Round buoyant laminar and turbulent plumes

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Two approximate solutions, which become exact for Prandtl numbers 1 and 2, are given for laminar round plumes for arbitrary Prandtl number σ. The first of these gives a very accurate solution for laminar found plumes in air, for which $\sigma = 0.73$, and the second provides a good solution for laminar round plumes in water, for which $\sigma = 6.7$. Two approximate solutions, which become exact for turbulent Prandtl numbers 1.1 and 2, respectively, are given for turbulent round plumes. The one which becomes exact at $\sigma = 1.1$ is used to give a highly accurate solution for $\sigma = 1$, which, for an eddy-viscosity coefficient λ equal to 0.0156, provides a remarkably good agreement with the experimental data of Beuther, Capp, and George, Jr.

I. INTRODUCTION

The problem of round laminar buoyant plumes was in an unstratified environment studied as early as 1937 by Zel'dovich.[1] Independently, Yih[2,3] developed the differential equations for a similarity solution of the problem, and he gave two exact solutions for the differential system consisting of those equations, the boundary conditions, and an integral condition; one for Prandtl number $\sigma = 1$, and one for $\sigma = 2$. Later, other researchers, unaware of Yih's work also studied the problem, but produced nothing new. Thus to this day, solutions for round laminar buoyant plumes for other Prandtl numbers are still unavailable. Especially desirable are the solutions for air, with $\sigma = 0.73$, and for water, with $\sigma = 6.7$.

The problem of round turbulent buoyant plumes was investigated by Schmidt[4], and later Rouse, unaware of Schmidt's work, suggested the problem to this writer. Rouse was the first to give, explicitly and completely, the dimensionless parameters which, when experimentally determined, would give the description of the phenomenon. The Iowa measurements were recorded in three papers,[2,3,5] and it was only in recent years that other measurements with more elaborate and therefore, presumably, more accurate instruments were made by George and his associates.[6] We shall later take the most recent measurements of Beuther, Capp, and George, Jr.[7] and compare them with our analytical results.

The first analytical results for turbulent plumes were given by Yih,[8] and these were shown to agree well with the Iowa measurements made nearly three decades ago. However, Yih's "exact" solutions for round turbulent plumes are for (turbulent) Prandtl numbers 1.1 and 2, and although the turbulent Prandtl number must be around 1, there is no reason to expect it to be 1.1. It is desirable to have a solution for turbulent Prandtl numbers in the neighborhood of 1, in order to provide an analytical basis for comparison with the recent measurements of Beuther, Capp, and George, Jr.[7]

In this paper, we shall present approximate solutions for laminar round plumes for any Prandtl number.

These are especially accurate when the Prandtl number is near 1, as for air. We shall also present approximate solutions for turbulent round plumes for (turbulent) Prandtl numbers around 1, and compare our analytical results with the measurements of Beuther *et al.*[7]

Furthermore, we shall give rigorous results for the asymptotic behavior of the angle of spread of laminar round plumes for large Prandtl numbers, and we shall give a set of transformations that leave the differential system (excepting the integral condition) invariant, which are useful for any accurate numerical calculation for a given Prandtl number, since one can ignore the integral condition first, carry out the numerical calculation, and then apply the set of transformations to satisfy the integral conditions. All the results given in this paper, excepting the existing results explicitly cited, are new and not previously available.

We note here that plumes in stratified environments were investigated by Morton, Taylor, and Turner[9] using the concept of entrainment.

II. ROUND LAMINAR PLUMES

A brief account of the formulation of the problem is necessary in order to facilitate the presentation of the new results. We shall take the heat source as the origin, use x and r as vertical and radial coordinates, and denote the velocity components in the directions of increasing x and r by u and v, respectively. The gravitational acceleration is denoted by g, and acts in the direction of decreasing x. If $\Delta\rho$ is the variation of the density of the fluid as a result of temperature rise due to the heat source, we can, for temperature rises small compared with the absolute temperature of the surrounding fluid, use $\Delta\rho$ instead of the temperature rise ΔT as a dependent variable. For convenience, we choose to use

$$\Delta\gamma = g\Delta\rho$$

instead of ΔT. The density and viscosity of the ambient fluid will be denoted by ρ and μ, respectively, and the kinematic viscosity μ/ρ will be denoted by ν, as usual.

Since the pressure variation in the plume is assumed to be small compared with the pressure (assumed hydrostatic) in the ambient fluid, we can use the thermal diffusivity at constant pressure. We denote it by α, and the Prandtl number ν/α by σ.

Assuming the hydrostatic pressure of the atmosphere to be essentially undisturbed by the flow caused by the heat source and $\partial^2 u/\partial x^2$ to be negligible as compared with $r^{-1}\partial(r\,\partial u/\partial r)/\partial r$, one can write the equation of motion as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - g\frac{\Delta\gamma}{\gamma_0}, \tag{1}$$

where $\gamma_0 = \rho g$. The equation for heat diffusion (the energy equation), can be written

$$u\frac{\partial\Delta\gamma}{\partial x} + v\frac{\partial\Delta\gamma}{\partial r} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Delta\gamma}{\partial r}\right), \tag{2}$$

if $\partial^2\Delta\gamma/\partial x^2$ is neglected in comparison with $r^{-1}\partial(r\,\partial\Delta\gamma/\partial r)/\partial r$. The equation of continuity is

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(rv) = 0. \tag{3}$$

Equations (1)–(3) are to be solved with the boundary conditions that; u, v, and $\Delta\gamma$ vanish at $r = \infty$; u and v vanish at $x = 0$ except at the origin, and v, $\partial\Delta\gamma/\partial r$, and $\partial u/\partial r$ vanish at $r = 0$. Equation (3) permits the use of Stokes' stream function ψ so that

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r}, \quad v = -\frac{1}{r}\frac{\partial\psi}{\partial x}. \tag{4}$$

Multiplying (2) by $2\pi r\,dr$ and integrating from zero to infinity, using the boundary conditions that $\partial\Delta\gamma/\partial r$ is zero at $r = 0$ and $\Delta\gamma = 0$ at infinity, we find that the following quantity is independent of x:

$$G = -\int_0^\infty 2\pi ru\Delta\gamma\,dr. \tag{5}$$

Indeed, G is a measure of the strength of the heat source. Then, with the substitutions

$$\Delta\gamma = -(G\rho/x\mu)\theta(\eta), \tag{6}$$

and

$$\psi = 4\nu xf(\eta), \tag{7}$$

in which

$$\eta = (\rho^2 G/4\mu^3)^{1/4}(r/x^{1/2}), \tag{8}$$

Eqs. (1) and (2) become

$$(1 - 4f)\frac{d}{d\eta}\left(\frac{f'}{\eta}\right) = f''' + \eta\theta, \tag{9}$$

and (after one integration and application of the boundary conditions)

$$f = -(1/4\sigma)(\theta'/\theta)\eta. \tag{10}$$

In obtaining (9), γ_0/g has been equated to the ambient density ρ. The boundary conditions are now

$$\theta(\infty) = 0,$$
$$f(0) = f'(0) = \theta'(0) = 0, \tag{11}$$
$$f(\infty) = C,$$

where C is any finite number. The integral condition for the heat flux as expressed by (5) now assumes the form

$$\int_0^\infty f'\theta\,d\eta = \frac{1}{8\pi}. \tag{12}$$

We note in passing that the definition of η by (8) shows that the term $\nu\partial^2 u/\partial x^2$ neglected in comparison with the first term of the right-hand side of (1) is justified for $Gx^2/\mu\nu^2 \gg 1$, as can readily be shown.

A. Exact solutions for round laminar plumes

For Prandtl numbers 1 and 2, Yih[2,3] obtained exact solutions of the differential system consisting of (9) to (12). For $\sigma = 1$:

$$\frac{\psi}{4\nu x} = f(\eta) = \frac{3}{2}\frac{\eta^2}{6\sqrt{2\pi} + \eta^2}, \tag{13}$$

$$-\frac{\mu x\Delta\gamma}{G\rho} = \theta(\eta) = \frac{\eta^2}{3\pi(1 + \eta^2/6\sqrt{2\pi})^3}, \tag{14}$$

$$\left(\frac{\mu}{G}\right)^{1/2}u = \frac{2f'}{\eta} = \frac{1}{\sqrt{2\pi}(1 + \eta^2/6\sqrt{2\pi})^2}, \tag{15}$$

$$\left(\frac{\rho^2 x^2}{\mu G}\right)^{1/4}v = \sqrt{2}\left(f' - \frac{2f}{\eta}\right) = -\frac{3\sqrt{2}\eta^3}{(6\sqrt{2\pi} + \eta^2)^2}. \tag{16}$$

For $\sigma = 2$:

$$\frac{\psi}{4\nu x} = f(\eta) = \frac{\sqrt{5}\eta^2}{8\sqrt{2\pi} + \sqrt{5}\eta^2}, \tag{17}$$

$$-\frac{\mu x\Delta\gamma}{G\rho} = \theta(\eta) = \frac{5}{8\pi[1 + (\sqrt{5}/8\sqrt{2\pi})\eta^2]^4}, \tag{18}$$

$$\left(\frac{\mu}{G}\right)^{1/2}u = \frac{2f'}{\eta} = \frac{\sqrt{5}}{2\sqrt{2\pi}[1 + (\sqrt{5}/8\sqrt{2\pi})\eta^2]^2}, \tag{19}$$

$$\left(\frac{\rho^2 x^2}{\mu G}\right)^{1/4}v = \sqrt{2}\left(f' - \frac{2f}{\eta}\right) = -\frac{10\sqrt{2}\eta^3}{(8\sqrt{2\pi} + \sqrt{5}\eta^2)^2}. \tag{20}$$

B. New solutions

The new solutions we shall present here are not exact, but they cover the entire range of Prandtl numbers. When the Prandtl number is near 1 or 2, for which exact solutions exist, they give a high degree of accuracy. But, the term neglected is a fixed small portion of a term retained, while both are proportional to $\Delta\sigma$, which is the difference between the actual Prandtl number σ and 1, or 2, as the case may be. In this sense, the relative error is of the same order of magnitude for all Prandtl numbers, and the solutions given here are reliable approximate solutions, which, at the least, would assist in any accurate numerical solution of the problem for any Prandtl number.

First, we shall briefly describe the obvious approach, which was the first that came to mind but which in the end proved laborious and inferior to the vastly simpler and reliable method finally used. Expanding f and θ in a power series of $\Delta\sigma$ and using f_0 and θ_0 to denote the exact solution at $\sigma = 1$ or 2, as the case may be, we have

$$f = f_0 + \Delta\sigma f_1 + (\Delta\sigma)^2 f_2 + \ldots,$$
$$\theta = \theta_0 - \Delta\sigma\theta_1 + (\Delta\sigma)^2\theta_2 + \ldots. \tag{21}$$

Then, substituting these expansions in (9) and (10) and

collecting equal powers of $\Delta\sigma$, we obtained a sequence of simultaneous equations for (f_1, θ_1), (f_2, θ_2), etc. For f_1 and θ_1, we found, after considerable effort and trials, that the following expansions are appropriate:

$$f_1 = \ln X\left(\sum_{n=1}^{\infty} A_n X^{-n-1}\right) + \sum_{n=1}^{\infty} B_n(1-X^{-1})X^{1-n},$$

$$\theta_1 = \ln X\left(\sum_{n=1}^{\infty} C_n X^{-n-4}\right) + \sum_{n=1}^{\infty} D_n X^{-n-2}, \tag{22}$$

where

$$X = 1 + A\eta^2, \tag{23}$$

A being a constant to be determined by the integral condition (12) afterward. (In practice, we used $A = 1$ for the determination of the coefficients in the expansions, and determined the actual A by a transformation to be presented later in this paper.) But, the first several coefficients obtained for each series are large compared with the corresponding coefficient in the exact solution, and both series converge very slowly. We calculated up to 22 terms. Even then, the results obtained by taking only terms of order $\Delta\sigma$ are already not satisfactory for $\Delta\sigma = 0.25$, indicating that terms of order $(\Delta\sigma)^2$ or even $(\Delta\sigma)^3$ are necessary. This outcome discouraged us from proceeding further with this approach.

The final method adopted is simple, direct, and, if one accepts the errors the order of magnitude of which is known *a priori*, reliable.

1. Solution based on exact solution for $\sigma = 1$

For any σ, let

$$f = (3/2\sigma)(1-X^{-1}), \tag{24}$$

where X is given by (23), in which A is to be determined later. Substituting (24) into (10) and integrating, we have

$$\theta = CX^{-3}. \tag{25}$$

Substituting (24) and (25) into (12), we find

$$C = \sigma/3\pi. \tag{26}$$

To determine A, we substitute (24) and (25) into (9), and obtain

$$(24/\sigma^2)[3 - 2\sigma + 3(\sigma-1)X^{-1}]A^2X^{-3} = CX^{-3}, \tag{27}$$

from which it is obvious that an exact solution exists for $\sigma = 1$. For any σ, we write (27) as

$$(24A^2/\sigma^2)[(3-2\sigma) + 3(\sigma-1)\beta]X^{-3} + R = CX^{-3}, \tag{28}$$

where

$$R = (72A^2/\sigma^2)(\sigma-1)(X^{-4} - \beta X^{-3}) \tag{29}$$

is the residual, and β is a constant to be determined so as to make

$$|\beta X^{-3} - X^{-4}|$$

as small as possible in the core of the plume.

The temperature distribution is given by (25).
The point of inflection of X^{-3} is at $X = 8/7$, or $A\eta^2 = 1/7$.

If we require that the *relative* (to X^{-4}) error $\beta X - 1$ at the origin be equal and of opposite sign to its value at $A\eta^2 = 2/7$ (twice its value at the point of inflection of the plume), we have

$$\beta = \frac{7}{8}. \tag{30}$$

The difference between βX^{-3} and X^{-4} is shown in Fig. 1.

Then, ignoring the residue R in (28), we have

$$C = [(9 + 15\sigma)/\sigma^2]A^2, \tag{31}$$

which, in combination with (26), gives

$$A = \sigma[\sigma/9\pi(5\sigma+3)]^{1/2}. \tag{32}$$

Then (23)–(26), and (32) constitute the solution.

2. Solution based on exact solution for $\sigma = 2$

For σ near 2 or greater than 2, a more accurate approximate solution can be obtained if it is based on the exact solution for $\sigma = 2$.

With X defined by (23), we take

$$f = (2/\sigma)(1 - X^{-1}). \tag{33}$$

Then, the solution of (10) is

$$\theta = CX^{-4}. \tag{34}$$

Substituting (33) and (34) into (12), we obtain, whatever the value of A may be in the definition (23) of X,

$$C = 5\sigma/16\pi. \tag{35}$$

To find A as a function of σ, we substitute (33)–(35) into (9), and obtain

$$-\frac{16}{\sigma}A^2\left[4\left(1-\frac{2}{\sigma}\right)X^{-3} + \left(\frac{8}{\sigma}-6\right)X^{-4}\right] = \frac{5\sigma}{16\pi}X^{-4}. \tag{36}$$

Again, equating X^{-3} to $\beta^{-1}X^{-4}$, where β is now $10/11$, we obtain from (36)

$$A^2 = \frac{25\sigma^3}{1024\pi(2\sigma+1)}. \tag{37}$$

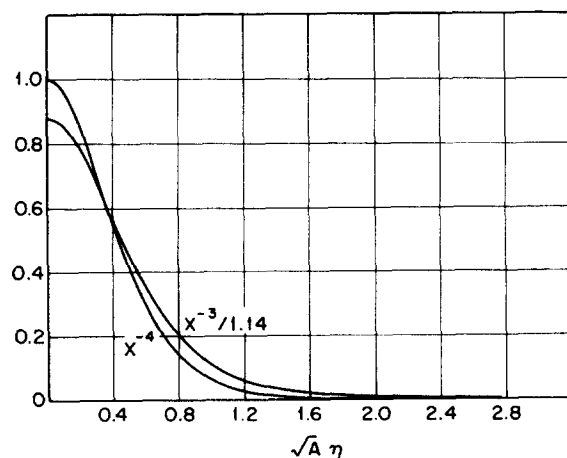Then (23), (33), (34), (35), and (37) constitute the solution.

FIG. 1. Approximation of X^{-4} by $\beta^{-1}X^{-3}$, $\beta = 8/7$.

C. Comparisons of approximate and exact solutions

In the approximate solutions just presented, the error committed arises from, and only from, approximating X^{-4} by $7X^{-3}/8$, or X^{-3} by $1.1X^{-4}$. If we call the approximate solution based on the exact solution for $\sigma=1$ the first approximate solution and that based on the exact solution for $\sigma=2$ the second approximate solution, then the first approximate solution is increasingly accurate as σ approaches 1 and is exact at $\sigma=1$, and the second approximate solution is increasingly accurate as σ approaches 2 and is exact at $\sigma=2$.

In Fig. 2, the first approximate solution evaluated at $\sigma=2$ is compared with the exact solution for $\sigma=2$, and the second approximate solution evaluated at $\sigma=1$ is compared with the exact solution for $\sigma=1$. It is seen that as σ is increased from 1, the first approximate solution overestimates the temperature and underestimates the velocity, and that as σ decreases from 2, the second approximate solution underestimates the temperature and (except for a small region near the axis of symmetry) overestimates the velocity. Or, to put it loosely, the approximate solutions overestimate the temperature and underestimate the velocity as the Prandtl number σ increases from the σ on which they are based, and the opposite is true when σ decreases.

But, the striking feature of Fig. 2 is that the approximate solutions give rather good predictions for both the velocity and the temperature, and that the temperature predictions are especially accurate. The rather good approximations obtained when σ varies from 1 to 2 are all the more remarkable when one notes that at $\eta=0$ (or $X=1$) the R in (28), which is neglected, is, for $\sigma=2$, 18% of the entire left-hand side of (28), and yet in Fig. 2 the maximum discrepancy between the exact and approximate solutions for $\sigma=2$ is only about 7% for $\eta \leqslant 1.5$, and that the absolute value of the error is never more than 7% of the maximum value for velocity or for temperature (at the axis of symmetry). When σ varies from 2 to 1 and (36) is used, the left-hand side of (36) varies, at $\eta=0$ (or $X=1$), by fully 28% when X^{-3} is replaced by $1.1X^{-4}$, and yet the agreement between the exact solution and the approximate solutions for $\sigma=1$ is even better, both for the velocity and the temperature distributions, than the agreement be-

tween the exact and approximate solutions for $\sigma=2$.

These comparisons of approximate and exact solutions at $\sigma=1$ and $\sigma=2$ indicate that there is good agreement between the approximate solutions even when the Prandtl number is doubled or halved, especially for the temperature distribution (Fig. 2). This agreement is due largely to the fact that the temperature equation (10) is integrated exactly, once one accepts (24), and that the integral conditions are exactly satisfied by (26). The only approximation is the neglect of R in (28), and we can give an estimate of the error in the results introduced by the neglect for the important case of $\sigma = 0.73$.

For $\sigma=0.73$ and at $\eta=0$, at which R is a maximum, the ratio of R to the entire left-hand side of (28) is 0.133. Thus, the maximum error in A computed from (31), where C is given by (26), is 6.4%, and the error in $A^{1/2}$ is 3.2%, at most. The actual error is probably smaller, since R changes sign at $A^{1/2}\eta=0.374$, and since R can be regarded as a vertical momentum source, so that from the physical point of view the effects due to negative values of R and those due to its positive values are mutually compensating to a certain extent. The constant $A^{1/2}$ determines the angle of spread of the plume, and is the only constant to be determined by the approximate solution. That it can be in error by 3.2% at most for $\sigma=0.73$ is certainly reassuring, in spite of the arbitrariness of the approximation of X^{-4} by $7X^{-3}/8$. For σ nearer 1 than 0.73, the results are even more accurate.

D. Approximate solutions for air and for water

Since the Prandtl number of air is 0.73, which is near 1, a good approximate solution for air is the first approximate solution, which is shown in Fig. 3 together with the exact results for $\sigma=1$, for comparison. From our discussion of the results shown in Fig. 2, the true velocity curve for $\sigma=0.73$ should lie somewhat below the velocity curve shown in Fig. 3 for $\sigma=0.73$, and the true temperature curve should be somewhat above the temperature curve shown in Fig. 3 for $\sigma=0.73$. But these corrections, especially for the temperature curve, are small because σ is quite near 1. We refrain from making further refinements and consider the
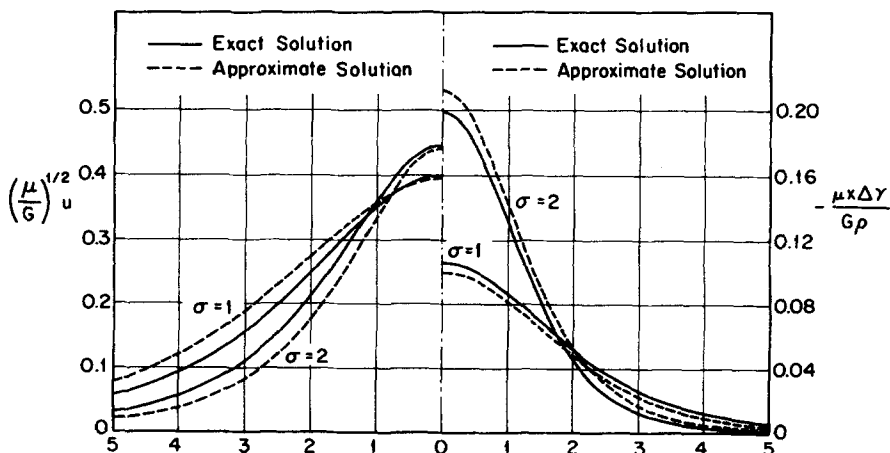

FIG. 2. Comparison of approximate and exact solutions for laminar round plumes for $\sigma=1$ and $\sigma=2$. The abscissa is η.
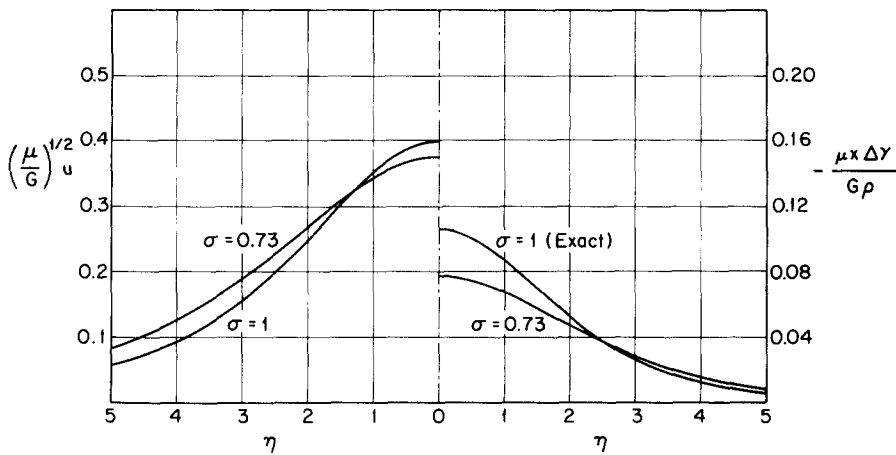
FIG. 3. Velocity and temperature (represented by $\Delta\gamma$) distributions in laminar round plumes for $\sigma = 1$ and $\sigma = 0.73$ (for air).

results given in Fig. 3 as sufficiently accurate.

For water $\sigma = 6.7$, the left-hand side of (36) varies by 20% when X^{-3} is replaced by $1.1 X^{-4}$ in our approximate solution. Judging by the corresponding percentage (28%) when the same is done for $\sigma = 1$, and the rather good agreement shown in Fig. 2 between the exact and approximate solutions for $\sigma = 1$, we venture to assert with some confidence that the second approximate solution (the one that becomes exact for $\sigma = 2$) can be used for $\sigma = 6.7$. Upon using (33)–(35), and (37), as well as (6) and (7), we obtain Fig. 4 for $\sigma = 6.7$. From our discussion in Sec. II C, we can expect the true velocity curve for $\sigma = 6.7$ to be somewhat above the one shown, and the true temperature curve to be somewhat below the one shown. In the core of the plume, we do not expect a relative error of more than a few percent.

There being no exact solution for σ greater than 2, we are obliged to give an error estimate for the approximate solution given by (33)–(35), and (37), when it is applied to water, of which $\sigma = 6.7$.

We note again that, once (33) is accepted, (34) is an exact solution of (10) and the integral condition (12) is

exactly satisfied by (35). These facts contribute to the accuracy of the approximate solution mentioned here. The only approximation is made when we substitute X^{-4} for $X^{-3}/1.1$ in (36). The ratio of the residue

$$R = -\frac{64}{\sigma} A^2 \left(1 - \frac{2}{\sigma}\right)(X^{-3} - \beta^{-1} X^{-4})$$

to the entire left-hand side of (36) is, for $\sigma = 6.7$ and at the axis of symmetry, where R is largest, equal to -0.196. The error introduced in the determination of A^2 is then 24.5% at most, giving an error in $A^{1/2}$, which determined the spread angle, of 5.5% at most. The velocity u is proportional to A, and the error in A is 11.6% at most. For a more than threefold increase in σ (from 2–6.7), an approximate solution with these maximum errors (the actual errors must be less due to the compensative effects of positive and negative values of R) is certainly useful in giving a good estimate of the velocity and temperature fields.

E. Behavior of the solution for large σ

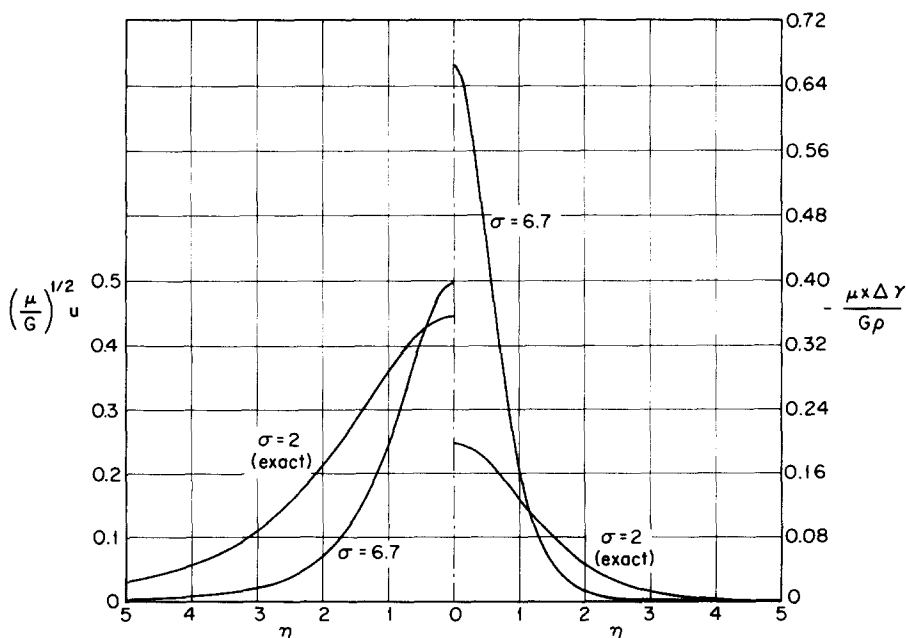For large values of σ we can make the transformations

FIG. 4. Velocity and temperature (represented by $\Delta\gamma$) distributions in laminar round plumes for $\sigma = 2$ and $\sigma = 6.7$ (for water).

C. -S. Yih and F. Wu

$$\hat{f}=\sigma f, \quad \hat{\theta}=\theta/\sigma, \quad \hat{\eta}=\sigma^{1/2}\eta .$$

Then, Eqs. (9), (10), and (12) become

$$\left(\frac{\hat{f}'}{\hat{\eta}}\right)' = \hat{f}''' + \hat{\eta}\hat{\theta}, \quad \hat{f} = -\frac{1}{4}\frac{\hat{\theta}'}{\hat{\theta}}\hat{\eta},$$

$$\int_0^\infty \hat{f}\hat{\theta}d\hat{\eta} = \frac{1}{8\pi},$$

where the primes indicate differentiation with respect to $\hat{\eta}$. Since $\hat{f}$ and $\hat{\theta}$ depend only on $\hat{\eta}$, we can say that at any value of x if r is reduced by the factor $\sigma^{1/2}$, the function f is reduced by the factor σ and θ is increased by the factor σ. The rate of widening of the plume is, in this sense, inversely proportional to $\sigma^{1/2}$. All this is in agreement with (32) or (37), if we recall the definition (23) of X and the role of A in X.

F. A transformation indispensable for numerical calculations

If a numerical solution of the differential system consisting of (9)–(12) is attempted, one quickly sees that (12), being an integral condition the satisfaction of which can only be tested *after* the computation, would give rise to great difficulties. It would be nice if a solution satisfying (9)–(11) can be made to satisfy (12) by some transformation. Such a transformation will be given.

Suppose that a solution (f, θ) satisfies (9)–(11), but gives

$$\int_0^\infty f'\theta d\eta = \frac{1}{8\pi}\alpha^4.$$

Then, let

$$f = f(\hat{\eta}), \quad \theta = \alpha^4 \hat{\theta}(\hat{\eta}), \quad \hat{\eta}=\alpha\eta .$$

These substitutions leave (9) to (11) invariant in form, but give

$$\int_0^\infty f'\hat{\theta} d\hat{\eta} = \frac{1}{8\pi},$$

where $f' = df/d\hat{\eta}$. Thus, $f(\hat{\eta})$ and $\hat{\theta}(\hat{\eta})$ are the solutions of the differential system. With this transformation available, we can impose the following conditions at $\eta = 0$:

$$f(0) = 0 = f'(0), \quad \theta(0) = 1, \quad f''(0) = a .$$

The constant a is chosen to satisfy the conditions at $\eta = \infty$, and different values of it must be tried before one is found for which the conditions at infinity are satisfied. Then, the transformation just given is applied to get $f(\hat{\eta})$ and $\hat{\theta}(\hat{\eta})$, which satisfy the integral condition (12) as well as (9)–(11).

This transformation is also useful in any numerical solution of the differential system governing turbulent round plumes. But, as will be seen, we shall have a very good analytical solution of that system for the turbulent plume for which the (turbulent) Prandtl number is near 1, and thus shall have no need for a numerical solution at all.

III. THE ROUND TURBULENT PLUME

We shall retain the use of the coordinates and the velocity components as hitherto in this paper, but we shall use an eddy viscosity ϵ which is supposed to be only a function of x and not of r. The form of the eddy viscosity can be found by a dimensional analysis to be

$$\epsilon = \lambda(Gx^2/\rho)^{1/3}F(r/x),$$

where λ is a dimensionless coefficient. (The Reynolds stress calculated from this expression for ϵ, with $\lambda = 0.016$ obtained from the mean velocity profile, has been compared with the Reynolds-stress measurements of Beuther *et al.*[7] and excellent agreement has been found.) In ignoring the transverse variation of ϵ, which we cannot determine analytically in any case, we are simply dropping $F(r/x)$ and assuming[8]

$$\epsilon = \lambda (Gx^2/\rho)^{1/3} . \tag{38}$$

Then, understanding u and v to be the time-mean velocity components at any point in space, the equation of motion and the diffusion equation are, respectively,

$$uu_x + vu_r = \frac{\epsilon}{r}\frac{\partial}{\partial r}(ru_r) - g\frac{\Delta\gamma}{\gamma_0} , \tag{39}$$

$$u\frac{\partial}{\partial x}\Delta\gamma + v\frac{\partial}{\partial r}\Delta\gamma = \frac{\epsilon}{\sigma r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\Delta\gamma\right) . \tag{40}$$

The equation of continuity is (3) and (4) can again be used, as well as the boundary conditions stated just after (3).

We note that, since the flow is turbulent, strictly speaking G is now given by

$$G = -2\pi\int_0^\infty r(u\Delta\gamma + \overline{u'\Delta\gamma'})dr , \tag{41}$$

where the primed quantities denote turbulent fluctuations. We assume the contribution of the turbulent part to be a definite fraction of G defined by (41), which indeed it is, and continue to use the nominal G defined by (5) to represent the strength of the source. This practice has the advantage not only of simplicity of exposition, but is convenient when the analytical results to be given in this paper are compared with available experimental results, which have all been given in terms of the G defined by (5).

Then, the appropriate forms for ψ and $\Delta\gamma$, arrived at by a dimensional analysis, are

$$\psi = 3\lambda(Gx^5/\rho)^{1/3}f(\eta) , \tag{42}$$

$$-\Delta\gamma = 3\lambda^2(\rho G^2/x^5)^{1/3}\theta(\eta) , \tag{43}$$

where

$$\eta = r/x . \tag{44}$$

The velocity components are then given by

$$u = 3\lambda(G/\rho x)^{1/3}f'/\eta , \tag{45}$$

$$v = \lambda(G/\rho x)^{1/3}(3f' - 5f/\eta) , \tag{46}$$

and the equations of motion and of diffusion assume the following dimensionless forms[8]

$$(1 - 5f)(f'/\eta)' - f'^2/\eta = f''' + \eta\theta , \tag{47}$$

$$-5\sigma(f\theta)' = (\eta\theta')' , \qquad (48)$$

where σ denotes the turbulent Prandtl number. The boundary conditions remain (11), and the integral condition (5) becomes

$$18\pi\lambda^3 \int_0^\infty f'\theta\,d\eta = 1 . \qquad (49)$$

A. Exact solutions for the round turbulent plume

The differential system consisting of (47) to (49) was solved exactly by Yih[8] for $\sigma = 1.1$ and $\sigma = 2$. The results are

$$f = B[1 - (1 + A\eta^2)^{-1}] , \qquad (50)$$

$$\theta = C/(1 + A\eta^2)^m , \qquad (51)$$

in which, for $\sigma = 1.1$,

$$B = 12/11, \quad m = 3, \quad C = (1536/121)A^2 . \qquad (52)$$

From (49) we obtain

$$C\lambda^3 = 11/54\pi, \quad A^2\lambda^3 = 1331/82944\pi . \qquad (53)$$

Thus, only λ needs to be determined experimentally.

For $\sigma = 2$,

$$B = \tfrac{4}{5}, \quad m = 4, \quad C = (256/25)A^2 , \qquad (54)$$

and use of (49) gives

$$C\lambda^3 = 25/72\pi, \quad A^2\lambda^3 = 625/18432\pi . \qquad (55)$$

For comparison of experimental results, we note that

$$u\left(\frac{\rho x}{G}\right)^{1/3} = \frac{6\lambda AB}{(1 + A\eta^2)^2} = \frac{3\lambda f'}{\eta} . \qquad (56)$$

B. Approximate solutions for the round turbulent plume

We can build approximate solutions upon the exact solutions for turbulent plumes in the same way as for laminar plumes. Expansions like (21) and (22), but without the logarithm terms, can be used, in principle, but converge very slowly and are, therefore, very cumbersome in practice. Instead, we approximate X^{-4} by βX^{-3} or X^{-3} by $\beta^{-1}X^{-4}$, as the case may be, where $\beta = 7/8$ or $\beta = 10/11$, respectively. With this approxi-

mation, we obtain two approximate solutions.

1. *The first approximate solution*—based on the exact solution for $\sigma = 1.1$. With a method strictly similar to that used in Sec. IIB, we obtain the approximate solution

$$f = \frac{6}{5\sigma}(1 - X^{-1}), \quad \theta = \frac{5\sigma}{27\pi\lambda^3} X^{-3}, \quad X = 1 + A\eta^2 ,$$

$$A^2 = \frac{5}{27\pi\lambda^3}\frac{\sigma^3}{2.16 + 12\sigma} , \qquad (57)$$

where η is defined by (44). This solution is exact for $\sigma = 1.1$.

2. *The second approximate solution*—based on the exact solution for $\sigma = 2$. With the same definition for X, and by a method similar to that employed in Sec. IIB, we also have the approximate solution

$$f = \frac{8}{5\sigma}(1 - X^{-1}), \quad \theta = \frac{25\sigma}{144\pi\lambda^3} X^{-4}, \quad A^2 = \frac{625\sigma^2}{73728\pi\lambda^3} . \qquad (58)$$

This solution is exact for $\sigma = 2$.

C. Comparison of approximate and exact solutions

In Fig. 5, the first approximate solution evaluated at $\sigma = 2$ is compared with the exact solution for $\sigma = 2$, and the second approximate solution evaluated at $\sigma = 1.1$ is compared with the exact solution for $\sigma = 1.1$. Again, although the change in σ is nearly double (or one-half) that for which the approximate solution is exact, it still gives satisfactory results. This gives us confidence that for σ equal to 1 or near 1, as it should be for turbulent flows, the first approximate solution will give very accurate results.

We can, indeed, give an estimate of the error introduced by the approximation of X^{-4} by $7/8X^3$. We note first that the solution (57) satisfies (48) and (49) exactly, and that the approximation only affects (47). This approximation introduces a residue on the left-hand side of (47), after division by η, and for $\sigma = 1$ the ratio of the maximum value of this residue (at the axis of symmetry) to the value of the left-hand of (47) can be shown to be -0.051. The error in A^2 determined by
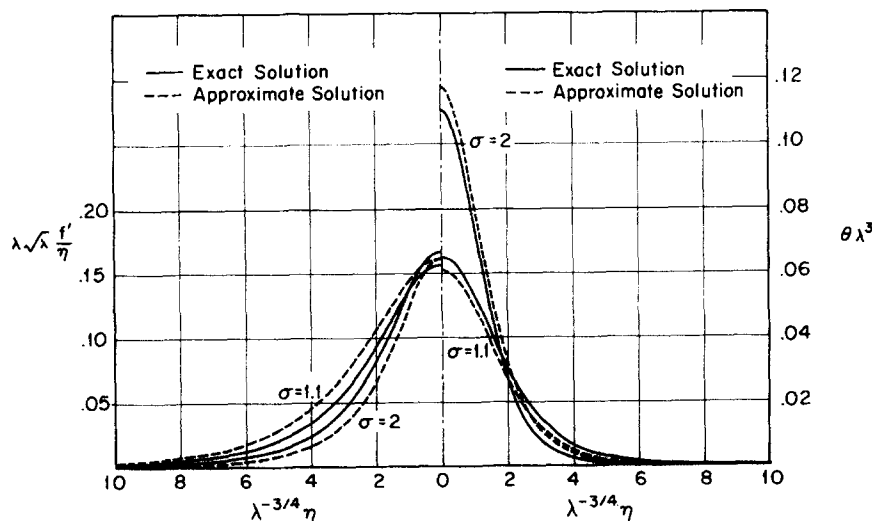

FIG. 5. Comparison of approximate and exact solutions for turbulent round plumes, for turbulent Prandtl numbers $\sigma = 1.1$ and $\sigma = 2$.
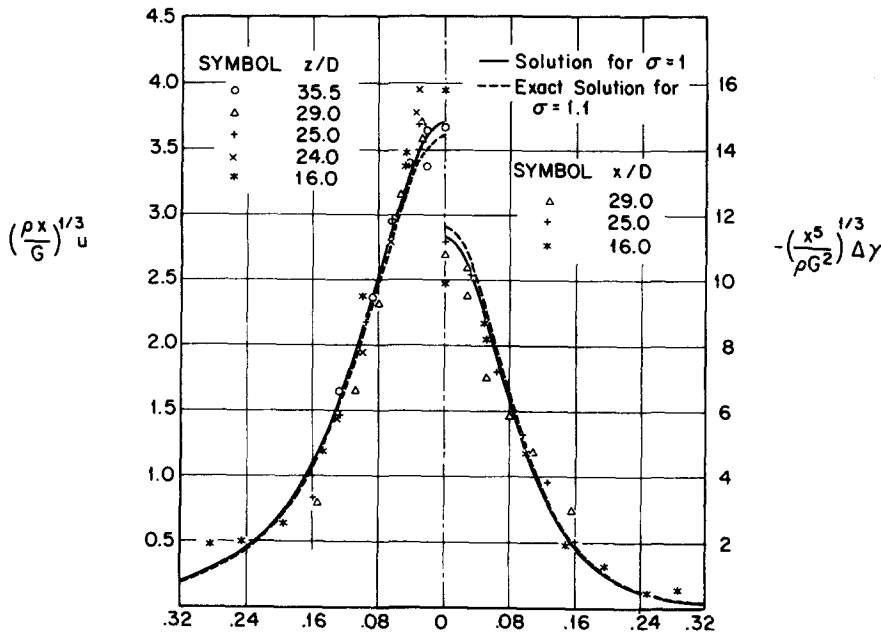
The y-axis left label: $\left(\dfrac{\rho x}{G}\right)^{1/3} u$

The y-axis right label: $-\left(\dfrac{x^5}{\rho G^2}\right)^{1/3} \Delta \gamma$

FIG. 6. Comparison of analytical results for turbulent Prandtl number 1 and λ = 0.0156 with the experimental data of Beuther et al.[7] The abscissa is η.

the last equation in (57) is thus 5.4% at most, and the error in $A^{1/2}$, which determines the spread angle of the plume is only 1.3% at most. The error in the vertical velocity is 2.7% at most. These maximum errors are certainly within the range of errors of any experimental data.

D. The proposed solution for round turbulent plumes

Taking solution (57) and fitting it to the data of Beuther, Capp, and George Jr.[7], we found that the best fit gives $\sigma = 1$, $\lambda = 0.0156$, $A = 33$. [In our comparison with the experimental data, we have taken into account the statement written on the reprint of their paper, which they kindly sent to us, that their F_0 (corresponding to our G) is 20% too high.] The comparison between the analytical curves for velocity and temperature distributions with their experimental data is shown in Fig. 6. The dotted curves are for $\sigma = 1.1$ and the same values of λ and A. It seems that the curves for $\sigma = 1$ give a slightly better fit than those for $\sigma = 1.1$, but that both the solid curves and the dotted ones fit the experimental data very well. We propose, then, that (57) with the above-mentioned values for λ and A and with $\sigma = 1$ be adopted as the solution for the problem of the round turbulent plume.

ACKNOWLEGMENT

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