Some Results on the Nonoscillation of Salt Fingers

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Salt-finger convection in water discovered by Stommel, Arons, and Blanchard is discussed, and conditions under which the principle of exchange of stabilities holds are given.

I. INTRODUCTION

Stommel, Arons, and Blanchard[1] discovered the salt-finger phenomenon, or the phenomenon of convection in water with a temperature gradient and a salinity gradient in such a way that the water becomes hotter and saltier with higher and higher elevation. The fascinating feature of the phenomenon is that convection can happen even if the density of water, as a result of the temperature and salinity gradients, actually decreases with height. Stommel *et al.* described an experiment with water with such gradients. A glass tube draws the bottom water slowly upward, which warms up but keeps its freshness as it rises, producing a column of fresh water with nearly the same temperature as outside the tube, thus, by buoyancy, keeps rising. Since the salinity diffusivity in water is less than one one-hundredth of the thermal diffusivity, the same phenomenon can happen even without the glass tube. Columns of fresh and originally cold water will, when the configuration is unstable, rise while columns of salty and originally hot water will sink. The salt-finger phenomenon was suggested as one possible means of mixing in the ocean, the top layer of which is warmer but due to evaporation may be saltier than the water below. Somehow the saltier water must transmit its salinity downward by a process other than molecular diffusion, since it is known that salinity in the ocean does not decrease monotonically with depth.

The problem of stability of water with temperature and salinity gradients was formulated by Stern,[2] who also discussed the so-called principle of exchange of stabilities for the simple but rather unrealistic case of free upper and lower boundaries. For a fixed lower boundary, it has not been shown that instability, if it occurs, is not characterized by growing oscillations. The purpose of this paper is to give some conditions under which the principle of exchange of stabilities holds.

II. EQUATIONS GOVERNING STABILITY

The equations governing the stability against salt-finger convection were given by Stern.[2] The following equations are taken from Yih,[3] and differ from Stern's only in notation. If T_0 is a reference temperature, c_0 is a reference concentration, and ρ_0 is the corresponding density, then for small deviations from the reference state

$$\rho = \rho_0[1 - \alpha(T - T_0) + \alpha'(c - c_0)], \qquad (1)$$

in which ρ, T, and c are, respectively, the density, the temperature, and the salinity concentration, and α is the coefficient of thermal expansion under the prevailing pressure, the effect of the pressure variation on α being neglected. Since the addition of salt in solution does not appreciably affect the volume of water, to which it is added, α' is simply $1/\rho_0$, if c is measured in mass per unit volume.

If d is the depth of water, κ is the thermal diffusivity, t is the time, and x_i, for $i = 1, 2, 3$, are Cartesian coordinates, the dimensionless coordinates and time are defined by

$$(x, y, z) = \left(\frac{x_1}{d}, \frac{x_2}{d}, \frac{x_3}{d}\right), \quad \tau = \frac{\kappa}{d^2}\, t, \qquad (2)$$

in which x_3 is measured in the direction of the vertical. The perturbation quantities are then assumed to have the forms

$$(u_3, T', c') = \left(\frac{k}{d}\, w(z),\ -\beta\, d\theta(z),\ -\beta'\, dc(z)\right) \cdot f(x, y) \exp(\sigma\tau), \qquad (3)$$

in which u_3 is the velocity component in the z direction, T' and c' denote temperature and concentration perturbations, β and β' are temperature and salinity gradients, both in the z direction, defined by

$$\beta = \frac{T_1 - T_0}{d}, \qquad \beta' = \frac{c_1 - c_0}{d},$$

$\sigma = \sigma_r + i\sigma_i$ is the growth rate, assumed complex *a priori*, and $f(x, y)$ satisfies

$$f_{xx} + f_{yy} + a^2 f = 0, \tag{4}$$

a being the wavenumber, and subscripts denoting partial differentiations.

Then, the differential equations governing stability are

$$\left(D^2 - a^2 - \frac{\sigma}{\mathrm{Pr}}\right)(D^2 - a^2)\, w = -Ra^2\theta + R'a^2\gamma, \tag{5}$$

$$(D^2 - a^2 - \sigma)\theta = -w, \tag{6}$$

$$\left(D^2 - a^2 - \frac{\sigma}{k}\right)\gamma = -w, \tag{7}$$

in which $\gamma = kc(z)$, Pr is the Prandtl number ν/κ, R is the Rayleigh number

$$R = \frac{g\alpha\beta\, d^4}{\kappa\nu}, \tag{8}$$

and

$$k = \frac{\kappa'}{\kappa}, \quad R' = \frac{g\beta'\, d^4}{\rho_0\kappa'\nu}, \quad D = \frac{d}{dz}, \tag{9}$$

κ' being the salinity diffusivity, g being the gravitation acceleration, and ν being the kinematic viscosity. Note that the R' here is the R_1 in Yih[3] and the $c(z)$, $\gamma(z)$, and w here are, respectively, the $\gamma(z)$, $\gamma_1(z)$, and $-w$ there.

In this paper we shall consider rigid boundaries only, for which the boundary conditions for w are

$$w(0) = 0 \quad \text{and} \quad Dw(0) = 0 \tag{10}$$

at the bottom where $z = 0$, and

$$w(1) = 0 \quad \text{and} \quad Dw(1) = 0, \tag{11}$$

and the boundary conditions for θ and γ are

$$\theta(0) = 0 \quad \text{and} \quad \gamma(0) = 0 \tag{12}$$

and (see Yih,[3] pp. 152, 153)

$$\theta(1) = 0 \quad \text{and} \quad \gamma(1) = 0 \tag{13}$$

for maintained temperature and salinity at the upper rigid surface.

III. CLASSIFICATION OF CASES

There are four cases, given for convenience in Table I. It is physically obvious and mathematically readily demonstrable (as will be seen) that case 2 corresponds to stability. By following the approach of Pellew and Southwell,[4] it can be proved for case 3 that if the imaginary part of σ is not zero, then its real part must be negative. Case 4 is similar to case 1, and the treatment for case 4 follows much the same line as the treatment of case 1. For this reason, we shall concentrate on case 1 (and incidentally case 2).

Table I. The four possible cases.

Case	1	2	3	4
R	+	+	−	−
R'	+	−	+	−

Before presenting the analysis we note that for the case of the rigid upper surface with maintained constant temperature and concentration, the boundary conditions for θ and γ are the same. Furthermore, if we put σ equal to zero and let

$$\gamma = \theta, \tag{14}$$

(6) and (7) become identical. Upon substituting θ for γ in (5), Eqs. (5) and (6), with $\sigma = 0$, together with the boundary conditions, constitute precisely the system governing Bénard cells at neutral stability. The solution of that system for $\sigma = 0$ is known to exist. However, this is not to say that if $\sigma_i \neq 0$, then $\sigma_r < 0$. In fact, it does not even say that if $\sigma_r = 0$, then σ_i must *necessarily* be (although it can be) zero.

We note here the peculiarity that while for $\sigma \neq 0$ the order of the differential equation in w is eight after θ and γ have been eliminated from (5), (6), and (7), the order of that equation with $\sigma = 0$ is only six provided the boundary conditions for θ and γ are identical. The case for $\sigma = 0$ is a degenerate case.

IV. CONDITIONS ON R, R′, AND k

Let R be positive, so that case 1 or 2 is considered, and let the upper boundary be rigid and the boundary conditions for θ and γ on it be given by (13). Multiplying (5) by w^*, the complex conjugate of w, and integrating between $z = 0$ and $z = 1$, using (6) and (7) to evaluate the w^* on the right-hand side, we have, upon repeated use of (10), (11), (12), and (13),

$$I_2 + \left(2a^2 + \frac{\sigma}{\mathrm{Pr}}\right)I_1 + a^2\left(a^2 + \frac{\sigma}{\mathrm{Pr}}\right)I_0$$
$$= -Ra^2[K_1 + (a^2 + \sigma^*)K_0]$$
$$+ R'a^2\left[J_1 + \left(a^2 + \frac{\sigma^*}{k}\right)J_0\right], \tag{15}$$

in which

$$I_n = \int |D^n w|^2, \quad K_n = \int |D^n \theta|^2, \quad J_n = \int |D^n \gamma|^2, \tag{16}$$

n being 0, 1, or 2, with $D^0 = 1$. The real and imaginary parts of (15) are

$$I_2 + 2a^2 I_1 + a^4 I_0 + Ra^2(K_1 + a^2 K_0) - R'a^2(J_1 + a^2 J_0) + \sigma_r\left(\frac{1}{\Pr}(I_1 + a^2 I_0) + Ra^2 K_0 - \frac{R'a^2}{k} J_0\right) = 0, \tag{17}$$

and

$$\sigma_i\left(\frac{1}{\Pr}(I_1 + a^2 I_0) - Ra^2 K_0 + \frac{R'a^2}{k} J_0\right) = 0. \tag{18}$$

If $\sigma_i \neq 0$, (18) shows that either R' is negative or

$$RK_0 > \frac{R'}{k} J_0. \tag{19}$$

If R' is negative, we have case 2 of Sec. III, and (17) shows that $\sigma_r < 0$. If (19) holds, due to the presence of the terms RK_1 and $R'J_1$ in (17), we cannot as yet reach the desired conclusion that

$$\sigma_i = 0 \quad \text{if} \quad \sigma_r \geq 0. \tag{20}$$

For convenience, we shall write

$$M = \frac{1}{\Pr}(I_1 + a^2 I_0) - Ra^2 K_0 + R'a^2 k^{-1} J_0. \tag{21}$$

We shall now give some conditions on R, R', and k under which $\sigma_r < 0$ or $\sigma_i = 0$. We can expand w in a convergent series:

$$W = \sum_{n=1}^{\infty} A_n \sin n\pi z, \tag{22}$$

which satisfies

$$w(0) = 0 = w(1).$$

Then, (6) and (7) give

$$\theta = -\sum_{n=1}^{\infty} \frac{A_n}{n^2\pi^2 + a^2 + \sigma} \sin n\pi z, \tag{23}$$

$$\gamma = -\sum_{n=1}^{\infty} \frac{A_n}{n^2\pi^2 + a^2 + \sigma k^{-1}} \sin n\pi z, \tag{24}$$

which satisfy (12) and (13).

From (23) and (24) it follows that

$$2K_0 = \sum_{n=1}^{\infty} \left|\frac{A_n}{n^2\pi^2 + a^2 + \sigma}\right|^2, \tag{25}$$

$$2J_0 = \sum_{n=1}^{\infty} \left|\frac{A_n}{n^2\pi^2 + a^2 + \sigma k^{-1}}\right|^2. \tag{26}$$

The ratio of the nth terms in (25) and (26) is

$$r_n = \left|\frac{n^2\pi^2 + a^2 + \sigma k^{-1}}{n^2\pi^2 + a^2 + \sigma}\right|^2. \tag{27}$$

If $\sigma_r \geq 0$ and $k > 1$,

$$k^{-2} < r_n < 1,$$

so that

$$k^{-2} J_0 < K_0 < J_0. \tag{28}$$

Furthermore, (23) and (24) can be differentiated at least once (actually twice) term by term, since (22) is convergent, and similar arguments lead to

$$k^{-2} J_1 < K_1 < J_1. \tag{29}$$

From (17), (18), (28), and (29) we deduce the following results for $k > 1$:

(a) $\sigma_i = 0$ if $\sigma_r \geq 0$, provided $kR \leq R'$,

(b) $\sigma_r < 0$ if $k^2 R' \leq R$.

Result (b) is obtained by showing that the assumption $\sigma_r \geq 0$ leads to a contradiction.

Similarly, if $k \leq 1$ and $\sigma_r \geq 0$, we have

$$1 < r_n < k^{-2},$$

and

$$J_0 \leq K_0 \leq k^{-2} J_0, \qquad J_1 \leq K_1 \leq k^{-2} J_1. \tag{30}$$

Thus, if $R \geq R'$, from (17) and (30) it follows that $\sigma_r \geq 0$ and $\sigma_i \neq 0$ lead to a contradiction, so that

$$\sigma_r < 0 \quad \text{if} \quad R \geq R', \quad \text{and} \quad \sigma_i \neq 0, \tag{31}$$

which is not without interest. Furthermore, (18) and (30) show that

$$\sigma_i = 0 \quad \text{if} \quad kR' \geq R. \tag{32}$$

Thus, for $k \leq 1$ (which is the interesting case, since it contains the case of water), it is only in the range

$$k < \frac{R}{R'} < 1 \tag{33}$$

that (20) is in question. Outside of the range (33), either $\sigma_r < 0$ or $\sigma_i = 0$, or possibly both hold.

V. A CIRCLE OF VALIDITY FOR THE PRINCIPLE OF EXCHANGE OF STABILITIES

So far we have not shown that (20) holds. Since for salt-finger convection in water $k < 1$, we shall henceforth concentrate on the case $k < 1$. First, we shall present the lemma.

Lemma: If

$$\frac{J_1K_0}{J_0K_1} < \frac{1}{k} \quad \text{for} \quad \sigma_r \geq 0, \tag{34}$$

then (20) holds.

The proof of the lemma is simple. If $\sigma_i \neq 0$, M defined by (21) vanishes, by virtue of (18). Thus

$$\frac{RK_0}{R'J_0} > \frac{1}{k}. \tag{35}$$

Then if $\sigma_r \geq 0$, (34) and (35) give

$$\frac{RK_1}{R'J_1} > \frac{kRK_0}{R'J_0} > 1, \tag{36}$$

and with (35) and (36) we deduce from (17) that $\sigma_r < 0$. Hence, the assumption $\sigma_r \geq 0$ is incompatible with the assumption $\sigma_i = 0$, and (20) holds.

Our task then is to show that (34) holds. Since

$$2K_1 = \pi^2 \sum_{n=1}^{\infty} \left|\frac{nA_n}{n^2\pi^2 + a^2 + \sigma}\right|^2, \tag{37}$$

$$2J_1 = \pi^2 \sum_{n=1}^{\infty} \left|\frac{nA_n}{n^2\pi^2 + a^2 + \sigma k^{-1}}\right|^2, \tag{38}$$

we obtain from (25), (26), (37), and (38)

$$\frac{J_1K_0}{J_0K_1} = \frac{\sum_{m,n} a_{mn} |A_m|^2 |A_n|^2}{\sum_{m,n} b_{mn} |A_m|^2 |A_n|^2}, \tag{39}$$

in which

$$a_{mn} = \left|\frac{m}{(m^2\pi^2 + a^2 + \sigma k^{-1})(n^2\pi^2 + a^2 + \sigma)}\right|^2, \tag{40}$$

$$b_{mn} = \left|\frac{m}{(n^2\pi^2 + a^2 + \sigma k^{-1})(m^2\pi^2 + a^2 + \sigma)}\right|^2. \tag{41}$$

Obviously, $a_{nn} = b_{nn}$. We shall determine the limits of

$$r_{mn} = \frac{a_{mn} + a_{nm}}{b_{mn} + b_{nm}} \tag{42}$$

for m not equal to n. It is clear from (39) and $a_{nn} = b_{nn}$ that if

$$r_{mn} < \frac{1}{k} \quad \text{for} \quad \sigma_r \geq 0, \tag{43}$$

for all m and n not equal to m, then (34) must hold. Our effort will then be directed toward the establishment of (43).

Using (40) and (41), we obtain

$$r_{mn} = \frac{M}{D}, \tag{44}$$

where

$$M = m^2 \left|(m^2\pi^2 + a^2 + \sigma)(n^2\pi^2 + a^2 + \sigma k^{-1})\right|^2 + n^2 \left|(n^2\pi^2 + a^2 + \sigma)(m^2\pi^2 + a^2 + \sigma k^{-1})\right|^2, \tag{45}$$

$$D = m^2 \left|(n^2\pi^2 + a^2 + \sigma)(m^2\pi^2 + a^2 + \sigma k^{-1})\right|^2 + n^2 \left|(m^2\pi^2 + a^2 + \sigma)(n^2\pi^2 + a^2 + \sigma k^{-1})\right|^2. \tag{46}$$

We shall rewrite (44) in the form

$$r_{mn} = 1 + \frac{N}{D}, \tag{47}$$

where $N = M - D$. On expanding (45) and (46), we find that

$$N = \left(\frac{1}{k} - 1\right) \sum_{h=1}^{7} N_h, \qquad D = \sum_{h=1}^{16} D_h, \tag{48}$$

where

$$\begin{aligned}
N_1 &= 2(m^2 - n^2)^2 m^2 n^2 \pi^6 \sigma_r,\\
N_2 &= 2(m^2 - n^2)^2 (m^2 + n^2)\pi^4 a^2 \sigma_r,\\
N_3 &= (m^2 - n^2)^2 (m^2 + n^2)\pi^4 (k^{-1} + 1)\sigma_r^2,\\
N_4 &= 2(m^2 - n^2)^2 \pi^2 AB\sigma_r,\\
N_5 &= (m^2 - n^2)^2 (m^2 + n^2)(k^{-1} + 1)\pi^4 \sigma_i^2,\\
N_6 &= 2(m^2 - n^2)^2 (k^{-1} + 1)\pi^2 a^2 \sigma_i^2,\\
N_7 &= 2(m^2 - n^2)^2 \pi^2 k^{-1} \sigma_r \sigma_i^2,\\
A &= a^2 + \sigma_r, \quad B = a^2 + \sigma_r k^{-1},
\end{aligned} \tag{49}$$

$$\begin{aligned}
&D_1 = m^4 n^4 (m^2 + n^2)\pi^8, \qquad D_2 = 4m^4 n^4 \pi^6 B,\\
&D_3 = m^2 n^2 (m^2 + n^2)\pi^4 B^2,\\
&D_4 = 2m^2 n^2 (m^4 + n^4)\pi^6 A,\\
&D_5 = 4m^2 n^2 (m^2 + n^2)\pi^4 AB, \qquad D_6 = 4m^2 n^2 \pi^2 AB^2,\\
&D_7 = (m^6 + n^6)\pi^4 A^2, \quad D_8 = 2(m^4 + n^4)\pi^2 A^2 B,\\
&D_9 = (m^2 + n^2)A^2 B^2, \qquad D_{10} = (m^6 + n^6)\pi^4 \sigma_i^2,\\
&D_{11} = 2(m^4 + n^4)\pi^4 B\sigma_i^2, \qquad D_{12} = (m^2 + n^2)B\sigma_i^2,\\
&D_{13} = m^2 n^2 (m^2 + n^2)\pi^4 k^{-2} \sigma_i^2,\\
&D_{14} = 2m^2 n^2 \pi^2 A k^{-2} \sigma_i^2, \qquad D_{15} = (m^2 + n^2)A^2 k^{-2} \sigma_i^2,\\
&D_{16} = (m^2 + n^2)k^{-2} \sigma_i^4.
\end{aligned} \tag{50}$$

By direct comparison it can be easily verified that, provided $k\pi^4 \geq |\sigma|^2$,

$$N_1 + N_6 + N_7 < D_4 + D_{11},$$

$$N_2 + N_3 + N_5 < D_1 + D_7 + D_{10},$$

$$N_4 < D_8.$$

Addition of these inequalities gives

$$\sum_{h=1}^{7} N_h < D_1 + D_4 + D_7 + D_8 + D_{10} + D_{11} < D.$$

Hence, from (48) we have

$$N < \left(\frac{1}{k} - 1\right)D,$$

and from (47) we have, finally

$$r_{mn} < 1 + \left(\frac{1}{k} - 1\right) = \frac{1}{k},$$

which is (43). Hence (34) holds, and by the lemma we conclude that (20) holds for $k < 1$, provided σ is on or within the circle $k\pi^4 = |\sigma|^2$ in the complex σ plane. [We know that (20) holds for $k = 1$.]

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