Reflection of Strong Blast Waves

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The reflection of strong point-source blast waves is studied using the continuum concept of flow of ideal gases. Methods of obtaining the upper and lower bounds as well as a Taylor series expansion of the position of the reflected shock is considered. The procedure of studying the flow variables near the reflected shock is also described.

I. INTRODUCTION

THE purpose of this investigation is to study the reflection of strong, point-source blast waves. This study was motivated by the recent interests in underground nuclear explosions and by the laboratory experiments of exploding wires in cylindrical tubes where the reflections of blast waves were observed by means of smear camera pictures.

The propagation of a strong, point-source blast wave in an ideal gas medium is known to be self-similar. This property was first observed by Taylor[1] in 1941. Closed form solutions describing the flow variables for such a blast wave in n dimensions have been obtained independently by Bethe, *et al.*,[2] and Sedov.[3] The concept of self-similarity, however, may no longer be applied to the propagation of the reflection of blast waves. Due to the nonisentropic flow variables ahead of and behind the reflected shock, there does not exist an obvious or convenient procedure to reduce the number of independent variables. The problem of blast wave reflection is further complicated by its associated initial and boundary conditions. Such a nonlinear problem cannot be treated by means of the conventional methods of considering the changes of the flow variables across a shock wave alone, since the description of the position, speed, and acceleration of the reflected shock is also of prime importance.

In this paper, methods of obtaining the upper and lower bounds of the position $\bar{R}(t)$ of the reflected shock and the flow variables near it are introduced. A series expansion of the position of the reflected shock in powers of the elapsed time is also considered.

[1] G. I. Taylor, Report of Civil Defence Research Committee of the Ministry of Home Security, RC-210, 12 (1941); also, Proc. Roy. Soc. (London) **A201**, 159 (1950).

[2] H. A. Bethe, K. Fuchs, J. von Neumann, R. Peierls, and W. G. Penny; U. S. Atomic Energy Commission Report AECD-2860, (1944).

[3] L. I. Sedov, Prikl. Math. Mekh. **10**, 241 (1946); also, *Similarity and Dimensional Methods in Mechanics* (Academic Press Inc., New York, 1959).

II. SELF-SIMILAR SOLUTION OF A POINT-SOURCE BLAST WAVE

Blast waves are produced in a gas medium due to the sudden release of a large amount of energy in a relatively small region of space. The propagation of a strong, point source, planar ($n = 1$), cylindrical ($n = 2$), or spherical ($n = 3$) blast wave in an ideal gas medium may be described in terms of a single similarity parameter

$$y = r/R, \qquad (1)$$

where r is the radial distance from the point of explosion, and

$$R = K_n(E_0/\rho_0)^{1/(2+n)} t^{2/(2+n)} \qquad (2)$$

is the shock radius measured from the point of explosion, E_0 is the energy released per unit area for a planar wave, per unit length for a cylindrical wave, and the total energy released for a spherical wave, ρ_0 is the initial density in the gas medium, t is the elapsed time, and K_n is a constant yet to be determined.

We follow Ref. 3 and express the flow variables behind the blast wave in dimensionless forms as follows:

$$\hat{u} = u_1 t/r, \qquad \hat{p} = p_1 t^2/\rho_0 r^2, \qquad \hat{\rho} = \rho_1/\rho_0, \qquad (3)$$

where u_1 is the flow velocity, p_1 is the fluid pressure, and ρ_1 is the fluid density behind the blast wave. The solution of the flow variables may be expressed in the following closed form:

$$f = u_1/u' = A_1 \hat{u} y,$$

$$g = p_1/p' = (A_1 \hat{u})^{A_2 n} [A_3(1 - \hat{u}/A_2)]^{A_4}$$
$$\cdot [A_1(1 - A_5 \hat{u})/(A_1 - A_5)]^{2A_6},$$

$$h = \rho_1/\rho'$$
$$= [A_3(\gamma \hat{u}/A_2 - 1)]^{A_7} [A_3(1 - \hat{u}/A_2)]^{A_4 - 1}$$
$$\cdot [A_1(1 - A_5 \hat{u})/(A_1 - A_5)]^{2A_6}, \qquad (4)$$

where

$$y = r/R = (A_1\hat{u})^{-A_2}[A_3(\gamma\hat{u}/A_2 - 1)]^{A_8}$$
$$\cdot[A_1(1 - A_5\hat{u})/(A_1 - A_5)]^{A_6-A_9}, \qquad (5)$$

and

$$A_1 = \frac{(\gamma + 1)(2 + n)}{4}, \quad A_2 = \frac{2}{2 + n}, \quad A_3 = \frac{\gamma + 1}{\gamma - 1},$$

$$A_4 = \frac{\gamma}{\gamma - 2}, \qquad A_5 = \frac{2 + n(\gamma - 1)}{2},$$

$$A_6 = \frac{n(2\gamma + n - 2)}{(2 + n)(n\gamma - n + 2)} - \frac{(2 + n)\gamma(1 - \gamma)}{2(2 - \gamma)(n\gamma - n + 2)},$$

$$A_7 = \frac{n}{2\gamma + n - 2}, \quad A_8 = \frac{\gamma - 1}{2\gamma + n - 2},$$

$$A_9 = \frac{n}{n\gamma - n + 2}$$
$$- \frac{(2 + n)^2\gamma(1 - \gamma)}{2(2 - \gamma)(n\gamma - n + 2)(2\gamma + n - 2)}. \qquad (6)$$

The flow velocity u', fluid pressure p', and fluid density ρ' just behind the blast wave are:

$$u' = \frac{2}{\gamma + 1}U, \quad p' = \frac{2}{\gamma + 1}\rho_0 U^2, \quad \rho' = \frac{\gamma + 1}{\gamma - 1}\rho_0,$$
$$\qquad (7)$$

where $U = DR/Dt$ is the speed of propagation of the self-similar blast wave, and the symbol D/Dt denotes differentiation with respect to time following the blast wave.

The constant K_n in Eq. (2) may be evaluated from the energy integral

$$E_0 = \int_0^R \rho\tau_n r^{n-1}\left(u_1^2 + \frac{2\gamma}{\gamma - 1}\frac{p_1}{\rho_1}\right) dr \qquad (8)$$

where $\tau_n = 1, \pi, 2\pi$ for $n = 1, 2, 3$, respectively.

The expressions f, g, h given in Eqs. (4) may be regarded as functions of the similarity parameter y which has a convenient range of $0 \leq y \leq 1$.

III. GOVERNING EQUATIONS FOR THE PROPAGATION OF A VARIABLE-STRENGTH SHOCK DISCONTINUITY

The jump condition across a shock discontinuity moving with speed $\bar{U}$ in an ideal gas may be written as:

$$(u_I - \bar{U})(u_{II} - \bar{U})$$
$$= \frac{\gamma - 1}{\gamma + 1}(u_I - \bar{U})^2 + \frac{2}{\gamma + 1}a_I^2 \qquad (9)$$

where u_I, u_{II} are the flow velocities just ahead of and behind the shock discontinuity, $a_I^2 = \gamma p_I/\rho_I$ is the

square of the sonic velocity just ahead of the shock wave, and γ is the adiabatic index. Equation (9) may be applied to the planar, cylindrical, or spherical propagations of shock waves. Denoting the shock position in these cases by $\bar{R}(t)$ and the differentiation operator with respect to time along the shock discontinuity by $\bar{D}/Dt$, Eq. (9) may be written as

$$\bar{U} = \bar{D}\bar{R}/Dt$$
$$= -\tfrac{1}{4}(\gamma - 3)u_I + \tfrac{1}{4}(\gamma + 1)u_{II} \pm \tfrac{1}{2}\sigma \qquad (10)$$

where

$$\sigma = [\tfrac{1}{4}(\gamma + 1)^2(u_I - u_{II})^2 + 4a_I^2]^{\frac{1}{2}}. \qquad (11)$$

Before proceeding to discuss the problem of blast wave reflection and its upper and lower bounds, it is appropriate here to derive an expression describing the acceleration of a shock discontinuity separating two nonisentropic, unsteady flow regions described by the flow variables u_1, p_1, ρ_1, and u_2, p_2, ρ_2, respectively. Later, this result will be applied to the determination of the leading terms of a Taylor series expansion for the position of the reflected shock.

Consider a flow variable $G(r, t)$ either just ahead of or just behind the shock discontinuity. Its variation along the shock can be expressed as follows:

$$dG(r, t) = (\partial G/\partial t)\, dt + (\partial G/\partial r)\, dr \qquad (12)$$

where $dr = d\bar{R} = \bar{U}\, dt$. Therefore, we have

$$\bar{D}G/Dt = \partial G/\partial t + \bar{U}\, \partial G/\partial r \qquad (13)$$

where the expression is understood to be evaluated at $r = \bar{R}$ and at $t = t(\bar{R})$. Equation (13) indicates that the differentiation operator $\bar{D}/Dt$ along the shock discontinuity may be expressed as $\partial/\partial t + \bar{U}\, \partial/\partial r$.

Differentiating Eq. (10) along the shock discontinuity with respect to time and utilizing the above operator, we obtain an expression describing the acceleration of the shock discontinuity as follows:

$$\frac{\bar{D}^2\bar{R}}{Dt^2} = \left[-\frac{\gamma - 3}{4} \pm \frac{(\gamma + 1)^2(u_I - u_{II})}{8\sigma}\right]$$
$$\cdot\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial r}\right)u_1 + \left[\frac{\gamma + 1}{4} \mp \frac{(\gamma + 1)^2(u_I - u_{II})}{8\sigma}\right]$$
$$\cdot\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial r}\right)u_2 \pm \frac{1}{\sigma}\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial r}\right)a_I^2. \qquad (14)$$

Equations (9) or (10) may be applied to the problem of blast wave reflection.[4] Consider a strong, point-source blast wave impinging against a rigid

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[4] The sign in front of $\tfrac{1}{2}\sigma$ must be chosen to be negative for blast wave reflection to insure negative propagation speeds.

boundary. This rigid boundary is assumed to be planar, cylindrical, or spherical, depending upon the type of the blast wave considered. The kinematics and notations used for such a blast wave reflection may best be illustrated in an r–t diagram as shown in Fig. 1. To solve either one of these equations, it is necessary to couple it with the flow variables in regions ahead of and behind the reflected shock subjected to the given initial and boundary conditions.

The flow variables u_1, p_1, ρ_1 (and therefore a_1) in "region 1" ahead of the reflected shock are given by the self-similar solution described in Sec. II. Writing out specifically for the reflected shock, we have

$$u_1 = \frac{2}{\gamma + 1} U f(y), \quad p_1 = \frac{2}{\gamma + 1} \rho_0 U^2 g(y),$$

$$\rho_1 = \frac{\gamma + 1}{\gamma - 1} \rho_0 h(y), \tag{15}$$

where $y = r/R$, R, U refer to the primary blast wave, as if it were not reflected by the rigid boundary (see Fig. 1). Therefore,

$$a_1^2 = \frac{\gamma p_1}{\rho_1} = \frac{2\gamma(\gamma - 1)U^2 g(y)}{(\gamma + 1)^2 h(y)}. \tag{16}$$

The differential equations governing the flow variables u_2, p_2, ρ_2, and a_2 in "region 2" behind the reflected shock are

$$\frac{\partial \rho_2}{\partial t} + \frac{\partial \rho_2 u_2}{\partial r} + (n - 1) \frac{\rho_2 u_2}{r} = 0,$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial r} + \frac{1}{\rho_2} \frac{\partial p_2}{\partial r} = 0, \tag{17}$$

$$\frac{\partial}{\partial t}\left(\frac{p_2}{\rho_2^\gamma}\right) + u_2 \frac{\partial}{\partial r}\left(\frac{p_2}{\rho_2^\gamma}\right) = 0,$$

where $a_2^2 = \gamma p_2/\rho_2$. The function u_2 must satisfy the boundary condition of

$$u_{20} = 0 \tag{18}$$

where the subscript 0 denotes evaluation at the rigid boundary of $r = r_0$.

In addition, the following conditions (internal boundary conditions) along the reflected shock must be satisfied

$$(u_1, p_1, \rho_1, a_1^2) = (u_1, p_1, \rho_1, a_1^2)\Big|_{\substack{r = \bar{R} \\ t = t(\bar{R})}},$$

$$(u_{II}, p_{II}, \rho_{II}, a_{II}^2) = (u_2, p_2, \rho_2, a_2^2)\Big|_{\substack{r = \bar{R} \\ t = t(\bar{R})}}. \tag{19}$$

Finally, the initial condition of the reflected shock must be satisfied. This means that at the instant the reflection originates, $t = t_0$, we have
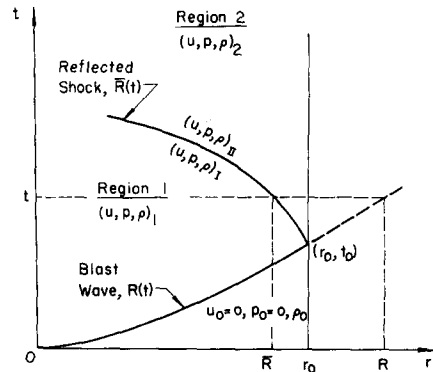

FIG. 1. Notations used for the reflection of blast waves.

$$r = r_0 = R_0 = \bar{R}_0. \tag{20}$$

From Eq. (2), this implies that

$$t = t_0 = (\rho_0 r_0^{2+n}/K_n^{2+n} E_0)^{\frac{1}{2}}. \tag{21}$$

IV. UPPER BOUND OF THE REFLECTED SHOCK

The main purpose of this section is to determine a criterion of establishing an upper bound of the position $\bar{R}(t)$ of the reflected shock. As it can be observed from Eqs. (9) or (10),[5] the coupling of the governing equation along the reflected shock are through the flow variables u_I, a_I, u_{II} only. If it is assumed that u_{II} vanishes everywhere behind the reflected shock, the integral curve of Eq. (9) or (10) can be obtained when considered with the description of the flow variables ahead of the reflected shock given by Eqs. (15) and (16).

Due to the sharp variation of the fluid pressure ahead of the reflected shock (behind the blast wave), it is reasonable to expect that the resultant flow with the assumption of $u_{II} = 0$ will introduce a sharp rate of change of pressure gradient behind the reflected shock. This means that in all probability, a reverse flow ($u_{II} < 0$) will be induced behind the reflected shock to counter-balance this effect. This will therefore increase the strength of the reflected shock as it propagates inward causing it to bend in a more horizontal direction in the r–t diagram, (see Fig. 2).

In other words, the assumption of $u_{II} = 0$ will result in a lesser pressure ratio across the reflected shock, which therefore appears weaker than the actual shock and travels more slowly. This means that the integral curve of Eq. (10) for $u_{II} = 0$ will lie above that of the true curve for shock reflection in the r–t diagram as depicted in Fig. 2. Therefore, the integral curve for the estimated reflected shock by assuming $u_{II} = 0$ will form an upper bound of the

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[5] As mentioned previously, the negative sign in front of $\frac{1}{2}\sigma$ in Eq. (10) must be used.
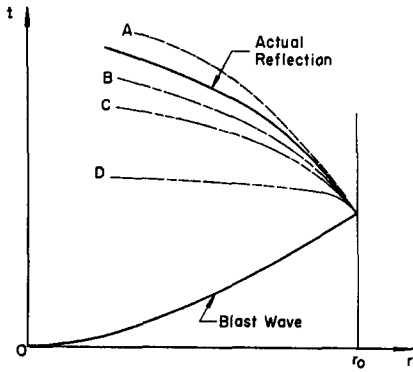
FIG. 2. Figure depicting the various approximations of the reflected shock. (A) Upper bound, $u_{II} = 0$, (B) Lower bound, $d^2\zeta/dy_0^2 = 0$, (C) Lower bound, $d^2p_{II}/dy_0^2 = 0$, (D) Almost "strong" shock, $p_{II} = p_{II0}$.

exact location of the reflected shock in the r–t diagram. The above consideration is justified by the numerical calculations for a planar blast wave reflection. It should be noted that due to the arbitrary assumption of $u_{II} = 0$, the flow variables just behind the reflected shock may not be compatible with the equations and boundary conditions governing them. The values of the flow variables obtained in this section form bounds of the actual values. It is shown that these bounds do not deviate too much from the true values.

Consider a given point $(\bar{R}, t)$ on the reflected shock. It is readily shown that the shock speed $\bar{U} = \bar{D}\bar{R}/Dt$ is completely determined by Eqs. (10), (15), and (16), if u_{II} is neglected.

To find the flow variables u_I, p_I, ρ_I, and a_I^2 just ahead of the reflected shock at $(\bar{R}, t)$, note the relation $y = \bar{R}/R$, where y is the similarity parameter for $r = \bar{R}$, and R is the shock radius of the blast wave at time t as if it were not reflected by the rigid bound-

ary, Fig. 1. Define a new dimensionless variable $y_0 = \bar{R}/r_0$ characterizing the location of the position of the reflected shock. We have

$$y = y_0(r_0/R). \qquad (22)$$

Thus, for each value of y_0 (or $\bar{R}$) and the corresponding time t, the value of the similarity parameter y may be calculated from Eqs. (20) and (2). This means that the values of the flow variables u_I, p_I, ρ_I, and a_I^2 just ahead of the reflected shock at $(\bar{R}, t)$ may be obtained by means of Eqs. (15) and (16).

The equation for the propagation of the reflected shock, Eq. (10), with $u_{II} = 0$ becomes

$$\bar{D}\bar{R}/Dt = -\tfrac{1}{4}(\gamma - 3)u_I$$
$$- \tfrac{1}{2}[\tfrac{1}{4}(\gamma + 1)^2 u_I^2 + 4a_I^2]^{\frac{1}{2}}. \qquad (23)$$

Therefore, the value of $\bar{D}\bar{R}/Dt$ may be calculated for each value of y_0.

To find a neighboring point $(\bar{R}', t')$ on the integral curve for the upper limit, the following relations may be applied:

$$t' = t + \Delta\bar{R}/(\bar{D}\bar{R}/Dt), \qquad \Delta\bar{R} = \bar{R}' - \bar{R}. \qquad (24)$$

Following the procedure outlined above, the entire integral curve of the upper bound of the reflected shock may be calculated. The starting point on the integral curve is given by Eqs. (20) and (21).

Such an integral curve for a planar blast wave reflection with $\gamma = 1.4$ is obtained and presented in Fig. 3. The numerical values of u_I, p_I, ρ_I, and a_I^2 corresponding to this integral curve for the upper bound are also calculated. These values are plotted in dimensionless form in Fig. 4.
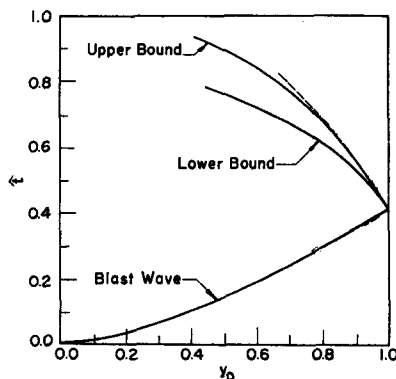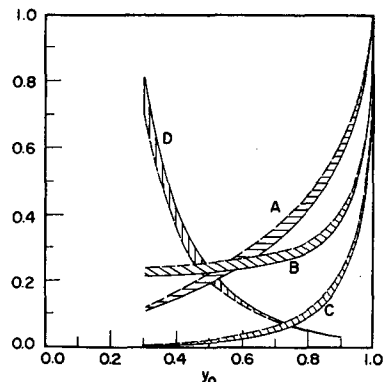

FIG. 3. Upper and lower bound integral curves of the reflected shock for $n = 1$ and $\gamma = 1.4$. Dashed curve is the parabolic approximation of the upper bound of the reflected shock.

FIG. 4. Flow variables ahead of the reflected shock for $n = 1$ and $\gamma = 1.4$. The ordinates are: (A) u_I/u_{I0} where $u_{I0} = 0.556 \ (K_I^3 E_0/\rho_0 r_0)^{\frac{1}{2}}$, (B) p_I/p_{I0} where $p_{I0} = 0.371 \ K_I^3 E_0/r_0$, (C) ρ_I/ρ_0 where $\rho_{I0} = 6\rho_0$, (D) a_I^2/a_{I0}^2 where $a_{I0}^2 = 0.259 \ K_I^3 E_0/\rho_0 r_0$. Solid and dashed curves refer to the upper and lower bound integral curves of the reflected shock, respectively.

It is seen from Fig. 4 that the pressure gradient is indeed very large near the wall and decays rapidly as the reflected shock propagates inward. This effect was assumed from the onset in the derivation of the general theory.

The remaining task in this section is to present a method of calculating the flow variables just behind the reflected shock for the integral curve of the upper bound. This may be accomplished by applying the well-known Rankine–Hugoniot relations which state that

$$\frac{1}{\xi} = \zeta = \frac{\gamma - 1 + (\gamma + 1)\eta}{\gamma + 1 + (\gamma - 1)\eta}, \qquad (25)$$

where $\xi = (u_{\mathrm{I}} - \bar{U})/(u_{\mathrm{II}} - \bar{U})$ is the ratio of the relative flow velocity just ahead of the reflected shock to the flow velocity just behind it, $\eta = p_{\mathrm{I}}/p_{\mathrm{II}}$ is the ratio of the fluid pressure just ahead of the reflected shock to the fluid pressure just behind it, and $\zeta = \rho_{\mathrm{I}}/\rho_{\mathrm{II}}$ is the ratio of the fluid density just ahead of the reflected shock to the fluid density just behind it.

From the assumption of $u_{\mathrm{II}} = 0$, Eq. (25), and the definitions of ξ, η, ζ, it follows that

$$\rho_{\mathrm{II}} = -\frac{u_{\mathrm{I}} - \bar{U}}{\bar{U}}\, \rho_{\mathrm{I}},$$

$$p_{\mathrm{II}} = -\frac{(\gamma + 1)(u_{\mathrm{I}} - \bar{U}) + (\gamma - 1)\bar{U}}{(\gamma + 1)\bar{U} - (\gamma - 1)(u_{\mathrm{I}} - \bar{U})}\, p_{\mathrm{I}}. \qquad (26)$$

Equations (26) completely determine the values of the flow properties just behind the upper bound of the reflected shock in terms of the flow properties u_{I}, p_{I}, ρ_{I} and the propagation speed of the reflected shock $\bar{U}$. As mentioned earlier, these calculated values form bounds of the values of the flow variables behind the actual reflected shock. In fact, the calculated values of p_{II}, ρ_{II} and the assumed value of $u_{\mathrm{II}} = 0$ should be the lower bounds of their actual values.

For the upper bound of the reflected shock of a planar blast wave against a rigid boundary with $\gamma = 1.4$, the values of p_{II}, ρ_{II}, a_{II}^2, and η, ζ are calculated. These values are plotted in dimensionless form along the position of the reflected shock in Figs. 5–7.

It is seen that the rate of change of the pressure gradient as well as of η and ζ decrease rapidly along the reflected shock. These are the results anticipated in the general discussion.

V. LOWER BOUND OF THE REFLECTED SHOCK

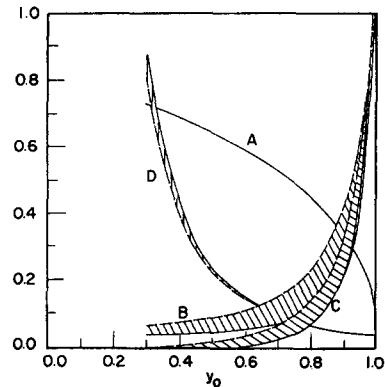In the previous section, an upper bound of the reflected shock was obtained by assuming the flow

FIG. 5. Flow variables behind the reflected shock for $n = 1$ and $\gamma = 1.4$. The ordinates are: (A) $u_{\mathrm{II}}/u_{\mathrm{I}0}$ where $u_{\mathrm{I}0} = 0.556(K_1{}^3 E_0/\rho_0 r_0)^{\frac{1}{2}}$, (B) $p_{\mathrm{II}}/p_{\mathrm{I}0}$ where $p_{\mathrm{I}0} = 0.371\, K_1{}^3 E_0/r_0$, curves start at 8.0 for $y_0 = 1$, (C) $\rho_{\mathrm{II}}/\rho_{\mathrm{I}0}$ where $\rho_{\mathrm{I}0} = 6\rho_0$, curves start at 3.5 for $y_0 = 1$, (D) $a_{\mathrm{II}}{}^2/a_{\mathrm{I}0}{}^2$ where $a_{\mathrm{I}0}{}^2 = 0.259\, K_1{}^3 E_0/\rho_0 r_0$. Solid and dashed curves refer to the upper and lower bound integral curves of the reflected shock, respectively.
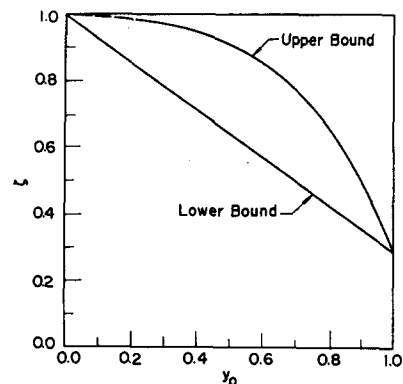

FIG. 6. Variation of the pressure ratio across the reflected shock corresponding to the upper and lower bound integral curves for $n = 1$ and $\gamma = 1.4$.
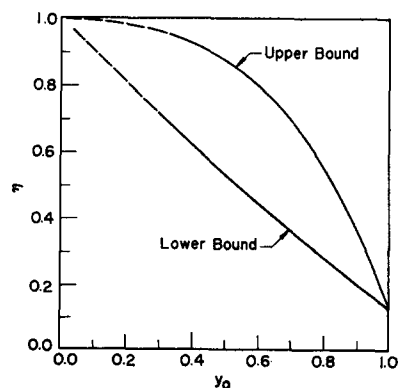

FIG. 7. Variation of the density ratio across the reflected shock corresponding to the upper and lower bound integral curves for $n = 1$ and $\gamma = 1.4$.

velocity behind the reflected shock to vanish. It is the purpose of this section to present a method which yields a reasonable lower bound of the same reflected shock.

If the assumption of negligible after-flow ($u_{II} = 0$) behind the reflected shock is correct, the pressure gradient behind it should also be equal to zero. Due to the large rate of change of the pressure gradient ahead of the reflected shock in the vicinity of the rigid boundary, there will exist correspondingly a large rate of change of the pressure gradient behind it in the same neighborhood. This means that the assumption of negligible after-flow can at most be a reasonable upper bound of the reflected shock. There must exist a backward flow u_{II} behind the reflected shock in an effort to balance the difference of pressure p_{II} in the same region.

There appear to be at least four approaches for the determination of the lower bound of the reflected shock:

(i) It appears reasonable that a lower bound of the reflected shock may be obtained if the fluid pressure p_{II} is assumed to take on the same value as that at the point of reflection. Due to the fast rate of decrease of pressure ahead of the reflected shock, however, this assumption is almost equivalent to the assumption of "strong" shock waves as the reflection propagates inward.

(ii) A more realistic assumption is to consider a constant pressure behind the reflected shock taken as the average of the value of the pressure p_{II} near the rigid boundary and its value far away from the rigid boundary. Since at distances far away from the rigid boundary, the pressure ratio across the shock is nearly unity,[6] it is possible to obtain this average value $\langle p_{II} \rangle$ as follows:

$$\langle p_{II} \rangle = \tfrac{1}{2}(p_{II0} + \lim_{y_0 \to 0} p_I). \quad (27)$$

Such an assumption may fit the actual reflected shock reasonably well, but it bounds the actual shock from below at far distances away from the rigid boundary, and from above near the point where the reflection originates.

(iii) Due to the fast rate of decay of the pressure gradient ahead of the reflected shock as the shock propagates inward, the rate of change of this gradient should also be of the same trend behind it. Thus, a lower bound can be obtained by assuming the decay rate of the pressure gradient behind the reflected shock to vanish, i.e.,

$$d^2 p_{II}/d\bar{R}^2 = 0 \quad \text{or} \quad d^2 p_{II}/dy_0^2 = 0. \quad (28)$$

(iv) Although (iii) is a plausible assumption for a lower bound, a still less generous lower bound may be deduced. Again, due to the fast decay rate of the pressure gradient, the pressure ratio η or density ratio ζ must increase faster near the rigid boundary and this rate must decrease as the reflected shock is propagated inward. The reasonable criterion for a lower bound is then given by a linear variation of the pressure or density ratio with respect to the shock radius R or y_0, i.e.,

$$d^2\eta/dy_0^2 = 0, \quad \text{or} \quad d^2\zeta/dy_0^2 = 0. \quad (29)$$

Since the strength of the reflected shock does not approach that of a "strong" shock, the criterion based on the pressure ratio η and density ratio ζ differ only slightly. In this paper, the criterion based on the density ratio ζ is presented. Therefore, for the location of the integral curve of the lower bound of the reflected shock, the criterion is

$$d\zeta/dy_0 = \zeta_0 - 1 = \text{const},$$
or
$$\zeta = 1 + (\zeta_0 - 1)y_0. \quad (30)$$

The above considerations of the lower bounds of the reflected shock and other approximations are shown in Fig. 2.

From Eq. (30) and the Rankine–Hugoniot relation, the flow velocity u_{II} behind the reflected shock corresponding to this lower bound may be obtained.

$$u_{II} = \bar{U} + [1 + (\zeta_0 - 1)y_0](u_I - \bar{U}). \quad (31)$$

Due to the dependence of u_{II} on $\bar{U}$, the more convenient expression for the determination of the speed of the reflected shock is obtained from Eq. (9). After slight rearrangement, this becomes

$$\bar{U} = \bar{D}\bar{R}/Dt$$
$$= u_I - \left\{ \frac{2a_I^2}{(\gamma + 1)[1 + (\zeta_0 - 1)y_0] - (\gamma - 1)} \right\}^{\frac{1}{2}}. \quad (32)$$

For each value of y_0 and the corresponding time t, the values of u_I and a_I as well as the entire integral curve of the lower bound may be obtained by applying Eq. (32), the initial conditions at the point where the reflection originates, and the results of the self-similar solution for the blast wave ahead of the reflected shock. The procedure is quite similar to that outlined for the upper bound integral curve of the reflected shock given in the previous section provided Eq. (32) is used in place of Eq. (10). The corresponding values of u_{II}, p_{II}, ρ_{II}, a_{II}^2, η, and ζ may be obtained from the Rankine–Hugoniot relations,

Eqs. (25), and the basic definitions of the pressure and density ratios across the reflected shock.

The numerical results for $n = 1$ and $\gamma = 1.4$ are plotted in Figs. 5–7 where the results of the upper bound of the same reflection were presented. The values of the flow variables u_{II}, p_{II}, ρ_{II} behind lower bound of the reflected shock form upper bounds of the actual values. The curve of ζ vs. y_0 is linear due to the basic assumption of the lower bound. The curve of η vs. y_0 is also very close to a straight line indicating that the criteria for linear variation of the pressure ratio and density ratio across the reflected shock are almost equivalent. It is expected that the same conclusion may be reached for the cases of cylindrically and spherically symmetric reflection of strong blast waves.

VI. TAYLOR SERIES EXPANSION OF THE REFLECTED SHOCK

In the preceeding sections, methods of obtaining the upper and lower bounds of the position $\bar{R}(t)$ of the reflected shock were presented. As is shown in this section, there exists a simple and effective procedure for the determination of an analytical expression of the position of the reflected shock by means of a Taylor series expansion in powers of elapsed time from the instant of reflection of the form:

$$\bar{R}(t) = r_0 + \left(\frac{\bar{D}\bar{R}}{Dt}\right)_0 (t - t_0) + \frac{1}{2!}\left(\frac{\bar{D}^2\bar{R}}{Dt^2}\right)_0 (t - t_0)^2$$
$$+ \frac{1}{3!}\left(\frac{\bar{D}^3\bar{R}}{Dt^3}\right)_0 (t - t_0)^3 + \cdots . \quad (33)$$

The procedure of finding the first derivative $(\bar{D}\bar{R}/Dt)_0$ is quite straightforward. To find the second derivative $(\bar{D}^2\bar{R}/Dt^2)_0$ of the reflected shock, we use Eq. (14) with the lower sign to insure negative propagation speeds. Near the rigid boundary where the reflection originates, the flow velocity immediately behind the reflected shock must vanish, i.e., $u_{II0} = 0$. To calculate the second derivative near the rigid boundary, however, it also is necessary to determine the contribution due to the change of u_{II} along the reflected shock. If the contribution is left within the expression of $(\bar{D}^2\bar{R}/Dt^2)_0$ as a parameter, it may be later adjusted such that the flow field behind the reflected shock along the rigid wall will vanish at any subsequent time.[7] Or, if the analytical expression (in terms of the Taylor series expansion) is needed to approximate the upper or lower bounds of the reflected shock, then the assumption used for

the upper or lower bound may be used to determine the value of the rate of change of u_{II} along the reflected shock near the point of reflection.

To demonstrate the usefulness of the Taylor series expansion technique, the assumption of the upper bound is used in this discussion. Setting the values of u_{II} and its rate of change along $\bar{R}(t)$ equal to zero[8] in Eq. (14) and using the lower signs, we obtain

$$\left(\frac{\bar{D}^2\bar{R}}{Dt^2}\right)_0 = \frac{3 - \gamma}{4}\left(\frac{Du_I}{Dt}\right)_0 - \left[\left(\frac{\gamma + 1}{2}\right)^2 u_{I0}^2 + 4a_{I0}^2\right]^{-\frac{1}{2}}$$
$$\cdot \left[\frac{(\gamma + 1)^2}{8} u_{I0}\left(\frac{Du_I}{Dt}\right)_0 + \left(\frac{Da_I^2}{Dt}\right)_0\right]. \quad (34)$$

It is now necessary to calculate the values of u_I, a_i^2, and their time derivatives along the reflected shock near the rigid boundary. At the rigid boundary, we have $R = \bar{R} = r_0$ and $y = 1$. Thus, from Eqs. (2) and (15), we obtain

$$u_{I0} = \frac{4}{(2 + n)(\gamma + 1)}\frac{K_n^{2+n}E_0}{\rho_0 r_0^n},$$

$$p_{I0} = \frac{8}{(2 + n)^2(\gamma + 1)}\frac{K_n^{2+n}E_0}{r_0^n},$$

$$\rho_{I0} = \frac{\gamma + 1}{\gamma - 1}\rho_0, \quad (35)$$

$$a_{I0}^2 = \frac{8\gamma(\gamma - 1)}{(2 + n)^2(\gamma + 1)^2}\frac{K_n^{2+n}E_0}{\rho_0 r_0^n},$$

where we have used the fact that $f(1), g(1), h(1) = 1$.

To find the time derivative of u_I along $\bar{R}(t)$, we apply the operator $(\partial/\partial t + \bar{U}\,\partial/\partial r)$ to the first expression in Eqs. (15). Thus,

$$\left(\frac{\bar{D}u_I}{Dt}\right)_0 = \left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial r}\right)2Uf(y)$$
$$= \frac{2}{\gamma + 1}\left[\frac{DU}{Dt} - \frac{U(U - \bar{U})}{r_0}f'(1)\right]_0, \quad (36)$$

where the value of $f'(1)$ may be evaluated from Eqs. (4).

It is still necessary to find U, $\bar{U}$, DU/Dt near the rigid boundary before Eq. (36) can be evaluated. The expressions for U_0 and $(DU/Dt)_0$ may be obtained by Eq. (2). They may be reduced to the following forms:

$$U_0 = \frac{2}{2 + n}\left(\frac{K_n^{2+n}E_0}{\rho_0 r_0^n}\right)^{\frac{1}{2}},$$

$$\left(\frac{DU}{Dt}\right)_0 = -\frac{2n}{(2 + n)^2}\frac{K_n^{2+n}E_0}{\rho_0 r_0^{1+n}}. \quad (37)$$

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[7] It is the authors' intention to describe the exact expressions of the characteristics of the blast wave and the derivatives of u_{II} near the rigid boundary in a subsequent paper.

[8] This assumption is less restrictive than that for the complete integral curve of the upper bound where the derivatives of u_{II} of any order are set equal to zero everywhere along the reflected shock.

To evaluate $\bar{U}_0$, Eqs. (10), and (35) may be applied. Thus, we find that

$$\bar{U}_0 = -\frac{4(\gamma - 1)}{(2 + n)(\gamma + 1)}\left(\frac{K_n^{2+n}E_0}{\rho_0 r_0^n}\right)^{\frac{1}{2}}. \quad (38)$$

Combining Eqs. (36)–(38), the expression for $(\bar{D}u_I/Dt)_0$ is finally obtained.

$$\left(\frac{\bar{D}u_I}{Dt}\right)_0 = -\frac{4n(\gamma + 1) + 8(3\gamma - 1)f'(1)}{(2 + n)^2(\gamma + 1)^2}\frac{K_n^{2+n}E_0}{\rho_0 r_0^{1+n}}. \quad (39)$$

To find the time derivative of a_I^2 along the reflected shock near the point of reflection, we apply the operator $(\partial/\partial t + \bar{U}\,\partial/\partial r)$ to the expression given in Eq. (16). After combining the result with Eqs. (37) and (38), and considerable rearrangement, we find that

$$\left(\frac{\bar{D}a_I^2}{Dt}\right)_0 = \frac{16\gamma(1 - \gamma)}{(2 + n)^3(\gamma + 1)^3}\,\{n(\gamma + 1)$$

$$+ (3\gamma - 1)\,[g'(1) - h'(1)]\}\left(\frac{K_n^{2+n}E_0}{\rho_0 r_0^{1+2n/3}}\right)^{\frac{1}{2}}, \quad (40)$$

where $g'(1)$ and $h'(1)$ may be calculated from Eqs. (4).

Substituting the results given in Eqs. (39) and (40) into the expression given by Eq. (34), we obtain

$$\left(\frac{\bar{D}^2\bar{R}}{Dt^2}\right)_0 = \frac{\gamma - 1}{(2 + n)^2(\gamma + 1)^2(3\gamma - 1)}$$

$$\cdot\{(\gamma - 1)[4n(\gamma + 1) + 8(3\gamma - 1)f'(1)]$$

$$+ 8\gamma[n(\gamma + 1) + (3\gamma - 1)(g'(1) - h'(1))]\}\frac{K_n^{2+n}E_0}{\rho_0 r_0^{1+n}}. \quad (41)$$

This type of analysis may be carried over for the determination of the higher order derivatives of $\bar{R}(t)$ at $r = r_0$ as well. To find $(\bar{D}^3\bar{R}/Dt^3)_0$, etc., it is necessary to differentiate the general expression for $(\bar{D}^2\bar{R}/Dt^2)$ along the reflected shock. In the resulting expression, one encounters higher order derivatives of u_{II} along $\bar{R}(t)$ near the point of reflection. These are parameters which have to be determined by the flow field condition, $u_{20} = 0$.

To estimate the value of $(\bar{D}^3\bar{R}/Dt^3)_0$ for the upper bound expansion, we may neglect all the terms pertaining to u_{II} in Eq. (14) before carrying out the differentiation. In the process of the evaluation of this expression, we need to find out the values of $(\bar{D}^2u_I/Dt^2)_0$, $(\bar{D}^2a_I^2/Dt^2)_0$, and $(\bar{D}^3R/Dt^3)_0$. This means that among other things, it is necessary to

obtain from Eqs. (4), the expressions of $f''(1)$, $g''(1)$, and $h''(1)$. This can be done in a rather straightforward manner.

With the derivatives, $(\bar{D}\bar{R}/Dt)_0$, $(\bar{D}^2\bar{R}/Dt^2)_0$, etc., calculated in terms of the parameters $(\bar{D}u_{II}/Dt)_0$, $(\bar{D}^2u_{II}/Dt^2)_0$, etc., it is a simple matter to write down the leading terms of the Taylor series expansion given in Eq. (33). With the position $\bar{R}(t)$ of the reflected shock given in an analytical form, the flow variables behind the reflected shock may be calculated. These form the given conditions along the initial curve $\bar{R}(t)$. The flow field behind the reflected shock now may be calculated. The parameters, $(\bar{D}u_{II}/Dt)_0$, $(\bar{D}^2u_{II}/Dt^2)_0$, etc., are determined by the condition, $u_{20} = 0$ along the rigid boundary.

If the upper, or lower bound, or any other assumption is made pertaining to the derivatives of u_{II} near the rigid boundary, the values $(\bar{D}\bar{R}/Dt)_0$, $(\bar{D}^2\bar{R}/Dt^2)_0$, etc., may be estimated without the calculations of the flow field behind the reflected shock. For the Taylor series expansion of the upper bound of the reflected shock, this becomes in dimensionless form,

$$y_0(\hat{t}) = 1 - \frac{4(\gamma - 1)}{(2 + n)(\gamma + 1)}\,(\hat{t} - 1)$$

$$+ \frac{(\gamma - 1)}{2(2 + n)^2(\gamma + 1)^2(3\gamma - 1)}$$

$$\cdot\{(\gamma - 1)[4n(\gamma + 1) + 8(3\gamma - 1)f'(1)]$$

$$+ 8\gamma[n(\gamma + 1) + (3\gamma - 1)$$

$$\cdot(g'(1) - h'(1))]\}(\hat{t} - 1)^2 + \cdots, \quad (42)$$

where $\hat{t} = t/t_0$. Equation (40) is the integral of Eq. (9) or (10) for the reflected shock while assuming $u_{II} = 0$. As an approximation, this series may be cutoff at the term $(\hat{t} - 1)^2$. Such an approximation is *parabolic* in the $y_0 - \hat{t}$ plane. For the reflection of a planar wave with $\gamma = 1.4$, the parabolic approximation of the upper bound of the reflected shock is presented in Fig. 3. The agreement with the upper bound integral curve is quite good. Due to the decrease of the curvature of the parabolic approximation away from the wall, the parabola begins to deviate from the integral curve at large distances away from the wall.

For other choices of the value of the rate of change of the flow velocity u_{II}, other series expansions may be obtained. The exact expression is the one which meets the condition of $u_{20} = 0$. It is expected that such an expansion corresponds to the variation of u_{II} somewhere between the limiting conditions set by the upper and lower bounds.