Research Notes

Research Notes published in this Section include important research results of a preliminary nature which are of special interest to the physics of fluids and new research contributions modifying results already published in the scientific literature. Research Notes cannot exceed five printed columns in length including space allowed for title, abstract, figures, tables, and references. The abstract should have three printed lines. Authors must shorten galley proofs of Research Notes longer than five printed columns before publication.

Curved Characteristics Behind Blast Waves

O. LAPORTE
Department of Physics, University of Michigan, Ann Arbor, Michigan 48104
AND
T. S. CHANG*
Center for Space Research, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 22 December 1970; final manuscript received 6 August 1971)

Exact solutions, expressed in closed form in terms of elementary functions, are presented for the three sets of curved characteristics behind a self-similar, strong blast wave.

Blast waves are produced in gaseous media due to the sudden deposition of large amounts of energy in relatively small regions. The propagation of a point-source blast wave into an ideal gas, whose initial pressure is assumed to be negligibly low, is known to be self-similar. This property can be deduced using either dimensional arguments[1] or invariant theorems of continuous groups of transformations.[2] Closed form solutions describing the flow variables in the nonisentropic region behind such a blast wave in $n=1, 2, 3$ dimensions were obtained independently by von Neumann[3] and Sedov.[4] The reflection of strong blast waves was discussed in an earlier paper[5] by the authors.

It is known that the basic equations governing the nonisentropic flow behind a propagating blast wave admit three distinct characteristic directions given by

$$\frac{dr}{dt}=u, \qquad u\pm a, \tag{1}$$

where r is the radial distance from the point of explosion, t is the elapsed time, u is the velocity, and a is the isentropic speed of sound. The first characteristic direction coincides with the local particle velocity and the family of characteristics is, therefore, composed of particle lines. This direction corresponds to the speed of propagation of entropy disturbances. The other two characteristic directions correspond to the local speeds of propagation of pressure, density, or velocity disturbances. The purpose of this note is to report the interesting result that these three families of curved characteristics can also be represented in closed form, and remarkably, still in terms of elementary functions.

First, consider the (u) characteristics or particle lines. According to its definition and the self-similar solution of Ref. 5, it may be easily deduced that along a (u) characteristic,

$$dr/dt=(2+n)U\hat{u}y(\hat{u})/2, \tag{2}$$

where

$$U=[2K_n/(2+n)](E_0/\rho_0)^{1/(2+n)}t^{-n/(2+n)} \tag{3}$$

is the shock speed, $y(\hat{u})$ is given by Eq. (5) of Ref. 5, $\hat{u}=ut/r$, E_0 is the energy released per unit area for a planar wave, per unit length for a cylindrical wave, and the total energy released for a spherical wave, ρ_0 is the initial density, and K_n is determined by the condition of conservation of total energy.

But, from the definition of the similarity parameter and the expression for the shock radius of Ref. 5, it can be shown that, in general,

$$dr/dt=K_n(E_0/\rho_0)^{1/(2+n)}t^{-n/(2+n)}[2y/(2+n)+t\,dy/dt]. \tag{4}$$

Therefore, from Eq. (5) of Ref. 5 and Eqs. (2)–(4), it is found that,

$$d\ln t=\frac{d\ln y}{d\hat{u}}\,\frac{d\hat{u}}{\hat{u}-[2/(2+n)]}, \tag{5}$$

along a particle line, where,

$$\frac{d\ln y}{d\hat{u}}=\frac{(\gamma-1)\hat{u}^2/2+[\hat{u}-2/(2+n)][\hat{u}-2/\gamma(2+n)]}{\hat{u}[2/\gamma(2+n)-\hat{u}][(2-n+n\gamma)\hat{u}/2-1]}, \tag{6}$$

and γ is the adiabatic index. Thus, Eq. (5) can be integrated in terms of elementary functions. The result is

$$\hat{t}=\alpha\hat{u}(A_2-\hat{u})^{\beta_1}(\hat{u}-A_2/\gamma)^{\beta_2}(\hat{u}-A_5^{-1})^{\beta_3}, \tag{7}$$

where

$$\alpha = A_1(A_2 - A_1^{-1})^{-\beta_1}(A_1^{-1} - A_2/\gamma)^{-\beta_2}(A_1^{-1} - A_5^{-1})^{-\beta_3},$$
$$\beta_1 = (2\gamma + n - 2)[n(\gamma - 2)]^{-1}\beta_2,$$
$$\beta_2 = n\gamma(2+n)(2-\gamma)(\gamma-1)[2(2n\gamma+2\gamma-n+2) \times (2-n-2\gamma) + 2n\gamma(2-\gamma)(n\gamma+4) + 2(\gamma+1)(2+n) \times (n\gamma^2 - 2n\gamma + n + 2\gamma - 2)]^{-1},$$
$$\beta_3 = (2+n)(n\gamma^2 - 2n\gamma + n + 2\gamma - 2)[n(2-\gamma) \times (n\gamma - n + 2)]^{-1}\beta_2 - 1, \quad (8)$$

$\hat{t} = t/t_0$ is a dimensionless time normalized with respect to some convenient time scale t_0, and the A's are given by Eqs. (6) of Ref. 5.

To complete the solution for the particle lines, an expression relating $(r, t, \hat{u})$ is obtained from Eqs. (2) and (5) of Ref. 5. In dimensionless form, this expression becomes,

$$\hat{r} = (A_1\hat{u})^{-A_2}[A_3(\gamma\hat{u}/A_2 - 1)]^{A_8} \times [A_1(1 - A_5\hat{u})/(A_1 - A_5)]^{A_6 - A_9}\hat{t}^{2/(2+n)}, \quad (9)$$

where $\hat{r} = r/R(t_0)$. Equations (8) and (9) form the closed form parametric solution for the family of (u) characteristics or particle lines behind a self-similar blast wave.

For the $(u \pm a)$ characteristics, we have

$$\frac{d_{\pm}r}{dt} = u'f \pm \left[\frac{\gamma p' g}{\rho' h}\right]^{1/2} = [U/(\gamma+1)]\{2A_1\hat{u}y(\hat{u}) \pm [2\gamma(\gamma-1)g(\hat{u})/h(\hat{u})]^{1/2}\}, \quad (10)$$

where $g(\hat{u})$, $h(\hat{u})$, and $y(\hat{u})$ are given by Eqs. (4) and (5) of Ref. 5, and the shock speed U by Eq. (3). Combining with Eq. (4) and after considerable manipulation, it is found that

$$d\ln t = G(\hat{u})\, d\hat{u}, \quad (11)$$

where,

$$G(\hat{u}) = (d\ln y/d\hat{u})\{[\hat{u} - 2/(2+n)] \pm [\gamma(\gamma-1)(2 - 2\hat{u} - n\hat{u})/2(2\gamma\hat{u} + n\gamma\hat{u} - 2)]^{1/2}\hat{u}\}^{-1}. \quad (12)$$

The expression $d\ln y/d\hat{u}$ in $G(\hat{u})$ is known and given by Eq. (6). Therefore, $G(\hat{u})$ may be reduced to an algebraic function involving square roots of second degree polynomials of $\hat{u}$ as radicals. This means that Eq. (11) can be integrated in terms of elementary, albeit transcendental, functions. The results are

$$\ln\hat{t} = K_{\pm} + \ln|A_5\hat{u}/(1 - A_5\hat{u})| \pm (m/l^{1/2}) \times \sin^{-1}\{[l/(A_5^{-1} - \hat{u}) - k]/(k^2 - l)^{1/2}\}, \quad (13)$$

where,

$$k = A_5^{-1} - (\gamma+1)[\gamma(2+n)]^{-1},$$
$$l = A_5^{-2} - 2A_5^{-1}(\gamma+1)[\gamma(2+n)]^{-1} + 4[\gamma(2+n)^2]^{-1},$$
$$m = -A_5^{-1}[(\gamma-1)/2]^{1/2}. \quad (14)$$

Equation (13) and the parametric expression for $(\hat{r}, \hat{t}, \hat{u})$ given by Eq. (9) form the closed form parametric solutions for the two families of $(u \pm a)$ characteristics in the dimensionless $\hat{r}$–$\hat{t}$ plane. The constant $K_{\pm}$ in Eq. (13) must be evaluated for each characteristic from a set of known values of $(\hat{r}, \hat{t}, \hat{u})$.

Figure 1 displays a typical set of solution curves expressing the dimensionless time $\hat{t}$ as functions of the dimensionless distance $\hat{r}$ for $n=3$ and $\gamma=\frac{5}{3}$. The terminating curve A is the path of the front of the blast wave. Behind A, there are three families of characteristics: (1) The solid curves are the (u) characteristics or particle lines; (2) the dashed curves are the $(u+a)$ characteristics; and (3) the dot-dashed curves are the $(u-a)$ characteristics.

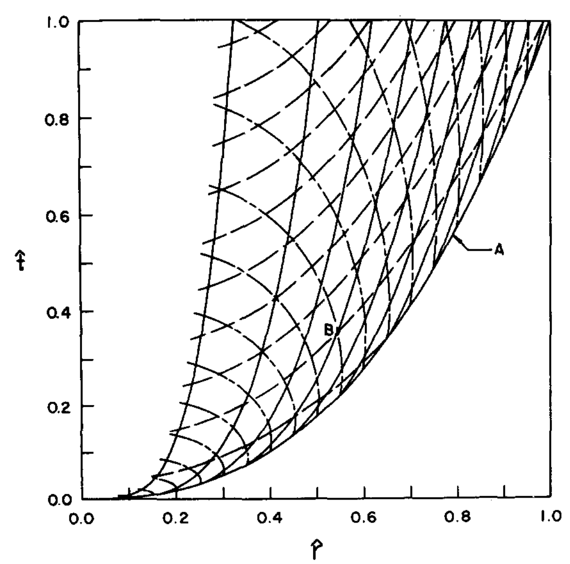

FIG. 1. Curved characteristics behind a strong blast wave for $n=3$, $\gamma=5/3$.

The authors are grateful to Mr. T. Tucker of Oak Ridge National Laboratory for computational assistance.

This research was supported in part by the Department of Defense Project Themis at Lehigh University under Contract Number DAAD05-69-C-0053 monitored by the Ballistic Research Laboratories, and by the National Aeronautics and Space Administration Grant number NASA-NGL34-002-084.

A portion of this manuscript was completed when one of us (T.S.C.) was visiting Cornell University under the auspices of the Army Research Office, Durham.

* Permanent address: North Carolina State University, Raleigh, North Carolina 27607.

[1] G. I. Taylor, Report of Civil Defense Research Committee of the Ministry of Home Security, RC-210, 12 (1941); also, Proc. Roy. Soc. (London) **A201**, 159 (1950).

[2] O. Laporte and T. S. Chang, Ballistic Research Laboratories Technical Report, CAM-110-14 (1971).

[3] J. von Neumann, *Collected Works of von Neumann*, edited by A. H. Taub (Pergamon, New York, 1963), Vol. 6, p. 219.
[4] L. I. Sedov, Prikl. Mat. Mekh. **10**, 241 (1946); also, *Similarity and Dimensional Methods in Mechanics* (Academic, New York, 1959), Chap. IV.
[5] T. S. Chang and O. Laporte, Phys. Fluids **7**, 1225 (1964).

THE PHYSICS OF FLUIDS VOLUME 15, NUMBER 3 MARCH 1972

Solution of Forced Burgers Equation

D. T. Jeng and W. C. Meecham
Mechanics and Structures Department, University of California, Los Angeles, California 90024
(Received 26 January 1971; final manuscript received 1 October 1971)

Burgers' model equation of turbulence is studied for the case when the driving force is periodic. Analytical solution is obtained for the inviscid flow and numerical integration is performed for the viscous flow.

As is known, the extreme complexity of the equations governing incompressible fluid flow has led to the extensive use of model equations to examine certain characteristics of the processes involved in turbulence. One most useful such model equation is due to Burgers.[1] The proposed forced model equation[2] is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}+F. \tag{1}$$

This equation can be thought of as a model for many physical problems. For instance, it could be said to be somewhat analogous to a problem of wind forcing the buildup of waterwaves where the wind drag is the function F. The model is written here with a forcing term F. This equation has many of the properties of real fluid flow although, of course, it is a one-dimensional model, lacks a pressure term, and has no associated equation of continuity. The solutions do have the property that the nonlinear transfer term, $u\,\partial u/\partial x$, conserves energy. It is known that the solutions of the equation, at least for a decaying process (where $F=0$), behave as follows. Any large, initial velocity function quickly forms a system of waves with shock-like structure, the dissipation being concentrated primarily in the thin shock region. The thickness of the shock region is known, at least in simple examples, to be of order,[3]

$$\delta=(2/R')\ \ln 2\pi R' \tag{2}$$

thus, approximately inversely proportional to the fluctuation Reynolds number.

We investigate the question of whether or not δ is altered by the forcing function F. We restrict to forcing functions those F which are sinusoidal, although the remarks presented here are readily extended to other forces. Furthermore, we shall be interested in steady-state solutions of the nonlinear equation (1). Thus, suppose

$$F=-A\sin\xi, \qquad A>0, \tag{3}$$

with

$$\xi=kx-\omega t.$$

We look for a solution of the form

$$u(x,t)=f(\xi). \tag{4}$$

We can find analytic solutions of this form for the simplified inviscid problem, $\nu=0$ in Eq. (1). Substituting Eqs. (3) and (4) in Eq. (1) and solving we find

$$f_{\pm}=c\pm c(\alpha\cos\xi+K)^{1/2}, \qquad \alpha=2A/kc^2, \tag{5}$$

where c, the speed of the moving force, is given by

$$c=\omega/k. \tag{6}$$

The constant K remains to be determined. It is easily shown that for a force of the type (3) the integral of the velocity over one period of the wave must vanish. This condition can be used to determine the choice of sign and of K in (5). The restriction on the integral of u leads to the solution of the following transcendental equation for the value of K:

$$\int_0^{2\pi}\left(1+\frac{\alpha}{K}\cos\xi\right)^{1/2}d\xi=\frac{2\pi}{K^{1/2}}. \tag{7}$$

This equation has a real root if

$$\alpha(\equiv 2A/kc^2)<\pi^2/8. \tag{8}$$

The solution is $u=f_-$. Thus, for a force small enough so that (8) is satisfied, the velocity field is a continuous function. Derivatives of the velocity are small so that the dissipation function analogous to Φ, is small.

For values of the force larger than (8) there is a discontinuous solution to the inviscid form of Eq. (1). For such larger values of force, the solution becomes

$$u=c-c(2\alpha)^{1/2}\cos\frac{\xi}{2}, \qquad -\xi_0\leq\xi\leq 2\pi-\xi_0 \tag{9}$$

and periodic outside this range, where using the property that

$$\int_0^{2\pi}u\,d\xi=0$$

we have