composed of 0.1-in. squares. Instead of measuring the vector direction of the current it was found, in this case, more informative to measure the horizontal and vertical components. The reason is that off the principal axes on this surface the current becomes elliptically polarized and in some regions is nearly circular. Thus, there is no vector direction that can be assigned to the current in these regions although the horizontal and vertical components are unambiguously defined.

In Fig. 7(a) are plotted the constant amplitude contours for the vertically polarized component of the induced surface current. The strong tendency toward a dipole distribution is clearly indicated. Slight deviations from symmetry are due to minute misalignment of the horn while the ripples in the contours can be traced to interference from backscattering at the horn edges. These details have been reproduced to show the resolution possible. The constant amplitude contours for the horizontal component are shown in Fig. 7(b). These exhibit the quadrupole distribution expected from the type of excitation employed. Relative amplitude is normalized to the same reference as that for the vertical component in Fig. 7(a).

Equiphase contours for these distributions are displayed in Fig. 8(a) and 8(b) for the vertical and horizontal components, respectively. Since the vertical polarization is the first-order component of the induced surface current, the equiphase contours are smooth and regular, but the horizontal component is a second-order effect. Accordingly, the ragged appearance of the equiphase contours is to be expected. The diagram shows, however, the usual 180° phase change in passing from one side of a null to the other.

ACKNOWLEDGMENT

Much of the data obtained with this probe was recorded by Mr. Robert Guyer who served as an undergraduate research assistant. His great care and painstaking effort is especially appreciated.

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Optical Measurement of Spectra of Two-Dimensional Functions*

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The principle of an optical computer for the measurement of two-dimensional spectral and cross-spectral densities is described. The performance of one of the experimental arrangements is checked by measuring the spectral densities of a simple two-dimensional function. The apparatus may be used to compute the Fourier transform of a two-dimensional function.

INTRODUCTION

CONSIDER a one-dimensional function $u(x)$ such that its correlation

$$R(\xi)=\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}u(x+\xi)u(x)dx=\langle u(x+\xi)u(x)\rangle_{\text{av}} \quad (1)$$

exists and is independent of the shift in the origin of x, i.e., it is statistically homogeneous. The spectral density $E(k)$ of u is the Fourier transform of $R(\xi)$,

$$E(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R(\xi)e^{i\xi k}d\xi. \quad (2)$$

The main result of the generalized harmonic analysis is that

$$\lim_{L\to\infty}\frac{1}{4L\pi}\left|\int_{-L}^{L}u(x)e^{ikx}dx\right|^2=E(k), \quad (3)$$

which provides a means for direct measurement of $E(k)$. Here we are assuming that u has continuous but no line spectrum.[1] Similarly, for two-dimensional random functions we may define the corresponding correlation and two-dimensional spectral density,

$$R(\xi_1,\xi_2)=\lim_{L\to\infty}\frac{1}{4L^2}\int_{-L}^{L}\int_{-L}^{L}u(x_1+\xi_1,\ x_2+\xi_2)u(x_1,x_2)dx_1dx_2$$
$$=\langle u(x_1+\xi_1,\ x_2+\xi_2)u(x_1,x_2)\rangle_{\text{av}}, \quad (4)$$

and

$$E(k_1,k_2)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R(\xi_1,\xi_2)e^{i(k_1\xi_1+k_2\xi_2)}d\xi_1d\xi_2. \quad (5)$$

The spectral density is also given by the relation

$$E(k_1,k_2)=\lim_{L\to\infty}\frac{1}{(4L\pi)^2}\left|\int_{-L}^{L}\int_{-L}^{L}u(x_1,x_2)e^{i(k_1x_1+k_2x_2)}dx_1dx_2\right|^2. \quad (6)$$

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[1] In order to include functions with periodic or almost periodic

If we further assume that $u(x_1,x_2)$ is statistically isotropic in addition to being homogeneous, then it is only necessary to consider one-dimensional correlation and spectrum,

$$R(\xi_1)=\langle u(x_1+\xi_1,\ x_2)u(x_1,x_2)\rangle_{\text{av}}, \tag{7}$$

$$E_1(k_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty} R(\xi_1)e^{ik_1\xi_1}d\xi_1. \tag{8}$$

The inverse transform is

$$R(\xi_1)=\int_{-\infty}^{\infty} E_1(k_1)e^{-ik_1\xi_1}dk_1, \tag{9}$$

which may be written as

$$R(\xi_1)=R(\xi_1,0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(k_1,k_2)e^{-i(k_1\xi_1+k_2\cdot 0)}dk_1dk_2$$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} E_1(k_1,k_2)dk_2\right]e^{-ik_1\xi_1}dk_1. \tag{10}$$

It follows from Eqs. (9) and (10) that

$$E_1(k_1)=\int_{-\infty}^{\infty} E(k_1,k_2)dk_2.$$

For statistically isotropic functions $E(k_1,k_2)=E(k)$ where $k^2=k_1^2+k_2^2$ so that one may write

$$E_1(k_1)=2\int_{k_1}^{\infty} E(k)\frac{kdk}{(k^2-k_1^2)^{\frac{1}{2}}}. \tag{11}$$

The solution of the above integral equation is[2]

$$E(k)=-\frac{1}{\pi}\int_{k}^{\infty}\frac{d}{dk_1}[E_1(k_1)k_1]\frac{dk_1}{(k_1^2-k^2)^{\frac{1}{2}}}, \tag{12}$$

so that, for statistically isotropic function, the two-dimensional spectral density $E(k)$ can be recovered from the one-dimensional spectral density $E_1(k_1)$.

MEASUREMENT OF CORRELATIONS AND SPECTRA

The one-dimensional random functions are usually given as electrical signals and the correlation can be measured by using electronic multipliers and the spectrum by using selective filters. For the two-dimensional functions it is necessary to use numerical computations. However, in many cases, such as the solar granules[3] and the shadowgraph of the turbulent wake of a bullet,[4] the function is given as a random picture on a photographic plate such that the local transparency is proportional to the function, i.e.,

$$T(x_1,x_2)=T_0+cu(x_1,x_2),$$

where T is the local transparency, T_0 the average transparency of the plate, and c a constant which depends on the photographic process. The combined transparency of two identical plates placed back to face and displaced from the matched position by amounts ξ_1 and ξ_2 is

$$T(x_1+\xi_1,\ x_2+\xi_2)T(x_1,x_2)$$
$$=[T_0+cu(x_1+\xi_1,\ x_2+\xi_2)][T_0+cu(x_1,x_2)]. \tag{13}$$

The correlation can be determined by measuring the average combined transparency of sufficiently large overlapping area of the plates thus:

$$\langle T(x_1+\xi_1,\ x_2+\xi_2)T(x_1,x_2)\rangle_{\text{av}}$$
$$=T_0^2+c^2\langle u(x_1+\xi_1,\ x_2+\xi_2)u(x_1,x_2)\rangle_{\text{av}}. \tag{14}$$

The spectral density $E(k_1,k_2)$ being the Fourier transform of the correlation $R(\xi_1,\xi_2)$ contains no new information. However, it is often more easily subject to physical interpretation, and its direct measurement is desirable. For the case of statistically isotropic functions we can easily measure the one-dimensional spectrum by scanning the photographic plate with a narrow beam of light and converting the transmitted light into an electrical signal by using a photocell. The electrical signal can be analyzed by using selective filters. The two-dimensional spectral density can be obtained from thus determined one-dimensional spectral density by using Eq. (12). In the more general case of statistically anisotropic functions it is necessary to measure the two-dimensional spectral density $E(k_1,k_2)$ given by Eq. (6). For a fixed point (k_1,k_2) or (k,θ) in the wave number plane we may integrate the latter equation with respect to the orthogonal coordinates y_1 and y_2 such that y_2 is perpendicular to $\mathbf{k}$ (see Fig. 1) thus:

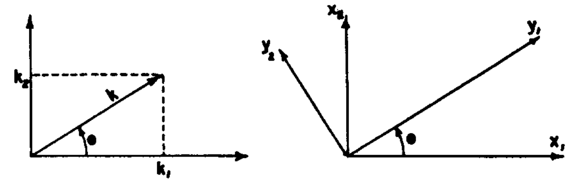

FIG. 1. Transformation of the coordinates.

$$E(k_1,k_2)=E(k,\theta)$$
$$=\lim_{L\to\infty}\frac{1}{(4L\pi)^2}\left|\int_{-L}^{L}\int_{-L}^{L} u(y_1,y_2)e^{i(ky_1+0y_2)}dy_1dy_2\right|^2$$
$$=\lim_{L\to\infty}\frac{1}{(4L\pi)^2}\left\{\left|\int_{-L}^{L}\left[\int_{-L}^{L} u(y_1,y_2)dy_2\right]\cos ky_1dy_1\right|^2\right.$$
$$\left.+\left|\int_{-L}^{L}\left[\int_{-L}^{L} u(y_1,y_2)dy_2\right]\sin ky_1dy_1\right|^2\right\}. \tag{15}$$

components (line spectrum) it is necessary to use Fourier-Stieltjes integrals, which add nothing new. The almost periodic components can be handled in a manner similar to that given above.

[2] M. S. Uberoi, Astrophys. J. 122, 466 (1955).
[3] M. S. Uberoi, Astrophy. J. 121, 400 (1955).
[4] M. S. Uberoi and L. S. G. Kovasznay, J. Appl. Phys. 26, 19 (1955).

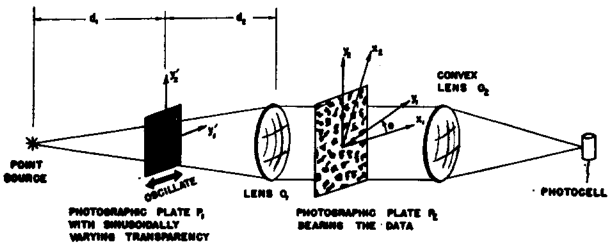

FIG. 2. Schematic representation of arrangement I.

The determination of the two-dimensional spectral density is reduced to the one-dimensional case.

Experimental Arrangement I

The operation indicated by Eq. (15) may be performed optically as shown in Fig. 2. The photographic plate P_2 bearing the data is placed between the two convex lenses O_1 and O_2. A point source and a photocell are placed at the foci of the lenses O_1 and O_2, respectively. The transparency of the photographic plate P_1 is

$$T_0+c\sin k_0y_2',$$

where k_0 is a constant, and we are assuming that the same photographic process is used to develop P_1 and P_2. If I_0 is the intensity of light at the plate P_2 in absence of P_1, the intensity with P_1 is

$$I_0(T_0+c\sin ky_2),$$

where

$$k=d_2k_0/(d_1+d_2),$$

and we are assuming that the solid angle of the light beam is small. The total light transmitted to the photocell or its output is

$$e_1=\int_{-L}^{L}\int_{-L}^{L}I_0[T_0+c\sin ky_2][T_0+cu(y_1,y_2)]dy_1dy_2. \quad (16)$$

For the sake of convenience we have assumed that there is a square aperture with sides of length $2L$ parallel to y_1 and y_2. We further assume that $L=n\pi/k$, where n is a positive integer so that

$$\int_{-L}^{L}\sin ky_2dy_2=0, \quad (17)$$

and

$$e_1=\int_{L}^{L}\int_{L}^{L}I_0(T_0+c\sin ky_2)cu(y_1,y_2)dy_1dy_2. \quad (18)$$

The essential term

$$I_0c^2\int_{-L}^{L}\int_{-L}^{L}\sin ky_2u(y_1,y_2)dy_1,dy_2$$

is dominated by the term I_0T_0, and it is necessary to subtract the latter from the total output in order to obtain reasonable accuracy. This is accomplished by oscillating the plate P_1. Let $\delta(t)$ be the displacement of the plate. The photocell output becomes

$$e_1=\int_{-L}^{L}\int_{-L}^{L}I_0[T_0+c\sin(ky_2+k_0\delta(t))]cu(y_1,y_2)dy_1dy_2. \quad (19)$$

It is passed through a filter which rejects the direct current and the temporally varying part is squared and averaged with respect to time thus:

$$\langle e_1^2\rangle_{av}-\bar{e}_1^2=[\langle\{\sin k_0\delta(t)\}^2\rangle_{av}-\{\langle\sin k_0\delta(t)\rangle_{av}\}^2]\times\left[\int_{-L}^{L}\int_{-L}^{L}I_0c^2u(y_1,y_2)\cos ky_2dy_1dy_2\right]^2+[\langle\{\cos k_0\delta(t)\}^2\rangle_{av}-\{\langle\cos k_0\delta(t)\rangle_{av}\}^2]\times\left[\int_{-L}^{L}\int_{-L}^{L}I_0c^2u(y_1,y_2)\sin ky_2dy_1dy_2\right]^2, \quad (20)$$

where av denotes an average with respect to time.

We displace the plate P_1 by a quarter wavelength relative to the rest of the equipment so that its transparency becomes

$$T_0+c\cos k_0y_2',$$

and in this case the photocell output is

$$e_2=\int_{-L}^{L}\int_{-L}^{L}I_0[T_0+c\cos(ky_2+k_0\delta(t))]cu(y_1,y_2)dy_1dy_2, \quad (21)$$

and its variance is

$$\langle e_2^2\rangle_{av}-\bar{e}_2^2=[\langle\cos k_0\delta^2(t)\rangle_{av}-\{\langle\cos k_0\delta(t)\rangle_{av}\}^2]\times\left[\int_{-L}^{L}\int_{-L}^{L}I_0c^2u(y_1,y_2)\sin ky_2dy_1dy_2\right]^2+[\langle\sin k_0\delta^2(t)\rangle_{av}-\{\langle\sin k_0\delta(t)\rangle_{av}\}^2]\times\left[\int_{-L}^{L}\int_{-L}^{L}I_0c^2u(y_1,y_2)\sin ky_2dy_1dy_2\right]^2. \quad (22)$$

For certain periodic displacements $\delta(t)$ Eqs. (20) and (22) may coincide. In any case their sum is proportional to the spectral density $E(k,\theta)$ for $k=n\pi/L$, and the constant of proportionality can be determined by calibrating the equipment. The dependence of $E(k,\theta)$ on θ is obtained by rotating the data bearing plate, and that on k by moving the plate P_1 in discrete steps such that Eq. (17) is always satisfied. This amounts to the assumption that $u(x_1,x_2)$ has a period $2L$ in x_1 and x_2 direction. The envelope of the line spectrum of this periodic function approaches the true spectrum as $L\rightarrow\infty$.

Experimental Arrangement II

The second experimental arrangement is shown in the Fig. 3. The photographic plate bearing the data is scanned with uniform speed α by a series of periodically spaced narrow slits of length $2L$ each. One side of the square aperture is parallel to the slits and is of length $2L$. As one slit leaves the aperture, another appears, and in the absence of the data bearing plate, the light measured by the photocell is constant. This requires an accurate adjustment of slits. There are a number of actual arrangements which may be used for scanning. The apparatus used is shown in Fig. 4. For convenience front surfaced mirrors are used to bend the light beam through 90°. The scanning is done at 57 cps by a 24-in. rotating drum with 22 slits of width 0.010 in. each. The periodic scanning is equivalent to the assumption of Eq. (17). The scanned area or the aperture is 3.4 in. square and is fixed in position. The 5×7 in. film bearing the data is larger in size and may be rotated around an axis perpendicular to the aperture and passing through its center. If I^* is the line intensity of the slit, then output of the photocell is

$$I^*\int_{-L}^{L}[T_0+cu(y_1,\alpha t)]dy_1,$$

where $y_2=\alpha t$ and α is the slit speed. The dc component is rejected, and the ac component

$$I^*c\int_{-L}^{L}u(y_1,\alpha t)dy_1$$

is periodic with a frequency of $\alpha/2L$. It is passed through a frequency selective filter which in this case is Hewlett-Packard model 300 A wave analyzer. The mean square output gives the spectral density $E(k,\theta)$ at multiples of the basic frequency $\alpha/2L$ or the basic wave number π/L. The envelope of the line spectrum approaches the true spectral density as $L\rightarrow\infty$. The dependence of E on θ is obtained by rotating the photographic plate bearing the data.

MEASUREMENT OF CROSS-SPECTRAL DENSITY OF TWO-DIMENSIONAL FUNCTIONS

Consider two functions $u(x_1,x_2)$ and $v(x_1,x_2)$ such that their cross correlation

$$R(\xi_1,\xi_2)=\langle u(x_1,x_2)_v(x_1+\xi_1,\ x_2+\xi_2)\rangle_{\text{av}} \qquad (23)$$

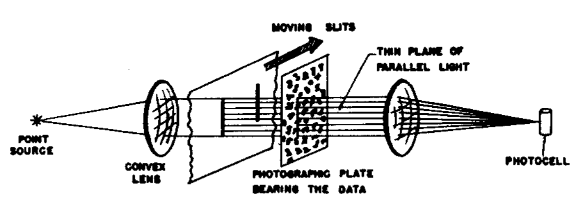

FIG. 3. Schematic representation of arrangement II.

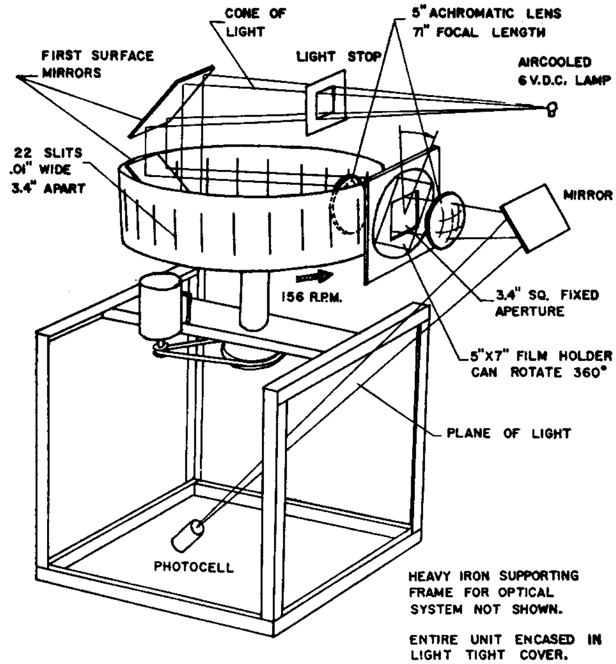

FIG. 4. The apparatus used to measure the spectral densities.

exists. The cross-spectral density of u and v is

$$E(k_1,k_2)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R(\xi_1,\xi_2)e^{i(k_1\xi_1+k_2\xi_2)}d\xi_1d\xi_2. \qquad (24)$$

For example, we may be interested in the cross-spectral density of the solar granule pattern at a given time with that at a later time.

The measurement of the two-dimensional cross-spectral density can be reduced to the one-dimensional case by either using the arrangement I or II. For example in the II arrangement we use two sets of identical slits, one exactly below the other to simultaneously scan the two photographic transcriptions of u and v using two independent photocells. It is necessary to preserve proper phase relation between the two functions u and v. The outputs of the photocells are fed to a cross-spectral density analyzer[5] which gives the sine and cosine components of the cross spectral $E(k_1,k_2)$ or $E(k,\theta)$ [see Fig. 1 for the relation between (k_1,k_2) and (k,θ)] for a one value of θ. The dependence on θ is obtained by rotating the photographic transcriptions of u and v by the same amount relative to the rest of the equipment.

We need hardly add that any of the experimental arrangements may be used to measure the Fourier transform of a two-dimensional function.

An Example

The operation of the experimental arrangement II was checked by measuring the spectral densities of a simple

[5] M. S. Uberoi and E. G. Gilbert, Rev. Sci. Instr. 30, 176 (1959).

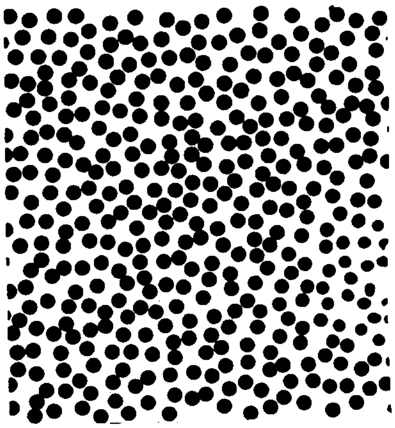

FIG. 5. Photograph of the test random function. The diameter of dots $D=0.185$.

function. Black circles were placed at random on white paper such that the resulting pattern was as statistically homogeneous and isotropic as possible. Obviously, no calibration of the photographic process is necessary to transcribe this random function. A photograph of the pattern was taken on a high contrast film (Fig. 5). The two- and one-dimensional spectral densities of this function were obtained by scanning the data bearing film with a slit and a narrow hole, respectively. It was found that the spectral densities varied as the plate was rotated, i.e., the random function was not statistically isotropic. This is partly due to the finite size of the sample. Since it is not our purpose here to generate an isotropic random function, the measured spectra were averaged with respect to the angle of rotation and are shown in Fig. 6. We note that the measured $E_1(k_1)$ shows considerably more scatter than $E(k)$ which is due to the finite field of the function used. The scanning with a slit uses more of the available information than scanning with a dot. The one-dimensional spectral density was computed from the two-dimensional spectral density using Eq. (11) and shows good agreement with the measured one-dimensional spectral density, as shown in Fig. 6. This is taken to be a test of the performance of the apparatus.

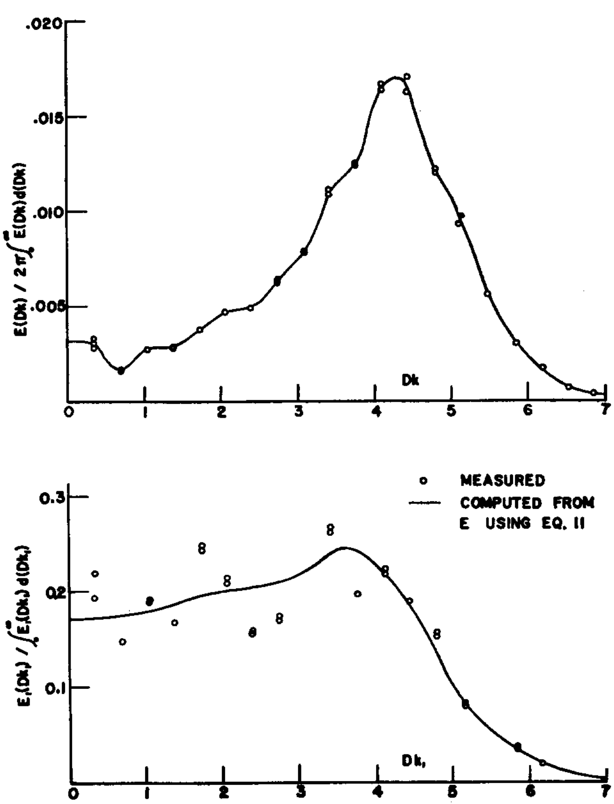

FIG. 6. Measured and computed spectral densities of the test random function.

Sources of Error

Available photocells showed some variation of sensitivity across their surfaces so that it was necessary to focus the entire light to a point on the photocell surface, otherwise the image moves across the surface as the slit moves across the aperture. An RCA No. 5819 photocell was used in arrangement II.

All slits must have the same width and be equally spaced so that in absence of the data bearing plate the light intensity remains constant as one slit moves out of the aperture and another appears. This was done with an accuracy of $\pm 1\%$. The finite size of the slit washes out the fine details for $k > \pi/\text{slitwidth}$. It is possible to partially compensate for this loss, but it was not necessary for the purpose of checking the operation of the equipment.

The reader may consult references given below[6] about mapping and measurements of random fields and the mathematical aspects of the estimation of the spectral density of an isotropic process.

ACKNOWLEDGMENTS

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[6] M. S. Uberoi and L. S. G. Kovasznay, Quart. Appl. Math. 4, 375 (1953). L. J. Tick, "Estimation of the spectral density of an isotropic process," Sci. Paper 13, Engineering Statistic Laboratory, New York University.