2j-\tfrac{1}{2})\pi, \quad j = 1, 2, 3, \ldots,$$
$$k_{j2} \sim \ln[(4j+1)\pi] + i(2j+\tfrac{1}{2})\pi, \quad j = 1, 2, 3, \ldots. \tag{12}$$

For all roots except the first one for each δ, the values of $k_{j\delta}$ given by Eq. (12) are valid to better than 2% of the actual value. The vorticity distribution along $y = 0$ for this problem is

$$\omega(x > 0, 0) = -\tfrac{1}{4}(U+P)\sum_{j=1}^{\infty}\sum_{\delta=1}^{2}\exp(-ixk_{j\delta}^{-}),$$
$$\omega(x < 0, 0) = \tfrac{1}{4}(U+P)\left[4 + \sum_{j=1}^{\infty}\sum_{\delta=1}^{2}\exp(-ixk_{j\delta}^{+})\right],$$

both series diverging at $x = 0$.

It remains to ascertain whether the conditions for large $|x|$ are satisfied. The behavior of a Fourier integral at large $|x|$ is indicated by the behavior of the kernel for small k. Now for small k, $B(k)G(k, y)$ behaves as

$$\frac{3i(U+P)}{4(2\pi)^{\frac{1}{2}}k^5}\,\frac{k^4 y(y-1)^2}{3}.$$

Thus the integral in Eq. (9) is approximated for large $|x|$ by

$$\frac{i(U+P)}{8\pi}\,y(1-y)^2\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\frac{\exp(-ikx)}{k}\,dk,$$

which is zero for negative x and $\tfrac{1}{4}(U+P)y(1-y)^2$ for positive x, ensuring that Eqs. (3) and (4) are satisfied.

SOLUTION OF THE PROBLEM FOR THE SEMI-INFINITE PLATE

Let

$$\psi(x, y) = \psi_0(x, y) + \psi_1(x, y).$$

Since ψ_0 will take care of the flow at large x, ψ_1 must be a vortex distribution which modifies ψ_0 near the edge. Taking $h(x)$ as the Heaviside step function $[h(x > 0) = 1,\ h(x < 0) = 0]$ and letting

$$\psi_1(x, y) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} A(k)G(k, y)\exp(-ikx)\,dk,$$

$$\bar{u}_0(k) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\left[\frac{\partial\psi}{\partial y}(x, 0) - \frac{U+P}{4}h(x)\right]\cdot\exp(-ikx)\,dk,$$

$$\bar{\omega}_0(k) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\left[\frac{\partial^2\psi}{\partial y^2}(x, 0) - (U+P)h(-x)\right]\cdot\exp(-ikx)\,dk,$$

($\bar{u}_0$, $\bar{\omega}_0$ are the transforms of the velocity and vorticity minus their asymptotic values, evaluated on the real axis) boundary conditions (6) and (7) now give $A(k)$ in terms of $\bar{u}_0$ and $\bar{\omega}_0$; thus,

$$\bar{u}_0(k) = A(k)(\sinh^2 k - k^2), \tag{13}$$

$$\bar{\omega}_0(k) = A(k)k(2k - \sinh 2k) + \{i(U+P)[4 + T(k)]/4k(2\pi)^{\frac{1}{2}}\}, \tag{14}$$

where

$$T(k) = k(2k - \sinh 2k)/(\sinh^2 k - k^2). \tag{15}$$

Upon eliminating A the relation between $\bar{\omega}_0$ and $\bar{u}_0$ is

$$\bar{\omega}_0(k) = \bar{u}_0(k)T(k) - \{(U+P)[4 + T(k)]/4ik(2\pi)^{\frac{1}{2}}\}. \tag{16}$$

Since $\bar{u}_0$ is the Fourier transform of a function which is zero along the negative real axis, $\bar{u}_0$ is analytic in the upper half-k plane including the real axis. Similarly, $\bar{\omega}_0$ is analytic in the lower half-plane including the real axis. The determination of $\bar{u}_0$ and $\bar{\omega}_0$ can proceed by the Wiener–Hopf method. (For a discussion of the method, see Noble[4] or Morse and Feshbach.[5])

The function T is first split into T_+ and T_-, $T = T_+/T_-$, where T_+ and T_- are analytic with no zeros in the upper and lower half-planes, respectively. Also T_+ and T_- must be analytic and nonzero along the real k axis and be of algebraic growth as $|k| \to \infty$. A suitable factorization is obtained by expressing the hyperbolic functions in terms of gamma functions; thus upon investigation of the zeros of the gamma functions,

$$T_+(k) = \frac{\Gamma[1 - f(1, 1)]\Gamma[1 - f(1, -1)]\Gamma[\frac{1}{2} - f(1, 1)]\Gamma[\frac{1}{2} - f(1, -1)]}{\pi\Gamma[1 - f(2, -1)]\Gamma[\frac{1}{2} - f(2, 1)]\exp(-X)} \tag{17}$$

and

$$T_-(k) = \frac{2\pi\Gamma[f(2, -1)]\Gamma[\frac{1}{2} + f(2, 1)]\exp X}{-ik\Gamma[f(1, 1)]\Gamma[f(1, -1)]\Gamma[\frac{1}{2} + f(1, 1)]\Gamma[\frac{1}{2} + f(1, -1)]}, \tag{18}$$

[4] B. Noble, *Methods Based on the Wiener–Hopf Technique for the Solution of Partial Differential Equations* (Pergamon Press, Inc., New York, 1958).

[5] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 978.

where

$$f(a, b) = \frac{i}{2\pi}(ak + b \sinh^{-1} ak).$$

The factor exp X is used to assure correct behavior for large $|k|$; using Stirling's formula for the gamma functions,

$$\ln T_+(k) \xrightarrow[|k|\to\infty]{} X + \frac{2ki}{\pi}\ln 2 + \frac{1}{2}\ln\left(\frac{-ik}{\pi}\right) + O\left(\frac{1}{k}\right),$$

$$\ln T_-(k) \xrightarrow[|k|\to\infty]{} X + \frac{2ki}{\pi}\ln 2 - \frac{1}{2}\ln\left(\frac{-ik}{\pi}\right) + \ln\left(\frac{-1}{2\pi i}\right) + O\left(\frac{1}{k}\right),$$

so that by choosing

$$X = -(2ki/\pi)\ln 2,$$

$$T_+ \xrightarrow[|k|\to\infty]{} \left(\frac{-ik}{\pi}\right)^{\frac{1}{2}}, \qquad T_- \xrightarrow[|k|\to\infty]{} \frac{i}{2\pi}\left(\frac{\pi}{-ik}\right)^{\frac{1}{2}},$$

which satisfies the requirement.

The determination of $\bar{u}_0$ and $\bar{\omega}_0$ can now be completed. Rearrangement of Eq. (16) in the form

$$T_-(k)\bar{\omega}_0(k) + \frac{(U + P)[T_+(0) + 4T_-(k)]}{4ik(2\pi)^{\frac{1}{2}}} = T_+(k)\bar{u}_0(k) + \frac{(U + P)[T_+(0) - T_+(k)]}{4ik(2\pi)^{\frac{1}{2}}} \qquad (19)$$

makes the left side analytic and nonzero in the lower half-plane including the real axis and the right side analytic and nonzero in the upper half-plane including the real axis. Equation (19) then represents the analytic continuation of a function which is analytic and has no zeros in the entire plane. This function is therefore a constant, say C. Thus,

$$\bar{u}_0(k) = \frac{(U + P)[Ck + T_+(k) - T_+(0)]}{4ik(2\pi)^{\frac{1}{2}}T_+(k)} \qquad (20)$$

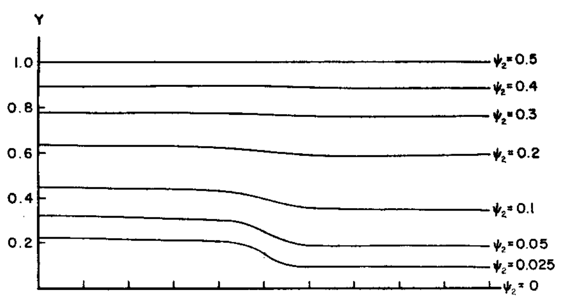

FIG. 2. Streamline pattern.

and

$$\bar{\omega}_0(k) = \frac{(U + P)[Ck - 4T_-(k) - T_+(0)]}{4ik(2\pi)^{\frac{1}{2}}T_-(k)}, \qquad (21)$$

where

$$T_+(0) = -4T_-(0) = 1/\pi^{\frac{1}{2}}.$$

Determination of C is accomplished by imposing the condition that ω_0 be integrable at the origin; hence,

$$\bar{\omega}_0(k) \xrightarrow[|k|\to\infty]{} 0,$$

which is satisfied only if C is zero.

The solution to the problem now is available upon solving for $A(k)$, whence using Eqs. (13), (14), (20), and (21),

$$A(k) = \frac{(U + P)[T_+(k) - T_+(0)]}{4ik(2\pi)^{\frac{1}{2}}T_+(k)(\sinh^2 k - k^2)} = \frac{(U + P)}{4ik(2\pi)^{\frac{1}{2}}}\left[\frac{1}{\sinh^2 k - k^2} - \frac{T_+(0)}{T_-(k)k(2k - \sinh 2k)}\right];$$

the first form being convenient for negative values of x, the second for positive values of x. Combining this result with Eq. (11),

$$\psi = -\frac{1}{3}Py^3 + \frac{U + P}{4} \cdot \left[2y^2 + \frac{i}{2\pi^{\frac{1}{2}}}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{G(k, y)\exp(-ikx)}{T_-(k)k^2(2k - \sinh 2k)}\,dk\right]$$

$$= -\frac{1}{3}Py^3 + \frac{U + P}{4} \cdot \left[2y^2 + \frac{i}{2\pi^{\frac{1}{2}}}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{G(k, y)\exp(-ikx)}{T_+(k)k(\sinh^2 k - k^2)}\,dk\right],$$

with the residue theorem giving

$$\psi(x > 0, y) = \tfrac{1}{4}(U + P)(y + y^3) - \tfrac{1}{3}Py^3 - \frac{U + P}{8\pi^{\frac{1}{2}}}\sum_{j=1}^{\infty}\frac{G(\frac{1}{2}k_{j2}^-, y)\exp(-\frac{1}{2}ixk_{j2}^-)}{(\frac{1}{2}k_{j2}^-)^2 T_-(\frac{1}{2}k_{j2}^-)(\cosh k_{j2}^- - 1)},$$

$$\psi(x < 0, y) = \tfrac{1}{2}(U + P)y^2 - \tfrac{1}{3}Py^3 - \frac{U + P}{4\pi^{\frac{1}{2}}}\sum_{j=1}^{\infty}\sum_{\delta=1}^{2}\frac{G(k_{j\delta}^+, y)\exp(-ixk_{j\delta}^+)}{k_{j\delta}^+ T_+(k_{j\delta}^+)(\sinh 2k_{j\delta}^+ - 2k_{j\delta}^+)}.$$

The location of lines of constant

$$\psi_2 = (\psi + \tfrac{1}{3}Py^3)/(U + P),$$

as well as profiles of $u_2 = \partial\psi_2/\partial y$, $v_2 = -\partial\psi_2/\partial x$, and $\omega_2 = \nabla^2\psi_2$, are shown in Figs. 2–6. The variation of u_2 and ω_2 along the x axis is shown in Fig. 7.

The behavior of u and ω for small x, y is governed

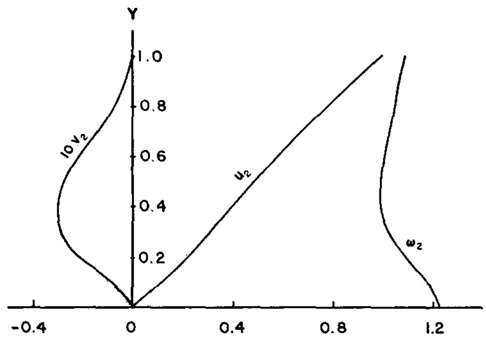

FIG. 3. Velocities and vorticity at $x = -0.2$.

by the behavior of the transform at large k. Expanding the integrand in powers of $k^{\frac{1}{2}}$, the behavior near the origin is given to the first approximation by

$$u(0 \leq x \ll 1, 0) \sim (U + P)(\tfrac{1}{2}x)^{\frac{1}{2}}, \tag{22}$$

$$\omega(-1 \ll x < 0, 0) \sim (U + P)/(-2x)^{\frac{1}{2}}. \tag{23}$$

These are shown by dashed lines on Fig. 7. The behavior of ψ near the origin is more difficult to determine; it is likely to be of the form

$$\psi(|x| \ll 1, |y| \ll 1) \sim \tfrac{1}{4}(U + P)r(2r)^{\frac{1}{2}}(\sin \tfrac{3}{2}\theta + \sin \tfrac{1}{2}\theta), \tag{24}$$

since this satisfies the biharmonic equation and the boundary conditions along $y = 0$ and agrees with Eqs. (22) and (23). Equation (24) corresponds also to the form given by Carrier and Lin[6] using a Stokes approximation near the leading edge of the flat plate in an unbounded flow.

The result given by Eq. (23) is in general agreement with Koiter's result, obtained upon approximating $T(k)$ by $-2(k^2 + 4)^{\frac{1}{2}}$, which has the correct numerical value at $k = 0$ as well as the correct asymptotic behavior. He suggests an improved approximation based on k coth k, which requires

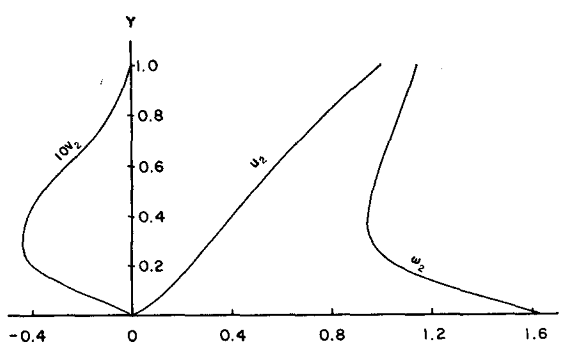

FIG. 4. Velocities and vorticity at $x = -0.1$.

[6] G. F. Carrier and C. C. Lin, Quart. Appl. Math. **6**, 63 (1948).

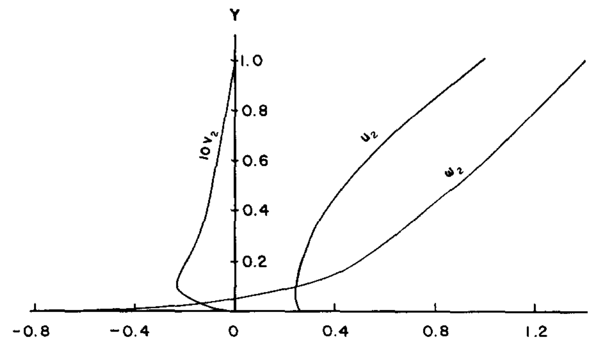

FIG. 5. Velocities and vorticity at $x = 0.1$.

a degree of computation of the same order as the exact solution. The first approximation then $[-2(k^2 + 4)^{\frac{1}{2}}]$ seems to be the only one which is more convenient than the exact solution. Even here the introduction of a branch point by the square root sign renders exact computation of the integrals impossible.

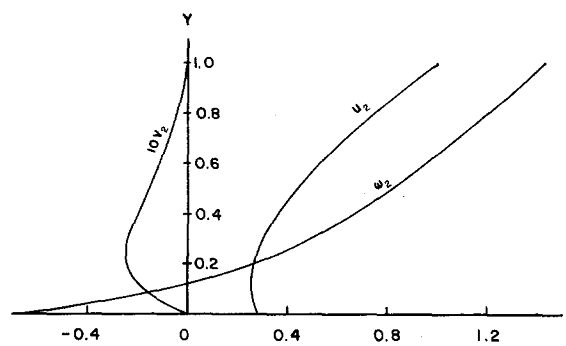

FIG. 6. Velocities and vorticity at $x = 0.2$.

THE FINITE PLATE

For a plate of length L placed at $y = 0$, $-L < x < 0$, the stream function is found in a manner similar to the preceding to be

$$\psi = Uy + 3P(3y - y^3) + \int_{-\infty}^{\infty} G(k, y)e^{-ikx}H(k)\, dk, \tag{25}$$

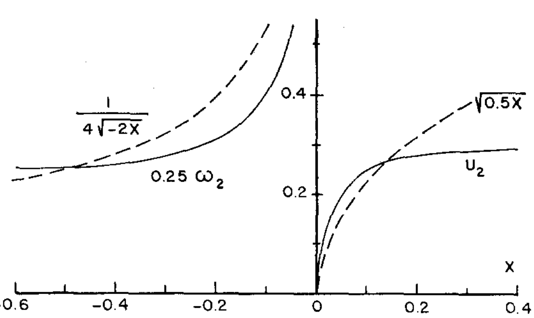

FIG. 7. Velocity and vorticity on $y = 0$.

where U, P, and G are as previously defined and

$$H(k) = \frac{\int_{-\infty}^{\infty} (\partial^2\psi/\partial y^2)_{y=0} e^{ikx}\, dx}{2\pi k(2k - \sinh 2k)}$$
$$= \frac{\int_{-\infty}^{\infty} [(\partial\psi/\partial y) - U - 9P]_{y=0} e^{ikx}\, dx}{2\pi(\sinh^2 k - k^2)} \qquad (26)$$

takes the place of $A(k)$. Equation (25) automatically satisfies all boundary conditions except for

$$\partial\psi/\partial y = 0 \quad \text{on} \quad y = 0, \qquad -L < x < 0,$$

and

$$\partial^2\psi/\partial y^2 = 0 \quad \text{on} \quad y = 0, \qquad x < -L \text{ and } x > 0.$$

For this case there seems to be no method available for finding an exact solution. However, because of the formal analogy between the present problem and the plane problems of elasticity, approximate methods are avilable which can be utilized to bound the drag on the plate from above and below, and which in the mean square sense can be made as accurate as desired. The infinite domain of the present problem requires only a slight modification of earlier theorems given by Hill and Power,[7] which in turn are simply a recasting of the well-known theorems of minimum complementary energy and minimum potential energy of elasticity theory.

To render the various integrals which will arise finite, let

$$\Psi = \psi - (U + 9P)y + 3Py^3,$$

with $\nabla^4\Psi = 0$ in a region R. Let the velocity vector $\mathbf{u}$ be specified on the boundaries S_u and the stress vector $\mathbf{T}$ on the boundaries S_T. For the analog of the theorem of minimum potential energy, let Ψ' be a stream function which satisfies the boundary conditions on S_u but such that $\nabla^4\Psi'$ is not necessarily zero and the boundary conditions on S_T are not necessarily satisfied. Letting $\mathbf{u}'$ be the velocity field found from Ψ' and defining

$$V(\Psi') = E(\Psi') - \int_{S_T} \mathbf{T}\cdot\mathbf{u}'\, dS,$$

where

$$E(\Psi) = \frac{1}{2}\mu \iint_R \left[4\left(\frac{\partial^2\Psi}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi}{\partial y^2}\right)^2\right] dx\,dy,$$

by simple rearrangement and use of the conditions on Ψ and Ψ', one finds that $V(\Psi') - V(\Psi) = E(\Psi' - \Psi)$; thus, $V(\Psi') - V(\Psi)$ is positive definite.

For the analog of the theorem of minimum complementary energy, choose this time a stream function Ψ'' such that $\nabla^4\Psi'' = 0$ and also such that the stress vector $\mathbf{T}''$ computed from Ψ'' is equal to $\mathbf{T}$ on S_T. (The use of a stream function here is convenient but not necessary. Choosing a stress field satisfying equilibrium and the stress boundary conditions would be sufficient.) Letting

$$V^*(\Psi'') = E(\Psi'') - \int_{S_u} \mathbf{u}\cdot\mathbf{T}''\, dS$$

and proceeding as in the previous case, $V^*(\Psi'') - V^*(\Psi) = E(\Psi'' - \Psi)$; therefore, $V^*(\Psi'') - V^*(\Psi)$ is positive definite also.

For the present problem S_u is made up of $y = 1$, $-\infty \leq x \leq \infty$, and $y = 0$, $-L < x < 0$, and S_T by $y = 0$, $x < -L$ or > 0, and $x = \pm\infty$. If the additional requirement is imposed that $\Psi = \Psi' = \Psi''$ on $y = 0$, then

$$\int_{S_T} \mathbf{T}\cdot\mathbf{u}'\, dS = 0 = \int_{S_T} \mathbf{T}\cdot\mathbf{u}''\, dS \qquad (27)$$

and

$$\int_{S_u} \mathbf{u}\cdot\mathbf{T}''\, dS = (U + 9P)\mu \int_{-L}^{0} \left(\frac{\partial^2\Psi''}{\partial y^2}\right)_{y=0} dx$$
$$= (U + 9P)F(\Psi''),$$

$F(\Psi)$ being the force exerted on one side of the plate. For any stream function satisfying equilibrium,

$$\int_{S_T+S_u} \mathbf{u}\cdot\mathbf{T}\, dS = 2E(\Psi); \qquad (28)$$

hence, from Eq. (27)

$$E(\Psi'') = \frac{1}{2}\int_{S_u} \mathbf{u}''\cdot\mathbf{T}''\, dS,$$

and also, $F(\Psi) = 2E(\Psi)/(U + 9P)$. Putting these results together it follows that

$$-\frac{1}{2}\mu \int_{-\infty}^{\infty} \left(\frac{\partial\Psi''}{\partial y}\frac{\partial^2\Psi''}{\partial y^2}\right)_{y=1} dx$$
$$+ \mu \int_{-L}^{0} \left[\left(U + 9P + \frac{1}{2}\frac{\partial\Psi''}{\partial y}\right)\frac{\partial^2\Psi''}{\partial y^2}\right]_{y=0} dx$$
$$= -V^*(\Psi'') \leq \tfrac{1}{2}(U + 9P)F(\Psi) \leq E(\Psi').$$

It remains now to choose suitable Ψ' and Ψ''. Choices which are suggested by Eq. (26) and which exceed the minimum requirements are

$$\Phi' = \int_{-\infty}^{\infty} G(k, y)e^{-ikx}H'(k)\, dk$$

[7] R. Hill and G. Power, Quart. J. Mech. Appl. Math. 9, 313 (1956).

and

$$\Psi'' = \int_{-\infty}^{\infty} G(k, y)e^{-ikx}H''(k)\, dk,$$

with

$$H'(k) = \frac{(\int_{-\infty}^{-L} + \int_0^{\infty})s(x)e^{ikx}dx - (U+9P)(1-e^{-ikL})/ik}{2\pi(\sinh^2 k - k^2)}$$

and

$$H''(k) = \frac{\int_{-L}^{0} t(x)e^{ikx}\, dx}{2\pi k(2k - \sinh 2k)},$$

s (representing velocity) and t (representing vorticity) being arbitrary functions of x except for obvious smoothness and integrability conditions. If s and t are the exact velocity and vorticity on $y = 0$, they give the exact solution. Since this choice of Ψ' satisfies equilibrium, the expression for $E(\Psi')$ can be simplified using Eq. (28). Since Ψ'' satisfies the velocity conditions at $y = 1$, $V^*(\Psi'')$ can be simplified also. Thus

$$\mu \int_{-L}^{0} \left[\left(U + 9P + \frac{1}{2}\frac{\partial \psi''}{\partial y}\right)\frac{\partial^2 \psi''}{\partial y^2}\right]_{y=0} dx$$
$$\leq \tfrac{1}{2}(U + 9P)F(\Psi)$$
$$\leq \tfrac{1}{2}(U + 9P)\int_{-L}^{0}\left(\frac{\partial^2 \Psi'}{\partial y^2}\right)_{y=0} dx, \qquad (29)$$

and so only single integrals need be carried out for each choice of H' and H''. This and the automatic satisfaction of the conditions on Ψ' and Ψ'' make this form for the approximation very attractive.

As an example of the calculations needed for simple choices of $s(x)$ and $t(x)$, let

$$t(x) = \begin{cases} B(U + 9P) & \text{for } -L \leq x \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$s(x) = \begin{cases} -(U + 9P)e^{-ax} & \text{for } x = 0 \\ -(U + 9P)e^{+a(x+L)} & \text{for } x = -L. \end{cases}$$

Then from Eq. (29), it follows that

$$E(\Psi') = (U + 9P)^2\mu a D/\pi$$

and

$$V^*(\Psi'') = (U + 9P)^2\mu B(L - BC/\pi),$$

where

$$C = -\int_{-\infty}^{\infty} T^{-1}(k)k^{-2}\sin^2(0.5kL)\, dk$$

and

$$D = -\int_{-\infty}^{\infty} T(k)k^{-2}(a^2 + k^2)^{-1} \cdot[k \sin kL + a(1 - \cos kL)]\, dk,$$

$T(k)$ being given as in the previous section. The minimum of V^* occurs when $B = 0.5\pi L/C$; hence,

$$\frac{L^2\pi}{2C} < \frac{F}{\mu(U + 9P)} < \frac{2Da}{\pi},$$

where a can be chosen to minimize aD. C, D, and a are functions of L which can be evaluated by approximate or numerical means for a given value of L.

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