CONDITIONS ON STRUCTURE MATRICES ON N-PORT, IMMITTANCE-EQUIVALENT ELECTRIC NETWORKS

Anthony J. Pennington

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
Department of Electrical Engineering

November, 1963

IP-640

ACKNOWLEDGEMENTS

The author thanks his doctoral committee chairman, Professor E. L. McMahon for his inspiration, advice, and encouragement throughout this research effort. He also thanks Professor G. J. Minty of the Department of Mathematics for his careful review of Chapters I-III, and for his many valuable suggestions for improvements. He thanks Professors Brown, Cutrona, and Scott for their interest in and encouragement of this study. Finally, he thanks Mrs. J. M. Ireland for her superb work in preparing the final manuscript.

The major part of this work was supported by the U. S. Air Force, Wright Air Development Command, under Contract No. AF 33(616)-7630, administered by the Radar Laboratory, Institute of Science and Technology, The University of Michigan. Partial support was provided by the U. S. Army Electronics Command, under Contract No. DA-36-039 sc-89227, administered by the Cooley Electronics Laboratory, Department of Electrical Engineering, The University of Michigan.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

Symbol	Meaning	Page of First Appearance
a_{ui}, a_{vj}	general entries of $K_{ee}\bar{C}_{ee}$	50
b	number of branches of a t-graph	9
b'	number of branches adjoined to a t-graph	41
b_{ui}, b_{vj}	general entries of $C_{ee}M_{ee}$	50
c_{ij}	general entry of C_{ee}	10
C_{ee}	fundamental connection matrix of a t-graph of a given network	10
$\bar{C}_{ee}$	formal representation for fundamental connection matrix of a t-graph of a new network	47
C_{bk}, C_{bn}, etc.	submatrices of C_{ee}	11
$\bar{C}_{bk}$, $\bar{C}_{bm}$, etc.	submatrices of $\bar{C}_{ee}$	47
d	number of degrees of freedom	60
D	an expression	67
e	number of arcs of a t-graph	9
E_{el}, E_{kl}, etc.	e-port, k-port, etc. potential matrices	19
i	general matrix entry row subscript	10
$\bar{I}_1$, $\bar{I}_2$, etc.	current-space basis vectors	24
I_{el}, I_{kl}, etc.	e-port, k-port, etc. current matrices	19
j	general matrix entry column subscript	10
J_{el}	element current matrix	5
J_{kl}, J_{bl}, etc.	submatrices of J_{el}	20
k	number of links of a t-graph	9
k'	number of links adjoined to a t-graph	41

LIST OF SYMBOLS (Cont.)

Symbol	Meaning	Page of First Appearance
k_{11}, k_{12}, etc.	general entries of K_{ee}	65
K_{ee}, K'_{ee}	matrices representing element current coordinate transformations	38
K_{nn}, K'_{pp}, etc.	submatrices of K_{ee} and K'_{ee}	53
L_{ee}, L'_{ee}	matrices representing element potential coordinate transformations	38
L_1, L_2, etc.	inductive element values	72
m	number of m-ports	16
M_{ee}, M'_{ee}	matrices representing e-port current coordinate transformations	31
M_{nn}, M'_{kk}, etc.	submatrices of M_{ee} and M'_{ee}	32
n	number of n-ports	16
N	general number of ports, as in "N-port"	title
N_{ee}, N'_{ee}	matrices representing e-port potential coordinate transformations	31
N_{nn}, N'_{kk}, etc.	submatrices of N_{ee} and N'_{ee}	33
p	number of p-ports	16
q	number of q-ports	16
R_1, R_2, etc.	resistive element values	61
s	complex frequency	6
$(\ )^t$	transpose of matrix in parentheses	21
u	general matrix entry row subscript	50
U_{kk}, U_{bb}, etc.	unit matrices	11
v	(i) number of vertices of a t-graph (ii) general matrix entry column subscript	9 50
$\bar{v}_1$, $\bar{v}_2$, etc.	potential-space basis vectors	25

LIST OF SYMBOLS (Cont.)

Symbol	Meaning	Page of First Appearance
V_{el}	element potential matrix	6
V_{kl}, V_{bl}, etc.	submatrices of V_{el}	21
Y^c_{pp}	p-port admittance matrix of a passive network	28
Y^e_{ee}	element admittance matrix	6
Y^f_{ee}	e-port admittance matrix	23
Y^f_{bb}, Y^f_{bk}, etc.	submatrices of Y^f_{ee}	23
Z^c_{nn}	n-port impedance matrix of a passive network	27
Z^e_{ee}	element impedance matrix	6
Z^e_{kk}, Z^e_{kb}, etc.	submatrices of Z^e_{ee}	22
Z^f_{ee}	e-port impedance matrix	22
Z^f_{kk}, Z^f_{kb}, etc.	submatrices of Z^f_{ee}	23
0_{kb}, 0_{kq}, etc.	zero matrices	11
$(\bar{\ })$, $(\bar{\bar{\ }})$	transformed matrices or element values	31

CHAPTER I

INTRODUCTION

Two electric networks are said to be equivalent with respect to a set of electrical characteristics if these characteristics of both networks are identical. This study treats the problem of finding passive networks equivalent to a given passive network with respect to immittance characteristics at selected terminal-pairs (ports), when both networks are composed of passive, linear, time-invariant, bilateral, two-terminal, electrical elements. We further restrict the study to connected networks without self-loops. The ports permitted in this analysis are

(i) subsets of the terminal-pairs created by cutting elements of a cotree associated with a specified tree of the network, or

(ii) subsets of the terminal-pairs created by connecting to the ends of the elements of a specified tree of the network.

We shall show in Chapter II that more general port configurations can be accommodated in this scheme by the introduction of "dummy" elements.

An equivalent network can differ from the associated given network in regard to its

(i) electromagnetic structure, or number and type of electrical elements,

(ii) topological structure (in the sense of graph theory), or pattern of connection of the elements, and

(iii) port structure, or locations of accessible terminal-pairs.

The problem of finding networks equivalent to a given one is important for both analysis and synthesis and has received considerable attention in the literature. The first general approach was W. Cauer's (Ref. 1) method of linear transformation of port currents and potentials, here called "port coordinates." The immittance matrices relating the new port coordinates are found, but the method does not realize structures (i), (ii), and (iii) individually, and yet these are required to construct the network. G. Kron (Ref. 7) discussed equivalent network problems from the point of view of his "orthogonal" method of analysis, and obtained Cauer's results as a special case. He also introduced the concept of element current and voltage transformations or "element coordinate" transformations. The immittance matrices relating the new element coordinates can be considered to represent new electromagnetic structures, so that this transformation permits direct specification of the equivalent network's electromagnetic structure. Recently J. D. Schoeffler (Ref.10) discussed element coordinate transformations leading to equivalent networks whose topological and port structures are identical to those of the given network. E. A. Guillemin (Ref. 4) and other workers in modern network theory have also discussed the equivalence problem, generally from the point of view of Cauer's method.

This study pursues Kron's approach based on element coordinate transformations. The major result is the determination of necessary and sufficient conditions on the matrices which represent the electromagnetic, topological, and port structures of networks equivalent to a given network. Additional conditions are imposed to make the new element and port immittance matrices symmetric. This result advances the theory of equivalent networks by permitting, for the first time, direct specification

of the matrices representing the electromagnetic, topological, or port structures of equivalent networks, followed by formal solution for the unspecified matrices. The major unsolved problem is to guarantee that these formal solution matrices always correspond to realizable structures. We show in Chapter V, however, that preliminary specification of realizable topological and port structure matrices leaves some freedom for the determination of electromagnetic structure matrices of equivalent networks. This freedom makes it possible to find example problems in which completely realizable equivalent networks are obtained.

The method used in Chapter IV to obtain the conditions described above is based on a new version of Kron's (Ref. 8) "orthogonal" method of analysis. The significant feature of the method is that the matrices representing topological and port structures are nonsingular. This method is presented in Chapter II, while Chapter III is devoted to element and port coordinate transformations. The necessary and sufficient conditions on equivalent network structure matrices are derived in Chapter IV. Chapter V contains examples showing the application of the conditions to finding networks equivalent to a given one. Chapter VI contains conclusions and a discussion of possible approaches to some unsolved problems suggested by this study.

CHAPTER II

NETWORK ANALYSIS

The port immittance characteristics of a network composed of passive, linear, time-invariant, bilateral, two-terminal, electrical elements are determined by its

(i) electromagnetic structure,

(ii) topological structure, and

(iii) port-structure.

The electromagnetic structure can be represented by element immittance matrices relating the currents through and potentials across the elements, as described in Section 2.1. The topological structure can be represented by a nonsingular "fundamental connection matrix," described in Section 2.2, which is a modification of Kron's "connection matrix." The port structure can be represented by row-column ordering and partitioning of a "fundamental connection matrix," as described in Section 2.3.

By network analysis we mean the determination of an immittance matrix relating the currents into and potentials across a designated set of ports from a knowledge of the electromagnetic, topological, and port structures of a given network. The method presented in Sections 2.4 and 2.5 for finding port immittance matrices is a modified version of Kron's "orthogonal" method of network analysis.

2.1. Representation of Electromagnetic Structures

The following terms are defined for a set of passive, linear, time-invariant, bilateral, two-terminal electrical elements, henceforth called "elements."

Element current coordinates. The element current coordinates are the set of steady-state, frequency-dependent, signed, positive charge currents flowing through a set of electrical elements. The positive current directions are assigned arbitrarily. The element current coordinates for a set of e elements can be represented by the

Element current matrix, J_{e1}. The element current matrix, J_{e1}, is an e x 1 matrix whose entries are the element current coordinates of a set of elements.

Throughout this study we shall indicate matrix row and column dimensions by means of subscripts, as above, and, when necessary to avoid ambiguity, the matrix type by means of superscripts. These conventions enable us to distinguish matrices identified by the same letter and provide a quick check of the dimensional validity of matrix expressions. Often the letters used for subscripts and superscripts will also be used as variables with numerical values. These numerical values should not be substituted for the letter subscripts and superscripts since these are a part of the matrix symbol. Thus, the symbol for the element current matrix of a five element network (e = 5) is J_{e1}, and not J_{51}. Although this usage may seem confusing at first it will prove to be very helpful in keeping track of various matrices and the physical quantities they represent.

Element potential coordinates. The element potential coordinates are the set of steady-state, frequency-dependent, signed potential-rises

across a set of elements. We choose to let the negative to positive potential-rise be in the direction opposite to the assigned positive element current direction. The element potential coordinates can be represented by the

<u>Element potential matrix,</u> V_{el}. The element potential matrix, V_{el}, is an e x 1 matrix whose entries are the element potential coordinates of a set of elements.

<u>Element impedance matrix,</u> Z^e_{ee}. The element impedance matrix, Z^e_{ee}, is an e x e matrix whose entries are constants times the impedance operators, s, 1, or s^{-1}, which relates J_{el} and V_{el} by the equation,

$$V_{el} = Z^e_{ee} J_{el} \quad . \tag{2.1}$$

The constants which multiply s, 1, or s^{-1} are the inductance, resistance, and elastance parameters or element-values, respectively. Non-zero off-diagonal entries indicate electromagnetic field coupling, which, in general, exists only among inductive elements, although capacitive element coupling is also possible (Ref. 3). The element impedance matrix of a set of bilateral elements is symmetric; passivity requires that the matrices of inductance, resistance, and elastance constants be positive-semidefinite. The restriction to constant multipliers is a consequence of element linearity and time-invariance.

<u>Element admittance matrix,</u> Y^e_{ee}. The element admittance matrix, Y^e_{ee}, is an e x e matrix whose entries are constants times the admittance operators, s^{-1}, 1, or s, which relates J_{el} and V_{el} by the equation,

$$J_{el} = Y^e_{ee} V_{el} \quad . \tag{2.2}$$

The constants which multiply s^{-1}, 1, or s are the reciprocal inductance, conductance, and capacitance parameters, or element-values,

respectively. The properties of Y^e_{ee} are the same as those of Z^e_{ee} in all respects. Also, if $(Y^e_{ee})^{-1}$ exists it is, by (2.1) and (2.2), clearly identical to Z^e_{ee}.

We note that the element immittance matrices defined here relate the element coordinates independently of any additional constraints which may be required by the connection of the elements. That is, Z^e_{ee} and Y^e_{ee} represent the electromagnetic structure independently of the topological structure.

2.2 Representation of Topological Structures

In order to represent its topological structure we can treat an electric network as an oriented linear graph whose arcs replace the electrical elements and whose vertices replace the nodes or connection points. It is convenient to let the arc orientations be the assigned positive current coordinate directions of the associated elements. We have stated in Chapter I that we shall treat only connected networks without self-loops, so that the corresponding oriented graphs are also connected and without self-loops. Figure 2.1 is an example of an oriented linear graph representing the topological structure of a Wheatstone Bridge network.

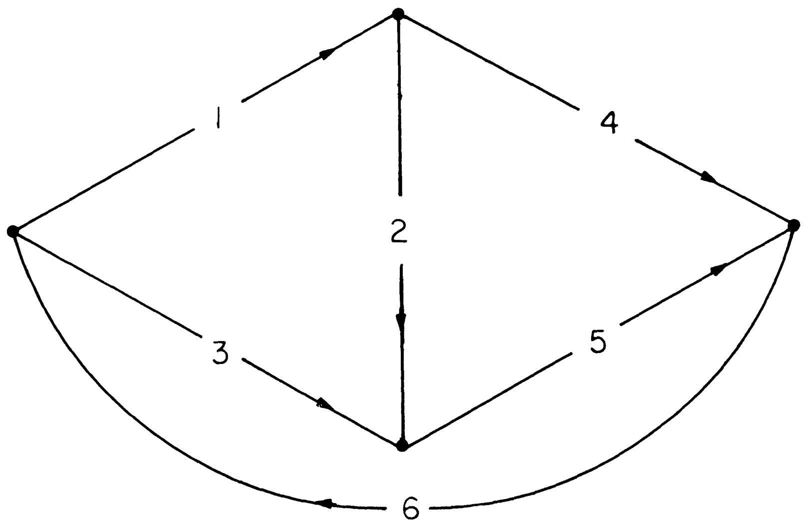


Fig. 2.1. Oriented linear graph (Wheatstone Bridge).

We shall assume that the reader is familiar with the concepts of a tree and its cotree associated with a connected graph, or can easily find discussions of them in the literature (Refs. 5 and 11). The following

terms are defined for the specific purposes of this study.

t-graph. A t-graph is an oriented linear graph with a distinguished (specified) tree.

b-path of a t-graph. A b-path of a t-graph is any arc of the distinguished tree.

k-path of a t-graph. A k-path of a t-graph is any arc of the cotree corresponding to the distinguished tree, plus the unique set of tree arcs which must be connected to join the terminal vertices of the defining cotree arc.

A k-path is often called a "fundamental circuit" in treatments of network topology (Ref. 11). The terms "branch" for arc of a tree, and "link" for arc of a cotree are also standard nomenclature, and shall be used here with the understanding that they always refer to the distinguished tree of a specific t-graph.

A connected t-graph with e arcs and v vertices has

$b = v - 1$ tree arcs, or branches, and

$k = e - b$ cotree arcs, or links.

The terms "b-path," "k-path," and other terms involving letter prefixes to be defined later are treated as the names of objects, so that it is improper to substitute the numerical values of variables designated by the same letters. Thus, if a t-graph has three branches ($b = 3$) its tree arcs are still called "b-paths," not "3-paths." This convention is similar to that for matrix subscripts and superscripts.

We can now define a "fundamental connection matrix of a t-graph." This object will be used throughout the study to represent the topological and port structures of networks.

Fundamental connection matrix of a t-graph, C_{ee}. The fundamental connection matrix of a t-graph, C_{ee}, is an e x e matrix whose entries, c_{ij} (in row i and column j), are given by the rule:

$$c_{ij} = \begin{cases} 0 \text{ if arc i is not a member of b- or k-path j;} \\ 1 \text{ if arc i is a member of b- or k- path j, and if traversal of the path in the positive direction of arc j results in positive traversal of arc i;} \\ -1 \text{ if arc i is a member of b- or k-path j and if traversal of the path in the positive direction of arc j results in negative traversal of arc i.} \end{cases}$$

It is always possible to arrange the rows and columns of C_{ee} in such a way that the first k rows correspond to the links (cotree arcs) and the first k columns correspond to the k-paths defined by the links in the same order. Likewise, the last b rows can be made to correspond to the branches (tree arcs) and the last b columns to the b-paths defined by the branches in the same order. Henceforth we shall assume that the rows and columns of all fundamental connection matrices, C_{ee}, are arranged in this way.

We emphasize that the terms "link" and "branch" must refer to a specific t-graph. For example, in the above paragraph they refer to the t-graph for which C_{ee} is written. Also, we shall consider that the phrase "of a t-graph" is implied whenever the terms "k-path," "b-path," and "fundamental connection matrix" are used.

The fundamental connection matrix defined here is a special case of Kron's "connection matrix," which is similar, except that the "paths" on which it is based need not be defined with respect to a tree of the graph. This specialization yields a particularly simple representation. The use of the word "fundamental" follows common practice for designating topological structure matrices based on a tree (Ref. 11).

We shall now consider some properties of fundamental connection matrices. First, we show that under the row-column ordering described above C_{ee} has the form

$$C_{ee} = \begin{bmatrix} U_{kk} & 0_{kb} \\ C_{bk} & U_{bb} \end{bmatrix}, \tag{2.3}$$

where U_{kk} and U_{bb} are unit matrices, 0_{kb} is a zero matrix, and C_{bk} is a matrix of 0's, 1's, and -1's. The upper-left submatrix is a unit matrix because each link is a positively oriented member of the k-path which it defines, and is a member of no other k-path; the lower-right submatrix is a unit matrix because each branch is a positively oriented member of the b-path which it defines, and is a member of no other b-path; the upper-right submatrix is a zero matrix because no link is a member of a b-path; the lower-left submatrix entries are 0, 1, or -1 depending on the presence and orientation of each branch in the k-paths. Since all submatrices except C_{bk} would be the same for any e arc, v vertex t-graph, the connection information in C_{ee} is carried, essentially, by C_{bk}.

The most significant property of C_{ee} for this study is that it is nonsingular. The nonsingularity can be established immediately by noting that C_{ee} is triangular with nonzero diagonal entries, so that the determinant, being the product of the diagonal entries, must be nonzero. Furthermore, the inverse matrix is simply

$$C_{ee}^{-1} = \begin{bmatrix} U_{kk} & 0_{kb} \\ -C_{bk} & U_{bb} \end{bmatrix}. \tag{2.4}$$

Equation (2.4) can be checked by multiplying both sides by C_{ee} using the usual rule for multiplying partitioned matrices (Ref. 12).

It is possible to interpret this inverse matrix in terms of the associated t-graph by defining the following terms.

<u>b-copath of a t-graph.</u> A b-copath of a t-graph is any branch plus the unique set of links which must be cut to separate the terminal vertices of the branch.

<u>k-copath of a t-graph.</u> A k-copath of a t-graph is any link.

A b-copath is also called a fundamental cut-set in treatments of network topology (Ref. 11).

The entries of C_{ee}^{-1} give the membership and orientation of each arc (indexed by the columns) in the b- and k-copaths (indexed by the rows). Positive orientation means that the member arc points in the same direction, with respect to the terminal vertices of the arc which defines the b- or k-copath, as the defining arc itself. The upper left submatrix of C_{ee}^{-1} is a unit matrix because each link is a positively oriented member of the k-copath which it defines, and is a member of no other k-copath; the lower right submatrix is a unit matrix because each branch is a positively oriented member of the b-copath which it defines, and is a member of no other b-copath; the upper right submatrix is a zero matrix because no branch is a member of a k-copath; the lower left submatrix entries are 0, 1, or -1 depending on the membership and orientation of the links in the b-copaths. The lower left submatrix is equal to $-C_{bk}$ because every branch which was a positively oriented member of a k-path must define a b-copath which includes the k-path defining link in the negative sense. This fact can be seen from Fig. 2.2 which shows a branch (b) and a link (k) with intervening arcs represented by the dotted lines. If branch b is a positively oriented member of the k-path defined by link k, then link k is a negatively oriented member of the b-copath defined by branch b.

Fig. 2.2. Relation between branch orientation in k-path and link orientation in b-copath.

To provide an example of a fundamental connection matrix we shall let the oriented graph of Fig. 2.1 be a t-graph by distinguishing (specifying) the tree composed of arcs 4, 5, and 6. The fundamental connection matrix of this t-graph is

$$C_{ee} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} .$$

All entries but those of the lower left 3 x 3 submatrix would be the same for any six arc, four vertex t-graph; the columns of this submatrix are obtained by noting the membership and orientation of the branches (arcs 4, 5, and 6) in the k-paths (defined by arcs 1, 2, and 3).

2.3 Representation of Port Structures

We shall now define various port structures, or sets of terminal-pairs, and show how they can be represented by means of row-column ordering and partitioning of a fundamental connection matrix. In the sections following this one we shall define the current and voltage coordinates associated with these ports and derive the immittance matrices which relate them.

k-ports of a t-graph. The k-ports of a t-graph are the terminal-pairs created by cutting the links.

b-ports of a t-graph. The b-ports of a t-graph are the terminal-pairs created by connecting to the terminal vertices of the branches.

e-ports of a t-graph. The e-ports of a t-graph are its combined k- and b-ports.

We shall generally drop the phrase, "of a t-graph," when referring to these objects, with the understanding that it is always implied. We also require that numerical values not be substituted for the letter prefixes. We shall refer to the physical ports of an electric network as k-, b-, and e-ports, the implication being that the elements involved correspond to a specific t-graph associated with the network.

We note that each network element is uniquely associated with an e-port of the network. (The prefix, "e," means "element," not "voltage.")

The special subsets of the e-ports of interest for equivalence studies are defined below.

n-ports of a t-graph. The n-ports of a t-graph are the members of a designated subset of the k-ports.

p-ports of a t-graph. The p-ports of a t-graph are the members of a designated subset of the b-ports.

It is also convenient to have special terms for the remaining subsets of the e-ports.

m-ports of a t-graph. The m-ports of a t-graph are the members of the subset of the k-ports complementary to the set of n-ports.

q-ports of a t-graph. The q-ports of a t-graph are the members of the subset of the b-ports complementary to the set of p-ports.

The usage described above for the terms k-, b-, and e-ports also applies for the n-, p-, m-, and q-ports.

This scheme of port classification is indicated schematically in Fig. 2.3.

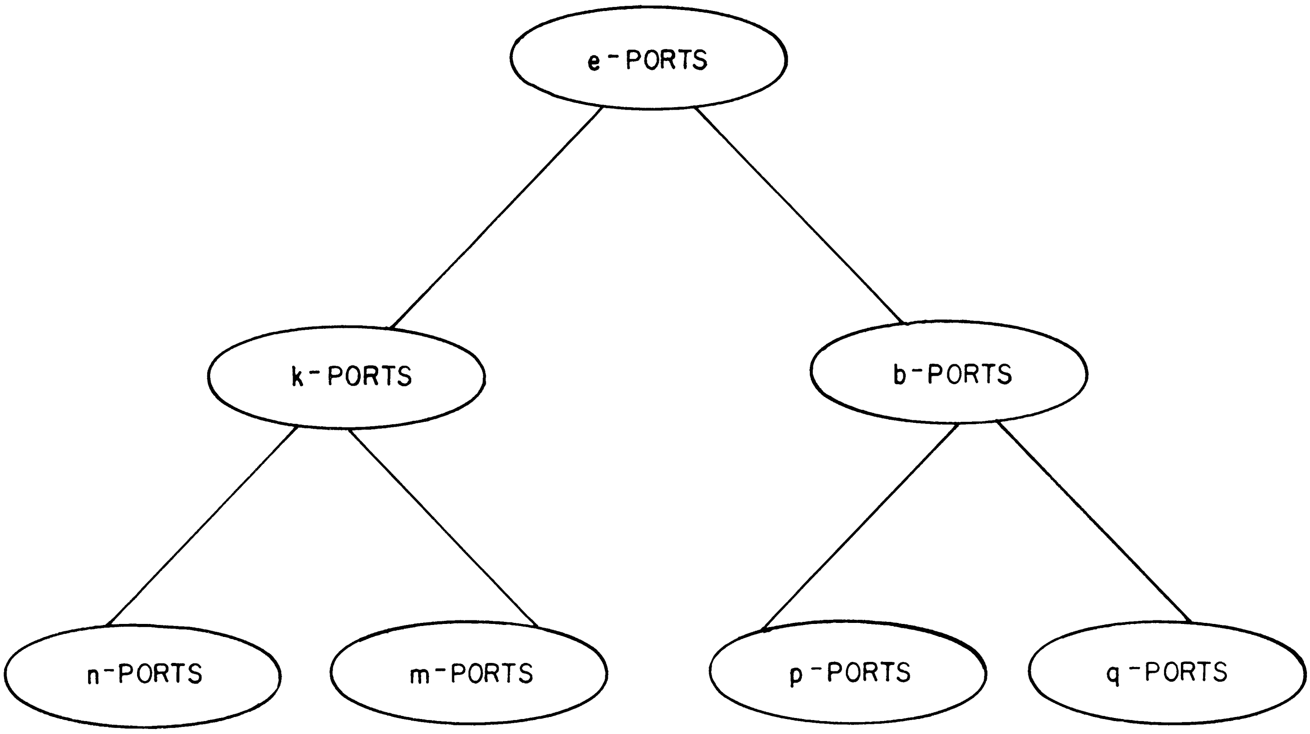


Fig. 2.3. Classification of e-ports.

We can now consider the problem of representing these port structures by means of row-column ordering and partitioning of a fundamental connection matrix, C_{ee}. We have seen that each network element has associated with it a k- or b-path and a k- or b-port. The ports are further subdivided into n-, m-, p-, and q-ports. This association permits us to

designate port locations in terms of C_{ee}, even though its entries do not refer to the ports directly. We choose the following row-column ordering of C_{ee} to represent port structures:

(i) the first n rows correspond to the elements which define the n-ports, and the first n columns correspond to the associated k-paths in the same order;

(ii) the next m rows correspond to the elements which define the m-ports, and the next m columns correspond to the associated k-paths in the same order;

(iii) the next q rows correspond to the elements which define the q-ports, and the next q columns correspond to the associated b-paths in the same order;

(iv) the last p rows correspond to the elements which define the p-ports, and the last p columns correspond to the associated b-paths in the same order.

This ordering does not conflict with that already established based on the k- and b-paths. C_{ee} can now be partitioned into submatrices corresponding to the various sets of ports. Henceforth, we shall assume that all fundamental connection matrices, C_{ee}, are arranged according to rules (i)-(iv) above.

Although the port structures described here may seem somewhat limited, they can treat a great many cases of interest; furthermore, it is possible to represent more general configurations by introducing "dummy" elements whose immittance values are equal to zero. For example, suppose we wish to represent a port created by connecting to the terminal vertices of two different elements, rather than a single one. We can introduce a zero admittance element, connected between the vertices, make it a branch

of the distinguished tree of a t-graph, and thereby treat the given terminal-pair as a b-port. This procedure can be continued for other vertex terminal-pairs as long as the dummy elements can be made part of a tree. If they cannot be made part of a tree, the dummy elements must form a closed loop, a situation which is inconsistent with the usual concept of a port (the port potentials could not take on arbitrary values without violating the Kirchhoff potential law).

Similarly, the terminal-pairs created by cutting elements, not all of which belong to a cotree, can be treated by introducing dummy zero impedance elements in series with the non-cotree elements, and then cutting the dummy elements to create k-ports. Zero impedance elements can also be connected across branches and then cut to convert b-ports into k-ports. A procedure of this sort will be used in the first example of Chapter V.

Undoubtedly other variations can be found. It seems likely that all legitimate port structures (those which do not lead to a violation of one of the Kirchhoff laws for arbitrary values of the port currents or voltages) can be represented by the introduction of dummy elements, although this conjecture has not been proved.

It is important to point out that the introduction of zero immittance elements creates singular element immittance matrices, so that special care must be taken to avoid manipulations requiring the existence of an inverse. This does not necessarily imply, however, that the associated k- or b-port immittance matrices are singular, as we shall see in Section 2.4.

2.4 e-Port Immittance Matrices

This section will be devoted to finding the immittance matrices relating the currents and potentials, or "port coordinates," of the e-ports defined above. We shall find that these equations constitute a complete set of independent Kirchhoff current and voltage law equations for the network.

We begin by defining the current and potential coordinates associated with the various port sets defined above. In the following, "( )-port" may be read as "e-, k-, b-, n-, m-, p-, or q-port."

( )-port current coordinates. The ( )-port current coordinates are the steady-state, frequency-dependent, signed, positive-charge currents flowing into the ( )-ports. We choose the port current direction in such a way that a positive port current tends to produce a positive current in the element which defines the ( )-port. The ( )-port currents can be represented by the

( )-port current matrix, $I_{(\ )1}$. The ( )-port current matrix, $I_{(\ )1}$, is an ( ) x 1 matrix whose entries are the ( )-port current coordinates of a set of ports.

( )-port potential coordinates. The ( )-port potential coordinates are the steady-state, frequency-dependent, signed potential-rises across the ( )-ports. We choose the negative to positive port potential-rise polarity direction to be the positive port current direction. The ( )-port potential coordinates can be represented by the

( )-port potential matrix, $E_{(\ )1}$. The ( )-port potential matrix, $E_{(\ )1}$, is an ( ) x 1 matrix whose entries are the ( )-port potential coordinates of a set of ports.

We are now ready to present the network analysis method used in this study to find the immittance matrices relating the e-port coordinates.

The essence of network analysis is to combine the information about the electromagnetic, topological, and port structures in such a way that an immittance matrix relating the port coordinates is determined. The various analysis methods differ only in the means by which this task is accomplished. The topological and port structures impose conditions on the element coordinates in addition to those arising from the electromagnetic structure. These conditions are the Kirchhoff current and potential law constraints. We shall now write a set of independent Kirchhoff law equations in terms of the element and e-port coordinates, making use of the properties of the fundamental connection matrix, C_{ee}, and its inverse, C_{ee}^{-1}. We require that the element coordinate matrices, J_{el} and V_{el}, and the element immittance matrices, Z_{ee}^{e} and Y_{ee}^{e}, be arranged to correspond to the element order used for indexing the rows of C_{ee}. The Kirchhoff current law constraints can be written

$$I_{el} = C_{ee}^{-1} J_{el} , \tag{2.5}$$

or, in partitioned form,

$$\begin{bmatrix} I_{kl} \\ I_{bl} \end{bmatrix} = \begin{bmatrix} U_{kk} & 0_{kb} \\ -C_{bk} & U_{bb} \end{bmatrix} \begin{bmatrix} J_{kl} \\ J_{bl} \end{bmatrix} . \tag{2.6}$$

This equation states that the k-port currents, I_{kl}, are identical to the currents in the links which were cut to create the k-ports, and that the b-port currents, I_{bl}, are equal to algebraic sums of the unique, signed element currents flowing between the terminal pairs created by connecting to the branches. The correctness of the signs can be verified by recalling that positive port currents tend to produce positive currents in the port-defining elements, and that the positive b-copath link currents are in a direction opposite to the currents in the branches which define

the b-copaths. Equation (2.6) includes the maximal independent set of Kirchhoff current law equations for b fundamental cut-sets.

The Kirchhoff potential law can be written

$$E_{el} = (C_{ee})^t V_{el} , \tag{2.7}$$

where $(\ )^t$ is used to denote the transpose of the matrix inside the parentheses. In partitioned form we have

$$\begin{bmatrix} E_{kl} \\ E_{bl} \end{bmatrix} = \begin{bmatrix} U_{kk} & (C_{bk})^t \\ 0_{bk} & U_{bb} \end{bmatrix} \begin{bmatrix} V_{kl} \\ V_{bl} \end{bmatrix} . \tag{2.8}$$

This equation states that the k-port potentials, E_{kl}, are equal to the algebraic sums of the unique, signed element-potentials around the paths between the terminal-pairs created by cutting the links, and that the b-port potentials, E_{bl}, are identical to the potentials across the branches which define the b-ports. The correctness of the signs can be verified by recalling that the directions of negative to positive port potentials are opposite to the directions of negative to positive potentials of the port-defining elements, and that the negative to positive k-path branch potentials are in a direction opposite to the negative to positive port potential direction. Equation (2.8) includes the maximal independent set of Kirchhoff potential law equations for k fundamental circuits.

If we multiply both sides of (2.5) by C_{ee}, we obtain

$$J_{el} = C_{ee} I_{el} .$$

Substitution of this form of the Kirchhoff current law equation in the element impedance equation, (2.1), gives the element potentials, V_{el}, in terms of the port currents, I_{el}:

$$V_{el} = Z^e_{ee} C_{ee} I_{el} \quad . \tag{2.9}$$

We now combine (2.9) with the Kirchhoff potential law equation, (2.7), to obtain

$$E_{el} = (C_{ee})^t Z^e_{ee} C_{ee} I_{ee} \quad . \tag{2.10}$$

The matrix

$$Z^f_{ee} = (C_{ee})^t Z^e_{ee} C_{ee} \tag{2.11}$$

relates the e-port coordinate matrices, I_{el} and E_{el}, and so represents the impedance matrix sought in the analysis. We shall call Z^f_{ee} the

<u>e-port impedance matrix.</u>

We note that once the matrix representations of the three structures have been established the analysis procedure is quite straightforward. Also, Equation (2.11) represents a complete set of independent Kirchhoff current and voltage law constraints.

In partitioned form the e-port impedance matrix, Z^f_{ee}, is

$$Z^f_{ee} = \begin{bmatrix} U_{kk} & (C_{bk})^t \\ 0_{bk} & U_{bb} \end{bmatrix} \begin{bmatrix} Z^e_{kk} & Z^e_{kb} \\ Z^e_{bk} & Z^e_{bb} \end{bmatrix} \begin{bmatrix} U_{kk} & 0_{kb} \\ C_{bk} & U_{bb} \end{bmatrix}$$

$$= \begin{bmatrix} [Z^e_{kk} + (C_{bk})^t Z^e_{bb} C_{bk} + (C_{bk})^t Z^e_{bk} + Z^e_{kb} C_{bk}] & [Z^e_{kb} + (C_{bk})^t Z^e_{bb}] \\ [Z^e_{bk} + Z^e_{bb} C_{bk}] & Z^e_{bb} \end{bmatrix} \quad . \tag{2.12}$$

The upper left k x k submatrix, Z^f_{kk}, relates the k-port coordinates. If all the k-ports have been created by the introduction of dummy zero impedance elements we have

$$Z^e_{kk} = 0_{kk} \quad . \tag{2.13}$$

Clearly, (2.13) does not imply that Z_{kk}^{f} is singular; except for very special values of C_{bk}, Z_{bb}^{e}, Z_{bk}^{e}, and Z_{kb}^{e} it is a nonsingular matrix function of s whose entries have isolated poles and zeros. This property of Z_{kk}^{f} will be used in the following section to determine the n-port impedance matrix of a passive network.

The e-port admittance matrix can be found by first solving the potential law equation, (2.7), for V_{e1} as

$$V_{e1} = [(C_{ee})^{t}]^{-1}E_{e1} .$$

This result can be combined with the element admittance equation, (2.2), to obtain the element currents in terms of the port potentials:

$$J_{e1} = Y_{ee}^{e}[(C_{ee})^{t}]^{-1}E_{e1} . \tag{2.14}$$

Application of the current law equation, (2.5), now gives

$$I_{e1} = [C_{ee}]^{-1}Y_{ee}^{e}[(C_{ee})^{t}]^{-1}E_{e1} , \tag{2.15}$$

so that

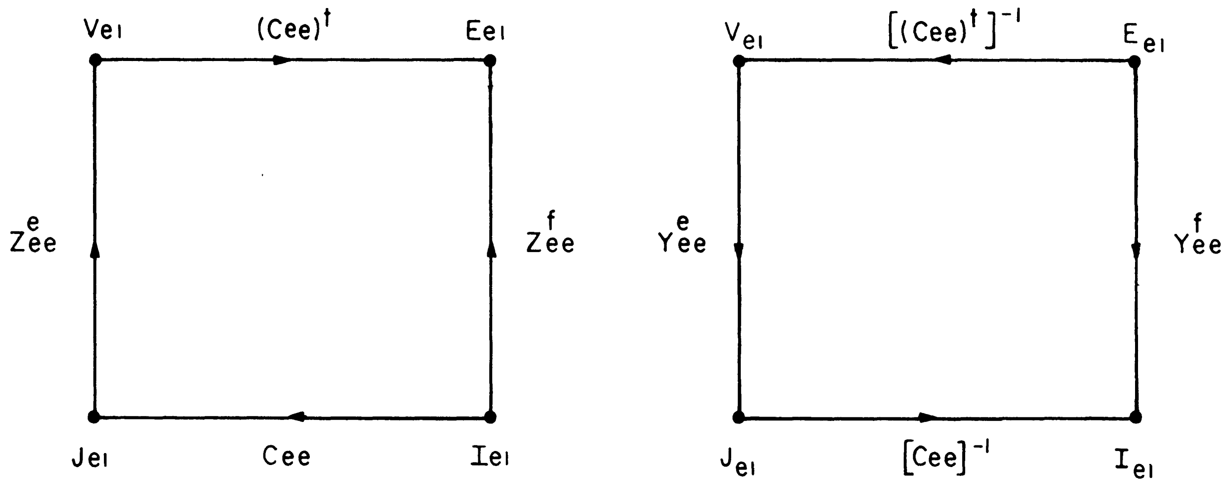
$$Y_{ee}^{f} = [C_{ee}]^{-1}Y_{ee}^{e}[(C_{ee})^{t}]^{-1} \tag{2.16}$$

is the desired

e-port admittance matrix.

By arguments parallel to those used to show that Z_{kk}^{f} is generally nonsingular it is possible to show that the submatrix Y_{bb}^{f} of Y_{ee}^{f} is generally a nonsingular matrix function of s whose entries have isolated poles and zeros.

These analysis procedures leading to port immittance matrices can be represented by means of the coordinate relation diagrams shown in Fig. 2.4. The vertices of these diagrams represent the various sets of coordinates and the lines represent the matrices which relate them. From



(a) Impedance Diagram (b) Admittance Diagram

Fig. 2.4. Coordinate relation diagrams.

a slightly different point of view, the vertices represent the linear vector spaces of currents and voltages which, in turn, are represented with respect to the sets of basis vectors,

$$\bar{i}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \bar{i}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \ldots \quad , \quad \bar{i}_e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix},$$

and

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \cdots \quad, \bar{v}_e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix},$$

by means of the coordinate current and potential matrices. Similarly, the lines can represent the linear transformations, between the spaces, which are represented by the structure matrices, Z^e_{ee}, Y^e_{ee}, and C_{ee}. The diagram indicates, schematically, why the e-port immittance matrices are triple products of these structure matrices. For example, the transformation from the space of port current coordinates to that of port potential coordinates in Diagram (a) can be traced from I_{el} to J_{el} to V_{el}, and then to E_{el}.

2.5 n- and p-Port Immittance Matrices of Passive Networks

In Chapters III and IV we shall determine the necessary and sufficient conditions on the structure matrices of networks which make the represented networks equivalent to a given network at special subsets of the e-ports. These special sets of e-ports were defined in Section 2.3 as the n-ports and p-ports. As stated in Chapter I we also require that all networks be passive. Since the electrical elements are passive, the network passivity condition actually refers to the remaining e-ports, and requires that they exchange zero power, instantaneously, with the network. This condition can hold if and only if the remaining e-port current or potential coordinates are zero at all instants of time. It is necessary, however, that the passivity conditions also preserve the electromagnetic and topological structures of the network; otherwise we are defining a <u>new</u> passive network, rather than making the given one passive. The passivity conditions which do preserve the network structures are:

(i) that the remaining k-port voltage coordinates be identically equal to zero, and,

(ii) that the remaining b-port current coordinates be identically equal to zero.

These conditions correspond, physically, to short-circuiting the remaining k-ports and open-circuiting the remaining b-ports. The effect of these operations is to obtain zero power exchange, instantaneously, and yet preserve the k-paths defined by the links, and the b-paths defined by the branches. It is clear that the opposite procedure of open-circuiting the remaining links and short-circuiting the remaining branches would effectively remove these elements from the network.

The appropriate passivity conditions for determining the n-port impedance characteristics are:

$$E_{m1} = 0_{m1} , \tag{2.17}$$

and

$$I_{b1} = 0_{b1} . \tag{2.18}$$

Substitution of these conditions into the appropriately partitioned form of the e-port impedance matrix yields

$$\begin{bmatrix} E_{n1} \\ 0_{m1} \\ E_{b1} \end{bmatrix} = \begin{bmatrix} Z^f_{nn} & Z^f_{nm} & Z^f_{nb} \\ Z^f_{mn} & Z^f_{mm} & Z^f_{mb} \\ Z^f_{bn} & Z^f_{bm} & Z^f_{bb} \end{bmatrix} \begin{bmatrix} I_{n1} \\ I_{m1} \\ 0_{b1} \end{bmatrix} . \tag{2.19}$$

The second row of this equation can be expanded as

$$0_{m1} = Z^f_{mn} I_{n1} + Z^f_{mm} I_{m1} . \tag{2.20}$$

As indicated in Section 2.4 submatrix Z^f_{kk} is nonsingular except for very special choices of electromagnetic and topological structure matrices. For this reason we can generally assume that submatrix Z^f_{mm} of Z^f_{kk} is a nonsingular matrix function of s whose entries have isolated poles and zeros. If this condition holds it is possible to solve (2.20) for I_{m1} as

$$I_{m1} = -[Z^f_{mm}]^{-1} Z^f_{mn} I_{n1} . \tag{2.21}$$

If we now expand the first row of (2.19) to obtain

$$E_{n1} = Z^f_{nn} I_{n1} + Z^f_{nm} I_{m1} , \tag{2.22}$$

and substitute the solution for I_{m1}, (2.21), we get

$$E_{n1} = (Z^f_{nn} - Z^f_{nm}[Z^f_{mm}]^{-1} Z_{mn}) I_{n1} . \tag{2.23}$$

Thus, the impedance matrix relating the n-port current and potential coordinates of a passive network is

$$Z^c_{nn} = Z^f_{nn} - Z^f_{nm}[Z^f_{mm}]^{-1} Z^f_{mn} . \tag{2.24}$$

We shall call Z^c_{nn} the

<u>n-port impedance matrix of a passive network.</u>

The proper passivity conditions for determination of the p-port admittance characteristics of a passive network are,

$$E_{k1} = 0_{k1} , \tag{2.25}$$

and

$$I_{q1} = 0_{q1} . \tag{2.26}$$

A development parallel to the above, but beginning with the e-port admittance matrix, yields, if Y^f_{qq} is nonsingular,

$$I_{p1} = (Y^f_{pp} - Y^f_{pq}[Y^f_{qq}]^{-1}Y^f_{qp})E_{p1} , \tag{2.27}$$

so that

$$Y^c_{pp} = Y^f_{pp} - Y^f_{pq}[Y^f_{qq}]^{-1}Y^f_{qp} \tag{2.28}$$

is the

<u>p-port admittance matrix of a passive network.</u>

This completes the presentation of basic material on network analysis. In Chapter III we shall discuss linear transformations of port coordinates which preserve the n-port impedance or p-port admittance characteristics of passive networks, and shall also present the linear element coordinate transformation used to obtain new element immittance matrices.

CHAPTER III

COORDINATE TRANSFORMATIONS

In Chapter II we obtained the immittance matrices relating various port coordinates of passive networks from a knowledge of the network

(i) electromagnetic structure, represented by an element immittance matrix, Z^e_{ee} or Y^e_{ee},

(ii) topological structure, represented by a fundamental connection matrix, C_{ee}, and

(iii) port structure, represented by the row-column ordering and partitioning of a fundamental connection matrix, C_{ee}.

This study treats the problem of finding networks whose structures (i), (ii), or (iii) are different from those of a given network, but whose passive n- or p-port immittance characteristics are the same. This chapter presents two types of transformations, one from the original network's e-port coordinates to a new set of e-port coordinates, and one from the original network's element coordinates to a new set of element coordinates. The e-port transformations presented here are based on Cauer's method of linear transformation as extended by Kron. The demonstration that known conditions on the matrices of transformation are both necessary and sufficient for n- or p-port passive network equivalence with symmetric e-port matrices is a new result. The element coordinate transformations are similar to those introduced by Kron. Unlike the e-port coordinate transformations, these transformations cannot have equivalence properties because they operate before the introduction of topological and port structure information. Instead, they lead directly to new element immittance matrices which can represent new electromagnetic structures.

Both of these classes of transformations relate equal numbers of port or element coordinates. In order to permit equivalent networks to have a larger number of elements and e-ports than the given network, a method for adjoining additional elements is presented. This method makes it possible to continue using the transformations defined here for such problems. Similar methods have been used by Cauer, Kron, and others. The introduction of new elements by the method of adjoinment is distinct from the use of "dummy" zero immittance elements described in Chapter II for representing general port structures.

3.1 e-Port Coordinate Transformations

We begin by defining an

e-port coordinate transformation. An e-port coordinate transformation is a linear transformation between equal numbers of e-port current and potential coordinates. It is represented by the equations,

$$\text{(i)} \quad I_{el} = M_{ee}\bar{I}_{el} , \tag{3.1}$$

and

$$\bar{E}_{el} = N_{ee}E_{el} , \tag{3.2}$$

or

$$\text{(ii)} \quad \bar{I}_{el} = M'_{ee}I_{el} , \tag{3.3}$$

and

$$E_{el} = N'_{ee}\bar{E}_{el} . \tag{3.4}$$

where $\bar{I}_{el}$, $\bar{E}_{el}$ are the new e-port current and potential matrices and M_{ee}, N_{ee}, M'_{ee}, N'_{ee} are real, constant matrices of transformation. Representation (i) can be used to determine the e-port impedance matrix relating the new e-port coordinates. In terms of the original coordinates we have

$$E_{el} = Z^{f}_{ee}I_{el} , \tag{3.5}$$

so that substitution of (3.1) and (3.2) yields

$$\bar{E}_{el} = N_{ee}Z^{f}_{ee}M_{ee}\bar{I}_{el} . \tag{3.6}$$

Thus,

$$\bar{Z}^{f}_{ee} = N_{ee}Z^{f}_{ee}M_{ee} \tag{3.7}$$

is the e-port impedance matrix relating the new e-port coordinates. $\bar{Z}^{f}_{ee}$ can also be considered as the e-port impedance matrix of a new network with as yet undetermined electromagnetic, topological, and port structures.

The admittance equation relating the original coordinates is

$$I_{el} = Y^{f}_{ee}E_{el} , \tag{3.8}$$

so that substitution of (3.3) and (3.4) yields

$$\bar{I}_{e1} = M'_{ee} Y^{f}_{ee} N'_{ee} \bar{E}_{e1} , \tag{3.9}$$

and

$$\bar{Y}^{f}_{ee} = M'_{ee} Y^{f}_{ee} N'_{ee} \tag{3.10}$$

is the e-port admittance matrix relating the new e-port coordinates, or the e-port admittance matrix of a <u>new network.</u>

Associated with these new e-port immittance matrices are n- and p-port passive network immittance matrices. We now seek the conditions on transformation matrices M_{ee}, N_{ee}, M'_{ee}, and N'_{ee} which keep these immittance characteristics invariant under e-port coordinate transformations. The passivity conditions established in Chapter II for treating n-port impedance characteristics are

$$E_{m1} = 0_{m1} , \tag{2.17}$$

and

$$I_{b1} = 0_{b1} . \tag{2.18}$$

The corresponding conditions for the new e-port coordinates are

$$\bar{E}_{m1} = 0_{m1} , \tag{3.11}$$

and

$$\bar{I}_{b1} = 0_{b1} . \tag{3.12}$$

We shall use these conditions to determine the required forms of matrices M_{ee} and N_{ee} for passivity. When conditions (2.18) and (3.12) are substituted in the partitioned form of (3.1) we have

$$\begin{bmatrix} I_{n1} \\ I_{m1} \\ 0_{b1} \end{bmatrix} = \begin{bmatrix} M_{nn} & M_{nm} & M_{nb} \\ M_{mn} & M_{mm} & M_{mb} \\ M_{bn} & M_{bm} & M_{bb} \end{bmatrix} \begin{bmatrix} \bar{I}_{n1} \\ \bar{I}_{m1} \\ 0_{b1} \end{bmatrix} . \tag{3.13}$$

In order that the equation obtained from the third row of (3.13) be true *for general values of the* unconstrained current matrices, $\bar{I}_{nl}$ and $\bar{I}_{ml}$, it is necessary and sufficient that

$$M_{bn} = 0_{bn} , \tag{3.14}$$

and

$$M_{bm} = 0_{bm} . \tag{3.15}$$

Similarly, under conditions (2.17) and (3.11), the partitioned form of (3.2) becomes

$$\begin{bmatrix} \bar{E}_{nl} \\ 0_{ml} \\ \bar{E}_{bl} \end{bmatrix} = \begin{bmatrix} N_{nn} & N_{nm} & N_{nb} \\ N_{mn} & N_{mm} & N_{mb} \\ N_{bn} & N_{bm} & N_{bb} \end{bmatrix} \begin{bmatrix} E_{nl} \\ 0_{ml} \\ E_{bl} \end{bmatrix} . \tag{3.16}$$

The equation obtained from the second row of (3.16) can be satisfied for general values of the unconstrained potential matrices, E_{nl} and E_{bl}, if and only if

$$N_{mn} = 0_{mn} , \tag{3.17}$$

and

$$N_{mb} = 0_{mb} . \tag{3.18}$$

Conditions (3.14), (3.15), (3.17), and (3.18) are necessary and sufficient for passivity of the original and final networks. We must now find conditions which also preserve the impedance matrix relating the associated n-port coordinates. In terms of the coordinates themselves equivalence means that

$$E_{nl} = Z^{c}_{nn} I_{nl} , \tag{3.19}$$

and

$$\bar{E}_{nl} = Z^{c}_{nn} \bar{I}_{nl} , \tag{3.20}$$

where Z^{c}_{nn} is the n-port impedance of the original passive network as determined by Eq.(2.24). That is, the original n-port impedance matrix

must relate both sets of n-port coordinates. Expansion of the first row of (3.13) gives

$$I_{n1} = M_{nn}\bar{I}_{n1} + M_{nm}\bar{I}_{m1} , \tag{3.21}$$

so that the first equivalence condition, (3.19), can be written

$$E_{n1} = Z^c_{nn}M_{nn}\bar{I}_{n1} + Z^c_{nn}M_{nm}\bar{I}_{m1} . \tag{3.22}$$

Expansion of the first row of (3.16) gives

$$\bar{E}_{n1} = N_{nn}E_{n1} + N_{nb}E_{b1} , \tag{3.23}$$

which, after substitution of (3.22) becomes

$$\bar{E}_{n1} = N_{nn}Z^c_{nn}M_{nn}\bar{I}_{n1} + N_{nn}Z^c_{nn}M_{nm}\bar{I}_{m1} + N_{nb}E_{b1} . \tag{3.24}$$

In order that the second equivalence condition equation, (3.20), be true for general Z^c_{nn}, $\bar{I}_{m1}$, and E_{b1} it is necessary that

$$M_{nn} = U_{nn} , \tag{3.25}$$

$$N_{nn} = U_{nn} , \tag{3.26}$$

$$M_{nm} = 0_{nm} , \tag{3.27}$$

and

$$N_{nb} = 0_{nb} . \tag{3.28}$$

The combined passivity and equivalence conditions obtained here may be summarized by incorporating them in the partitioned forms of matrices M_{ee} and N_{ee}.

$$M_{ee} = \begin{bmatrix} U_{nn} & 0_{nm} & M_{nb} \\ M_{mn} & M_{mm} & M_{mb} \\ 0_{bn} & 0_{bm} & M_{bb} \end{bmatrix} , \tag{3.29}$$

and,

$$N_{ee} = \begin{bmatrix} U_{nn} & N_{nm} & O_{nb} \\ O_{mn} & N_{mm} & O_{mb} \\ N_{bn} & N_{bm} & N_{bb} \end{bmatrix} . \tag{3.30}$$

It now remains to determine whether or not any additional conditions are required. This can be done by writing the new e-port impedance matrix, and then determining the associated n-port impedance matrix of a passive network. In partitioned form Eq. (3.7) is

$$\bar{Z}^f_{ee} = \begin{bmatrix} U_{nn} & N_{nm} & O_{nb} \\ O_{mn} & N_{mm} & O_{mb} \\ N_{bn} & N_{bm} & N_{bb} \end{bmatrix} \begin{bmatrix} Z^f_{nn} & Z^f_{nm} & Z^f_{nb} \\ Z^f_{mn} & Z^f_{mm} & Z^f_{mb} \\ Z^f_{bn} & Z^f_{bm} & Z^f_{bb} \end{bmatrix} \begin{bmatrix} U_{nn} & O_{nm} & M_{nb} \\ M_{mn} & M_{mm} & M_{mb} \\ O_{bn} & O_{bm} & M_{bb} \end{bmatrix}$$

$$= \begin{bmatrix} [Z^f_{nn} + N_{nm}Z^f_{mm}M_{mn} + N_{nm}Z^f_{mn} + Z^f_{nm}M_{mn}] & (Z^f_{nm} + N_{nm}Z^f_{mm})M_{mm} & \bar{Z}^f_{nb} \\ N_{mm}(Z^f_{mn} + Z^f_{mm}M_{mn}) & N_{mm}Z^f_{mm}M_{mm} & \bar{Z}^f_{mb} \\ \bar{Z}^f_{bn} & \bar{Z}^f_{bm} & \bar{Z}^f_{bb} \end{bmatrix} . \tag{3.31}$$

The entries in the third row and column have not been written in expanded form since they do not participate in the determination of the associated n-port impedance matrix of a passive network. Using the method given in Section 2.5 of Chapter II we find that this new n-port matrix, $\bar{Z}^c_{nn}$, is

$$\bar{Z}^c_{nn} = Z^f_{nn} + N_{nm}Z^f_{mm}M_{mn} + N_{nm}Z^f_{mn} + Z^f_{nm}M_{mn}$$
$$- (Z^f_{nm} + N_{nm}Z^f_{mm})M_{mm}[N_{mm}Z^f_{mm}M_{mm}]^{-1}N_{mm}(Z^f_{mn} + Z^f_{mm}M_{mn}) . \tag{3.32}$$

In order that $\bar{Z}^c_{nn}$ be defined it is necessary that the matrices N_{mm} and M_{mm} be nonsingular. (We have already argued, in Section 2.4 of Chapter II, that Z^f_{mm} is generally nonsingular.) These conditions, along with those

expressed by Eqs. (3.29) and (3.30) are also sufficient for n-port impedance equivalence, as we can see by expanding (3.32) to get

$$\begin{aligned}\bar{Z}^c_{nn} &= Z^f_{nn} + N_{nm}Z^f_{mm}M_{mn} + N_{nm}Z^f_{mn} + Z^f_{nm}M_{mn} \\ &\quad - (Z^f_{nm}[Z^f_{mm}]^{-1} + N_{nm})\,(Z^f_{mn} + Z^f_{mm}M_{mn}) \\ &= Z^f_{nn} - Z^f_{nm}(Z^f_{mm})^{-1}Z^f_{mn} \quad . \end{aligned} \tag{3.33}$$

The right hand side of this equation is identical to the n-port impedance matrix, Z^c_{nn}, of the original network as determined from Eq. (2.24). This result establishes that Eqs. (3.29) and (3.20), along with the requirements that N_{mm} and M_{mm} be nonsingular, are both necessary and sufficient for n-port impedance equivalence of passive networks.

An additional condition can be imposed to make the new e-port matrix, $\bar{Z}^f_{ee}$, symmetric. This condition is

$$N_{ee} = (M_{ee})^t \quad . \tag{3.34}$$

It is evident that (3.34) does not conflict with the conditions expressed by (3.29) and (3.30).

The necessary and sufficient conditions that an e-port coordinate transformation represented by Eqs. (3.3) and (3.4) produce a new symmetric e-port admittance matrix, whose associated p-port passive network admittance matrix is identical to that of the original network, can be obtained by arguments similar to those used above for n-port impedance equivalence. The conditions can be summarized by

$$M'_{ee} = \begin{bmatrix} M'_{kk} & M'_{kq} & M'_{kp} \\ 0_{qk} & M'_{qq} & 0_{qp} \\ 0_{pk} & M'_{pq} & U_{pp} \end{bmatrix} \quad , \tag{3.35}$$

$$M'_{qq} \text{ is nonsingular,} \tag{3.36}$$

and

$$N'_{ee} = (M'_{ee})^t \; . \tag{3.37}$$

3.2 Element Coordinate Transformations

We define an

Element coordinate transformation. An element coordinate transformation is a linear transformation between equal numbers of element current and potential coordinates. It can be represented by the matrix equations

$$\text{(i)} \quad J_{el} = K_{ee}\bar{J}_{el} , \tag{3.38}$$

and

$$\bar{V}_{el} = L_{ee}V_{el} , \tag{3.39}$$

or

$$\text{(ii)} \quad \bar{J}_{el} = K'_{ee}J_{el} , \tag{3.40}$$

and

$$V_{el} = L'_{ee}\bar{V}_{el} , \tag{3.41}$$

where $\bar{J}_{el}$, $\bar{V}_{el}$ are the new element current and potential coordinate matrices, and K_{ee}, L_{ee}, K'_{ee}, L'_{ee} are real, constant matrices. The element impedance matrix relating the new coordinates can be determined from representation (i). The original element impedance equation is

$$V_{el} = Z^{e}_{ee}J_{el} , \tag{2.1}$$

so that substitution of (3.38) and (3.39) yields

$$\bar{V}_{el} = L_{ee}Z^{e}_{ee}K_{ee}J_{el} . \tag{3.42}$$

Thus,

$$\bar{Z}^{e}_{ee} = L_{ee}Z^{e}_{ee}K_{ee} \tag{3.43}$$

is the impedance matrix relating the new element coordinates. Z^{e}_{ee} can also be considered as the representation of a new electromagnetic structure. In this way an element coordinate transformation permits us to control the element values of the new network independently of its topological and port structures. By contrast, an e-port coordinate transformation

produces the e-port immittance matrix of a new network for which all three structures are changed. In order to make the new element impedance matrix, $\bar{Z}^e_{ee}$, symmetric we can require that

$$L_{ee} = (K_{ee})^t \quad . \tag{3.44}$$

Physically, this condition establishes that all transformed elements be bilateral. The new element impedance matrix now becomes

$$\bar{Z}^e_{ee} = (K_{ee})^t Z^e_{ee} K_{ee} \quad . \tag{3.45}$$

The corresponding transformed admittance matrix under the symmetry condition,

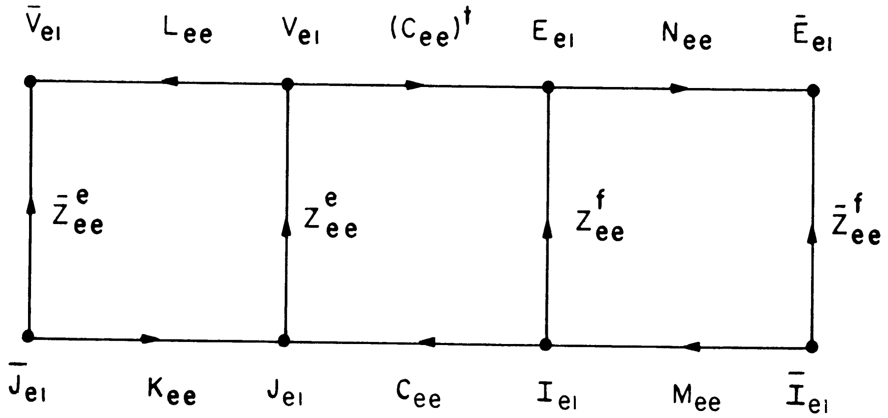
$$L'_{ee} = (K'_{ee})^t \ ,$$

is

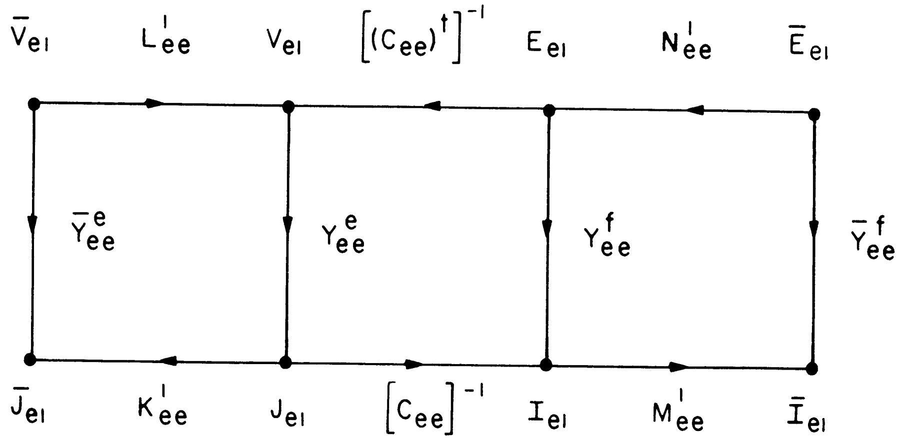
$$\bar{Y}^e_{ee} = K'_{ee} Y^e_{ee} (K'_{ee})^t \quad . \tag{3.46}$$

It should be emphasized that element coordinate transformations do not have equivalence properties since they operate before the introduction of topological and port structure information. Chapter IV will exploit this fact to establish independent control over the electromagnetic, topological, and port structures of equivalent networks.

The two classes of transformation defined in this chapter can be represented diagrammatically by extending Fig. 2.4 to include the new e-port and element coordinates and the associated transformation matrices as shown in Fig. 3.1. The center rectangle of each diagram corresponds to the original coordinate relation diagram. The extensions to the right represent e-port coordinate transformations, and those to the left element coordinate transformations.



(a) Impedance Diagram



(b) Admittance Diagram

Fig. 3.1. Transformed-coordinate relation diagrams.

3.3 The Number of Element and Port Coordinates

The transformation types defined above relate equal numbers of coordinates to one another. In equivalence studies, however, it is often desirable to permit the new network to have more elements and e-ports than the original one because of the greater freedom inherent in large networks. This goal can be achieved within the framework of the transformations already defined by adjoining new elements and corresponding e-paths and e-ports to the original network in such a way that the associated n-port immittance matrix is unchanged. In particular, k' cotree elements and associated k-paths and k-ports, and b' tree elements and associated b-paths and b-ports can be added by adjoining the arbitrary passive impedance matrices, $Z^e_{k'k'}$ and $Z^e_{b'b'}$ to the original element impedance matrix as follows:

$$Z^e_{ee} = \begin{bmatrix} Z^e_{kk} & 0_{kk'} & Z^e_{kb} & 0_{kb'} \\ 0'_{kk} & Z^e_{k'k'} & 0_{k'b} & 0_{k'b'} \\ Z^e_{bk} & 0_{bk'} & Z^e_{bb} & 0_{bb'} \\ 0_{b'k} & 0_{b'k'} & 0_{b'b} & Z^e_{b'b'} \end{bmatrix} . \qquad (3.47)$$

The new elements are not electromagnetically coupled to the original ones (as indicated by the zero off-diagonal matrices). We now impose the additional condition that the new elements be unconnected to those of the original network. This can be done by extending the definition of a fundamental connection matrix to accommodate elements "connected" in the "pseudo k- and b-paths" which they define. The "pseudo fundamental connection matrix" representing this mode of "connection" can be written

$$C_{ee} = \begin{bmatrix} U_{kk} & O_{kk'} & O_{kb} & O_{kb'} \\ O_{k'k} & U_{k'k'} & O_{k'b} & O_{k'b'} \\ C_{bk} & O_{bk'} & U_{bb} & O_{bb'} \\ O_{b'k} & O_{b'k'} & O_{b'b} & U_{b'b'} \end{bmatrix} . \tag{3.48}$$

The diagonal matrices, $U_{k'k'}$ and $U_{b'b'}$, express the "connection" of the new elements in the "pseudo k- and b- paths" which they define.

The e-port impedance matrix associated with these matrices is

$$Z^f_{ee} = \begin{bmatrix} Z^f_{kk} & O_{kk'} & Z^f_{kb} & O_{kb'} \\ O_{k'k} & Z^e_{k'k'} & O_{k'b} & O_{k'b'} \\ Z^f_{bk} & O_{bk'} & Z^f_{bb} & O_{bb'} \\ O_{b'k} & O_{b'k'} & O_{b'b} & Z^e_{b'b'} \end{bmatrix} . \tag{3.49}$$

The impedance n-port matrix associated with Z^f_{ee} may be obtained by partitioning Z^f_{kk} according to the n-ports and m-ports, and then using the method of Section 2.5, Chapter II, to obtain

$$\begin{aligned} Z^c_{nn} &= Z^f_{nn} - [Z^f_{nn} O_{nk'}] \begin{bmatrix} Z^f_{mm} & O_{mk'} \\ O_{k'm} & Z^e_{k'k'} \end{bmatrix}^{-1} \begin{bmatrix} Z^f_{mn} \\ O_{k'n} \end{bmatrix} \\ &= Z^f_{nn} - Z^f_{nm}(Z^f_{mm})^{-1} Z^f_{mn} \quad . \end{aligned} \tag{3.50}$$

The right hand side of (3.50) is identical to the n-port impedance matrix of the network before the adjoinment of new elements as found from Eq. (2.24). In effect, we have simply made additional elements available for later incorporation into the equivalent network. A similar adjoinment can be defined for admittance equivalence problems.

The method of adjoinment described here permits us to treat equivalence problems in terms of networks with the same numbers of elements. In the future the term "fundamental connection matrix" will stand for the

appropriate "pseudo fundamental connection matrix" when augmentation is required.

The method of adjoinment described here is distinct from, and in addition to, the introduction of "dummy" elements to represent general port configurations. Here, the new elements are general immittances, not zero-valued ones.

We are now ready to derive the necessary and sufficient conditions on structure matrices for n- and p-port immittance equivalence.

CHAPTER IV

EQUIVALENCE CONDITIONS

The goal of this study is to determine the necessary and sufficient conditions on matrices representing the electromagnetic, topological, and port structures of n- and p-port networks equivalent to a given network. The representations of electromagnetic, topological, and port structures presented in Chapter II, along with the element and e-port coordinate transformation methods presented in Chapter III, enable us to obtain these conditions. This result is a first step toward achievement of a general method for finding equivalent networks while independently controlling one of their structures. In Chapter V the conditions are used to obtain formal solutions for the structure matrices. The problem of ensuring that these formal solutions always correspond to realizable structures remains to be solved. Chapter V, however, contains some applications based on the discovery that, after specification of realizable topological and port structure matrices, C_{ee}, the equivalence conditions still allow some freedom for selection of element transformation matrices, K_{ee}.

In Chapter III we found that the e-port immittance matrix relating a transformed set of e-port current and potential coordinates can be found from the original e-port immittance matrix and an e-port coordinate transformation matrix. We also found the necessary and sufficient conditions under which the associated n- or p-port immittance matrix is identical to that of the original e-port immittance matrix. We shall call this procedure for finding new e-port matrices "Method I." If we assume that the new e-port matrix is associated with a new network, then, in a sense, we have solved the equivalence problem. The difficulty is that not enough

information has been obtained to construct the equivalent network. Cauer assumed the availability of ideal transformers, thereby removing constraints on the topological structure. He proved, by arguments based on the network energy functions, that under this assumption the new e-port matrix can be realized as a network as long as the e-port coordinate transformation matrix is positive semidefinite. Schoeffler, on the other hand, obtained new electromagnetic structures by means of element coordinate transformations but required that the topological and port structures of the networks be identical. Instead of modifying the requirements on the topological and port structures,we seek to obtain conditions which hold for general values of the matrices representing the three structures. When these matrices are known, and when they correspond to realizable structures, it is possible to construct the network without using ideal transformers, although electromagnetic field coupling may still be required.

In order to obtain the desired conditions we propose an alternate method, to be called "Method II," for the determination of e-port immittance matrices. First, we carry out an element coordinate transformation and find the new element immittance matrix relating the transformed coordinates. This new element immittance matrix directly represents a new electromagnetic structure. We now recognize that the new elements can be connected in <u>new</u> topological and port structures, represented by a new fundamental connection matrix. This is the major consequence of the ability to control element immittance matrices independently of the topological and port structures. Application of the Kirchhoff laws in terms of the new fundamental connection matrix by the method of Chapter II results in a new e-port matrix. In contrast to the e-port immittance

matrix obtained by Method I this e-port matrix involves electromagnetic, topological, and port structure matrices of the new network explicitly. In Method I this information is "mixed up" in an e-port coordinate transformation matrix. On the other hand, the necessary and sufficient conditions for n- and p-port equivalence for Method II are as yet unknown. Our goal is to obtain them.

The approach used is to equate the e-port matrices obtained by Methods I and II. Because the necessary and sufficient equivalence conditions on Method I are known we can, in this way, establish necessary and sufficient equivalence conditions on the matrices representing Method II as well. By itself, however, this procedure would yield conditions which involve the entries of an e-port coordinate transformation matrix, so that the electromagnetic, topological, and port structure information would still be unseparated. The existence of the inverse of the fundamental connection matrix, however, permits us to carry out manipulations which remove all entries of the e-port transformation matrix, so that the conditions can be expressed solely in terms of the structure matrices.

Since Method I was presented in Chapter III we begin the present chapter with a discussion of Method II. We then establish the basic relation between the two methods, and finally carry out the manipulations which remove the unwanted dependence on e-port coordinate transformation matrices.

4.1 e-Port Immittance Matrices from Structure Matrices

In Chapter III we found the symmetric immittance matrices relating sets of transformed element coordinates. Equations (3.45) and (3.46) read

$$\bar{Z}^e_{ee} = (K_{ee})^t Z^e_{ee} K_{ee} , \tag{3.45}$$

and

$$\bar{Y}^e_{ee} = K'_{ee} Y^e_{ee} (K'_{ee})^t . \tag{3.46}$$

These matrices represent new electromagnetic structures which can be directly specified by K_{ee} or K'_{ee}. In order that the new structure be realizable it is necessary and sufficient that the matrices of element inductance, resistance, and capacitance values be positive semidefinite, and further that the submatrices of resistance and capacitance values be diagonal.

We are now free to connect the transformed elements obtained above in new topological and port structures. That is, we can specify a fundamental connection matrix, $\bar{C}_{ee}$, which is different from that of the original network. The e-port immittance matrices associated with the new structure can be obtained by expressing the Kirchhoff laws in terms of $\bar{C}_{ee}$ as was done in Chapter II. The e-port impedance matrix is

$$\bar{\bar{Z}}^f_{ee} = (\bar{C}_{ee})^t \bar{Z}^e_{ee} \bar{C}_{ee} , \tag{4.1}$$

or, by substituting (3.45),

$$\bar{\bar{Z}}^f_{ee} = (\bar{C}_{ee})^t (K_{ee})^t Z^e_{ee} K_{ee} \bar{C}_{ee}$$

$$= (K_{ee}\bar{C}_{ee})^t Z^e_{ee} (K_{ee}\bar{C}_{ee}) . \tag{4.2}$$

The e-port admittance matrix is

$$\bar{\bar{Y}}^f_{ee} = \bar{C}^{-1}_{ee} \bar{Y}^e_{ee} [(\bar{C}_{ee})^t]^{-1} , \tag{4.3}$$

or, by substituting (3.46),

$$\bar{\bar{Y}}^f_{ee} = \bar{C}^{-1}_{ee} K'_{ee} Y^e_{ee} (K'_{ee})^t [(\bar{C}_{ee})^t]^{-1}$$

$$= (\bar{C}^{-1}_{ee} K'_{ee}) Y^e_{ee} (\bar{C}^{-1}_{ee} K'_{ee})^t \quad . \tag{4.4}$$

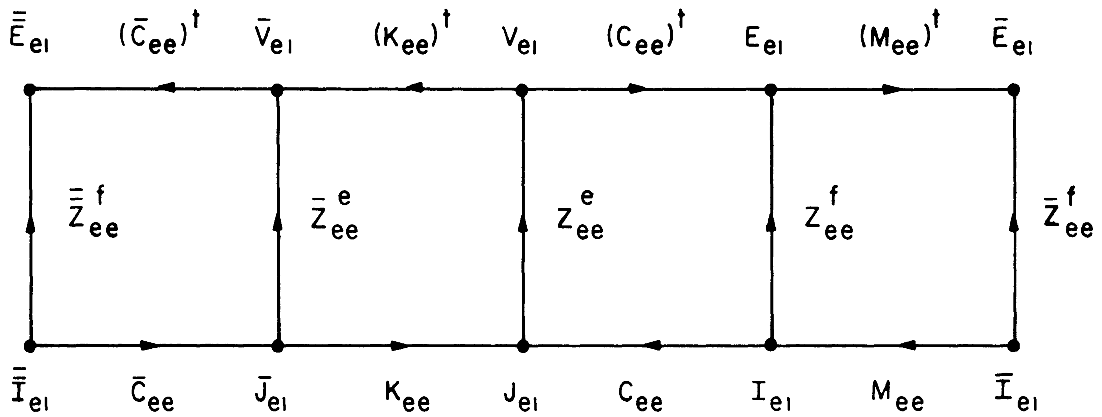
Equations (4.2) and (4.4) summarize Method II for obtaining new e-port immittance matrices. By introducing the new fundamental connection matrix, $\bar{C}_{ee}$, we have, in effect, defined a new set of e-port coordinates which are related by the immittance matrices, $\bar{\bar{Z}}^f_{ee}$ and $\bar{\bar{Y}}^f_{ee}$. We let the new coordinates be represented by the matrices $\bar{\bar{I}}_{e1}$ and $\bar{\bar{E}}_{e1}$ so that the new e-port immittance equations read

$$\bar{\bar{E}}_{e1} = \bar{\bar{Z}}^f_{ee} \bar{\bar{I}}_{e1} \; , \tag{4.5}$$

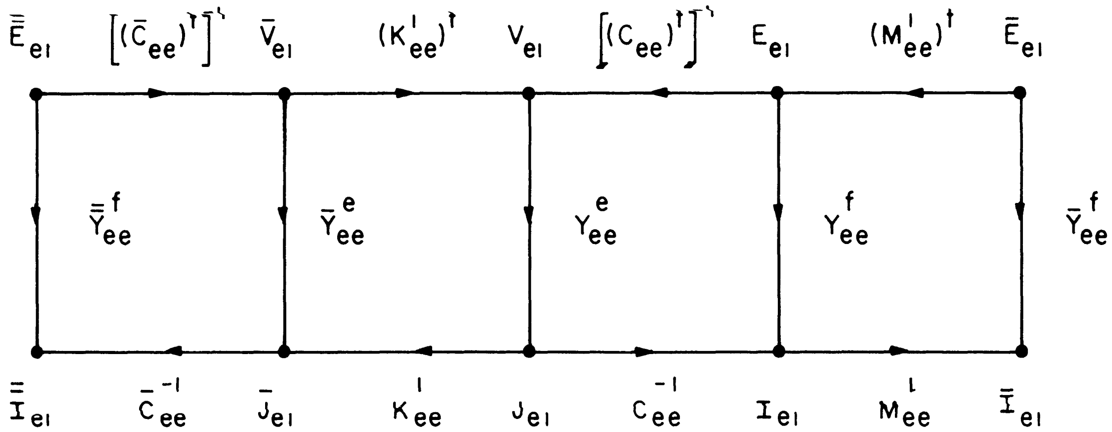
and

$$\bar{\bar{I}}_{e1} = \bar{\bar{Y}}^f_{ee} \bar{\bar{E}}_{e1} \quad . \tag{4.6}$$

The development of e-port immittance matrices in this way can be represented by a further extension of the diagrams of coordinate relations as shown in Fig. 4.1. The extensions of Fig. 3.1 represented by the new fundamental connection matrix, $\bar{C}_{ee}$, appear at the left of the diagrams of Fig. 4.1.



(a) Impedance Diagram



(b) Admittance Diagram

Fig. 4.1. Transformed-coordinate relation diagrams with new network topological and port structures.

4.2 Equivalence Conditions

We have now obtained new e-port immittance matrices by two methods. For impedance e-port matrices the methods can be summarized by the equations,

$$\bar{Z}^f_{ee} = (C_{ee}M_{ee})^t Z^e_{ee}(C_{ee}M_{ee}) , \tag{4.7}$$

for Method I, and

$$\bar{\bar{Z}}^f_{ee} = (K_{ee}\bar{C}_{ee})^t Z^e_{ee}(K_{ee}\bar{C}_{ee}) , \tag{4.2}$$

for Method II. In Chapter III we obtained the necessary and sufficient conditions under which M_{ee} produces a matrix $\bar{Z}^f_{ee}$ whose associated n-port matrix of a passive network is identical to that of the original network. In order to establish the corresponding conditions on matrices K_{ee} and $\bar{C}_{ee}$ we can require that the e-port impedance matrices obtained by Methods I and II be identical, or

$$\bar{\bar{Z}}^f_{ee} = \bar{Z}^f_{ee} . \tag{4.8}$$

Using (4.2) and (4.7) we have

$$(K_{ee}\bar{C}_{ee})^t Z^e_{ee}(K_{ee}\bar{C}_{ee}) = (C_{ee}M_{ee})^t Z^e_{ee}(C_{ee}M_{ee}) . \tag{4.9}$$

The known conditions on M_{ee} hold for general original network element impedance matrices, Z^e_{ee}. In order that Eq. (4.9) be true for general Z^e_{ee} it is necessary and sufficient that

$$K_{ee}\bar{C}_{ee} = \pm C_{ee}M_{ee} . \tag{4.10}$$

The sufficiency of (4.10) is obvious. Necessity can be proved by considering the set of Z^e_{ee} matrices for which the entry in row u and column v is unity and all other entries are zero. The condition that the (i,j)th entries on both sides of (4.9) be equal now reads

$$a_{ui}a_{vj} = b_{ui}b_{vj} , \tag{4.11}$$

where a_{ui} and a_{vj} are entries of $K_{ee}\overline{C}_{ee}$ and b_{ui} and b_{vj} are entries of $C_{ee}M_{ee}$. Equation (4.11) must hold for all possible values of the indices u, v, i, and j. In particular if u = v and i = j we have

$$(a_{ui})^2 = (b_{ui})^2 ,$$

so that the magnitudes of corresponding entries of $K_{ee}\overline{C}_{ee}$ and $C_{ee}M_{ee}$ are equal. Furthermore, if any corresponding entries, a_{vj} and b_{vj}, have the same sign then (4.11) requires that a_{ui} and b_{ui} have the same sign for all u and i. Likewise if any corresponding a_{vj} and b_{vj} have opposite signs then a_{ui} and b_{ui} must have opposite signs for all u and i. This completes the proof that (4.9) implies (4.10), or in other words, that (4.10) is a necessary condition for (4.9) to be true.

The corresponding condition for p-port admittance equivalence is obtained by setting

$$\bar{\bar{Y}}^f_{ee} = \bar{Y}^f_{ee} , \tag{4.12}$$

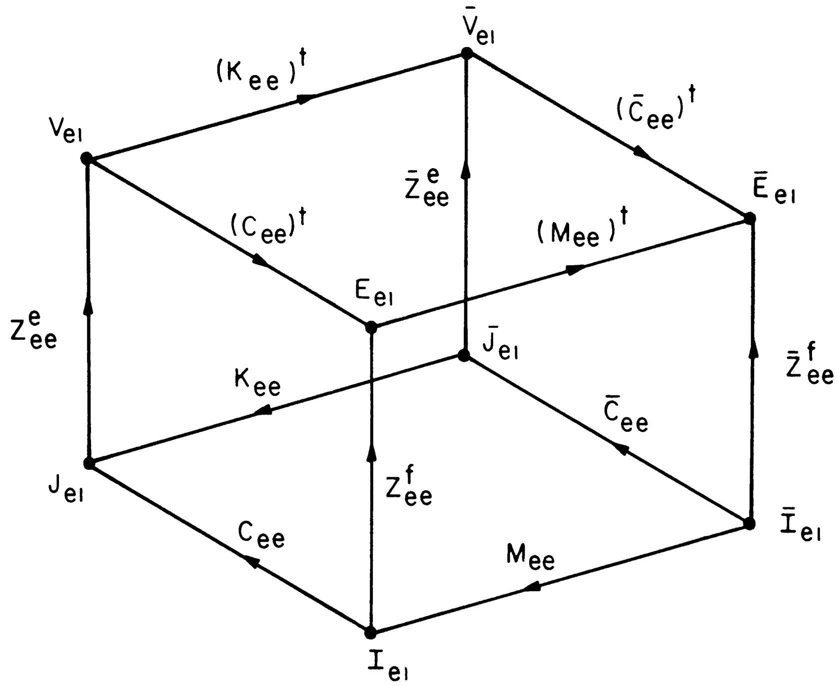
or

$$(\overline{C}^{-1}_{ee}K'_{ee})Y^e_{ee}(\overline{C}^{-1}_{ee}K'_{ee})^t = (M'_{ee}C^{-1}_{ee})Y^e_{ee}(M'_{ee}C^{-1}_{ee})^t .$$

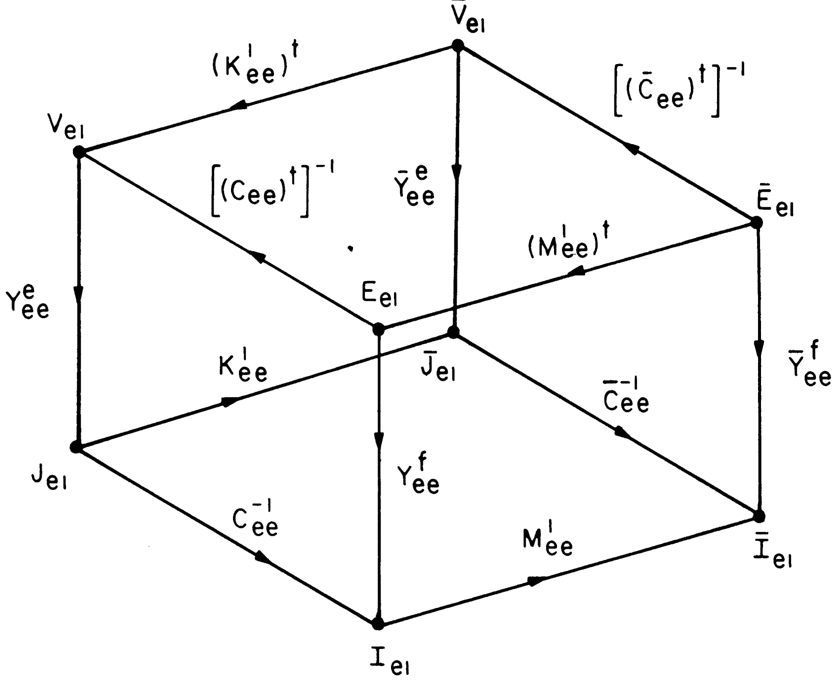
The necessary and sufficient condition that this equation be true for general Y^e_{ee} is that

$$\overline{C}^{-1}_{ee}K'_{ee} = \pm M'_{ee}C^{-1}_{ee} . \tag{4.13}$$

It is interesting to interpret the basic conditions represented by Eqs. (4.8) and (4.12) in terms of the coordinate relation diagrams presented up to now. The conditions require that the extreme left and right vertical lines of the diagrams in Fig. 4.1 represent the same matrix, thereby connecting the diagrams into three-dimensional figures as shown in Fig. 4.2. When e-port coordinate transformation matrices M_{ee} and



(a) Impedance Diagram



(b) Admittance Diagram

Fig. 4.2. Transformed-coordinate relation diagrams with new network topological and port structures, after application of equivalence conditions.

M'_{ee} meet the necessary and sufficient conditions for n- and p-port equivalence then Eqs. (4.10) and (4.13) represent corresponding necessary and sufficient conditions on element coordinate transformation matrices, K_{ee} and K'_{ee}, and the fundamental connection matrices, C_{ee} and $\bar{C}_{ee}$. Unfortunately, however, Eqs. (4.10) and (4.13) involve the e-port coordinate transformation matrices, M_{ee} and M'_{ee}, so that a different set of conditions results for each choice of M_{ee} and M'_{ee}. We will now show how this dependence can be removed to obtain conditions involving the structure matrices only.

The fact that C_{ee} is nonsingular enables us to isolate M_{ee} on the right of (4.10) by multiplying both sides by C_{ee}^{-1} to get

$$C_{ee}^{-1} K_{ee} \bar{C}_{ee} = \pm M_{ee} \quad . \tag{4.14}$$

When M_{ee} has the form required by n-port impedance equivalence, Eq. (4.14) becomes

$$C_{ee}^{-1} \begin{bmatrix} K_{nn} & K_{nm} & K_{nb} \\ K_{mn} & K_{mm} & K_{mb} \\ K_{bn} & K_{bm} & K_{bb} \end{bmatrix} \begin{bmatrix} U_{nn} & 0_{nm} & 0_{nb} \\ 0_{mn} & U_{mm} & 0_{mb} \\ \bar{C}_{bn} & \bar{C}_{bm} & U_{bb} \end{bmatrix} = \pm \begin{bmatrix} U_{nn} & 0_{nm} & M_{nb} \\ M_{mn} & M_{mm} & M_{mb} \\ 0_{bn} & 0_{bm} & M_{bb} \end{bmatrix} ,$$

or

$$\begin{bmatrix} U_{nn} & 0_{nm} & 0_{nb} \\ 0_{mn} & U_{mm} & 0_{mb} \\ -C_{bn} & -C_{bm} & U_{bb} \end{bmatrix} \begin{bmatrix} K_{nn}+K_{nb}\bar{C}_{bn} & K_{nm}+K_{nb}\bar{C}_{bm} & K_{nb} \\ K_{mn}+K_{mb}\bar{C}_{bn} & K_{mm}+K_{mb}\bar{C}_{bm} & K_{mb} \\ K_{bn}+K_{bb}\bar{C}_{bn} & K_{bm}+K_{bb}\bar{C}_{bm} & K_{bb} \end{bmatrix} = \pm \begin{bmatrix} U_{nn} & 0_{nm} & M_{nb} \\ M_{mn} & M_{mm} & M_{mb} \\ 0_{bn} & 0_{bm} & M_{bb} \end{bmatrix} ,$$

or

$$\begin{bmatrix} (K_{nn}+K_{nb}\bar{C}_{bn}) & (K_{nm}+K_{nb}\bar{C}_{bm}) & K_{nb} \\ (K_{mn}+K_{mb}\bar{C}_{bn}) & (K_{mm}+K_{mb}\bar{C}_{bm}) & K_{mb} \\ [-C_{bn}(K_{nn}+K_{nb}\bar{C}_{bn}) & [-C_{bn}(K_{nm}+K_{nb}\bar{C}_{bm}) & [-C_{bn}K_{nb} \\ -C_{bm}(K_{mn}+K_{mb}\bar{C}_{bn}) & -C_{bm}(K_{mm}+K_{mb}\bar{C}_{bm}) & -C_{bm}K_{mb} \\ +K_{bn}+K_{bb}\bar{C}_{bn}] & +K_{bm}+K_{bb}\bar{C}_{bm}] & +K_{bb}] \end{bmatrix} = \pm \begin{bmatrix} U_{nn} & 0_{nm} & M_{nb} \\ M_{mn} & M_{mm} & M_{mb} \\ 0_{bn} & 0_{bm} & M_{bb} \end{bmatrix} . \tag{4.15}$$

We can now make use of the fact that the certain submatrices of M_{ee} are arbitrary, so that the corresponding submatrices of the triple product, $C_{ee}^{-1}K_{ee}\overline{C}_{ee}$, on the left of (4.15) are also arbitrary. The conditions on the remaining submatrices on the left side of (4.15) for n-port equivalence are:

(i) the upper left submatrix must be a unit matrix multiplied by $\pm$ 1, or

$$K_{nn} + K_{nb}\overline{C}_{bn} = \pm U_{nn} ; \tag{4.16}$$

(ii) the upper center submatrix must be a zero matrix, or

$$K_{nm} + K_{nb}\overline{C}_{bm} = 0_{nm} ; \tag{4.17}$$

(iii) the lower left submatrix must be a zero matrix, or

$$- C_{bn}(K_{nn}+K_{nb}\overline{C}_{bn}) - C_{bm}(K_{mn}+K_{mb}\overline{C}_{bn}) + K_{bn} + K_{bb}\overline{C}_{bn} = 0_{bn} ; \tag{4.18}$$

(iv) the lower center submatrix must be a zero matrix, or

$$- C_{bn}(K_{nm}+K_{nb}\overline{C}_{bn}) - C_{bm}(K_{mm}+K_{mb}\overline{C}_{bm}) + K_{bm} + K_{bb}\overline{C}_{bm} = 0_{bm} . \tag{4.19}$$

Equation (4.18) can be simplified by observing that the parenthesized expression in the first term on the left side must be a unit matrix times $\pm$ 1, according to Eq. (4.16), so that

$$\mp C_{bn} - C_{bm}(K_{mn}+K_{mb}\overline{C}_{bn}) + K_{bn} + K_{bb}\overline{C}_{bn} = 0_{bn} . \tag{4.20}$$

Similarly, the parenthesized expression in the first term on the left side of (4.19) must, according to Eq. (4.17), be a zero matrix. Thus, (4.19) becomes

$$- C_{bm}(K_{mm}+K_{mb}\overline{C}_{bm}) + K_{bm} + K_{bb}\overline{C}_{bm} = 0_{bm} . \tag{4.21}$$

In addition, n-port impedance equivalence requires that the center submatrix be nonsingular, or

$$\boxed{K_{mm} + K_{mb}\bar{C}_{bm}\ ,\quad \text{nonsingular}\quad .} \tag{4.22}$$

Equations (4.16) and (4.17) can be treated as a single n x k dimensional matrix equation by writing

$$[K_{nn}\ K_{nm}] + K_{nb}[\bar{C}_{bn}\ \bar{C}_{bm}] = [\pm U_{nn}\ 0_{nm}]\ ,$$

or

$$\boxed{K_{nk} + K_{nb}\bar{C}_{bk} = [\pm U_{nn}\ 0_{nm}]\ .} \tag{4.23}$$

Equations (4.20) and (4.21) can be written as the single b x k dimensional matrix equation,

$$(K_{bb}-C_{bm}K_{mb})[\bar{C}_{bn}\ \bar{C}_{bm}] - C_{bm}[K_{mn}\ K_{mm}] + [K_{bn}\ K_{bm}] = [\pm C_{bn}\ 0_{bm}]\ ,$$

or

$$\boxed{(K_{bb}-C_{bm}K_{mb})\bar{C}_{bk} - C_{bm}K_{mk} + K_{bk} = [\pm C_{bn}\ 0_{bm}]\ .} \tag{4.24}$$

Equations (4.23) and (4.24), plus the nonsingularity condition, (4.22), are the necessary and sufficient conditions on K_{ee}, $\bar{C}_{ee}$ and C_{ee} which make the network with electromagnetic structure represented by $(K_{ee})^t Z^e_{ee} K_{ee}$ and the topological and port structures represented by $\bar{C}_{ee}$, n-port impedance equivalent to the original network for all values of the original element impedance matrix, Z^e_{ee}. All dependence of the conditions on entries of the e-port coordinate transformation matrix, M_{ee}, has been removed.

The corresponding conditions for p-port admittance equivalence are obtained by a parallel development beginning with the solution of (4.13) for M'_{ee},

$$\bar{C}_{ee}^{-1} K'_{ee} C_{ee} = \pm M'_{ee} \quad . \tag{4.25}$$

The conditions are

$$\boxed{K'_{qq} - \bar{C}_{qk} K'_{kq} \;, \quad \text{nonsingular} \quad .} \tag{4.26}$$

$$\boxed{K'_{bp} - \bar{C}_{bk} K'_{kp} = \begin{bmatrix} 0_{qp} \\ \pm U_{pp} \end{bmatrix} ,} \tag{4.27}$$

and

$$\boxed{- \bar{C}_{bk}(K'_{kk} + K'_{kq} C_{qk}) + K'_{bq} C_{qk} + K'_{bk} = \begin{bmatrix} 0_{qk} \\ \mp C_{pk} \end{bmatrix} .} \tag{4.28}$$

We have now obtained the major result of the study. Applications will be considered in the following chapter.

CHAPTER V

APPLICATIONS

Throughout this study we have focussed attention on the fact that the n- and p-port immittance characteristics of a passive network are determined by its

(i) electromagnetic structure,

(ii) topological structure, and

(iii) port structure.

Now that the necessary and sufficient n- and p-port equivalence conditions on the matrices representing these structures have been obtained, we can turn to the problem of finding equivalent networks themselves. The appearance of the element coordinate transformation matrices, K_{ee} and K'_{ee}, and the fundamental connection matrices, C_{ee} and $\bar{C}_{ee}$, in the equivalence conditions suggests two possible approaches to the problem:

(i) specify the electromagnetic structure of the equivalent network by selecting an element coordinate transformation matrix, K_{ee} or K'_{ee}, and use the conditions obtained in Chapter IV to solve <u>formally</u> for the associated fundamental connection matrix representing the topological and port structures of the equivalent network;

(ii) specify the topological and port structures of the equivalent network by selecting a t-graph with designated port locations, find the associated fundamental connection matrix, and use the conditions obtained in Chapter IV to solve <u>formally</u> for the element coordinate transformation matrix.

In principle, these methods permit us to control the structures of the equivalent network individually, in contrast to methods which determine new e-port immittance matrices whose three structures cannot be

separated. In practice, (i) is of little value because of the great difficulty of selecting element coordinate transformation matrices, K_{ee} or K'_{ee}, in such a way that the formal solutions for the fundamental connection matrix of the equivalent network actually correspond to ported t-graphs. It is not surprising that this difficulty exists since the prior specification of the element values of a network clearly restricts the available characteristics to a great extent. For completeness, however, we present the formal solution for $\bar{C}_{ee}$ in Section 5.1.

Problem (ii) is considerably more interesting since preliminary specification of the equivalent network topological and port structures is much less restrictive than specification of its electromagnetic structure. We will find that the number of conditions required by n- or p-port equivalence is always less than the number of entries of the element coordinate transformation matrix. The resulting freedom often makes it possible to find realizable element immittance matrices.

Preliminary specification of the topological and port structures is a feature of most synthesis methods. For example, the decision to expand a driving point reactance function as a continued fraction determines that it be realized in a ladder structure. Thus, it is not surprising that specification of topological and port structures also plays a role in equivalence problems. In Section 5.2 we shall discuss this approach further and in Sections 5.3 and 5.4 present examples leading to realizable structures.

5.1 Formal Solution for Fundamental Connection Matrices.

One of the necessary and sufficient conditions for n-port impedance equivalence found in Chapter IV is given by Eq. (4.24),

$$(K_{bb}-C_{bm}K_{mb})\bar{C}_{bk} - C_{bm}K_{mk} + K_{bk} = [\pm C_{bn} \;\; 0_{bm}] \quad . \tag{4.24}$$

If K_{bb} and K_{mb} are chosen in such a way that

$$(K_{bb}-C_{bm}K_{mb}) \text{ is nonsingular} \; , \tag{5.1}$$

then (4.24) can be solved formally for $\bar{C}_{bk}$ as

$$\bar{C}_{bk} = (K_{bb}-C_{bm}K_{mb})^{-1}([\pm C_{bn} \;\; 0_{bm}] + C_{bm}K_{mk}-K_{bk}) \quad . \tag{5.2}$$

If the formal solution, $\bar{C}_{bk}$, is a submatrix of a fundamental connection matrix of a t-graph, then it carries all the connection information required to construct an associated electrical network. In general, however, the formal solution will not be a legitimate submatrix of a fundamental connection matrix. This difficulty is comparable to the problem of interpreting the e-port immittance matrices resulting from Cauer's method of linear transformation of port variables. In addition to the solution, (5.2), it is necessary that the remaining two equivalence conditions be satisfied, i.e., that

$$K_{mm} + K_{mb}\bar{C}_{bm} \text{ be nonsingular} \; , \tag{4.22}$$

and that

$$K_{nk} + K_{nb}\bar{C}_{bk} = [\pm U_{nn} \;\; 0_{nm}] \quad . \tag{4.23}$$

We now turn to the more tractable problem which begins with a specification of the topological and port structures of the equivalent network.

5.2 Determination of Element Immittance Matrices

When the t-graph and associated fundamental connection matrix, $\bar{C}_{ee}$, of a network which is to be equivalent to a given one are specified in advance, Eqs (4.23) and (4.24) become a set of linear equations in the entries of an element coordinate transformation matrix, K_{ee}. There are nk such constraints from (4.23) and bk from (4.24), while the number of entries of K_{ee} to be selected is e^2. The minimum number of degrees of freedom remaining is

$$\begin{aligned} d &= e^2 - nk - bk \\ &= (k+b)^2 - nk - bk \\ &= k^2 + bk + b^2 - nk \, , \end{aligned} \tag{5.3}$$

a number which is always positive since n must be less than or equal to k. One convenient method of exploiting this freedom is to incorporate the constraints arising from the equivalence conditions into an element transformation matrix, K_{ee}, by defining a subset of its entries as "parameters," and expressing the remaining entries in terms of the parameters by means of the equivalence conditions. The only requirement on the parameters is that they be selected in such a way that the nonsingularity condition, (4.22), is satisfied. As an example of this procedure we will derive the familiar Tee-Pi two port transformation for a resistive network.

5.3 Example 1: The Tee-Pi Transformation

A resistive Tee network is shown in Fig. 5.1. Assigned directions of positive element-current flow are indicated by arrows. The two n-ports, designated as ports 1 and 2, have been created by cutting R_1 and R_2, so that R_4 must be a tree element. Two new elements, R_3 and R_5, have been adjoined. We will consider that R_3 is an additional cotree element, and R_5 an additional tree element. The branch numbering has been assigned to correlate with that of the Pi-network shown in Fig. 5.2. Here $\bar{R}_1$ and $\bar{R}_2$ define the two n-ports. The tree is chosen to consist of $\bar{R}_4$ and $\bar{R}_5$, so that these elements define the two b-ports, while the cotree element, $\bar{R}_3$, defines the m-port. This scheme provides that elements with the same numbers have corresponding functions in both networks. The conventional Pi-network has only three elements, $\bar{R}_1$ and $\bar{R}_2$ in Fig. 5.2 being zero. It is necessary that they be included, however, so that the n-ports can be defined. Later an element coordinate transformation will be used to set these elements to zero values. This procedure is related to the introduction of "dummy" zero immittance elements as described in Chapter II. Here, however, we create dummy elements in the new network by means of an element coordinate transformation, rather than introducing them in a given network to represent a general port configuration.

Since the determination of an equivalent network will involve, as a first step, transformation of the original element impedance matrix, and since the given network is resistive, it is possible to specify that the adjoined elements both be unit resistors. In order to realize a general frequency-dependent equivalent network it would be necessary to adjoin a sufficient number of inductive and capacitive elements. Any number of elements can be adjoined, and, in general, there is more realization

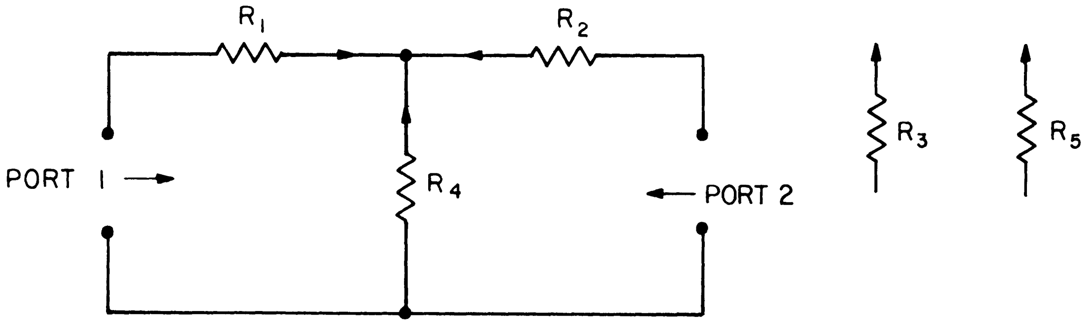


Fig. 5.1. Resistive Tee-network.

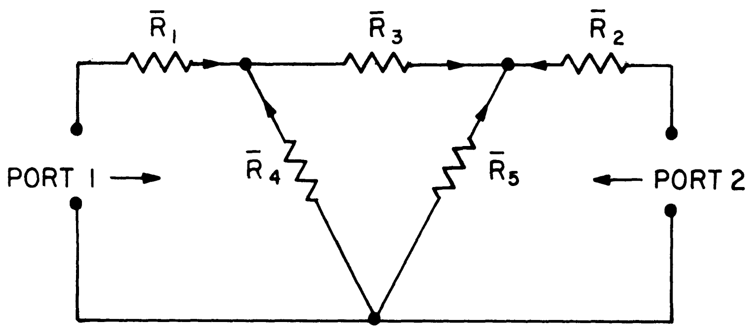


Fig. 5.2. Resistive Pi-network.

freedom in large networks than small ones; the computational difficulties are also greater. The given element impedance matrix, adjoined by unit resistance cotree and tree elements, is

$$Z^e_{ee} = \left[\begin{array}{cc|c|cc} R_1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & R_4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] . \tag{5.4}$$

The dotted lines indicate the partitioning according to the elements associated with the two n-ports, the one m-port, and the two b-ports. This partitioning follows that of the fundamental connection matrix of the t-graph chosen for analysis of the Tee-network. This matrix, C_{ee}, is

$$C_{ee} = \left[\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] . \tag{5.5}$$

The fundamental connection matrix of the t-graph chosen for analysis of the Pi network is

$$\overline{C}_{ee} = \left[\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{array}\right] . \tag{5.6}$$

Since submatrix C_{bm} of C_{ee} is a zero matrix, conditions (4.23) and (4.24) become

$$K_{nk} + K_{nb}\bar{C}_{bk} = [\pm U_{nn} \; 0_{nm}] \; , \tag{5.7}$$

and

$$K_{bb}\bar{C}_{bk} + K_{bk} = [\pm C_{bn} \; 0_{bm}] \; , \tag{5.8}$$

while condition (4.22) becomes

$$K_{mm} + K_{mb}\bar{C}_{bm} \; , \quad \text{nonsingular} \; .$$

We now choose to let the entries of submatrices K_{nb} and K_{bb} be the "parameters" of K_{ee} and express the remaining entries in terms of these parameters. This choice permits the direct expression of submatrices K_{nk} and K_{bk} as

$$K_{nk} = [\pm U_{nn} \; 0_{nm}] - K_{nb}\bar{C}_{bk} \; , \tag{5.9}$$

and

$$K_{bk} = [\pm C_{bn} \; 0_{bm}] - K_{bb}\bar{C}_{bk} \; , \tag{5.10}$$

so that K_{ee} can be written

$$K_{ee} = \begin{bmatrix} (\pm U_{nn} - K_{nb}\bar{C}_{bn}) & -K_{nb}\bar{C}_{bm} & K_{nb} \\ K_{mn} & K_{mm} & K_{mb} \\ (\pm C_{bn} - K_{bb}\bar{C}_{bn}) & -K_{bb}\bar{C}_{bm} & K_{bb} \end{bmatrix} . \tag{5.11}$$

We note that the equivalence conditions do not restrict submatrices K_{mn} and K_{mm}, except through the nonsingularity requirement, (4.22).

As indicated above, $\bar{R}_1$ and $\bar{R}_2$ of the final network are to be set to zero to obtain the usual Pi configuration. Since the transformed element impedance matrix is, from Chapter III,

$$\bar{Z}^e_{ee} = (K_{ee})^t Z^e_{ee} K_{ee} \; , \tag{3.45}$$

$\bar{R}_1$ and $\bar{R}_2$ can only be zero for general values of the original network

elements if the first two columns of K_{ee} are identically zero. In terms of the submatrices of (5.11) these additional conditions are

$$\pm U_{nn} - K_{nb}\bar{C}_{bn} = 0_{nn} , \tag{5.12}$$

$$K_{mn} = 0_{mn} , \tag{5.13}$$

$$\pm C_{bn} - K_{bb}\bar{C}_{bn} = 0_{bn} . \tag{5.14}$$

Equation (5.12) can be rewritten as

$$K_{nb}\bar{C}_{bn} = \pm U_{nn} , \tag{5.15}$$

or, in terms of the problem at hand,

$$\begin{bmatrix} k_{14} & k_{15} \\ k_{24} & k_{25} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{5.16}$$

The solution of this equation for K_{nb} is clearly

$$K_{nb} = \pm \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{5.17}$$

Condition (5.14) is

$$\begin{bmatrix} k_{44} & k_{45} \\ k_{54} & k_{55} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \pm \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} , \tag{5.18}$$

which has the solution

$$K_{bb} = \pm \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} . \tag{5.19}$$

Using the upper (+) signs for K_{nb} and K_{bb}, the remaining constrained submatrices of K_{ee} become

$$-K_{nb}\bar{C}_{bm} = - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} , \tag{5.20}$$

and

$$-K_{bb}\overline{C}_{bm} = -\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{5.21}$$

substitution of (5.12), (5.13), (5.14), (5.20), and (5.21) into (5.11) gives

$$K_{ee} = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & k_{33} & k_{34} & k_{35} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{5.22}$$

The new element impedance matrix obtained from this transformation matrix, K_{ee}, is

$$\overline{Z}^e_{ee} = (K_{ee})^t Z^e_{ee} K_{ee}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (R_1+R_2+k_{33}^2) & (-R_1+k_{33}k_{34}) & (R_2+k_{33}k_{35}) \\ 0 & 0 & (-R_1+k_{33}k_{34}) & (R_1+R_4+k_{34}^2) & (R_4+k_{34}k_{35}) \\ 0 & 0 & (R_2+k_{33}k_{35}) & (R_4+k_{34}k_{35}) & (R_2+R_4+k_{35}^2) \end{bmatrix} . \tag{5.23}$$

For general values of k_{33}, k_{34}, and k_{35} this element impedance matrix involves coupling. Since resistive coupling is not permitted we now wish to select these terms in such a way that the coupling is removed. The conditions are

$$-R_1+k_{33}k_{34} = 0 , \tag{5.24}$$

$$R_2+k_{33}k_{35} = 0 , \tag{5.25}$$

and

$$R_4+k_{34}k_{35} = 0 . \tag{5.26}$$

Solution for k_{33}, k_{34}, and k_{35} gives

$$k_{33} = \left(\frac{R_1R_2}{R_4}\right)^{\frac{1}{2}} , \tag{5.27}$$

$$k_{34} = \left(\frac{R_1R_4}{R_2}\right)^{\frac{1}{2}} , \tag{5.28}$$

and

$$k_{35} = -\left(\frac{R_2R_4}{R_1}\right)^{\frac{1}{2}} . \tag{5.29}$$

Substitution of these values into (5.23) produces the final element impedance matrix without coupling,

$$\bar{Z}^e_{ee} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D/R_4 & 0 & 0 \\ 0 & 0 & 0 & D/R_2 & 0 \\ 0 & 0 & 0 & 0 & D/R_1 \end{bmatrix} , \tag{5.30}$$

where $D = (R_1R_2+R_1R_4+R_2R_4)$.

The nonsingularity condition, (4.22), becomes

$$k_{33} + [k_{34}\ k_{35}] \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq 0 ,$$

or

$$k_{33} + k_{34} - k_{35} \neq 0 ,$$

or

$$\pm\sqrt{\frac{R_1R_2}{R_4}} \pm\sqrt{\frac{R_1R_4}{R_2}} \pm\sqrt{\frac{R_2R_4}{R_1}} \neq 0 , \tag{5.31}$$

which can be satisfied for all positive resistance values by choosing either the positive or negative roots of all terms on the left. This result is the well known Tee-Pi transformation. Even if this solution were not already known to be true we could be sure that it represented an

equivalence transformation as long as conditions (4.22), (4.23), and (4.24) had been incorporated correctly. That is, it is never necessary to actually obtain the accessible n-port immittance matrix of either the original or final networks to apply the theory.

5.4 Example 2: Star Tree to Linear Tree Transformation

Synthesis methods for single element kind networks are sometimes based on special topological and port structures (Ref. 6). Two important classes are those involving a star tree (all tree arcs have one vertex in common), and those involving a linear tree (the unique path between one pair of vertices includes all arcs of the tree). Examples of inductive three-ports with star and linear trees are shown in Figs. 5.3 and 5.4. Both are completely connected four node networks, and so differ only in the selection of the t-graphs and associated ports. There are no k-ports which are not n-ports so that $m = 0$ and all matrices with this subscript vanish from the equivalence conditions. Also, $k = n + m = n$. Equations (4.23) and (4.24) become

$$K_{nn} + K_{nb}\bar{C}_{bn} = \pm U_{nn} , \qquad (5.32)$$

and

$$K_{bn} + K_{bb}\bar{C}_{bn} = \pm C_{bn} . \qquad (5.33)$$

The nonsingularity condition, (4.22), no longer appears. As in Example 1 we let the entries of submatrices K_{nb} and K_{bb} be the "parameters" and solve for K_{nn} and K_{bn} in terms of them.

$$K_{nn} = \pm U_{nn} - K_{nb}\bar{C}_{bn} , \qquad (5.34)$$

and

$$K_{bn} = \pm C_{bn} - K_{bb}\bar{C}_{bn} . \qquad (5.35)$$

Substitution of these relations into K_{ee} gives

$$K_{ee} = \begin{bmatrix} (\pm U_{nn} - K_{nb}\bar{C}_{bn}) & K_{nb} \\ (\pm C_{bn} - K_{bb}\bar{C}_{bn}) & K_{bb} \end{bmatrix} . \qquad (5.36)$$

Every choice of K_{nb} and K_{bb} produces a new element impedance matrix

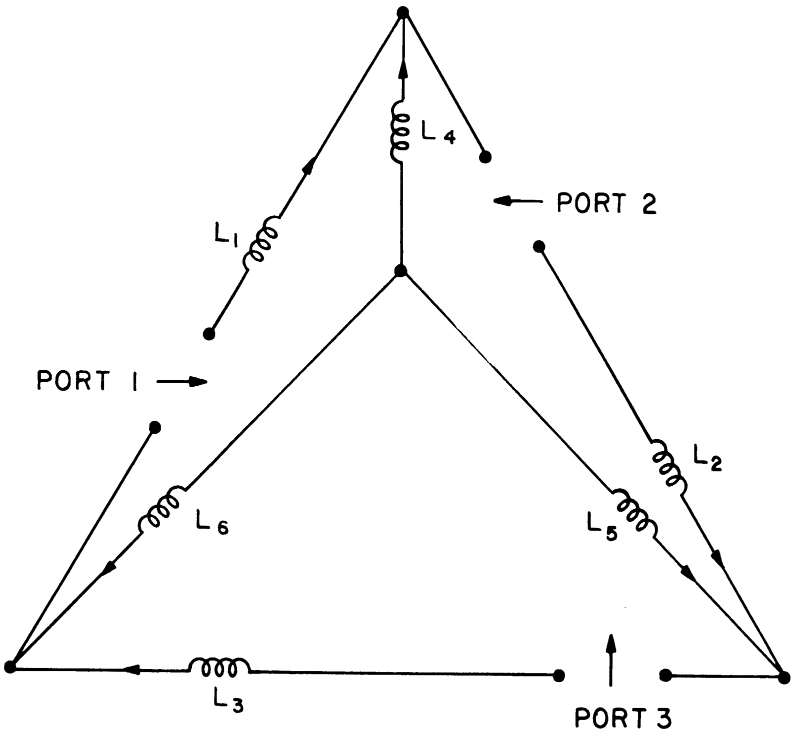


Fig. 5.3. Inductive 3-port with star tree.

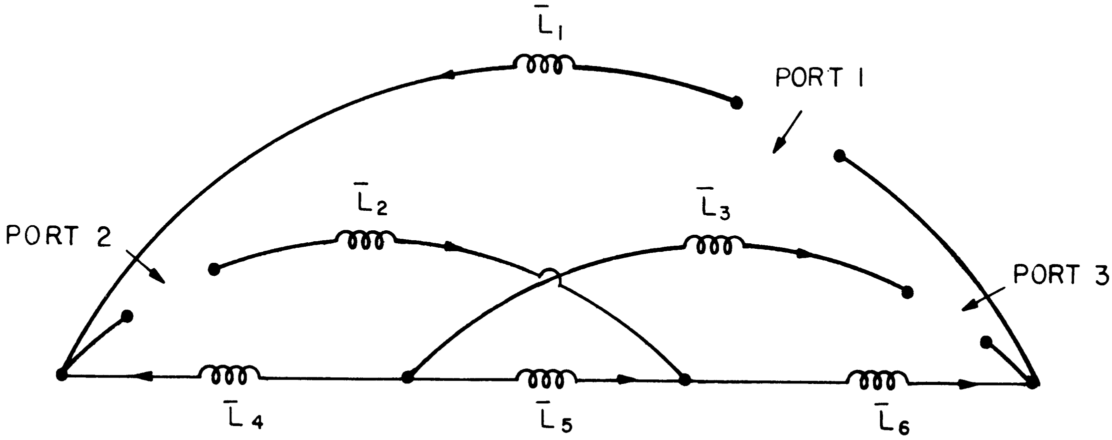


Fig. 5.4. Inductive 3-port with linear tree.

resulting in an equivalent three-port network. In the present case we have that submatrix C_{bn} of the fundamental connection matrix of the star tree network, Fig. 5.3, is

$$C_{bn} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \qquad (5.37)$$

while for the linear tree network, Fig. 5.4, we have

$$\bar{C}_{bn} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}. \qquad (5.38)$$

A particularly simple choice of parameter matrices, K_{nb} and K_{bb} is

$$K_{nb} = 0_{nb}, \qquad (5.39)$$

and

$$K_{bb} = U_{bb}. \qquad (5.40)$$

Using the upper (+) signs in (5.36), K_{ee} now becomes

$$K_{ee} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (5.41)$$

The original element impedance matrix is

$$Z_{ee}^{e} = \begin{bmatrix} L_1 s & 0 & 0 & 0 & 0 & 0 \\ 0 & L_2 s & 0 & 0 & 0 & 0 \\ 0 & 0 & L_3 s & 0 & 0 & 0 \\ 0 & 0 & 0 & L_4 s & 0 & 0 \\ 0 & 0 & 0 & 0 & L_5 s & 0 \\ 0 & 0 & 0 & 0 & 0 & L_6 s \end{bmatrix}, \tag{5.42}$$

so that the new element impedance matrix associated with the above element coordinate transformation matrix is

$$\bar{Z}_{ee}^{e} = (K_{ee})^t Z_{ee} K_{ee}$$

$$= \begin{bmatrix} (L_1+L_5)s & 0 & -2L_5 s & 0 & -L_5 s & 0 \\ 0 & L_2 s & 0 & 0 & 0 & 0 \\ -2L_5 s & 0 & (L_3+4L_5)s & 0 & 2L_5 s & 0 \\ 0 & 0 & 0 & L_4 s & 0 & 0 \\ -L_5 s & 0 & 2L_5 s & 0 & L_5 s & 0 \\ 0 & 0 & 0 & 0 & 0 & L_6 s \end{bmatrix}. \tag{5.43}$$

By forming the determinants of the principal minors it can be shown that this matrix is positive-definite as long as all inductance values of the original network are positive. This means that the equivalent network is realizable by means of coupled elements.

Although it is not necessary to determine the n-port impedance matrix we may do so in order to verify that the new network is equivalent to the original one. The n-port impedance matrix of the new network is

$$\bar{Z}_{nn}^{c} = \bar{Z}_{nn}^{e} + (\bar{C}_{bn})^t \bar{Z}_{bb}^{e} \bar{C}_{bn} + (\bar{C}_{bn})^t \bar{Z}_{bn}^{e} + \bar{Z}_{nb}^{e} \bar{C}_{bn}, \tag{5.44}$$

or

$$\bar{Z}^c_{nn} = \begin{bmatrix} (L_1+L_5)s & 0 & -2L_5s \\ 0 & L_2s & 0 \\ -2L_5s & 0 & (L_3+4L_5)s \end{bmatrix} + \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} L_4s & 0 & 0 \\ 0 & L_5s & 0 \\ 0 & 0 & L_6s \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$+ \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -L_5s & 0 & 2L_5s \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -L_5s & 0 \\ 0 & 0 & 0 \\ 0 & 2L_5s & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (L_1+L_4+L_6)s & -L_4s & -L_6s \\ -L_4s & (L_2+L_4+L_5)s & -L_5s \\ -L_6s & -L_5s & (L_3+L_5+L_6)s \end{bmatrix} . \qquad (5.45)$$

This result is identical with the usual mesh impedance matrix obtained by inspection of Fig. 5.3, or calculated from

$$Z^c_{nn} = Z^e_{nn} + (C_{bn})^t Z^e_{bb} C_{bn} , \qquad (5.46)$$

or

$$Z^c_{nn} = \begin{bmatrix} L_1s & 0 & 0 \\ 0 & L_2s & 0 \\ 0 & 0 & L_3s \end{bmatrix} + \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} L_4s & 0 & 0 \\ 0 & L_5s & 0 \\ 0 & 0 & L_6s \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (L_1+L_4+L_6)s & -L_4s & -L_6s \\ -L_4s & (L_2+L_4+L_5)s & -L_5s \\ -L_6s & -L_5s & (L_3+L_5+L_6)s \end{bmatrix} . \qquad (5.47)$$

This completes the presentation of applications of the conditions obtained in Chapter IV. We note that in Example 2 the equivalent network involved coupled elements. The problem of introducing the additional

constraint that the new element immittance matrices be diagonal appears to be quite difficult in general. The coupling removal equations are nonlinear in the elements of the transformation matrix, K_{ee}. Furthermore, there are $e(e - 1)/2$ such relations among only ek free entries of K_{ee}. This problem and others suggested by the results obtained here are discussed further in the following chapter.

CHAPTER VI

CONCLUSION

The major result of this study is the discovery of the conditions on the matrices which represent the electromagnetic, topological, and port structures of passive networks in order that they be n- or p-port equivalent to a given network. The networks considered are composed of passive, linear, time-invariant, bilateral electrical elements whose associated linear graphs are connected, and without self-loops. The development of these conditions contributes to the theory of equivalent networks by introducing the possibility of direct control over the three underlying structures.

The method used to obtain this result was made possible by the availability of Kron's "orthogonal" method involving a nonsingular matrix representation of topological structures. The successful exploitation of this analysis method partially verifies a frequent assertion by Kron (for example, Ref. 9) that nonsingular topological structure matrices should have wide usefulness in network studies.

In this study specification of individual structures is followed by formal solution for the remaining structure matrices using the necessary and sufficient equivalence conditions. An outstanding unsolved problem is that of ensuring that these formal solutions correspond to realizable electromagnetic, topological, or port structures. This problem appears to be quite difficult. One approach might be to seek the conditions under which special classes of specified structures lead to realizable equivalent networks. For example, one could consider topological and port structures based on star or linear trees, and find the conditions

on element coordinate transformation matrices required for equivalence and realizability of the new electromagnetic structure.

A second major problem is that of finding electromagnetic structures which do not require coupling. For general element immittance matrices of the given network, coupling removal requires the simultaneous solution of a set of nonlinear equations, since the off-diagonal entries of the new element immittance matrices are bilinear forms in the entries of the element coordinate transformation matrices. This sort of computational problem arises frequently in network synthesis studies. Cederbaum (Ref. 2) has recently suggested that a methematical programming technique be used to attack such problems; unfortunately, little is known about programming problems in which the constraints are nonlinear.

The development of realizability and coupling removal conditions are major tasks on which progress must be made before the conditions obtained here can provide a fully effective means for controlling the structures of equivalent networks. A <u>new</u> area of investigation suggested by the methods used here involves the problem of general synthesis of N-port networks.

In one sense a complete solution to the n- and p-port equivalence problems treated here would also completely resolve the synthesis problem, since it would then be possible to begin with an undesirable, or even unrealizable network with the desired immittance characteristics and transform it to a new network, if one exists, with the desired properties. For example, one could treat a given N x N immittance matrix as the element immittance matrix of a set of N coupled elements, adjoin a number of new elements, and seek new electromagnetic, topological, and port structures which preserve the given immittance matrix, but which do not require

coupling. Obviously, it would be necessary to have made progress on the realizability and coupling removal problems described above in order to carry out such a scheme.

This concludes our study of equivalent networks. We hope that the methods and results presented here will be useful to other workers interested in equivalence and related network problems.

REFERENCES

1. Cauer, Wilhelm. Synthesis of Linear Communication Networks. Vol. I and II. Translated from the German edition of 1940 by G. E. Knausenberger and J. N. Warfield. New York: McGraw-Hill Book Co., Inc., 1958, pp. 623-625.

2. Cederbaum, Israel. "The Realization of n-Port Networks Without Transformers--A Panel Discussion," IRE Trans. on Circuit Theory, Vol. CT-9, No. 3 (September, 1962), p. 213.

3. Gadsden, Christopher P. "The Anisotropy Transformer," Proc. IEEE (Correspondence), Vol. 51, No. 6 (April, 1963), p. 952.

4. Guillemin, Ernst A. Synthesis of Passive Networks. New York: John Wiley and Sons, Inc., 1957, pp. 141-157.

5. _______ . Introductory Circuit Theory. New York: John Wiley and Sons, Inc., 1953, pp. 8-10.

6. Kim, Wan H. and Chien, Robert T. Topological Analysis and Synthesis of Communication Networks. New York: Columbia University Press, 1962, pp. 157-163.

7. Kron, Gabriel. Tensor Analysis of Networks. New York: John Wiley and Sons, Inc., 1939, pp. 614-616.

8. _______ , op. cit., pp. 407-435.

9. _______ , "Diakoptics--The Piecewise Solution of Large-Scale Systems," The Electrical Journal (London), Vol. CLVIII, No. 23, (June 7, 1957), pp. 1673-1677.

10. Schoeffler, James D. Foundations of Equivalent Network Theory. Technical Report No. 2, Contract NONR-1141(iv), NR-375-370. Cleveland: Engineering Division, Case Institute of Technology, December 6, 1961.

11. Seshu, Sundaram and Reed, Myril B. Linear Graphs and Electrical Networks. Reading, Massachusetts: Addison-Wesley Publishing Co., Inc., 1961, pp. 24-33.

12. Tropper, A. M. Matrix Theory for Electrical Engineers. Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1962, pp. 37-39.