Representation of the five-dimensional harmonic oscillator with scalar-valued U(5) ⊃ SO(5) ⊃ SO(3)–coupled VCS wave functions

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Vector coherent state methods, which reduce the U(5) ⊃ SO(5) ⊃ SO(3) subgroup chain, are used to construct basis states for the five-dimensional harmonic oscillator. Algorithms are given to calculate matrix elements in this basis. The essential step is the construction of SO(5) ⊃ SO(3) irreps of type $[v,0]$. The methodology is similar to that used in two recent papers except that one-dimensional, as opposed to multidimensional, vector-valued wave functions are used to give conceptually simpler results. Another significant advance is a canonical resolution of the SO(5) ⊃ SO(3) multiplicity problem. © *1995 American Institute of Physics.*

I. INTRODUCTION

Two recent papers[1] (referred to herein as I and II), showed how VCS (Vector-Coherent-State) theory can be used to construct SO(3)-coupled basis states and calculate matrix elements for the five- and six-dimensional harmonic oscillators. The central result of these papers was the construction of SO(5) ⊃ SO(3) irreps of the types $[v,0]$ and $[v,v]$. Two subsequent advances make it possible to simplify the results of these papers and construct the generic representations of SO(5) of type $[v_1,v_2]$.

The first advance is a different choice of Cartan subalgebra for SO(5). With the new choice, the VCS irreps of type $[v,0]$ and $[v,v]$ are carried by one-dimensional, as opposed to multidimensional, vector-valued wave functions. As a result, the construction of these SO(5) irreps becomes of the same degree of complexity as for generic (λ,μ) irreps of SU(3) ⊃ SO(3). In this paper we focus on the representations of type $[v,0]$ needed in the solution of five-dimensional harmonic oscillator problems. The generic $[v_1,v_2]$ SO(5) irreps will be treated in a following paper.

The second advance is a development in K-matrix theory[2] which simplifies its application and yields a canonical orthonormal SO(5) ⊃ SO(3) basis, relative to which the VCS irreps are unitary.

A large body of literature has been addressed to the U(5) ⊃ SO(5) ⊃ SO(3) problem (cf., for example, Refs. 3,4 and many references quoted in I and II). Moreover, computer programs have been developed to calculate the matrix elements for practical applications.[5] The problem is of fundamental interest because of the occurrence of a non-trivial multiplicity. The problem was resolved in the early literature[4] by ingenious *ad hoc* devices, such as the use of "traceless bosons" and "elementary permissible diagrams." The present paper (together with papers I and II) brings a completely new approach to the problem. Among its achievements are the following: (i) Representations are constructed by systematic (albeit relatively new) vector coherent state methods. These methods were used in Ref. 6 to construct SU(3) ⊃ SO(3) representations and in Ref. 7 to construct SU(4) ⊃ SO(4) ~ SU(2)×SU(2) representations. (ii) The SO(5) ⊃ SO(3) multiplicity problem is given a canonical resolution. (iii) Exact, as opposed to numerical, results are obtained for all multiplicity-free cases; a simple computer algorithm gives matrix elements for other cases. (iv) The approach admits an extension (to be given in a following paper) to the generic SO(5) irreps of type $[v_1,v_2]$ to which the previous methods did not apply.

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II. BASIS STATES

Our first objective is the construction of orthonormal basis states $\{|nv\alpha LM\rangle\}$ for the five-dimensional harmonic oscillator that reduce the subgroup chain

$$\begin{array}{ccccccc} \mathrm{U}(5) & \supset & \mathrm{SO}(5) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2) \ , \\ N & & v & \alpha & L & & M \end{array} \tag{1}$$

where the quantum numbers $(NvLM)$ are representation labels, $N=2n+v$, and α indexes the multiplicity of SO(3) irreps in the SO(5) irrep $[v,0]$. To construct such a basis, we use the duality[8] between U(5) $\supset$ SO(5) and U(1) $\subset$ SU(1,1) to replace the subgroup chain of Eq. (1) by the equivalent chain

$$\begin{array}{ccccc} \mathrm{SU}(1,1)\times\mathrm{SO}(5) & \supset & \mathrm{U}(1)\times\mathrm{SO}(3) & \supset & \mathrm{SO}(2) \ . \\ v \qquad\qquad \alpha & & n \quad L & & M \end{array} \tag{2}$$

The infinitesimal generators of the groups appearing in these chains are defined as follows. First, raising and lowering operators of the 5-dimensional harmonic oscillator are defined such that

$$(d^m)^\dagger = d_m^\dagger, \quad d^m = (-1)^m d_{-m} \tag{3}$$

and

$$[d^m, d_n^\dagger] = \delta_{mn} . \tag{4}$$

The operators,

$$C_{mn} = d_m^\dagger d^n, \tag{5}$$

are then infinitesimal generators of U(5) and span the u(5) Lie algebra. The so(5) Lie algebra is spanned by three components of angular momentum and seven components of an octupole tensor:

$$\begin{aligned} L_k &= \sqrt{10}(d^\dagger \times d)_{1k}, \quad k=0,\pm 1, \\ O_\nu &= \sqrt{10}(d^\dagger \times d)_{3\nu}, \quad \nu=0,\pm 1,\pm 2,\pm 3. \end{aligned} \tag{6}$$

The Lie algebra su(1,1) is spanned by operators

$$\begin{aligned} X_+ &= \frac{\sqrt{5}}{2}(d^\dagger \times d^\dagger)_0 = \frac{1}{2}\sum_m (-1)^m d_m^\dagger d_{-m}^\dagger, \\ X_0 &= \frac{\sqrt{5}}{4}[(d^\dagger \times d)_0 + (d \times d^\dagger)_0] = \frac{1}{4}\sum_m [d_m^\dagger d^m + d^m d_m^\dagger], \\ X_- &= \frac{\sqrt{5}}{2}(d \times d)_0 = \frac{1}{2}\sum_m (-1)^m d^m d^{-m}, \end{aligned} \tag{7}$$

which obey the commutation relations

$$[X_-, X_+] = 2X_0, \quad [X_0, X_\pm] = \pm X_\pm . \tag{8}$$

The U(1)⊂SU(1,1) subgroup has a single infinitesimal generator given by the SU(1,1) operator X_0. A U(5) irrep is labeled by the value N of the d-boson number operator

$$N_d = \sum_m d_m^\dagger d^m. \tag{9}$$

Our construction of a basis starts with a sequence of states $\{|v\rangle; v=0,1,2,...\}$ each of which is simultaneously a lowest weight state for an SU(1,1) irrep (v) and a highest weight state for an SO(5) irrep $[v,0]$. Each state $|v\rangle$ is then extended to a basis

$$|nv\rangle = \sqrt{\frac{(v+\frac{3}{2})!}{n!(n+v+\frac{3}{2})!}}(X_+)^n|v\rangle \tag{10}$$

for an SU(1,1) irrep with

$$\begin{aligned} X_+|nv\rangle &= \sqrt{(n+v+\tfrac{5}{2})(n+1)}\,|n+1,v\rangle, \\ X_-|nv\rangle &= \sqrt{(n+v+\tfrac{3}{2})n}\,|n-1,v\rangle, \\ X_0|nv\rangle &= \tfrac{1}{2}(2n+v+\tfrac{5}{2})|nv\rangle. \end{aligned} \tag{11}$$

The states $\{|nv\rangle\}$ are identified with five-dimensional harmonic oscillator states as follows.

Let $\eta^\dagger$ denote the linear combination of d-boson raising operators which is of highest weight relative to an ordered basis for a Cartan subalgebra for so(5). Then the states

$$|v\rangle = \frac{1}{\sqrt{v!}}(\eta^\dagger)^v|0\rangle, \quad v=0,1,2,... \tag{12}$$

are SO(5) highest weight states.

A Cartan subalgebra for so(5) was chosen in I with basis elements $\{L_0, O_0\}$. For this choice, $\eta^\dagger$ is equal to $d_2^\dagger$. The state $|v\rangle$ is the state $|n=0, L=2v, M=2v\rangle$ and the states $|nv\rangle$ are the states $|n, L=2v, M=2v\rangle$. Here we choose a Cartan subalgebra which contains no component of the so(3) angular momentum. As a consequence, the highest weight state $|v\rangle$ becomes a state of mixed angular momentum. Moreover, one can show that a complete basis of states $\{|0v\alpha LM\rangle\}$ for an SO(5) irrep $[v,0]$ can be generated by projecting out states of good angular momentum from such a highest weight state. This property is particularly useful in constructing basis states which reduce the SO(3) ⊂ SO(5) subgroup. Among the infinite number of equivalent Cartan subalgebras having this property, we arbitrarily choose the one spanned by the operators

$$\begin{aligned} H_1 &= \frac{1}{2}(C_{2-1}+C_{1-2}+C_{-12}+C_{-21}+C_{22}-C_{-2-2}-C_{11}+C_{-1-1}), \\ H_2 &= \frac{1}{2}(C_{2-1}+C_{1-2}+C_{-12}+C_{-21}-C_{22}+C_{-2-2}+C_{11}-C_{-1-1}). \end{aligned} \tag{13}$$

Consider, the fundamental irrep of SO(5) spanned by the five $(n=0, v=1, L=2, M=m)$ states

$$|012m\rangle = d_m^\dagger|0\rangle, \quad m=0,\pm1,\pm2. \tag{14}$$

Diagonalizing the operators $[H_1,H_2]$ in this basis, yields the eigenstates and eigenvalues

$$\begin{aligned} &\frac{1}{\sqrt{2}}(d_2^\dagger+d_{-1}^\dagger)|0\rangle, \quad [N_1,N_2]=[1,0],\\ &\frac{1}{\sqrt{2}}(d_2^\dagger-d_{-1}^\dagger)|0\rangle, \quad [0,1],\\ &d_0^\dagger|0\rangle, \quad [0,0],\\ &\frac{1}{\sqrt{2}}(d_{-2}^\dagger+d_1^\dagger)|0\rangle, \quad [0,-1],\\ &\frac{1}{\sqrt{2}}(d_{-2}^\dagger-d_1^\dagger)|0\rangle, \quad [-1,0], \end{aligned} \tag{15}$$

where N_1 and N_2 are, respectively, the eigenvalues of H_1 and H_2. The pair $[N_1,N_2]$ is the *weight* of the corresponding state. Thus, the state $|1\rangle=\eta^\dagger|0\rangle$, with

$$\eta^\dagger=\frac{1}{\sqrt{2}}(d_2^\dagger+d_{-1}^\dagger) \tag{16}$$

is the highest weight state of the fundamental irrep and $[1,0]$ is the corresponding highest weight. Similarly, a state $|v\rangle$, defined by Eq. (12), is a highest weight state for an SO(5) irrep $[v,0]$. We also note that

$$X_-|v\rangle=0, \tag{17}$$

which implies that $|v\rangle$ is simultaneously a lowest weight state for su(1,1), as required. Since the operators of the su(1,1) and so(5) Lie algebras commute, the states $\{|nv\rangle\}$ are all highest weight states for equivalent so(5) $[v,0]$ irreps.

The next step is to extend each state $|nv\rangle$ to an orthonormal basis $\{|nv\alpha LM\}$ for an SO(5) irrep and complete the construction of an orthonormal basis for the five-dimensional harmonic oscillator. This is conveniently done in a VCS representation.

III. VCS WAVE FUNCTIONS

The states $|nv\alpha LM\rangle$, $|nv\rangle$ and $|v\rangle$ can be represented by (one-dimensional) vector-valued coherent-state wave functions

$$|nv\alpha LM\rangle\rightarrow\varphi_{nv\alpha LM}, \quad |nv\rangle\rightarrow\chi_{nv}, \quad |v\rangle\rightarrow\zeta_v, \tag{18}$$

defined by the expressions

$$\varphi_{nv\alpha LM}(\Omega)=|nv\rangle\langle nv|R(\Omega)|nv\alpha LM\rangle, \tag{19}$$

$$\chi_{nv}(z)=|v\rangle\langle v|e^{zX_-}|nv\rangle=|v\rangle\sqrt{\frac{(n+v+\frac{3}{2})!}{n!(v+\frac{3}{2})!}}z^n, \tag{20}$$

$$\zeta_v(x)=\langle 0|e^{x\eta}|v\rangle=\frac{1}{\sqrt{v!}}x^v, \tag{21}$$

in which $R(\Omega)$ is an SO(3) rotation, and x and z are complex variables. Now, if in Eq. (20) we replace the vector $|v\rangle$ by its coherent-state wave function ζ_v and, in Eq. (19), we replace the vector $|nv\rangle$ by its coherent-state wave function χ_{nv}, then the wave function $\varphi_{nv\alpha LM}$ for the state $|nv\alpha LM\rangle$ is replaced by the wave function $\Psi_{nv\alpha LM}$ with

$$\Psi_{nv\alpha LM}(x,z,\Omega)=\langle 0|e^{x\eta}|v\rangle\langle v|e^{zX_-}|nv\rangle\langle nv|R(\Omega)|nv\alpha LM\rangle$$
$$=\langle 0|e^{xa+zX_-}R(\Omega)|nv\alpha LM\rangle. \tag{22}$$

Thus, any five-dimensional harmonic oscillator state $|\Psi\rangle$ has coherent state wave function

$$\Psi(x,z,\Omega)=\langle 0|e^{x\eta+zX_-}R(\Omega)|\Psi\rangle. \tag{23}$$

In particular, the orthonormal basis states $\{|nv\alpha LM\rangle\}$ have coherent state wave functions $\{\Psi_{nv\alpha LM}\}$ with

$$\Psi_{nv\alpha LM}(x,z,\Omega)=\sqrt{\frac{(n+v+\frac{3}{2})!}{n!(v+\frac{3}{2})!}}z^n\psi^v_{\alpha LM}(x,\Omega) \tag{24}$$

and

$$\psi^v_{\alpha LM}(x,\Omega)=\langle 0|e^{x\eta}R(\Omega)|0v\alpha LM\rangle. \tag{25}$$

If we make the expansion

$$\psi^v_{\alpha LM}(x,\Omega)=\sum_K \langle 0|e^{x\eta}|0v\alpha LK\rangle\mathscr{D}^L_{KM}(\Omega), \tag{26}$$

we obtain

$$\psi^v_{\alpha LM}(x,\Omega)=\xi^v(x)\sum_K a^v_K(\alpha L)\mathscr{D}^L_{KM}(\Omega), \tag{27}$$

where

$$\xi^v(x)=\frac{1}{\sqrt{2^v v!}}x^v \tag{28}$$

and

$$a^v_K(\alpha L)=\frac{1}{\sqrt{v!}}\langle 0|(d^2+d^{-1})^v|0v\alpha LK\rangle. \tag{29}$$

For each value of v, the $n=0$ wave functions $\{\psi^v_{\alpha LM}\}$ are an orthonormal basis for an SO(5) irrep $[v,0]$. Thus, it remains to determine the $\{a^v_K(\alpha L)\}$ coefficients to have a complete orthonormal basis of wave functions for the five-dimensional harmonic oscillator.

In constructing a basis for an SO(5) irrep $[v,0]$, it is helpful to know what angular momentum values appear. Williams and Pursey[9] showed that the range of L values in the irrep $[v,0]$ is given by

$$L=2K,\ 2K-2,\ 2K-3,\ \ldots,\ K,$$

$$K=v,\ v-3,\ v-6,\ \ldots,\ \nu, \qquad (30)$$

$$\nu=0,1,\ \text{or}\ 2.$$

One sees from this that L has maximum value $L_{\max}=2v$. One also sees that L has minimum value $L_{\min}=0$ when v is a multiple of three ($v=3k$) and $L_{\min}=2$ otherwise ($v=3k+1$ or $3k+2$).

The construction of basis wave functions is facilitated by the following theorems:

Theorem 1: The SO(3)-coupled product of $[v_1,0]$ and $[v_2,0]$ wave functions $[\psi^{v_1}_{\alpha L_1}\times\psi^{v_2}_{\beta L_2}]_{LM}$ belongs to the SO(5) irrep $[v_1+v_2,0]$.

Proof: Wave functions of the $[1,0]$ irrep are given by

$$\psi^1_{2m}(x,\Omega)=\langle 0|e^{x\eta}R(\Omega)d^\dagger_m|0\rangle=\xi^1(x)[\mathscr{D}^2_{2m}(\Omega)+\mathscr{D}^2_{-1m}(\Omega)]. \qquad (31)$$

The state $d^\dagger_m|0v\alpha LM\rangle$ is a linear combination of states of type $|1,v-1,\beta L'M'\rangle$ and $|0,v+1,\beta L'M'\rangle$. However, the matrix elements $\langle 0|e^{x\eta}R(\Omega)|1,v-1,\beta L'M'\rangle$ are all zero. Consequently, only the $v+1$ component of the state $d^\dagger_m|0v\alpha LM\rangle$ contributes to the matrix element

$$\begin{aligned}&\langle 0|e^{x\eta}R(\Omega)d^\dagger_m|0v\alpha LM\rangle\\&=\langle 0|e^{x\eta}R(\Omega)d^\dagger_mR(\Omega^{-1})e^{-x\eta}e^{x\eta}R(\Omega)|0v\alpha LM\rangle\\&=\frac{1}{\sqrt{2}}x[\mathscr{D}^2_{2m}(\Omega)+\mathscr{D}^2_{-1m}(\Omega)]\langle 0|e^{x\eta}R(\Omega)|0v\alpha LM\rangle\\&=\psi^1_{2m}(x,\Omega)\psi^v_{\alpha LM}(x,\Omega).\end{aligned} \qquad (32)$$

Thus the product $\psi^1_{2m}\psi^v_{\alpha LM}$ is the VCS wave function for the $(v+1)$ component of the state $d^\dagger_m|0v\alpha LM\rangle$. Similarly, if $Z^v_{\alpha LM}(d^\dagger)$ is the creation operator for the state $|0v\alpha LM\rangle$, i.e.,

$$|0v\alpha LM\rangle=Z^v_{\alpha LM}(d^\dagger)|0\rangle, \qquad (33)$$

only the v_1+v_2 component of the state $Z^{v_1}_{\alpha L_1M_1}(d^\dagger)|0v_2\beta L_2M_2\rangle$ contributes to the matrix element

$$\langle 0|e^{x\eta}R(\Omega)Z^{v_1}_{\alpha L_1M_1}(d^\dagger)|0v_2\beta L_2M_2\rangle=\psi^{v_1}_{\alpha L_1M_1}(x,\Omega)\psi^{v_2}_{\beta L_2M_2}(x,\Omega). \qquad (34)$$

We conclude that the product $\psi^{v_1}_{\alpha L_1M_1}\psi^{v_2}_{\beta L_2M_2}$ is a wave function of the $[v_1+v_2,0]$ irrep and the SO(3)-coupled product $[\psi^{v_1}_{\alpha L_1}\times\psi^{v_2}_{\beta L_2}]_{LM}$ is a linear combination of $[v_1+v_2,0]$ basis wave functions of angular momentum LM of the form $\Sigma_\gamma c_\gamma\psi^{v_1+v_2}_{\gamma LM}$.

Theorem 2: The expansion coefficients $\{a^v_K(2v)\}$ for the multiplicity-free states of $L=L_{\max}=2v$ satisfy the recursion relation

$$\begin{aligned}a^v_K(2v)=&a^{v-1}_{K-2}(2v-2)(2v-2,K-2,2,2|2v,K)\\&+a^{v-1}_{K+1}(2v-2)(2v-2,K+1,2,-1|2v,K).\end{aligned} \qquad (35)$$

Proof: The harmonic oscillator states $|0v,L=2v,M=2v\rangle$ satisfy the relationship

$$|0v,L=2v,M=2v\rangle=\frac{1}{\sqrt{v+1}}d^\dagger_2|0v,L=2v-2,M=2v-2\rangle. \qquad (36)$$

The theorem then follows directly from Theorem 1 and the identity

TABLE I. Expansion coefficients $\{a_K^v(2v)\}$ for the multiplicity-free wave functions $\{\psi_{2v,M}^v\}$ of $L=L_{\max}=2v$ for $v=1,\ldots,8$. The K quantum number is expressed in terms of κ by $K=2v-3\kappa$. The $k(v,2v)$ coefficients are defined in Equation (82).

$v\backslash\kappa$	0	1	2	3	4	5	6	7	8	$k(v,2v)$
1	1	1								$\sqrt{2}$
2	1	$\sqrt{\frac{2}{7}}$	$\frac{2}{\sqrt{7}}$							$\sqrt{\frac{13}{7}}$
3	1	$\frac{3}{\sqrt{55}}$	$2\sqrt{\frac{3}{77}}$	$\frac{4}{\sqrt{55}}$						$2\sqrt{\frac{31}{77}}$
4	1	$\frac{2}{\sqrt{35}}$	$3\sqrt{\frac{8}{1001}}$	$\frac{8}{\sqrt{715}}$	$\frac{8}{\sqrt{455}}$					$\sqrt{\frac{7089}{5005}}$
5	1	$\sqrt{\frac{5}{57}}$	$\sqrt{\frac{40}{969}}$	$\sqrt{\frac{160}{4199}}$	$\sqrt{\frac{640}{12597}}$	$\frac{8}{\sqrt{969}}$				$\sqrt{\frac{16174}{12597}}$
6	1	$3\sqrt{\frac{2}{253}}$	$\frac{30}{\sqrt{33649}}$	$\frac{40}{\sqrt{81719}}$	$\frac{120}{\sqrt{676039}}$	$\frac{48}{\sqrt{81719}}$	$\frac{32}{\sqrt{33649}}$			$\sqrt{\frac{1272053}{1062347}}$
7	1	$\frac{1}{3}\sqrt{\frac{7}{13}}$	$2\sqrt{\frac{7}{1495}}$	$4\sqrt{\frac{7}{9867}}$	$\frac{16}{3}\sqrt{\frac{35}{96577}}$	$8\sqrt{\frac{7}{37145}}$	$\sqrt{\frac{14336}{937365}}$	$\frac{1}{3}\sqrt{\frac{2048}{16445}}$		$\frac{1}{3}\sqrt{\frac{54565792}{5311735}}$
8	1	$\sqrt{\frac{8}{155}}$	$\frac{4}{3}\sqrt{\frac{7}{899}}$	$\frac{7}{5}\sqrt{\frac{128}{35061}}$	$\sqrt{\frac{4480}{806403}}$	$\frac{7}{3}\sqrt{\frac{2048}{1964315}}$	$\frac{448}{5\sqrt{1178589}}$	$\sqrt{\frac{32768}{4032015}}$	$\frac{128}{15\sqrt{11687}}$	$\frac{1}{5}\sqrt{\frac{140292947}{5107219}}$

$$\sum_{M_1 m}(LM_1,\ 2m|L+2,M)[\mathscr{D}^L_{K_1M_1}\times\mathscr{D}^2_{km}]=(LK_1,\ 2k|L+2,K_1+k)\mathscr{D}^{L+2}_{K_1+k,M}. \tag{37}$$

Table I gives the $\{a_K^v(2v)\}$ coefficients for the $L=2v$ wave functions $\{\psi^v_{2v,M}\}$ for $v=1,\ldots,8$ obtained by means of this recursion relation.

In determining the $\{a_K^v(\alpha L)\}$ coefficients for other L values, it is useful to first construct an arbitrary linearly independent set. Transformation to the coefficients of an orthonormal set of wave functions can then be made subsequently such that the corresponding SO(5) irreps are unitary. Thus, we construct a set of $(n=0)$ basis wave functions

$$\varphi^v_{\alpha LM}=\xi^v\sum_K b_K^v(\alpha L)\mathscr{D}^L_{KM}, \tag{38}$$

that are orthonormal relative to the convenient (but arbitrary) inner product

$$(\varphi^v_{\alpha LM}|\varphi^{v'}_{\beta L'M'})=\delta_{vv'}\delta_{LL'}\delta_{MM'}\sum_K b_K^{v*}(\alpha L)b_K^v(\beta L). \tag{39}$$

We shall refer to this inner product as the *rotor inner product.*

Theorem 3: Basis states of lowest L $(L=L_{\min})$ of an SO(5) irrep $[v=3k+\nu,0]$ have wave functions given by

$$\varphi_0^{3k}=\xi^{3k}, \tag{40}$$

$$\varphi_{2M}^{3k+1}=\xi^{3k+1}\frac{1}{\sqrt{2}}[\mathscr{D}^2_{2M}+\mathscr{D}^2_{-1M}], \tag{41}$$

TABLE II. Expansion coefficients $\{b_K^v(\alpha L)\}$ multiplied by a factor $\mathcal{N}$, given in the last column, for $v=3k$ basis wave functions $\{\varphi_{\alpha LM}^v\}$.

$L_\alpha \backslash K$	9	6	3	0	-3	-6	-9	$\mathcal{N}$
0				1				1
3			2	$-\sqrt{5}$	-1			$\sqrt{10}$
4			$2\sqrt{5}$	$\sqrt{7}$	$-\sqrt{5}$			$\sqrt{32}$
6_1		$\sqrt{385}$	$3\sqrt{7}$	$2\sqrt{15}$	$4\sqrt{7}$			$2\sqrt{155}$
6_2		$30\sqrt{5}$	$-82\sqrt{11}$	$4\sqrt{1155}$	$-6\sqrt{11}$	$31\sqrt{5}$		$\sqrt{102145}$
7		$8\sqrt{11}$	$-\sqrt{104}$	$-\sqrt{273}$	$\sqrt{26}$	$2\sqrt{11}$		$\sqrt{1151}$
8		$8\sqrt{13}$	$\sqrt{280}$	$\sqrt{33}$	$-\sqrt{70}$	$2\sqrt{13}$		$\sqrt{1267}$
9_1	$4\sqrt{4641}$	$\sqrt{91}$	-37	$-2\sqrt{1155}$	-92	$-4\sqrt{91}$		$4\sqrt{5641}$
9_2	21748	$-28209\sqrt{51}$	$3535\sqrt{4641}$	$-866\sqrt{12155}$	$252\sqrt{4641}$	$16\sqrt{51}$	-45128	$\sqrt{110498074144}$
10_1		$2\sqrt{10659}$	$31\sqrt{11}$	$22\sqrt{11}$	$4\sqrt{195}$	$8\sqrt{11}$	$-16\sqrt{11}$	$\sqrt{65171}$

$$\varphi_{2M}^{3k+2}=\xi^{3k+2}\frac{1}{\sqrt{5}}[2\mathscr{D}_{1M}^{2}-\mathscr{D}_{-2M}^{2}]. \tag{42}$$

Proof: Eq. (40) follows directly from Eq. (25) and the fact that $\langle v|R(\Omega)|0vL=0,\ M=0\rangle$ is independent of Ω. Eqs. (41) and (42) follow from the first by successive application of Theorem 1.

Wave functions for states of other L values can be constructed by application of Theorem 1, using the wave functions of Theorems 2 and 3 as building blocks. Basis wave functions, expressed in terms of $\{b_K^v(\alpha L)\}$ coefficients which satisfy the orthogonality relation

$$\sum_K b_K^{v*}(\alpha L)b_K^v(\beta L)=\delta_{\alpha\beta} \tag{43}$$

are given in Tables II–IV for all $(n=0)$ states of $L\leqslant 9$. The $b_K^{3k+\nu}$ coefficients for these wave functions are tabulated separately for the $\nu=0$, 1, and 2, categories. It can be seen that the

TABLE III. Expansion coefficients $\{b_K^v(\alpha L)\}$ multiplied by a factor $\mathcal{N}$, given in the last column, for $v=3k+1$ basis wave functions $\{\varphi_{\alpha LM}^v\}$.

$L_\alpha \backslash K$	8	5	2	-1	-4	-7	$\mathcal{N}$
2			1	1			$\sqrt{2}$
4			8	$-\sqrt{8}$	$\sqrt{7}$		$\sqrt{79}$
5		$-\sqrt{40}$	$\sqrt{3}$	$\sqrt{21}$	2		$\sqrt{68}$
6		$2\sqrt{11}$	$\sqrt{15}$	$\sqrt{6}$	$-\sqrt{8}$		$\sqrt{73}$
7		32	$-8\sqrt{22}$	$2\sqrt{33}$	-1	$-\sqrt{91}$	$4\sqrt{166}$
8_1	$\sqrt{5005}$	$2\sqrt{143}$	$6\sqrt{10}$	$8\sqrt{7}$	$8\sqrt{11}$		$\sqrt{7089}$
8_2	11224	$-5840\sqrt{35}$	$-216\sqrt{2002}$	$542\sqrt{715}$	$-161\sqrt{455}$	7089	$2\sqrt{421292181}$
9	$8\sqrt{34}$	$-2\sqrt{5}$	$-5\sqrt{13}$	$-\sqrt{286}$	$\sqrt{56}$	8	$\sqrt{2927}$
10_1	$16\sqrt{663}$	$22\sqrt{130}$	145	$19\sqrt{3}$	$-26\sqrt{13}$	$4\sqrt{442}$	$\sqrt{270616}$

TABLE IV. Expansion coefficients $\{b_K^v(\alpha L)\}$ multiplied by a factor $\mathcal{N}$, given in the last column, for $v=3k+2$ basis wave functions $\{\varphi_{\alpha LM}^v\}$.

$L_\alpha \backslash K$	10	7	4	1	−2	−5	−8	$\mathcal{N}$
2				2	−1			$\sqrt{5}$
4			$\sqrt{7}$	$\sqrt{2}$	2			$\sqrt{13}$
5			8	$-2\sqrt{21}$	$\sqrt{3}$	$\sqrt{10}$		$\sqrt{161}$
6			$8\sqrt{2}$	$\sqrt{24}$	$-\sqrt{15}$	$\sqrt{11}$		$\sqrt{178}$
7		$2\sqrt{91}$	−1	$-\sqrt{33}$	$-\sqrt{88}$	−4		$\sqrt{502}$
8_1		$2\sqrt{5005}$	$23\sqrt{11}$	$21\sqrt{7}$	$10\sqrt{10}$	$-4\sqrt{143}$		$\sqrt{32214}$
8_2		$304\sqrt{91}$	$-3832\sqrt{5}$	$264\sqrt{385}$	$-248\sqrt{22}$	$20\sqrt{65}$	$-177\sqrt{91}$	$\sqrt{112893963}$
9		64	$-8\sqrt{14}$	$-2\sqrt{286}$	$5\sqrt{13}$	$-\sqrt{5}$	$-2\sqrt{34}$	$\sqrt{6602}$
10_1	$\sqrt{12597}$	$\sqrt{1105}$	$2\sqrt{130}$	$4\sqrt{30}$	$8\sqrt{10}$	$8\sqrt{13}$		$\sqrt{16174}$

b_K^{3k+v} coefficients are independent of k. This is useful because it means that a complete set of so(5) basis wave functions for a finite range of L values, and any v, is a relatively small set. The possibility of constructing a k-independent basis in this way follows from the fact that the b_K^v coefficients for the $L_{\min}$ (building block) wave functions are independent of k.

Consider, for example, the two $L=6$ states that occur in a generic $v=3k$ irrep. For $v=3$, there is only one $L=6$ state. Thus, we set

$$b_K^{3k}(\alpha=1,6)=b_K^3(6), \quad k\geqslant 1,$$
$$b_K^{3k}(\alpha=2,6)=b_K^6(2,6), \quad k\geqslant 2. \tag{44}$$

For $L=12$, there is a triplet of states and we set

$$b_K^{3k}(1,12)=b_K^6(12), \quad k\geqslant 2,$$
$$b_K^{3k}(2,12)=b_K^9(2,12), \quad k\geqslant 3, \tag{45}$$
$$b_K^{3k}(3,12)=b_K^{12}(3,12), \quad k\geqslant 4,$$

and so on.

It may also be noted from Eq. (29) that the $a_K^v(\alpha L)$ coefficients, and hence the $b_K^v(\alpha L)$ coefficients, vanish unless K lies in the range

$$-v\leqslant K\leqslant 2v. \tag{46}$$

As a consequence of this and the way the $b_K^v(\alpha L)$ coefficients are defined, one finds, for example, that $b_K^{3k}(1,6)$ is zero when $K=-6$ for any k. Similarly, $b_K^{3k}(1,12)$ is zero when $K=-9$ and -12 and $b_K^{3k}(2,12)$ is zero when $K=-12$, for any k. As we shall show in section VII, wave functions with this property are characterized by a pseudo quantum number equal to the effective number of *zero-coupled triplets*.[4]

In the following sections, a systematic construction of basis wave functions will be given that is well suited to computer calculations. The construction uses elements of the SO(5) Lie algebra to step states up in angular momentum from $L_{\min}$ or down from $L_{\max}$.

IV. VCS REPRESENTATIONS OF SO(5) ⊃ SO(3)

The action of any operator X in the so(5) Lie algebra on the VCS wave functions $\{\psi^{v}_{\alpha LM}\}$ of Eq. (25) is defined by the equation

$$[\Gamma(X)\psi^{v}_{\alpha LM}](x,\Omega)=\langle 0|e^{x\eta}R(\Omega)X|0v\,\alpha LM\rangle. \tag{47}$$

Thus, the angular momentum operators act in the standard way:

$$\Gamma(L_0)\psi^{v}_{\alpha LM}=M\psi^{v}_{\alpha LM},$$
$$\Gamma(L_\pm)\psi^{v}_{\alpha LM}=\sqrt{(L\mp M)(L\pm M+1)}\,\psi^{v}_{\alpha LM\pm 1}. \tag{48}$$

The action of the octupole operators is determined by first expanding

$$R(\Omega)O_\nu|0v\,\alpha LM\rangle=\sum_{\mu K} O_\mu|0v\,\alpha LK\rangle\mathscr{D}^{3}_{\mu\nu}(\Omega)\mathscr{D}^{L}_{KM}(\Omega). \tag{49}$$

We then have

$$[\Gamma(O_\nu)\psi^{v}_{\alpha LM}](x,\Omega)=\sum_{\mu K}\langle 0|e^{x\eta}O_\mu|0v\,\alpha LK\rangle\mathscr{D}^{3}_{\mu\nu}(\Omega)\mathscr{D}^{L}_{KM}(\Omega), \tag{50}$$

and

$$[\Gamma(O)\times\psi^{v}_{\alpha L}]_{L'M'}(x,\Omega)=\sum_{\mu KK'}\langle 0|e^{x\eta}O_\mu|0v\,\alpha LK\rangle(LK,\,3\mu|L'K')\mathscr{D}^{L'}_{K'M'}(\Omega). \tag{51}$$

The matrix elements

$$\langle 0|e^{x\eta}O_\mu|0v\,\alpha LK\rangle=\langle 0|(O_\mu+x[\eta,O_\mu]+\tfrac{1}{2}x^2[\eta,[\eta,O_\mu]]+\ldots)e^{x\eta}|0v\,\alpha LK\rangle \tag{52}$$

can be expressed in terms of similar matrix elements of the angular momentum operators as follows. From Eq. (6), one infers that

$$\langle 0|e^{x\eta}L_0|0v\,\alpha LK\rangle=\frac{x}{\sqrt{2}}\langle 0|(2d^2-d^{-1})e^{x\eta}|0v\,\alpha LK\rangle,$$
$$\langle 0|e^{x\eta}L_+|0v\,\alpha LK\rangle=\frac{x}{\sqrt{2}}\langle 0|2(d^1+d^{-2})e^{x\eta}|0v\,\alpha LK\rangle, \tag{53}$$
$$\langle 0|e^{x\eta}L_-|0v\,\alpha LK\rangle=\frac{x}{\sqrt{2}}\langle 0|\sqrt{6}d^0e^{x\eta}|0v\,\alpha LK\rangle.$$

We also have

$$v\langle 0|e^{x\eta}|0v\,\alpha LK\rangle=x\frac{d}{dx}\langle 0|e^{x\eta}|0v\,\alpha LK\rangle=\frac{x}{\sqrt{2}}\langle 0|(d^2+d^{-1})e^{x\eta}|0v\,\alpha LK\rangle. \tag{54}$$

Expressing the O_μ matrix elements in a similar way, one finds that

$$
\begin{aligned}
\langle 0|e^{x\eta}O_0|0v\,\alpha LK\rangle &= \langle 0|e^{x\eta}\frac{1}{3}(5v-L_0)|0v\,\alpha LK\rangle,\\
\langle 0|e^{x\eta}O_1|0v\,\alpha LK\rangle &= \langle 0|e^{x\eta}\left(-\frac{\sqrt{3}}{2}L_+\right)|0v\,\alpha LK\rangle,\\
\langle 0|e^{x\eta}O_{-1}|0v\,\alpha LK\rangle &= \langle 0|e^{x\eta}\left(-\frac{1}{\sqrt{3}}L_-\right)|0v\,\alpha LK\rangle,\\
\langle 0|e^{x\eta}O_2|0v\,\alpha LK\rangle &= \langle 0|e^{x\eta}\left(\sqrt{\frac{5}{6}}L_-\right)|0v\,\alpha LK\rangle,\\
\langle 0|e^{x\eta}O_{-2}|0v\,\alpha LK\rangle &= 0,\\
\langle 0|e^{x\eta}O_3|0v\,\alpha LK\rangle &= \langle 0|e^{x\eta}\frac{\sqrt{5}}{3}(-2v+L_0)|0v\,\alpha LK\rangle,\\
\langle 0|e^{x\eta}O_{-3}|0v\,\alpha LK\rangle &= \langle 0|e^{x\eta}\frac{\sqrt{5}}{3}(v+L_0)|0v\,\alpha LK\rangle.
\end{aligned}
\tag{55}
$$

Thus, for example,

$$
\begin{aligned}
\langle 0|e^{x\eta}O_0|0v\,\alpha LK\rangle &= \frac{1}{3}(5v-K)\xi^v(x)a_K^v(\alpha L),\\
\langle 0|e^{x\eta}O_1|0v\,\alpha LK\rangle &= -\frac{1}{2}\sqrt{3(L-K)(L+K+1)}\,\xi^v(x)a_{K+1}^v(\alpha L).
\end{aligned}
\tag{56}
$$

We then have

$$
[\Gamma(O)\times\psi_{\alpha L}^v]_{L'M'} = \xi^v(x)\sum_{K'K} M_{K'K}^{(L'L)}a_K^v(\alpha L)\mathscr{D}_{K'M'}^{L'} \tag{57}
$$

where the $M^{(L'L)}$ matrix elements are the simple linear combinations of SO(3) Clebsch–Gordan coefficients

$$
\begin{aligned}
M_{KK}^{(L'L)} =& \frac{1}{3}(5v-K)(LK,\ 30|L'K)\\
&-\frac{1}{2}\sqrt{3(L+K)(L-K+1)}(LK-1,\ 31|L'K)\\
&-\sqrt{\frac{1}{3}(L-K)(L+K+1)}(LK+1,\ 3-1|L'K),\\
M_{K+3,K}^{(L'L)} =& -\frac{\sqrt{5}}{3}(2v-K)(LK,\ 33|L'K+3)\\
&+\sqrt{\frac{5}{6}(L-K)(L+K+1)}(LK+1,\ 32|L'K+3),
\end{aligned}
\tag{58}
$$

$$M^{(L'L)}_{K-3,K}=\frac{\sqrt{5}}{3}(v+K)(LK,\ 3-3|L'K-3).$$

These results show that it is only a matter of matrix multiplication to compute the coefficients $\Sigma_K M^{(L'L)}_{K'K} a^v_K(\alpha L)$ for the wave function $[\Gamma(O)\times\psi^v_{\alpha L}]_{L'M'}$ from the coefficients $a^v_K(\alpha L)$ for the wave function $\psi^v_{\alpha L}$. Similarly, one can compute the coefficients $\Sigma_K M^{(L'L)}_{K'K} b^v_K(\alpha L)$ for the wave function $[\Gamma(O)\times\varphi^v_{\alpha L}]_{L'M'}$ from the coefficients $b^v_K(\alpha L)$ for the wave function $\varphi^v_{\alpha L}$. Thus, one can start from the coefficients $b^v_K(L_{\min})$ for the $L_{\min}$ wave functions, given by Theorem 3, or the $L_{\max}$ wave functions, given by Theorem 2, and generate a complete basis $\{\varphi^v_{\alpha LM}\}$, such as given in Tables II–IV. This construction is generally simpler to implement on a computer than the building up procedure of the previous section.

It is sometimes useful to have explicit expressions for the $\Gamma(O_\nu)$ operators. Such expressions are easily obtained in terms of left angular momentum operators $\{\bar{L}_k\}$, defined by the equation

$$[\bar{L}_k\psi^v_{\alpha LM}](x,\Omega)=\langle 0|e^{x\eta}L_kR(\Omega)|0v\,\alpha LM\rangle$$
$$=\sum_K \langle 0|e^{x\eta}L_k|0v\,\alpha LK\rangle\mathscr{D}^L_{KM}(\Omega). \tag{59}$$

Left angular momentum operators transform rotational wave functions according to the equations

$$\bar{L}_0\mathscr{D}^L_{KM}=K\mathscr{D}^L_{KM},$$
$$\bar{L}_\pm\mathscr{D}^L_{KM}=\sqrt{(L\pm K)(L\mp K+1)}\mathscr{D}^L_{K\mp 1M}. \tag{60}$$

Thus, they are the so-called *intrinsic* angular momentum operators of the rotor model. Using these operators, we obtain

$$\Gamma(O_\nu)=\frac{1}{3}\mathscr{D}^3_{0\nu}(5\hat{v}-\bar{L}_0)-\frac{\sqrt{3}}{2}\mathscr{D}^3_{1\nu}\bar{L}_+-\frac{1}{\sqrt{3}}\mathscr{D}^3_{-1\nu}\bar{L}_-$$
$$+\sqrt{\frac{5}{6}}\mathscr{D}^3_{2\nu}\bar{L}_--\frac{\sqrt{5}}{3}\mathscr{D}^3_{3\nu}(2\hat{v}-\bar{L}_0)+\frac{\sqrt{5}}{3}\mathscr{D}^3_{-3\nu}(\hat{v}+\bar{L}_0), \tag{61}$$

where $\hat{v}=x\partial/\partial x$.

V. SO(5) MATRIX ELEMENTS

An SO(5) irrep with highest weight $[v,0]$ is known to be unitary relative to an appropriate inner product when v is a positive integer (i.e., when the highest weight is dominant integral). It is also known that all irreps of type $[v,0]$ with integer values of v occur in the space of the five-dimensional harmonic oscillator and are unitary relative to the inner product of the harmonic oscillator Hilbert space. It follows that, if $\{|0v\,\alpha LM\rangle\}$ is an orthonormal set of $n=0$ harmonic oscillator states, the matrix elements of the so(5) octupole operators satisfy the hermiticity relationships

$$\langle 0v\beta L'||O||0v\,\alpha L\rangle=(-1)^{L-L'}\langle 0v\,\alpha L||O||0v\beta L'\rangle^* \tag{62}$$

characteristic of a unitary representation. Conversely, a set of states which satisfy these relationships for some positive integer value of v and give the standard matrix elements of the angular momentum operators (cf. Eq. (48)), is an orthonormal basis for an SO(5) irrep $[v,0]$.

By definition, the VCS wave functions, $\{\psi^v_{\alpha LM}\}$, corresponding to an orthonormal set of $n=0$ harmonic oscillator states, $\{|0v\,\alpha LM\rangle\}$, are orthonormal relative to the VCS inner product inherited from the harmonic oscillator; i.e.,

$$\langle\psi^v_{\alpha LM}|\psi^v_{\beta L'M'}\rangle=\langle 0v\,\alpha LM|0v\,\beta L'M'\rangle=\delta_{\alpha\beta}\delta_{LL'}\delta_{MM'}. \tag{63}$$

Similarly, matrix elements in the VCS representation are identical (by definition) to their harmonic oscillator counterparts. It follows that the VCS wave function $[\Gamma(O)\times\psi^v_{\alpha L}]_{L'M'}$ can be expanded, using the Wigner-Eckart theorem,

$$[\Gamma(O)\times\psi^v_{\alpha L}]_{L'M'}=\sum_\beta \psi^v_{\beta L'M'}\frac{\langle 0v\,\beta L'||O||0v\,\alpha L\rangle}{\sqrt{2L'+1}}. \tag{64}$$

Equivalently, in terms of the $a^v_K(\alpha L)$ coefficients,

$$\sum_K M^{(L'L)}_{K'K}a^v_K(\alpha L)=\frac{1}{\sqrt{2L'+1}}\sum_\beta a^v_{K'}(\beta L')\mathscr{M}^{(L'L)}_{\beta\alpha}, \tag{65}$$

where

$$\mathscr{M}^{(L'L)}_{\beta\alpha}=\langle 0v\,\beta L'||O||0v\,\alpha L\rangle. \tag{66}$$

To derive the $\langle 0v\,\beta L'||O||0v\,\alpha L\rangle$ matrix elements from this expression, we need the $\{a^v_K(\alpha L)\}$ coefficients. We know the coefficients $\{a^v_K(L=2v)\}$ for the states of $L=L_{\max}=2v$. The remaining coefficients must lead to matrix elements that satisfy the unitarity conditions of Eq. (62) and, hence, correspond to an orthonormal basis, $\{\psi^v_{\alpha LM}\}$, of VCS wave functions. Such coefficients are conveniently derived by K-matrix methods.[10,2]

As shown in sections III and IV, it is easy to construct an arbitrary basis of VCS wave functions. The wave functions $\{\varphi^v_{\alpha LM}\}$ given in Tables II–IV, for example, are orthonormal relative to a convenient provisional inner product, the so-called rotor inner product. We then seek a map

$$\varphi^v_{\alpha LM}\rightarrow\psi^v_{\alpha LM}=\mathscr{K}(vL)\varphi^v_{\alpha LM}=\sum_\beta \varphi^v_{\beta LM}\mathscr{K}_{\beta\alpha}(vL) \tag{67}$$

to an orthornormal VCS basis with coefficients

$$a^v_K(\alpha L)=\sum_\beta b^v_K(\beta L)\mathscr{K}_{\beta\alpha}(vL). \tag{68}$$

Now, if matrix elements in the $\{\varphi^v_{\alpha L}\}$ basis are defined, in parallel with Eq. (65), by the expansion

$$\sum_K M^{(L'L)}_{K'K}b^v_K(\alpha L)=\frac{1}{\sqrt{2L'+1}}\sum_\beta b^v_{K'}(\beta L')O^{(L'L)}_{\beta\alpha}, \tag{69}$$

then, with $b^v_K(\alpha L)$ coefficients satisfying Eq. (39), they are given by

$$O^{(L'L)}_{\beta\alpha}=\sqrt{2L'+1}\sum_{KK'} b^{v*}_{K'}(\beta L')M^{(L'L)}_{K'K}b^v_K(\alpha L). \tag{70}$$

TABLE V. $O^{(L'L)}_{\beta\alpha}$ matrix elements for $v=6$.

$L'_\beta \backslash L_\alpha$	0	3	4	6_1	6_2	7	8	9	10	12
0	0	7.216	0	0	0	0	0	0	0	0
3	−37.42	−3.240	−31.71	15.85	18.10	0	0	0	0	0
4	0	33.26	3.784	27.37	−17.38	23.84	0	0	0	0
6_1	0	−16.28	25.51	28.99	2.003	−17.15	29.78	36.27	0	0
6_2	0	−11.75	−9.619	2.986	−32.08	−33.71	−26.08	7.040	0	0
7	0	0	−13.24	9.152	30.99	−0.7391	−37.36	24.8	22.85	0
8	0	0	0	13.09	−18.14	26.99	10.11	14.63	47.93	0
9	0	0	0	−16.94	−4.511	19.95	−16.29	32.12	−32.07	49.15
10	0	0	0	0	0	−8.290	24.08	14.47	26.55	41.16
12	0	0	0	0	0	0	0	−11.10	20.60	38.17

For example, the $v=6$ irrep contains the range of angular momentum values $L=\{0,3,4,6_1,6_2,7,8,9,10,12\}$. With the corresponding $b^v_K(\alpha L)$ coefficients of Table II, Eq. (70) gives the O matrix elements listed in Table V.

Comparison of eqs. (65) and (69), shows that the desired $\langle 0v\beta L'||O||0v\alpha L\rangle$ matrix elements are given by

$$\mathscr{M}^{(L'L)}=\mathscr{K}^{-1}(L')O^{(L'L)}\mathscr{K}(L), \tag{71}$$

where it is understood that $\mathscr{K}(L)=\mathscr{K}(vL)$.

Thus, it remains to determine the $\mathscr{K}(L)$ matrices. This is especially simple when the initial basis states are already orthogonal (relative to the inner product inherited from the harmonic oscillator). For in such a situation, one has only to renormalize the basis states to obtain an orthonormal set and this is accomplished with diagonal $\mathscr{K}(L)$ matrices.

Orthogonal states arise automatically for representations that are multiplicity free; basis states are then classified by complete sets of quantum numbers and are orthogonal to one another. On the other hand, when there are multiplicities, orthogonal states can be generated from the eigenvectors of the $M^{(LL)}$ matrices.[2] For, if the vector $a^v(\alpha L)$ is an eigenvector of $M^{(LL)}$, then

$$\sum_K M^{(LL)}_{K'K}a^v_K(\alpha L)=a^v_{K'}(\alpha L)\frac{\langle 0v\alpha L||O||0v\alpha L\rangle}{\sqrt{2L+1}}, \tag{72}$$

and, by Eq. (65),

$$\mathscr{M}^{(LL)}_{\beta\alpha}=\delta_{\alpha\beta}\langle 0v\alpha L||O||0v\alpha L\rangle. \tag{73}$$

Thus, the hermiticity relationships of Eq. (62) are automatically satisfied within the subspace of states of angular momentum L.

We therefore start by diagonalizing the matrix $O^{(LL)}$ for each L for which there is a multiplicity; i.e., we find the similarity transformations

$$O^{(LL)}\rightarrow\tilde{\mathscr{M}}^{(LL)}=\tilde{\mathscr{K}}^{-1}(L)O^{(LL)}\tilde{\mathscr{K}}(L) \tag{74}$$

such that the matrices $\tilde{\mathscr{M}}^{(LL)}$ are diagonal. We then seek the additional normalization factors $\{k(\alpha L)\}$, for which

$$\mathscr{M}^{(L'L)}_{\beta\alpha}=k^{-1}(\beta L')\tilde{\mathscr{M}}^{(L'L)}_{\beta\alpha}k(\alpha L). \tag{75}$$

One sees that the unitarity condition, Eq. (62), is satisfied if we set

TABLE VI. $\tilde{\mathscr{M}}^{(L'L)}_{\beta\alpha}$ matrix elements for $v=6$.

$L' \backslash L$	0	3	4	6_1	6_2	7	8	9	10	12
0	0	7.216	0	0	0	0	0	0	0	0
3	−37.42	−3.240	−31.71	16.71	17.57	0	0	0	0	0
4	0	33.26	3.784	26.49	−18.26	23.84	0	0	0	0
6_1	0	−16.66	25.18	29.09	0	−18.24	28.91	36.49	0	0
6_2	0	−10.95	−10.85	0	−32.17	−32.83	−27.50	5.263	0	0
7	0	0	−13.24	10.65	30.67	−0.7391	−37.36	24.80	22.85	0
8	0	0	0	12.20	−18.56	26.99	10.11	14.63	47.93	0
9	0	0	0	−17.14	−3.955	19.95	−16.29	32.12	−32.07	49.15
10	0	0	0	0	0	−8.290	24.08	14.47	26.55	41.16
12	0	0	0	0	0	0	0	−11.10	20.60	38.17

$$\left|\frac{k(\alpha L)}{k(\beta L')}\right|^2 = (-1)^{L-L'} \frac{\tilde{\mathscr{M}}^{(LL')*}_{\alpha\beta}}{\tilde{\mathscr{M}}^{(L'L)}_{\beta\alpha}}. \tag{76}$$

We then have an orthonormal basis of VCS wave functions $\{\psi^{v}_{\alpha LM}\}$, given by Eq. (67) with

$$\mathscr{K}_{\beta\alpha}(L) = \tilde{\mathscr{K}}_{\beta\alpha}(L) k(\alpha L), \tag{77}$$

where it is noted that $\tilde{\mathscr{K}}(L)=1$ for a multiplicity-free value of L.

For example, for the $v=6$ irrep, for which the $L=6$ states have multiplicity two, one sees from Table V that

$$O^{(6,6)} = \begin{pmatrix} 28.99 & 2.003 \\ 2.986 & -32.08 \end{pmatrix}. \tag{78}$$

The $\tilde{\mathscr{K}}(6)$ matrix which brings $\tilde{\mathscr{M}}^{(6,6)} = \tilde{\mathscr{K}}^{-1}(6) O^{(6,6)} \tilde{\mathscr{K}}(6)$ to diagonal form is the matrix

$$\tilde{\mathscr{K}}(6) = \begin{pmatrix} 0.9988 & -0.03273 \\ 0.04875 & 0.9996 \end{pmatrix}. \tag{79}$$

For all other multiplicity-free L values, $\tilde{\mathscr{K}}(L \neq 6) = 1$. Making the transformation of Eq. (74), we obtain the $\tilde{\mathscr{M}}$ matrix given in Table VI, for $v=6$. The $|k(\alpha L)/k(\beta L')|^2$ ratios can now be read from Table VI. For example,

$$\left|\frac{k(3)}{k(0)}\right|^2 = \frac{37.42}{7.216} = 5.185, \qquad \left|\frac{k(6_1)}{k(4)}\right|^2 = \frac{25.18}{26.49} = 0.9505. \tag{80}$$

TABLE VII. $\mathscr{K}(L)\mathscr{K}^{\dagger}(L)/|k(0)|^2$ for the $v=6$ irrep.

L	0	3	4	6_1	6_2	7	8	9	10	12
$\frac{\mathscr{K}(L)\mathscr{K}^{\dagger}(L)}{\lvert k(0)\rvert^2}$	1	5.18519	5.43791	$\begin{pmatrix} 5.16029 & 0.14603 \\ 0.14603 & 3.24044 \end{pmatrix}$		3.01844	2.18012	2.42761	1.09506	0.548055

TABLE VIII. $|k(vL)/k(vL_{\min}|^2$ for $v=3k$ irreps.

L	$\lvert k(vL)/k(vL_{\min}\rvert^2$
3	$\frac{70v}{9(v+3)}$
4	$\frac{32v(2v+1)}{3(v+3)(2v+5)}$
7	$\frac{5755v(v-3)(2v+1)}{243(v+3)(v+6)(2v+5)}$
8	$\frac{21539v(v-3)(2v-1)(2v+1)}{729(v+3)(v+6)(2v+5)(2v+7)}$
11	$\frac{1188916v(v-6)(v-3)(2v-1)(2v+1)}{19683(v+3)(v+6)(v+9)(2v+5)(2v+7)}$

From these results we obtain the ratios of the matrices

$$\frac{(\mathscr{K}(L)\mathscr{K}^\dagger(L))_{\alpha\beta}}{\mathscr{K}(L_{\min})\mathscr{K}^\dagger(L_{\min})}=\sum_\gamma \tilde{\mathscr{K}}_{\alpha\gamma}(L)\left|\frac{k(\gamma L)}{k(0)}\right|^2\tilde{\mathscr{K}}^\dagger_{\gamma\beta}(L) \tag{81}$$

given in Table VII for all L values in the $v=6$ irrep.

Eq. (76) gives analytical expressions for the ratio $|k(vL)/k(vL_{\min})|^2$ for all multiplicity-free values of L. These L values are determined by inspection of Eq. (30). The ratios $|k(vL)/k(vL_{\min}|^2$ for all of them are given in Tables VIII–X. Expressions can also be derived for the matrices $\mathscr{K}(vL)\mathscr{K}^\dagger(vL)/|k(vL_{\min})|^2$ for other L values. However, when the multiplicity exceeds two, the analytical expressions are complicated and, since the algorithm for generating the $\mathscr{K}$ matrices is simple, it is better to evaluate them numerically as needed. One will, in any case, need to have the results stored in a computer for any practical application.

Reduced matrix elements $\langle 0v\beta L'\|O\|0v\alpha L\rangle$ for the unitary $[v,0]$ so(5) irreps with $v=1,\ldots,5$, are given in Tables XI and, for $v=6$, in Table XII.

VI. OTHER MATRIX ELEMENTS

First observe that Eq. (76) gives only the ratios of norm factors of wave functions belonging to a common SO(5) irrep. This is sufficient for the purpose of calculating matrix elements of so(5) operators as is clear from Eq. (75). However, absolute norms are required to determine matrix elements between states belonging to different SO(5) irreps of operators outside of the so(5) Lie

TABLE IX. $|k(vL)/k(vL_{\min}|^2$ for $v=3k+1$ irreps.

L	$\lvert k(vL)/k(vL_{\min}\rvert^2$
4	$\frac{79(v-1)}{20(2v+5)}$
5	$\frac{374(v-1)}{135(v+5)}$
6	$\frac{949(v-1)(2v-1)}{270(v+5)(2v+5)}$
7	$\frac{664(v-4)(v-1)}{81(v+5)(2v+5)}$
9	$\frac{55613(v-4)(v-1)(2v-1)}{7290(v+5)(v+8)(2v+5)}$

TABLE X. $|k(vL)/k(vL_{\min}|^2$ for $v=3k+2$ irreps.

L	$\lvert k(vL)/k(vL_{\min}\rvert^2$
4	$\frac{26(2v+1)}{25(v+4)}$
5	$\frac{1771(v-2)}{675(v+4)}$
6	$\frac{2314(v-2)(2v+1)}{675(v+4)(2v+7)}$
7	$\frac{1004(v-2)(2v+1)}{405(v+4)(v+7)}$
9	$\frac{125438(v-5)(v-2)(2v+1)}{18225(v+4)(v+7)(2v+7)}$

algebra. These are readily obtained from Theorem 2 which gives the $a_K^v(\alpha L)$ coefficients for the properly normalized VCS wave functions $\{\psi_{L=2v,M}^v\}$ corresponding to the multiplicity-free ($n=0$) harmonic oscillator states $\{|0v,L=2v,M\rangle\}$. It can be seen (cf. Table I) that the coefficient $a_K^v(vL)$ always has value 1 when $L=2v$ and $K=L$. Thus, one has only to set

$$|k(v,L=2v)|^2=\sum_K |a_K^v(v,2v)|^2 \tag{82}$$

to give $a_K^v(v,L)$ coefficients with this property. Values of $k(v,2v)$ are listed in Table I.

Matrix elements of any operator, defined in terms of the harmonic oscillator d-boson operators, can now be evaluated by means of Theorems 4 and 5 which follow. Theorem 4 gives an explicit expression for the elementary d-boson matrix elements $\langle 0v+1,\beta L'||d^\dagger||0v\alpha L\rangle$ and Theorem 5 expresses all other d-boson matrix elements in terms of them.

Theorem 4: Elementary reduced matrix elements of the d-boson operators are given by

$$\langle 0v+1,\beta L'||d^\dagger||0v\alpha L\rangle=\sqrt{(2L'+1)(v+1)}\sum_{K'K} N_{K'K}^{(\beta L',\alpha L)}(v)$$
$$\times[(LK,\ 22|L'K')+(LK,\ 2,-1|L'K')], \tag{83}$$

where

$$N_{K'K}^{(\beta L',\alpha L)}(v)=\sum_{\delta\gamma} \mathscr{K}^{-1}_{\beta\delta}(v+1,\ L')b_{K'}^{v+1*}(\delta L')b_K^v(\gamma L)\mathscr{K}_{\gamma\alpha}(vL). \tag{84}$$

Proof: In the proof of Theorem I, it was observed that the $v+1$ component of the state $d_m^\dagger|0v\alpha LM\rangle$ has VCS wave function given by the product $\psi_{2m}^1\psi_{\alpha LM}^v$. It follows that

$$[\psi_2^1\times\psi_{\alpha L}^v]_{L'M'}=\sum_\beta \psi_{\beta L'M'}^{v+1}\frac{\langle 0v+1,\beta L'||d^\dagger||0v\alpha L\rangle}{\sqrt{2L'+1}}. \tag{85}$$

Combining the product wave functions on the left, using the identity $\xi^1\xi^v=\sqrt{v+1}\xi^{v+1}$, gives

$$\sqrt{v+1}\sum_K a_K^v(\alpha L)[(LK,\ 22|L'K')+(LK,\ 2,-1|L'K')]$$

TABLE XI. $\mathscr{M}^{(L'L)}=\langle 0vL'\|O\|0vL\rangle$ matrix elements for the multiplicity-free irreps with $v=1,\ldots,5$.

v	$L'\backslash L$	2
1	2	$\sqrt{70}$

v	$L'\backslash L$	2	4
2	2	$-8\sqrt{\frac{10}{7}}$	$\frac{30}{\sqrt{7}}$
	4	$\frac{30}{\sqrt{7}}$	$3\sqrt{\frac{110}{7}}$

v	$L'\backslash L$	0	3	4	6
3	0	0	$3\sqrt{10}$	0	0
	3	$-3\sqrt{10}$	$-\sqrt{\frac{21}{2}}$	$-7\sqrt{\frac{15}{2}}$	$\sqrt{78}$
	4	0	$7\sqrt{\frac{15}{2}}$	$3\sqrt{\frac{35}{22}}$	$\sqrt{\frac{2730}{11}}$
	6	0	$-\sqrt{78}$	$\sqrt{\frac{2730}{11}}$	$6\sqrt{\frac{91}{11}}$

v	$L'\backslash L$	2	4	5	6	8
4	2	$29\sqrt{\frac{5}{14}}$	$3\sqrt{\frac{130}{7}}$	$9\sqrt{\frac{5}{2}}$	0	0
	4	$3\sqrt{\frac{130}{7}}$	$-48\sqrt{\frac{10}{77}}$	$3\sqrt{30}$	$6\sqrt{\frac{105}{11}}$	0
	5	$-9\sqrt{\frac{5}{2}}$	$-3\sqrt{30}$	$\sqrt{\frac{2145}{14}}$	$-\sqrt{390}$	$4\sqrt{\frac{85}{7}}$
	6	0	$6\sqrt{\frac{105}{11}}$	$\sqrt{390}$	$4\sqrt{\frac{91}{11}}$	$\sqrt{408}$
	8	0	0	$-4\sqrt{\frac{85}{7}}$	$\sqrt{408}$	$2\sqrt{\frac{969}{7}}$

v	$L'\backslash L$	2	4	5	6	7	8	10
5	2	$-31\sqrt{\frac{5}{14}}$	$3\sqrt{\frac{330}{7}}$	$9\sqrt{\frac{5}{2}}$	0	0	0	0
	4	$3\sqrt{\frac{330}{7}}$	$60\sqrt{\frac{10}{77}}$	$-3\sqrt{\frac{110}{13}}$	$6\sqrt{\frac{85}{11}}$	$18\sqrt{\frac{15}{13}}$	0	0
	5	$-9\sqrt{\frac{5}{2}}$	$3\sqrt{\frac{110}{13}}$	$-23\sqrt{\frac{165}{182}}$	$-\sqrt{\frac{5610}{7}}$	$66\sqrt{\frac{5}{91}}$	$2\sqrt{\frac{1615}{91}}$	0
	6	0	$6\sqrt{\frac{85}{11}}$	$\sqrt{\frac{5610}{7}}$	$-2\sqrt{\frac{13}{77}}$	$15\sqrt{\frac{8}{7}}$	$2\sqrt{\frac{1254}{7}}$	0
	7	0	$-18\sqrt{\frac{15}{13}}$	$66\sqrt{\frac{5}{91}}$	$-15\sqrt{\frac{8}{7}}$	$\frac{47}{2}\sqrt{\frac{85}{91}}$	$-\frac{55}{2}\sqrt{\frac{51}{91}}$	$5\sqrt{14}$
	8	0	0	$-2\sqrt{\frac{1615}{91}}$	$2\sqrt{\frac{1254}{7}}$	$\frac{55}{2}\sqrt{\frac{51}{91}}$	$\frac{61}{2}\sqrt{\frac{51}{133}}$	$5\sqrt{\frac{462}{19}}$
	10	0	0	0	0	$-5\sqrt{14}$	$5\sqrt{\frac{462}{19}}$	$\sqrt{\frac{17710}{19}}$

TABLE XII. $\mathcal{M}_{\beta\alpha}^{(L'L)}=\langle 0v\beta L'||O||0v\alpha L\rangle$ matrix elements for $v=6$.

$L'_\beta \backslash L_\alpha$	0	3	4	6_1	6_2	7	8	9	10	12
0	0	16.4317	0	0	0	0	0	0	0	0
3	−16.4317	−3.24037	−32.4773	16.6851	13.8685	0	0	0	0	0
4	0	32.4773	3.78394	25.8288	−14.0782	17.7647	0	0	0	0
6_1	0	−16.6851	25.8288	29.0921	0	−13.9397	18.778	25.0041	0	0
6_2	0	−13.8685	−14.0782	0	−32.1737	−31.7351	−22.5917	4.56238	0	0
7	0	0	−17.7647	13.9397	31.7351	−0.739066	−31.7543	22.2417	13.7627	0
8	0	0	0	18.778	−22.5917	31.7543	10.1143	15.438	33.9711	0
9	0	0	0	−25.0041	−4.56238	22.2417	−15.438	32.1182	−21.5388	23.355
10	0	0	0	0	0	−13.7627	33.9711	21.5388	26.5482	29.1175
12	0	0	0	0	0	0	0	−23.355	29.1175	38.1670

$$=a_{K'}^{v+1}(\beta L')\frac{\langle 0v+1,\beta L'||d^\dagger||0v\alpha L\rangle}{\sqrt{2L'+1}}. \tag{86}$$

Expressing $a_K^v(\alpha L)$ in terms of $b_K^v(\alpha L)$ and using the orthogonality relationship of Eq. (43), the theorem follows.

Theorem 5: General reduced matrix elements of the d-boson operators are given by the equations:

$$\langle n'v'\beta L'||d||nv\alpha L\rangle=(-1)^{L-L'}\langle nv\alpha L||d^\dagger||n'v'\beta L'\rangle^*, \tag{87}$$

$$\langle nv+1,\beta L'||d^\dagger||nv\alpha L\rangle=\sqrt{\frac{n+v+\frac{5}{2}}{v+\frac{5}{2}}}\langle 0v+1,\beta L'||d^\dagger||0v\alpha L\rangle, \tag{88}$$

$$\langle n+1,v-1,\beta L'||d^\dagger||nv\alpha L\rangle=(-1)^{L-L'}\sqrt{\frac{n+1}{v+\frac{3}{2}}}\langle 0v\alpha L||d^\dagger||0v-1,\beta L'\rangle^*, \tag{89}$$

with the understanding that $\langle n'v'\beta L'||d^\dagger||nv\alpha L\rangle$ is zero unless $(n'=n,\ v'=v+1)$ or $(n'=n+1,\ v'=v-1)$.

Proof: Eq. (87) follows from standard angular momentum coupling theory. The other equations follow from the intermediate result

$$[\Gamma(d^\dagger)\times\Psi_{nv\alpha L}]_{L'M'}=\frac{n+v+\frac{5}{2}}{v+\frac{5}{2}}[\Psi_{012}\times\Psi_{nv\alpha L}]_{L'M'}$$
$$+(-1)^{L-L'}\sqrt{\frac{n+1}{v+\frac{3}{2}}}\sum_\beta\frac{\langle 0v\alpha L||d^\dagger||0v-1,\beta L'\rangle^*}{\sqrt{2L'+1}}\Psi_{n+1,v-1,\beta L'M'}. \tag{90}$$

To derive this intermediate result, consider the equation

$$[\Gamma(d_\nu^\dagger)\Psi_{nv\alpha LM}](x,z,\Omega)=\langle 0|e^{x\eta+zX_-}R(\Omega)d_\nu^\dagger|nv\alpha LM\rangle$$
$$=\langle 0|e^{x\eta}R(\Omega)(d_\nu^\dagger+zd_\nu)e^{zX_-}|nv\alpha LM\rangle. \tag{91}$$

We have

$$\langle 0|e^{x\eta}R(\Omega)d_\nu^\dagger e^{zX_-}|nv\alpha LM\rangle=\sqrt{\frac{(n+v+\frac{3}{2})!}{n!(v+\frac{3}{2})!}}z^n\langle 0|e^{x\eta}R(\Omega)d_\nu^\dagger|0v\alpha LM\rangle \tag{92}$$

and

$$\langle 0|e^{x\eta}R(\Omega)zd_\nu e^{zX_-}|nv\alpha LM\rangle=\sqrt{\frac{(n+v+\frac{3}{2})!}{n!(v+\frac{3}{2})!}}z^{n+1}\langle 0|e^{x\eta}R(\Omega)d_\nu|0v\alpha LM\rangle$$
$$+\sqrt{\frac{n(n+v+\frac{3}{2})!}{(n-1)!(v+\frac{5}{2})!}}z^n\langle 0|e^{x\eta}R(\Omega)d_\nu|1v\alpha LM\rangle$$
$$=\sqrt{\frac{(n+v+\frac{3}{2})!}{n!(v+\frac{3}{2})!}}\Bigg[z^{n+1}\langle 0|e^{x\eta}R(\Omega)d_\nu|0v\alpha LM\rangle$$
$$+\frac{n}{v+\frac{5}{2}}z^n\langle 0|e^{x\eta}R(\Omega)d_\nu^\dagger|0v\alpha LM\rangle\Bigg]. \tag{93}$$

It follows that

$$[\Gamma(d_\nu^\dagger)\Psi_{nv\alpha LM}](x,z,\Omega)=\sqrt{\frac{(n+v+\frac{3}{2})!}{n!(v+\frac{3}{2})!}}\Bigg[\frac{n+v+\frac{5}{2}}{v+\frac{5}{2}}z^n\langle 0|e^{x\eta}R(\Omega)d_\nu^\dagger|0v\alpha LM\rangle$$
$$+z^{n+1}\langle 0|e^{x\eta}R(\Omega)d_\nu|0v\alpha LM\rangle\Bigg]. \tag{94}$$

Expanding

$$d_\nu|0v\alpha LM\rangle=\sum_{\beta L'M'}|0v-1,\beta L'M'\rangle(-1)^\nu\langle 0v\alpha LM|d_{-\nu}^\dagger|0v-1,\beta L'M'\rangle^*, \tag{95}$$

and using eqs. (24) and (32), we obtain

$$\Gamma(d_\nu^\dagger)\Psi_{nv\alpha LM}=\frac{n+v+\frac{5}{2}}{v+\frac{5}{2}}\Psi_{012\nu}\Psi_{nv\alpha LM}+(-1)^\nu\sqrt{\frac{n+1}{v+\frac{3}{2}}}$$
$$\times\sum_{\beta L'M'}\langle 0v\alpha LM|d_{-\nu}^\dagger|0v-1,\beta L'M'\rangle^*\Psi_{n+1,v-1,\beta L'M'}, \tag{96}$$

from which Eq. (90) follows. Eq. (89) of the theorem follows directly from Eq. (90). To derive Eq. (88), expand

$$[\Psi_{012}\times\Psi_{nv\alpha L}]_{L'M'}=\sqrt{\frac{(n+v+\frac{3}{2})!}{n!(v+\frac{3}{2})!}}z^n[\psi_2^1\times\psi_{\alpha L}^v]_{L'M'}$$

$$=\sqrt{\frac{(n+v+\frac{3}{2})}{n!(v+\frac{3}{2})!}}z^n\sum_\beta \Psi_{0v+1\beta L'M'}\frac{\langle 0v+1,\beta L'||d^\dagger||0v\alpha L\rangle}{\sqrt{2L'+1}}$$

$$=\sqrt{\frac{v+\frac{5}{2}}{n+v+\frac{5}{2}}}\sum_\beta \Psi_{nv+1\beta L'M'}\frac{\langle 0v+1,\beta L'||d^\dagger||0v\alpha L\rangle}{\sqrt{2L'+1}}. \qquad (97)$$

VII. ALTERNATIVE ORTHONORMAL BASES

A. The inner product

In constructing UIR's (unitary irreducible representations) of a group G, the VCS method starts with a convenient basis $\{\varphi_i\}$ for a space which contains a module for the desired UIR as an invariant subspace. The basis $\{\varphi_i\}$ is conveniently chosen to be orthonormal relative to a natural but, in principle, arbitrary inner product. A transformation is then found such that the non-zero states $\{\psi_i=\Sigma_j\varphi_j\mathcal{K}_{ji}\}$ are a basis for an irreducible subspace relative to which the VCS representation is unitary; i.e., the non-zero states in the set $\{\psi_i\}$ are an orthonormal basis.

Note, however, that while an orthonormal basis defines an inner product, the converse is only true to within arbitrary unitary transformations. However, if $\psi_i\to\Sigma_j\psi_jU_{ji}$ is a unitary transformation of the non-zero states in the set $\{\psi_i\}$, one sees that the corresponding transformation of the $\mathcal{K}$ matrix, in which $\mathcal{K}\to\mathcal{K}U$, leaves the matrix $\mathcal{K}\mathcal{K}^\dagger$ invariant. Thus, as observed previously,[10,2] the matrix

$$S=\mathcal{K}\mathcal{K}^\dagger \qquad (98)$$

is uniquely defined by the inner product and, conversely, the inner product is uniquely defined by S.

In this paper, we have given a construction of $\mathcal{K}$ corresponding to a particular orthonormal basis. The above argument shows that the corresponding S matrix is independent of the choice and that it uniquely defines the inner product relative to which the VCS representation is unitary. Moreover, any basis $\{\psi_i'=\Sigma_j\varphi_j\mathcal{K}_{ji}'\}$, defined by a $\mathcal{K}'$ matrix for which $\mathcal{K}'\mathcal{K}'^\dagger=S$, is an orthonormal basis.

B. Canonical bases

It is useful if basis states can be uniquely defined, to within phase factors, independent of computational algorithms. Results obtained by different authors can then be compared. Such bases are said to be *canonical.* Basis states for the five-dimensional harmonic oscillator would be uniquely defined, by the representation labels of the subgroup chains (1) and (2), were it not for the missing label α needed to distinguish the multiplicity of SO(3) irreps that occur in an SO(5) irrep. We now give two canonical resolutions of the SO(5) ⊃ SO(3) multiplicity problem; the first is used (as far as we know for the first time) in this paper, the second has been used previously.[4] Both methods are easily realised within the framework of the VCS representation.

The orthonormal SO(5) ⊃ SO(3) basis states given in this paper have an additional quantum number similar to that introduced by Racah[11] to resolve the SU(3) ⊃ SO(3) multiplicity. We have shown how to construct states which satisfy the equation

$$\langle 0v\beta L||O||0v\alpha L\rangle=\delta_{\alpha\beta}\langle 0v\alpha L||O||0v\alpha L\rangle, \qquad (99)$$

cf. Eq. (73). Such states are eigenstates of the SO(3)-invariant Hermitian operator

$$X=(L\times O\times L\times L)_0+(L\times L\times O\times L)_0.$$

To see this, observe that the eigenstates of X satisfy the equation

$$\langle 0v\alpha L||X||0v\beta L'\rangle=\delta_{LL'}\delta_{\alpha\beta}C_L\langle 0v\alpha L||O||0v\beta L\rangle, \tag{100}$$

where C_L is a constant whose value depends only on L. Thus, states which satisfy Eq. (99) can be labelled by the value of $\langle 0v\alpha L||O||0v\alpha L\rangle$. More important is the fact that a basis of states labelled by a complete set of distinct quantum numbers is unique to within phase factors and therefore canonical.

An alternative resolution of the SO(5) ⊃ SO(3) multiplicity problem is given by considering zero-coupled triplets. Unfortunately, it is generally not possible to define basis states which have a precise number of zero-coupled triplets. The number cannot be defined, for example, as the eigenvalue of any Hermitian operator. Thus, it is not possible to have the number of zero-coupled triplets as a quantum number in the usual sense. However, it is possible to give an operational definition of an effective number of triplets in a state, thereby defining a pseudo quantum number t. It turns out that this is sufficient for the construction of an orthonormal basis. Let T_0 denote the angular-momentum-zero-coupled product of d-boson lowering operators

$$T_0=(d\times d\times d)_0 \tag{101}$$

and let

$$T_0^\dagger=(d^\dagger\times d^\dagger\times d^\dagger)_0 \tag{102}$$

be its Hermitian adjoint. If $T_0|v,t,L\rangle=0$, for some state $|v,t,L\rangle$ of an SO(5) irrep $[v,0]$ and angular momentum L, we say that the state $|v,t,L\rangle$ contains no zero-coupled triplets and assign t the value $t=0$. We also say that the state $(T_0^\dagger)^n|v,t=0,\ L\rangle$ "contains" n triplets and for such a state assign t the value $t=n$. In this way, the multiplet of states of a given angular momentum L in an SO(5) irrep $[v,0]$ contains a unique state of $t=t^v_{\max}$ generated by adding a zero-coupled triplet to the $t^{v-3}_{\max}=t^v_{\max}-1$ state of the same L of the irrep $[v-3,0]$. States of $t<t^v_{\max}$ can be generated in a similar recursive way. For example, if $|v-3,\ t^v_{\max}-2,\ L\rangle$ is a state of $t=t^{v-3}_{\max}-1$, the state

$$|v,t^v_{\max}-1,\ L\rangle=\alpha T_0^\dagger|v-3,\ t^v_{\max}-2,\ L\rangle+\sqrt{1-\alpha^2}|v,t^v_{\max},L\rangle, \tag{103}$$

with α chosen such that $|v,t^v_{\max}-1,\ L\rangle$ is orthogonal to $|v,t^v_{\max},L\rangle$ is assigned the value $t=t^v_{\max}-1$.

The construction of a basis labelled by the pseudo quantum number t is particularly natural and easy to implement within the framework of VCS theory. First observe that the basis states of section III are constructed in this way, albeit relative to the so-called rotor inner product of Eq. (39). Thus, within a multiplet of states of angular momentum L, the desired states are of the form

$$\begin{aligned}\psi^v_{1,L}&=\varphi^v_{1,L}\mathscr{K}_{11},\\ \psi^v_{2,L}&=\varphi^v_{1,L}\mathscr{K}_{12}+\varphi^v_{2,L}\mathscr{K}_{22},\\ \psi^v_{3,L}&=\varphi^v_{1,L}\mathscr{K}_{13}+\varphi^v_{2,L}\mathscr{K}_{23}+\varphi^v_{3,L}\mathscr{K}_{33},\quad \text{etc.,}\end{aligned} \tag{104}$$

where $\psi^v_{1,L}$ is the wave function for the state with $t=t_{\max}$, $\psi^v_{2,L}$ is the wave function for the state with $t=t_{\max}-1$, etc. One sees that states classified by zero-coupled triplets correspond, in VCS theory, to a solution of the equation $S=\mathscr{K}\mathscr{K}^\dagger$ with $\mathscr{K}$ lower triangular. In other words, they are obtained by a Gramm-Schmidt orthogonalization of the basis wave functions of section III.

VIII. CONCLUDING REMARKS

In papers I and II, the wave functions for a VCS representation of SO(5) were multidimensional vector-valued functions of the form

$$\psi^{v}_{\alpha LM}(\Omega)=\sum_{K} a^{v}_{K}(\alpha L)\xi^{v}_{K}\mathscr{D}^{L}_{KM}(\Omega), \tag{105}$$

where $\{\xi^{v}_{K}\}$ are basis vectors for an SU(2) irrep of spin $v/2$. In the present paper, the wave functions are one-dimensional of the form

$$\psi^{v}_{\alpha LM}(x,\Omega)=\xi^{v}(x)\sum_{K} a^{v}_{K}(\alpha L)\mathscr{D}^{L}_{KM}(\Omega). \tag{106}$$

Thus, in both cases the wave functions are defined by a set of $\{a^{v}_{K}(\alpha L)\}$ coefficients and can be put into one-to-one correspondence with one another. Thus, for all practical purposes, the two formulations are equivalent. However, the formulation of this paper, which uses scalar-valued wave functions, is conceptually simpler. Consequently, the physical content of the theory becomes more transparent and extending the theory to generic SO(5) representations is easier.

One can see that the multidimensional-vector character of VCS wave functions of the form given by Eq. (105) is not essential because the components of different K would already be orthogonal, without the ξ^{v}_{K} vectors, by virtue of the orthogonality properties of the $\mathscr{D}^{L}_{KM}$ functions. Thus, a VCS wave function is fully defined by the $\{a^{v}_{K}(\alpha L)\}$ coefficients, with or without the ξ^{v}_{K} vectors. However, as will be shown in a following paper, VCS wave functions for generic SO(5) irreps of type $v=[v_1,v_2]$ are expressible in the form

$$\psi^{v}_{\alpha LM}(\Omega)=\sum_{KK'} a^{v}_{K'K}(\alpha LM)\xi^{v}_{K'}\mathscr{D}^{L}_{KM}(\Omega). \tag{107}$$

Thus, the generic SO(5) irreps are defined by a matrix array $\{a^{v}_{K'K}(\alpha LM)\}$ of coefficients and it is clear that the vectors $\{\xi^{v}_{K'}\}$ then play an independent and essential role as basis vectors and no longer duplicate the parallel role of the $\mathscr{D}^{L}_{KM}$ functions. The generic SO(5) irreps will be treated in a following paper.

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