

Solutions of Penrose's equation

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(Received 21 July 1998; accepted for publication 22 September 1998)

The computational use of Killing potentials which satisfy Penrose's equation is discussed. Penrose's equation is presented as a conformal Killing–Yano equation and the class of possible solutions is analyzed. It is shown that solutions exist in space–times of Petrov type *O*, *D*, or *N*. In the particular case of the Kerr background, it is shown that there can be no Killing potential for the axial Killing vector. © 1999 American Institute of Physics. [S0022-2488(99)03001-7]

I. INTRODUCTION

In a space–time which admits a Killing vector k^a it is straightforward to find its Killing potential. Killing potentials are real bivectors Q^{ab} whose divergence returns the Killing vector $(1/3)\nabla_b Q^{ab} = k^a$. Killing potentials attain physical importance when they are used in the Penrose–Goldberg (PG)¹ superpotential for computing conserved quantities such as mass and angular momentum. The PG superpotential is

$$U_{PG}^{ab} = \sqrt{-g} \frac{1}{2} G^{ab}{}_{cd} Q^{cd}, \quad (1)$$

where $G^{ab}{}_{cd} = -*R^{*ab}{}_{cd}$, the negative right and left dual of the Riemann tensor. When the Ricci tensor is zero then $G^{ab}{}_{cd} = C^{ab}{}_{cd}$, the Weyl tensor. If Q^{ab} satisfies Penrose's equation (4) then

$$\nabla_b U_{PG}^{ab} = \sqrt{-g} G^{ab} k_b \quad (2)$$

for Einstein tensor G^{ab} . The current density

$$J^a = \sqrt{-g} G^{ab} k_b \quad (3)$$

is conserved independently of the left-hand side of Eq. (2). It is the PG superpotential that allows the Noether quantities to be computed by integrating over closed two-surfaces, which is Penrose's quasilocal construction.² If one views the Killing vector itself as a conserved current then its integral over a three-surface is identically equal to 1/3 the integral of its Killing potential over the bounding two-surface and no new information can be obtained.

The tensor version of Penrose's equation³ is

$$P^{abc} := \nabla^{(a} Q^{b)c} - \nabla^{(a} Q^{c)b} + g^{a[b} Q^{c]e}{}_{;e} = 0. \quad (4)$$

With $j^a := (1/3)\nabla_b Q^{*ab}$, and $k^a := (1/3)\nabla_b Q^{ab}$, an equivalent equation⁴ to $P^{abc} = 0$ is

$$\nabla_c Q^{ab} = -2\delta_c^{[a} k^{b]} + 2(\delta_c^{[a} j^{b]})^*. \quad (5)$$

If Q^{ab} is a solution of the Penrose equation then $k_{(b;c)} = -(1/2)Q_{a(b}R^a{}_{c)}$ with a similar relation connecting j^a and Q^{*ab} . For Ricci-flat space–times j^a and k^a are Killing vectors.

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For a particular space–time the number of independent Killing vectors is between zero and ten. Penrose³ gave the complete solution to Eq. (4) in Minkowski space for ten real independent Q^{ab} .

This work discusses the existence of Killing potentials which satisfy Penrose’s equation or equivalently the conformal Killing–Yano (CKY) equation for two-form Q . The fact that such tensors only exist in space–times of Petrov type D , N , or O is discussed in Sec. III B and Appendices C and D .

In the Kerr background, it has previously been shown that there is no Killing potential for the axial Killing vector.⁵ We show, in Sec. III C, how this can be anticipated from properties of the curvature and the fact that the axial Killing vector must vanish along the axis of symmetry.

We use both the abstract index notation familiar to relativists and some coordinate free notation for which we provide Appendix A as a reference. We use boldface characters for index free tensor notation, excepting differential forms which appear in calligraphic type. Appendix B describes some aspects of the Petrov classification in a way convenient for our purposes.

II. PREVIOUS RESULTS

An exact solution of the Penrose equation for Kerr’s vacuum solution is given in Eq. (8). This solution was first used in the context of the PG superpotential construction in Ref. 6. The Kerr solution has two Killing vectors (KVs), stationary $k_{(t)}$ and axial $k_{(\varphi)}$, and the metric is

$$g^{\text{Kerr}} = l \otimes n + n \otimes l - m \otimes \bar{m} - \bar{m} \otimes m, \quad (6)$$

where $\{l, n, m, \bar{m}\}$ is the Newman–Penrose principal null coframe, given in Boyer–Lindquist coordinates by

$$\begin{aligned} l &= dt - (\Sigma/\Delta) dr - a \sin^2 \theta d\varphi, \\ n &= \frac{\Delta}{2\Sigma} [dt + (\Sigma/\Delta) dr - a \sin^2 \theta d\varphi], \\ m &= \frac{1}{\sqrt{2R}} [ia \sin \theta dt - \Sigma d\theta - i(r^2 + a^2) \sin \theta d\varphi], \end{aligned} \quad (7)$$

where $R = r - ia \cos \theta$, $\Sigma = R\bar{R}$, and $\Delta = r^2 + a^2 - 2m_0 r$. The Killing potential for $k_{(t)}$ is the bivector $Q_{(t)}^{ab}$ obtained by raising the components of the two-form

$$Q_{(t)} = -(R\mathcal{M} + \bar{R}\bar{\mathcal{M}}), \quad (8)$$

where $\mathcal{M} := l \wedge n - m \wedge \bar{m}$ is an anti-self-dual two-form, that is $*\mathcal{M} = -i\mathcal{M}$. We mention that $Q_{(t)}^{ab}$ is a global solution since the quasilocal PG mass, resulting from integration of the PG superpotential over two-surfaces of constant t and r , is *independent of choice of two-surface*

$$\oint_{S^2} U_{\text{PG}}^{ab} dS_{ab} = -8\pi m_0 \quad (9)$$

for any r beyond the outer event horizon.

The next interesting result involves the axial Kerr symmetry. Goldberg¹ found asymptotic solutions of the Penrose equation for the Bondi–Sachs metric which includes the Kerr solution as a special case. But Glass⁵ showed that the axial Killing potential could not be a solution of the Penrose equation at finite r .

The bivector $Q_{(t)}^{ab}$ generally has six independent components and so enough information to describe two Killing vectors. Since the Kerr solution has two KVs, can the dual of $Q_{(t)}^{ab}$ yield $k_{(\varphi)}$? Direct differentiation shows

$$\nabla_b Q_{(t)}^{*ab} = 0, \tag{10}$$

and so $Q_{(t)}^{ab}$ can only yield $k_{(t)}$. In fact $Q_{(t)}^{*ab}$ satisfies the Killing–Yano (KY) equation, which for an antisymmetric tensor A_{ab} can be written as

$$A_{a(b;c)} = 0. \tag{11}$$

This generalizes Killing’s equation to antisymmetric tensors and can be further generalized to antisymmetric tensors of arbitrary valence. Modern usage reserves the name KY tensor for antisymmetric tensors. For the Kerr solution a symmetric tensor K_{ab} is constructed from the dual Killing potential by

$$K_{ab} = Q_{(t)a}^{*e} Q_{eb}^{*(t)} = 2 \Sigma l_{(a} n_{b)} - r^2 g_{ab}. \tag{12}$$

This ‘‘hidden’’ symmetry of the Kerr solution was discovered by Carter⁷ and later shown to be the ‘‘square’’ of a two-index Killing spinor,⁸ or equivalently, the ‘‘square’’ of a Killing–Yano tensor. Though K_{ab} satisfies Eq. (11) it is symmetric and generally referred to as a Killing tensor.

Collinson⁹ found that all vacuum metrics of Petrov type D , with the exception of Kinnersley’s type $IIIB$, possess a KY tensor. He gave an explicit expression for both the KY tensor and its associated Killing tensor.

III. EXISTENCE OF SOLUTIONS

A. Conformal Killing–Yano tensors

Many of the arguments in this work depend on the conformal covariance of Penrose’s equation. Penrose and Rindler¹⁰ established the conformal covariance of its spinor form $\nabla_A ({}^A \sigma^{BC}) = 0$ for a symmetric spinor σ^{BC} . The tensor version was previously discovered by Tachibana as the conformally covariant generalization of the KY equation.¹¹ In this paper it was written in the form

$$Q_{a(b;c)} = (1/3)[g_{bc} Q_{a;e}^e - g_{a(b} Q_{c);e}^e]. \tag{13}$$

In that same work Tachibana showed that in a Ricci-flat space, for Q_{ab} a CKY bivector satisfying Eq. (13), $(1/3)\nabla^b Q_{ab}$ is a Killing vector.

From Eq. (13) we can obtain an expression for $Q_{ab;c}$ by writing out the symmetrization brackets explicitly:

$$Q_{ab;c} = -Q_{ac;b} + \frac{2}{3}g_{bc} Q_{a;e}^e - \frac{1}{3}g_{ab} Q_{c;e}^e - \frac{1}{3}g_{ac} Q_{b;e}^e.$$

Now, since $Q_{ab;c}$ is antisymmetric in the first two indices, we have

$$\begin{aligned} 3Q_{ab;c} &= Q_{ab;c} + Q_{ab;c} - Q_{ba;c} \\ &= Q_{ab;c} - Q_{ac;b} + \frac{2}{3}g_{bc} Q_{a;e}^e - \frac{1}{3}g_{ab} Q_{c;e}^e - \frac{1}{3}g_{ac} Q_{b;e}^e \\ &\quad + Q_{bc;a} - \frac{2}{3}g_{ac} Q_{b;e}^e + \frac{1}{3}g_{ba} Q_{c;e}^e + \frac{1}{3}g_{bc} Q_{a;e}^e \end{aligned}$$

and so from (13) we can deduce that

$$3Q_{ab;c} = 3Q_{[ab;c]} - 2g_{c[a} Q_{b];e}^e. \tag{14}$$

It is easily verified that given Eq. (14) we recover Eq. (13) and hence Eq. (14) is an alternative form of the CKY equation. Furthermore Penrose’s Eq. (4) can easily be rewritten as Tachibana’s Eq. (13) and so is another form of the CKY equation.

Since Q is an antisymmetric tensor, it is natural to discuss its properties in the language of differential forms. Equation (14) is manifestly antisymmetric in the first two indices, and so it is straightforward to verify that it is the abstract index equivalent of the CKY two-form equation of Benn *et al.*,¹²

$$3\nabla_z Q = Z \lrcorner dQ - Z^b \wedge \delta Q, \quad \forall Z. \quad (15)$$

In this form, since $*$ commutes with ∇_z , it is readily verified using the identities given in Appendix A, that whenever Q is a CKY two-form so is $*Q$. Thus any solution to the CKY equation can be decomposed into self-dual and anti-self-dual CKY two-forms.

B. Existence of CKY two-forms

On a flat background the CKY equation has many solutions, while, as will be explained, in a more general space–time the curvature imposes tight consistency conditions and there can be at most two independent solutions, one self-dual and one anti-self-dual with respect to the Hodge star. This result appears to be closely tied to the four-dimensional nature of space–time and the properties of these solutions are almost universally discussed in their spinor form, where the utility of the two-component spinor formalism simplifies the calculations. A detailed discussion of this can be found in spinor form in Ref. 12 or in terms of differential forms in Ref. 13.

Since any CKY two-form can be decomposed into self-dual and anti-self-dual parts that are themselves CKY two-forms, in discussing their existence, it is sufficient to consider only two-forms of definite Hodge-duality.

In order to understand how the curvature of the underlying space–time restricts the solutions to Eq. (15) two steps are required. First, it can be shown directly from the CKY two-form equation that the real eigenvectors of (anti-) self-dual CKY two-forms are shear-free and hence principal null directions of the conformal tensor. Second, by differentiating Eq. (15) an integrability condition can be obtained that restricts the Petrov type by showing these eigenvectors to be *repeated* principal null directions.

In the case of non-null self-dual two-forms, Dietz and Rüdiger¹⁴ used spinor methods to obtain both of these results for a scaling covariant generalization of Eq. (15). It was later shown, again using spinor methods, that similar results can be obtained for the null case.¹²

An outline of these results in the notation of differential forms is given in Appendices C and D. It is shown that apart from conformally flat space–times, non-null (anti-) self-dual CKY two-forms can only exist in space–times of Petrov type D , while null (anti-) self-dual CKY two-forms require a background space–time of Petrov type N .

C. The divergence of a CKY two-form

In order to apply the PG superpotential method using a given CKY two-form Q , its divergence (coderivative) δQ must be dual to a Killing vector. Tachibana showed that this was always the case in a Ricci flat background¹¹ (the result also holds for the slightly more general case of an Einstein space–time).

In the Kerr background, there are two independent Killing vectors and two independent CKY two-forms (one of each Hodge-duality). However the divergence of either of these CKY two-forms is proportional to the timelike Killing vector, leaving the axial KV without a Killing potential. This allows a divergence free linear combination of the self-dual and anti-self-dual CKY two-forms to be found. The Hodge-dual of this two-form is known as a Killing–Yano two-form and satisfies the Killing–Yano equation (11), which can be written in a similar fashion to Eq. (15) as

$$3\nabla_x Q = X \lrcorner dQ. \quad (16)$$

However, this leaves open the question of why it is that the timelike rather than the axial KV possesses a Killing potential? To answer this question, we note that the axial Killing vector must vanish along the symmetry axis and we show that a Killing vector obtained as the divergence of a CKY two-form must be nowhere vanishing.

First consider a non-null anti-self-dual CKY two-form Q^- . From Eq. (15) we can write $d(Q^{-2})$ in terms of Q^- and δQ^- :

$$d(Q^{-2}) = \frac{4}{3}(\delta Q^-)^\# \lrcorner Q^-,$$

which after contracting with Q^- leads to

$$\delta Q^- = -\frac{3}{2}(d(Q^{-2}))^\# \lrcorner Q^-.$$

Hence δQ^- vanishes if and only if $d(Q^{-2})$ vanishes.

In a vacuum type D background we can deduce that Q^{-2} is a constant multiple of $\Psi_2^{-2/3}$ from the fact that Q^- is an eigen-two-form of C and both $(Q^{-2})^{-3/2}Q^-$ and CQ^- are Maxwell fields. Hence if Q^- vanishes, then so does Ψ_2 and the background becomes conformally flat.

Further, it can be deduced from the Bianchi identities that for a type D vacuum space-time, the gradient of Ψ_2 vanishes if and only if the Ψ_2 itself vanishes. [In the Newman-Penrose (NP) formalism, using a principal null tetrad, the vacuum type D condition implies that the only nonzero curvature component is Ψ_2 and $\kappa = \sigma = \nu = \lambda = 0$. Then, imposing $\nabla_{X_a} \Psi_2 = 0$, the Bianchi identities lead to either $\rho = \mu = \tau = \pi = 0$ or $\Psi_2 = 0$. If we assume the former, then the NP equations for the derivatives of the spin coefficients immediately force the conclusion that Ψ_2 vanishes.] We therefore conclude that Q^{-2} is nowhere constant and hence δQ^- is nowhere vanishing and Kerr's axial Killing vector *cannot* have a Killing potential.

IV. SUMMARY

We have shown here that Penrose's equation for Killing potentials is equivalent to the conformal Killing-Yano equation for two-forms. With no appeal to Ricci-flatness existence of solutions was proven for space-times of Petrov type D , N or O . It was further shown, for type D vacuum backgrounds possessing a Killing-Yano two-form, that Killing vectors with zeros cannot have Killing potentials.

ACKNOWLEDGMENTS

E.N. Glass was partially supported by an NSERC of Canada grant and thanks the Physics Department of the University of Michigan and Professor Jean Krisch for their hospitality. Jonathan Kress began this work at the University of Newcastle.

APPENDIX A: DIFFERENTIAL FORMS

We denote a basis for vector fields by $\{X_a\}$. The natural dual of this we denote by $\{e^a\}$, a basis for covector or one-form fields. A coordinate basis is $X_a = \partial/\partial x^a$ and $e^a = dx^a$. The metric gives a natural bijection between vector and one-form fields, which we denote by $^\#$ and $^\flat$; X^\flat is the one-form metric dual to the vector X and $\alpha^\#$ is the vector field metric dual to the one-form α .

The one-forms, along with the wedge product \wedge , generate the algebra of differential forms. The wedge product is antisymmetric and so the differential forms of degree p can be thought of as the subset of covariant tensors of valence p that are antisymmetric in their arguments. If α and β are one-forms with components $\alpha_a = \alpha(X_a)$ and $\beta_a = \beta(X_a)$, then

$$\alpha \wedge \beta = \alpha_{[a} \beta_{b]} e^a \otimes e^b = \alpha_a \beta_b e^a \wedge e^b. \tag{A1}$$

A vector can be contracted with the p -form \mathcal{P} to give a $(p-1)$ -form $X \lrcorner \mathcal{P}$ so that

$$(X \lrcorner \mathcal{P})(X_{a_1}, X_{a_2}, \dots, X_{a_{p-1}}) = p \mathcal{P}(X, X_{a_1}, X_{a_2}, \dots, X_{a_{p-1}}),$$

and so the components of a p -form can be expressed using the hook as

$$\mathcal{P}_{ab\dots c} = \mathcal{P}(X_a, X_b, \dots, X_c) = \frac{1}{p!} X_c \lrcorner \dots \lrcorner X_b \lrcorner X_a \lrcorner \mathcal{P}.$$

We can define an inner product between any pair of two-forms:

$$\mathcal{P} \cdot \mathcal{Q} = \frac{1}{2} X_a \lrcorner X_b \lrcorner \mathcal{P} X^a \lrcorner X^b \lrcorner \mathcal{Q} = 2 \mathcal{P}_{ab} \mathcal{Q}^{ab}.$$

For $\mathcal{P} \cdot \mathcal{P}$ we write \mathcal{P}^2 .

The metric defines a natural map from p -forms to $(n-p)$ -forms called the Hodge star. In four dimensions, this maps two-forms to two-forms, and is defined so that

$$\mathcal{P} \wedge * \mathcal{Q} = (\mathcal{P} \cdot \mathcal{Q}) * 1,$$

where $*1$ is the volume four-form. For a Lorentzian metric, this map squares to -1 and so has eigenvalues, $\pm i$. Elements of the eigenspace corresponding to $(-i) + i$ are called (anti-) self-dual two-forms. Any two-form can be decomposed into self-dual and anti-self-dual parts

$$\mathcal{P} = \mathcal{P}^+ + \mathcal{P}^- \quad \text{where } * \mathcal{P}^\pm = \pm i \mathcal{P}.$$

The Hodge star relates the hook and wedge operations by

$$X \lrcorner * \mathcal{P} = * (\mathcal{P} \wedge X^\flat). \tag{A2}$$

The two-form commutator is given by

$$[\mathcal{P}, \mathcal{Q}] = -2 X_a \lrcorner \mathcal{P} \wedge X^a \lrcorner \mathcal{Q} \tag{A3}$$

for two-forms \mathcal{P} and \mathcal{Q} . The Lie algebra of two-forms under commutation is the Lie algebra of the Lorentz group.

It is often useful to work with a null coframe (basis for one-forms) $\{l, n, m, \bar{m}\}$ dual to a Newman–Penrose tetrad, that is, one for which all inner products vanish except

$$l \cdot n = -m \cdot \bar{m} = 1. \tag{A4}$$

From this we can construct a basis for the anti-self-dual two-forms:

$$\mathcal{U} = -n \wedge \bar{m}, \quad \mathcal{M} = l \wedge n - m \wedge \bar{m}, \quad \mathcal{V} = l \wedge m \tag{A5}$$

with the property that all inner products vanish except

$$\mathcal{U} \cdot \mathcal{V} = 1, \quad \mathcal{M} \cdot \mathcal{M} = -2. \tag{A6}$$

In this basis, the two-form commutator can be calculated from

$$[\mathcal{M}, \mathcal{U}] = -4\mathcal{U}, \quad [\mathcal{M}, \mathcal{V}] = 4\mathcal{V}, \quad [\mathcal{U}, \mathcal{V}] = -\mathcal{M}. \tag{A7}$$

The null basis elements \mathcal{U} and \mathcal{V} for each have one two-dimensional eigenspace, with corresponding zero eigenvalue, spanned by $\{n^\#, \bar{m}^\#\}$ and $\{l^\#, m^\#\}$, respectively. These are also the eigenspaces of \mathcal{M} for which they have eigenvalues $+1$ and -1 . Note that choosing \mathcal{M} determines \mathcal{U} and \mathcal{V} up to their relative scaling or interchange.

We denote the torsion-free metric compatible covariant derivative of a two-form \mathcal{Q} with respect to a vector field Z by $\nabla_Z \mathcal{Q}$. In terms of this, the exterior derivative d and coderivative $\delta = *d*$ can be expressed:

$$d \equiv e^a \wedge \nabla_{X_a}, \quad \delta \equiv -X^a \lrcorner \nabla_{X_a}.$$

APPENDIX B: THE PETROV CLASSIFICATION

In a vacuum background, the Riemann curvature tensor \mathbf{R} is equal to the Weyl conformal curvature tensor \mathbf{C} . The symmetries of these tensors allow them to be written as the sum of terms made of symmetric tensor products of two-forms (i.e., terms like $\mathcal{P} \otimes \mathcal{Q} + \mathcal{P} \otimes \mathcal{Q}$). So, both can be considered as self-adjoint maps on two-forms; if C_{abcd} are components of \mathbf{C} and \mathcal{P}_{ab} the components of a two-form, then the definition

$$(\mathbf{C}\mathcal{P})_{ab} = \frac{1}{2}C_{abcd}\mathcal{P}^{cd}$$

gives the components of the two-form $\mathbf{C}\mathcal{P}$. As a map on two-forms, the conformal tensor preserves the eigenspaces of $*$ and so may be decomposed into a part made from self-dual two-forms alone and a part made from anti-self-dual two-forms. That is, we can write

$$\mathbf{C} = \mathbf{C}^{(+)} + \mathbf{C}^{(-)},$$

where $\mathbf{C}^{(\pm)}\mathcal{Q}^{\mp} = 0$. Note that since the conformal tensor is real, $\mathbf{C}^{(-)}$ is the complex conjugate of $\mathbf{C}^{(+)}$, and so it is sufficient to classify only one of these.

The action of $\mathbf{C}^{(-)}$ on the Newman–Penrose two-form basis described in Appendix A is the same as the action of \mathbf{C} on this basis and can be written as

$$\mathbf{C}^{(-)} \begin{bmatrix} \mathcal{U} \\ \mathcal{M} \\ \mathcal{V} \end{bmatrix} = \begin{bmatrix} -\Psi_2 & \Psi_3 & -\Psi_4 \\ -2\Psi_1 & 2\Psi_2 & -2\Psi_3 \\ -\Psi_0 & \Psi_1 & -\Psi_2 \end{bmatrix} \begin{bmatrix} \mathcal{U} \\ \mathcal{M} \\ \mathcal{V} \end{bmatrix}.$$

Note that the matrix of this transformation is trace-free and the mapping is self-adjoint (that is, $\mathcal{Q} \cdot \mathbf{C}\mathcal{P} = \mathbf{C}\mathcal{Q} \cdot \mathcal{P}$).

The Petrov classification is a classification of this mapping. The space–time is known as algebraically general when there are three distinct eigenvalues, and algebraically special otherwise. Two special cases of interest here are that of type D and N , for which a basis can be chosen so that the matrix above takes the forms,

$$\begin{bmatrix} -\Psi_2 & 0 & 0 \\ 0 & 2\Psi_2 & 0 \\ 0 & 0 & -\Psi_2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Psi_0 & 0 & 0 \end{bmatrix},$$

respectively.

The real null direction of a null anti-self-dual two-form \mathcal{Q} is said to be a *principal null direction* (PND) of the conformal tensor if $\mathcal{Q} \cdot \mathbf{C}\mathcal{Q} = 0$. We will call such a \mathcal{Q} , a *principal null* (PN) two-form. There can be at most four independent PNDs and their number and “multiplicities” provide another description of the Petrov types.³ The multiplicities can be determined in the present formulation by the following (with \mathcal{P} an anti-self-dual two-form):

Multiplicity	Equivalent conditions		
1	$\mathcal{Q} \cdot \mathbf{C}\mathcal{Q} = 0$		$\Psi_4 = 0$
2	$[\mathcal{Q}, \mathbf{C}\mathcal{Q}] = 0$	$\mathbf{C}\mathcal{Q} \propto \mathcal{Q}$	$\Psi_3 = \Psi_4 = 0$
3	$\mathcal{Q} \cdot \mathbf{C}\mathcal{P} = 0 \ \forall \mathcal{P}$	$\mathbf{C}\mathcal{Q} = 0$	$\Psi_2 = \Psi_3 = \Psi_4 = 0$
4	$[\mathcal{Q}, \mathbf{C}\mathcal{P}] = 0 \ \forall \mathcal{P}$	$\mathbf{C}\mathcal{P} \propto \mathcal{Q} \ \forall \mathcal{P}$	$\Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$

APPENDIX C: CKY TWO-FORMS AND SHEAR-FREE CONGRUENCES

Defining the shear of a null geodesic vector field requires the choice of a ‘‘screen space,’’ and so is not an intrinsic property of the vector field. However, if the shear vanishes for one choice of screen space, then it does for all and hence the notion of a shear-free null vector field is well defined. For definitions and discussion of optical scalars see Ref. 3.

Robinson¹⁵ showed that the real null eigenvector l of a (anti) self-dual null two-form ϕ is geodesic and shear-free if and only if ϕ is proportional to a source-free Maxwell field, that is $d\phi=0$. Note that the eigenspace of such a two-form is two-dimensional, isotropic and integrable. So we can use this fact or the Frobenius integrability condition, that $d\phi=\alpha\wedge\phi$ for some α , for the vanishing of the shear of l . It is convenient here to use these results interchangeably as our criterion for a shear-free null geodesic.

Note that a shear-free null geodesic is a PND of the conformal tensor.

1. Null CKY two-forms

Now, suppose that \mathcal{Q} is a null anti-self-dual CKY two-form. Since the right-hand side of CKY two-form Eq. (15) is simply the anti-self-dual part of $-2Z^b\wedge\delta\mathcal{Q}$, we have that

$$0 = \mathcal{Q} \cdot 3\nabla_Z \mathcal{Q} = -2(Z^b \wedge \delta\mathcal{Q}) \cdot \mathcal{Q} = 2Z \lrcorner (\delta\mathcal{Q})^\# \lrcorner \mathcal{Q}.$$

Hence we can find an α such that $\delta\mathcal{Q}=\alpha^\#\lrcorner\mathcal{Q}$ or equivalently $d\mathcal{Q}=-\alpha\wedge\mathcal{Q}$. So the real null eigenvector of \mathcal{Q} is shear-free.

2. Non-null CKY two-forms

We wish to show that the eigenspaces of a non-null CKY two-form \mathcal{Q} are integrable and hence contain a shear-free null geodesic vector field. That is, we want to show that if X and Y are elements of the same eigenspace of \mathcal{Q} with eigenvalue λ ($X\lrcorner\mathcal{Q}=\lambda X^b$ and $Y\lrcorner\mathcal{Q}=\lambda Y^b$), then so is $[X, Y]$. Since $[X, Y]=\nabla_X Y - \nabla_Y X$, we will show that $\nabla_X Y \lrcorner \mathcal{Q} = \lambda \nabla_X Y^b$. Note that this eigenspace is isotropic, that is $g(X, Y)=0$.

Since the map $\alpha \mapsto \alpha^\# \lrcorner \mathcal{Q}$ is of maximal rank for non-null \mathcal{Q} , it can always be inverted and a one-form α found such that $\delta\mathcal{Q}=-\alpha^\#\lrcorner\mathcal{Q}$ and $d\mathcal{Q}=\alpha\wedge\mathcal{Q}$. Using these expressions for $\delta\mathcal{Q}$ and $d\mathcal{Q}$, and the CKY two-form Eq. (15), we have

$$\nabla_X Y \lrcorner \mathcal{Q} = \nabla_X (Y \lrcorner \mathcal{Q}) - Y \lrcorner \nabla_X \mathcal{Q} = \lambda \nabla_X Y^b + X \lambda Y^b - \frac{1}{3} \lambda \alpha(X) Y^b.$$

Rearranging and writing the vector equation dual to this shows that

$$(\nabla_X Y \lrcorner \mathcal{Q})^\# - \lambda \nabla_X Y = (X \lambda - \frac{1}{3} \lambda \alpha(X)) Y. \quad (C1)$$

Note that the right-hand side is a multiple of Y and hence an eigenvector of \mathcal{Q} with eigenvalue λ . However, upon contracting the left-hand side with \mathcal{Q} , we find that it is an element of the other eigenspace, having eigenvalue $-\lambda$. Hence we must conclude that

$$\nabla_X Y \lrcorner \mathcal{Q} - \lambda \nabla_X Y^b = 0, \quad (C2)$$

and we have the required result.

Since each eigenspace of \mathcal{Q} is integrable they each give rise to a null self-dual two-form proportional to a Maxwell field, and hence the real eigenvectors of \mathcal{Q} are shear-free.

APPENDIX D: INTEGRABILITY OF CKY TWO-FORMS

Apart from conformally flat space-times, CKY two-forms can only exist in space-times of Petrov type D or N . To understand this it is sufficient to consider only CKY tensors of definite Hodge-duality, for which we give an integrability condition. For an anti-self-dual CKY two-form \mathcal{Q} ,

$$[Q, C\mathcal{P}] = \frac{1}{2}[\mathcal{P}, CQ], \quad \forall \text{ two-forms } \mathcal{P}. \quad (\text{D1})$$

If we let $\mathcal{P} = Q$, it follows that

$$[CQ, Q] = 0.$$

Then, from the commutator algebra of anti-self-dual two-forms Eq. (A7), it can be deduced that CQ must be proportional to Q , i.e.,

$$CQ = \mu Q, \quad (\text{D2})$$

where μ is a scalar. From this, we can deduce the Petrov type as described in Appendix B.

1. Null CKY two-forms

When Q is null this implies that the real null eigenvector of Q is a repeated principal null direction. However, if we write out Eq. (D1) in an anti-self-dual two-form basis chosen so that $\mathcal{U} = Q$ and $\mathcal{V} \propto \mathcal{P}$, we find that $\mu = -\Psi_2 = 0$. Not only does this immediately tell us that $CQ = 0$, but upon substitution into Eq. (D1) we have that $[Q, C\mathcal{P}] = 0$ for all anti-self-dual two-forms \mathcal{P} . Hence the real null direction defined by Q is a fourfold PND and the space-time is of Petrov type N .

2. Non-null CKY two-forms

When Q is non-null, we concluded in Appendix C that the real null eigenvectors of Q are shear-free. If we align our anti-self-dual two-form basis so that $\mathcal{M} \propto Q$ then \mathcal{U} and \mathcal{V} have shear-free eigenvectors and hence are PN two-forms. From this we conclude that $\Psi_0 = \Psi_4 = 0$. The integrability condition Eq. (D2) immediately requires that Ψ_1 and Ψ_3 vanish and hence the space-time is of Petrov type D .

This reasoning made no use of Ricci-flatness wherein the Goldberg-Sachs theorem¹⁶ would imply the same result.

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