A new proof of absence of positive discrete spectrum of the Schrödinger operator

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A new method to prove the absence of positive discrete spectrum of the Schrödinger operator is given.

1. INTRODUCTION

Consider the equation

$$(\nabla^2 + \lambda^2 - p(x))u = 0, \quad x\in\mathbb{R}^n, \lambda^2 > 0 . \qquad (1)$$

Let us assume that

$$u\in L^2(\equiv L^2(\mathbb{R}^n)) . \qquad (2)$$

One of the most important steps in spectral analysis of the Schrödinger operator (1) is to prove that the only solution of (1) and (2) under some assumptions about $p(x)$ is $u\equiv 0$. The first result of this kind in potential scattering was obtained by T. Kato.[1] It was generalized by many authors. In A.G. Ramm[2] the boundary value problem was considered in domains with infinite boundaries. In M. Reed and B. Simon[3] the result is obtained under more general assumptions about $p(x)$. Then, in Saito,[4] and in P. Mishnaevsky,[5] the case of the operator-valued Sturm–Liouville equation was treated; some results are known in case ∇^2 is replaced by an operator $Lu = -\Sigma_{i,j=1}^3 (\partial/\partial x_i)(a_{ij}(x)(\partial u/\partial x_j))$, with $a_{ij}(x) = \delta_{ij}(x)$, for $|x| > R$ and which satisfies some other assumptions (M. Shifrin[6]). It is not our purpose to mention all the papers which deal with the question under discussion. For us it is important that all known proofs of the above mentioned uniqueness theorem are rather complicated and technical. It is our purpose to outline a new approach to this question and to present a new proof of the known result. We shall not present the result in the most general form but try to explain a new idea in our approach. It should be mentioned that in Ref. 1 the absence of positive eigenvalues of the Schrödinger operator was proved under the assumption $|p(x)| \leqslant C(1+|x|)^{-a}$, $a > 1$, whereas here we handle only the case $|p(x)| \leqslant Ce^{-\epsilon|x|}$.

2. THE SIMPLEST CASE

First let us consider the simplest case when $p(x)$ is a bounded continuous function with compact support. Suppose that $u\in L^2(\mathbb{R}^n)$ and satisfies Eq. (1). Taking the Fourier transform of both sides of the equation, we find

$$(\lambda^2 - \zeta^2)\hat{u}(\zeta) = (\widehat{pu})(\zeta), \quad \zeta\in\mathbb{R}^n , \qquad (3)$$

where $\zeta^2 = \zeta_1^2 + \cdots + \zeta_n^2$ and the $\hat{}$ denotes the Fourier transform

$$\hat{u}(\zeta) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{-i\zeta\cdot x} u(x)\, dx . \qquad (4)$$

Since $pu\in L^2(\mathbb{R}^n)$ and has support in some compact set in $\mathbb{R}^n$, its Fourier transform $\widehat{pu}(\zeta)$ is an entire function of $\zeta\in\mathbb{C}^n$. Further, $\widehat{pu}$ is an entire function of exponential type and its restriction to each of the real subspaces of $\mathbb{C}^n$, $\text{Im}\zeta =$ constant, is in $L^2(\mathbb{R}^n)$. The class of all Fourier transforms of L^2 functions with compact support in $\mathbb{R}^n$ will be denoted $\hat{L}_0^2$. We then claim that Eq. (3) implies $\hat{u}\in\hat{L}_0^2$.

Lemma 1: *If* $u\in L^2(\mathbb{R}^n)$ *and satisfies* (1) *with* $\lambda^2 > 0$, *and if* $p(x)$ *is a bounded measureable function with compact support in* $\mathbb{R}^n$, *then* $\hat{u}\in\hat{L}_0^2$. *That is, u has compact support in* $\mathbb{R}^n$.

If we assume Lemma 1, then the conclusion $u\equiv 0$ follows from the unique continuation theorem for Schrödinger equation [3].

Proof of Lemma 1: We will check first that $\hat{u}$ is an entire function and, second, that $\hat{u}\in\hat{L}_0^2$. To see this note that $(\widehat{pu})(\zeta)/(\lambda^2 - \zeta^2) = \hat{u}\in L^2(\mathbb{R}^n)$. It follows that $(\widehat{pu})(\zeta) = 0$ on the sphere $S_\lambda = \{\zeta\in\mathbb{R}^n : \zeta^2 = \lambda^2\}$, since $1/(\lambda^2 - \zeta^2)$ is not square integrable over any neighborhood in $\mathbb{R}^n$ of a point in S_λ. Then, since $(\widehat{pu})(\zeta)$ vanishes on $\{\zeta\in\mathbb{R}^n : \zeta^2 = \lambda^2\}$, it follows by analytic continuation that $(\widehat{pu})(\zeta)$ vanishes on the analytic variety in $\mathbb{C}^n$, $V = \{\zeta\in C^n : \zeta^2 = \lambda^2\}$. But, the gradient of $\zeta^2 - \lambda^2$ does not vanish on V, so $(\widehat{pu})(\zeta)/(\zeta^2 - \lambda^2)$ is a smooth function and, therefore, analytic on $\mathbb{C}^n$. Thus, $\hat{u}(\zeta) = (\widehat{pu})(\zeta)/(\zeta^2 - \lambda^2)$ is an entire function.

To prove that $\hat{u}\in\hat{L}_0^2$, we will use a version of the Paley–Wiener theorem (Ref. 7, p. 20, 21). It is not hard to verify that each $\hat{v}\in\hat{L}_0^2$ satisfies for some $C,R > 0$,

$$\text{(i)} \quad |\hat{v}(\zeta)| \leqslant C\|v\|_{L^2(\mathbb{R}^n)} \exp(R\,|\text{Im}\zeta|) , \qquad (5)$$

(ii) the restriction of $\hat{v}$ to $\mathbb{R}^n$ belongs to $L^2(\mathbb{R}^n)$.

The Paley–Wiener theorem implies that the converse also holds.

Theorem 1: *If* $\hat{v}$ *is an entire function on* $\mathbb{C}^n$ *satisfying the growth conditions* (i) *and* (ii), *then* $\hat{v}\in\hat{L}_0^2$. *In fact,* (i) *can be replaced by the weaker growth condition,*

$$\text{(i}'\text{)} \quad |\hat{v}(\zeta)| \leqslant C\exp(R\,|\zeta|).$$

Further, v has support in $\{|x| \leqslant R\}$.

To conclude the proof of Lemma 1, we will prove that $\hat{u}$ satisfies (i) and (ii), assuming that $\Phi = \widehat{pu}$ satisfies (i) and (ii). Since $u\in L^2(\mathbb{R}^n)$, we already know that (ii) holds. We also know that $(\lambda^2 - \zeta^2)\hat{u}(\zeta)$ satisfies an estimate of the form (i). It is then an immediate consequence of the "division lemma," (see Corollary 1.3, p. 8 of L. Ehrenpreis[8]), that $\hat{u}$ also satisfies an estimate of the form (i). This completes the proof.

Remark: The hypothesis that p is bounded can be considerably relaxed in Lemma 1. All that is needed is that

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$\Phi = (\widehat{pu})$ is an entire function of exponential type. This will be true if pu is a distribution with compact support; for example, if $p\in L^{\epsilon}$ for some $\epsilon > 0$. However, to conclude the result of the theorem, $u\equiv 0$, we apply the unique continuation theorem which requires stronger hypotheses. See remark 1 of Sec. 5.

3. EXPONENTIALLY DECREASING POTENTIALS

In this section we show how the idea of Sec. 2 can be used to prove $u\equiv 0$ when the assumption on the potential $p(x)$ is relaxed to

$$|p(x)| \leqslant C\exp(-a|x|), \quad x\in\mathbb{R}^n \tag{6}$$

for some $C,a > 0$. The steps are the same except that an additional induction argument is required. First, recall that the main point was to prove $\hat{u}\in\hat{L}^2_0$, and the first step in this argument was to prove that $\hat{u}$ could be analytically continued from $\mathbb{R}^n$ to $\mathbb{C}^n$. To handle potentials of the form (6), we need a local version of this result. A more general version will be stated and proved, since it isolates the crucial property satisfied by the polynomial $\zeta^2 - \lambda^2$ and is therefore perhaps of independent interest.

Lemma 2: Let Ω be an open set in $\mathbb{C}^n$. Suppose Q is a polynomial on $\mathbb{C}^n$, Φ is analytic on Ω, f is locally square integrable on $\mathbb{R}^n\cap\Omega$, and

$$Qf = \Phi, \text{ on } \mathbb{R}^n\cap\Omega, \tag{7}$$

if Q satisfies the following condition:

Each irreducible component V of $\{\zeta\in\Omega:Q(\zeta) = 0\}$ intersects $\mathbb{R}^n\cap\Omega$ in a real analytic set of dimension $n-1$, (8)

then f has an analytic continuation to all of Ω.

We will postpone the proof of Lemma 2 to the next section. Let us remark, however, that Q need not be a polynomial—Q analytic on Ω is all that is needed. Further, the property (8) satisfied by Q is exactly what is needed to conclude that f is analytic, at least when Ω is a domain of holomorphy which is homeomorphic to a ball. That is, if Ω satisfies these conditions, if $\Omega\cap\mathbb{R}^n$ is not empty, and if each f satisfying the hypotheses of Lemma 2 is also analytic on Ω, then Q must also satisfy (8). We will not include a proof here, since it does not seem relevant to the problem.

The other ingredient needed for the proof is a version of the Paley–Wiener theorem for functions with exponential decay. However, we need to have fairly explicit estimates of the constants appearing in the theorem. The form stated below is adequate. We will not give the proof, since any standard proof of the Paley–Wiener theorem will give the result just by explicitly carrying through the estimates.

Lemma 3: Let $\epsilon > 0$, $a\geqslant 0$, and suppose that $e^{(a+2\epsilon)|x|}u(x)\in L^2(\mathbb{R}^n)$ has L^2 norm at most K. Then $\hat{u}\in L^2(\mathbb{R}^n)$, $\hat{u}$ is analytic in the strip $|\mathrm{Im}\zeta| < a + 2\epsilon$ and there is a constant $C = C(\epsilon,n)$ dependending only on ϵ and n, and independent of a, such that

$$|\hat{u}(\zeta)| \leqslant K\cdot C$$

for all ζ with $|\mathrm{Im}\,\zeta| < a + \epsilon$. In the converse direction, if $\hat{u}\in L^2(\mathbb{R}^n)$, $\hat{u}$ is analytic in the strip $|\mathrm{Im}\,\zeta| < a + 2\epsilon$ and satisfies $|\hat{u}(\zeta)| \leqslant K$ in that strip, then $e^{(a+\epsilon)|x|}u(x)\in L^2(\mathbb{R}^n)$ and has L^2 norm at most $K\cdot C_1$, where C_1 is a constant depending only on n and ϵ.

We can now prove the theorem.

Theorem 2: Suppose $p(x)$ is a locally bounded, measurable function such that for some $\epsilon > 0$,

$$|p(x)| \leqslant Ke^{-4\epsilon|x|}.$$

Suppose further that $u\in L^2(\mathbb{R}^n)$ satisfies for some $\lambda > 0$,

$$\Delta u(x) + \lambda^2 u(x) - p(x)u(x) = 0, \quad x\in\mathbb{R}^n.$$

Then $u\equiv 0$.

Proof: Taking the Fourier transform of the equation yields $(\lambda^2 - \zeta^2)\hat{u}(\zeta) = (\widehat{pu})(\zeta)$. Because of the estimate on p, we have from Lemma 3, with $a = 0$ that (pu) is analytic in the strip $|\mathrm{Im}\,\zeta| < 4\epsilon$ and bounded by $KC\|u\|_{L^2}$ in the strip $|\mathrm{Im}\,\zeta| < 3\epsilon$. Because $\lambda^2 - \zeta^2$ satisfies (8), it follows that $\hat{u}$ is analytic in $|\mathrm{Im}\,\zeta| < 4\epsilon$. Further, from a standard division lemma (see Corollary 1.3, p. 8 of L. Ehrenpreis[8]) it follows that $\hat{u}(\zeta)$ is bounded by $C_1K\|u\|_{L^2}$ in the strip $|\mathrm{Im}\,\zeta| < 2\epsilon$, where C_1 is a constant depending only on n and ϵ. Finally, from the converse part of Lemma 3, we deduce that $e^{\epsilon|x|}u(x) \in L^2(\mathbb{R}^n)$ and, further, has L^2-norm at most $C_2K\|u\|_{L^2(\mathbb{R}^n)}$; where C_2 is a constant depending only on n and ϵ.

We can then repeat the argument, except using $a = \epsilon$ in Lemma 3, to deduce

$$e^{2\epsilon|x|}u(x)\in L^2(\mathbb{R}^n),$$

and has L^2 norm at most $(C_2)^2K\|u\|_{L^2(\mathbb{R}^n)}$. Continuing in this fashion, we find that $e^{k\epsilon|x|}u(x)\in L^2(\mathbb{R}^n)$ and has L^2-norm at most equal to $(C_2)^kK\|u\|_{L^2(\mathbb{R}^n)}$, $k = 1,2,\ldots$. It therefore follows from Lemma 3 that

$$|\hat{u}(\zeta)| \leqslant K\|u\|_{L^2(\mathbb{R}^n)}\exp(C_3|\mathrm{Im}\,\zeta|),$$

for some constant $C_3 > 0$. Thus, by the Paley-Wiener theorem, u must have compact support in the ball $|x| < C_3$. Therefore, the problem has been reduced to the case treated in Sec. 2, and we conclude that $u\equiv 0$.

4. PROOF OF LEMMA 2

We will prove that if Φ/Q is locally in L^2 (or even in $L^{1+\epsilon}$) on $\Omega\cap\mathbb{R}^n$ and if Q satisfies (8), then Φ/Q is analytic in Ω. By factoring Q into a product of powers of prime factors (in the ring of functions analytic on Ω—not in the polynomial ring), it is seen that it is enough to consider the case when Q is irreducible. In this case, Φ/Q is analytic on Ω if and only if the variety $\{\zeta\in\Omega:\Phi(\zeta) = 0\}$ contains $V = \{\zeta\in\Omega:Q(\zeta) = 0\}$. Now, since $\dim_{\mathbb{R}}(V\cap\Omega\cap\mathbb{R}^n) = n-1$, we must have $\Phi = 0$ on $V\cap\Omega$. Otherwise, there exists $x_0\in V\cap\Omega$ and $\epsilon > 0$ such that

$$\int_{|x-x_0|\leqslant\epsilon}\frac{dx}{|Q(x)|^2} < +\infty.$$

But, the integral $\int_{|x-x_0|\leqslant\epsilon} dx/|Q(x)|^2$ must diverge. To see this, choose linear coordinates $(t_1,\ldots,t_n)$ near x_0 so that $t = 0$ corresponds to x_0 and for each choice of $(t_2,\ldots,t_n)$, $t_1 \to Q(t_1,\ldots,t_n)$ has a zero near $t_1 = 0$. This is possible since $\dim_{\mathbb{R}}(V\cap\mathbb{R}^n\cap\Omega) = n-1$. Then

$$\int\frac{dt}{|Q(t)|^2} = \int dt_2\ldots dt_n\int\frac{dt_1}{|Q(t_1,t_2,\ldots t_n)|^2} = +\infty.$$

Consequently, $\Phi = 0$ on $V\cap\mathbb{R}^n\cap\Omega$.

Next, since V is irreducible, to prove $V\subset\{\Phi(\zeta)=0\}$, all we have to show is that $\{\Phi(\zeta)=0\}\cap U\supset V\cap U$, where U is a neighborhood of a point $z_0\in V$. Thus, let $x_0\in V\cap\mathbb{R}^n\cap\Omega$. Then the germ of the real analytic set $\{\Phi(\zeta)=0\}\cap\mathbb{R}^n$ at x_0 contains the germ of $V\cap\mathbb{R}^n$ at x_0, which is of dimension $n-1$. It follows that the complex germ of the real analytic set $\{\Phi(\zeta)=0\}\cap\mathbb{R}^n$ at x_0 contains the complex germ of $Q=0$ at x_0 (a consequence of Proposition 2, p. 92 of R. Narasimhan,[9] which is also a good reference for the other facts about analytic sets which have been used in the proof). This completes the proof.

5. REMARKS

1. The method given in this paper can be used in many cases when the potential $p(x)$ has singular points. Our arguments require only that pu be a distribution of exponential rate of decrease and of finite order. For example, this will be the case if for some ϵ, $a>0$, $p(x)$ is measurable and $\int_{|x|>t}|p(x)|^{\epsilon}\,dx\leqslant e^{-a|t|}$. However, the proof also uses the unique continuation theorem for the Schrödinger operator in $\mathbf{R}^n$, which is only known for $p(x)\in L^{q}_{\text{loc}}(\mathbf{R}^3)$ for $q\geqslant 3/2$, and in $\mathbf{R}^n$ for $p(x)\in L^{q}$, $q=\frac{2}{3}(n-1)$, $n\geqslant 5$; $q=2$ for $n=4$; $q>1$ for $n=1,2$. (See Refs. 3, 7, and 10).

2. It would be interesting to extend the ideas of this paper in order to prove absence of the positive eigenvalues of Schrödinger operator $|p(x)|\leqslant c(1+|x|)^{-a}$, $a>1$. The Wigner–von Neumann example shows that there exists a potential $|p(x)|\leqslant c(1+|x|)^{-1}$, which has a positive eigenvalue [Ref. 3, p. 223]. It would be interesting to understand from the standpoint of our technique why $a=1$ plays the role of frontier.

APPENDIX

We give an example which shows the importance of condition (8). If $f\geqslant 0$, $f\in C_0^{\infty}(\mathbb{R}^3)$, $k>0$ and $u(x)=\int\exp(-k|x-y|)(4\pi|x-y|)^{-1}f(y)\,dy$, then $u(x)>0$, $u\in L^2(\mathbb{R}^3)$, $-\Delta u+k^2u=f(x)$, $\Delta^2u+q(x)u-k^4u=0$, $q(x)\equiv(k^2f+\Delta f)u^{-1}(x)$. Thus $k^4>0$ is the eigenvalue of $\Delta^2+q(x)$, $q(x)\in C_0^{\infty}$. In this case $Q(\zeta)=\zeta^4-k^4=(\zeta^2-k^2)(\zeta^2+k^2)$ and $V=V_1\cup V_2$, $V_1=\{\zeta:\zeta\in\mathbb{C}^3,\zeta_1^2+\zeta_2^2+\zeta_3^2-k^2=0\}$. Hence V_2 does not intersect $\mathbb{R}^3$, condition (8) is not fulfilled and hte operator $\Delta^2+q(x)$ has a positive eigenvalue. The potential $q(x)$ was used also in the review article, D. Eidus, "The principal of limit amplitude," Russ. Math. Surv. **24**, N3, 97 (1969).

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