

# The influence of damping on the ion hose instability

R. A. Bosch and R. M. Gilgenbach

*Intense Energy Beam Interaction Laboratory, Nuclear Engineering Department, The University of Michigan, Ann Arbor, Michigan 48109-2104*

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The ion hose instability in the ion-focused regime is analyzed using a rigid beam model with phenomenological damping terms. Values of the instability wavelength,  $e$ -folding length, and group velocity are calculated and compared with numerical results. The impulse response function for this model is also obtained.

## I. INTRODUCTION

Propagation of an electron beam in an ion channel in the ion-focused regime (IFR) allows the beam to propagate without expanding from space charge repulsion.<sup>1,2</sup> However, onset of the ion hose instability may disrupt propagation as transverse oscillations of the  $e$ -beam and channel couple and grow.<sup>3-5</sup>

A simple description of the ion hose instability is given by the rigid beam model.<sup>6</sup> However, this model predicts an absolute instability because it neglects the phase-mix damping of coherent oscillations and the resonance broadening that result from the radial variation in ion and electron densities. Numerical spread-mass models provide more realistic results since they include the damping of coherent beam oscillations.<sup>7</sup>

In order to describe the effects of phase-mix damping and the associated resonance broadening in a simple manner, we include phenomenological damping terms within the context of the rigid beam approach. The resultant model is sufficiently simple that the dispersion relation can be found analytically. When realistic damping parameters are used, the wavelength,  $e$ -folding length, and group velocity of the instability found by this approach are in good agreement with the values found with numerical techniques. In addition, we construct the impulse response function for the system.

## II. MATHEMATICAL MODEL

For the case of no magnetic field, the rigid beam model, with phenomenological damping terms, is

$$\begin{aligned} \frac{d^2 b}{dt^2} &= -\omega_e^2 (b - c) - 2\alpha_e \frac{db}{dt}, \\ \frac{d^2 c}{dt^2} &= -\omega_i^2 (c - b) - 2\alpha_i \frac{dc}{dt}, \end{aligned} \quad (1)$$

where  $b(z, t)$  is the beam displacement,  $c(z, t)$  is the channel displacement,

$$\omega_e = (n_i e^2 / 2\epsilon_0 \gamma m_e)^{1/2}, \quad \omega_i = (n_e e^2 / 2\epsilon_0 m_i)^{1/2},$$

$n_i$  and  $n_e$  are ion and electron densities at the center of the channel and beam, respectively, and  $\alpha_e$  and  $\alpha_i$  are damping coefficients.<sup>8</sup>

Radial density variations of the beam and channel give rise to damping through phase mixing.<sup>1,9</sup> Additional damping can arise from kinetic (Landau) damping. Thus the magnitude of the damping is expected to be dependent upon the radial density profiles and temperatures of the electron beam and ion channel. Since coherent betatron oscillations

are observed to damp within several wavelengths,<sup>1</sup> a realistic value of  $A_e \equiv \alpha_e / \omega_e$  will be on the order of unity.<sup>9</sup> The existence of betatron oscillations indicates that these oscillations are underdamped, so that  $A_e < 1$ . Radial variations in ion density and ion plasma frequency are expected to produce an analogous damping of coherent ion channel oscillations, so that  $A_i \equiv \alpha_i / \omega_i$  will also be on the order of unity, and  $A_i < 1$ .

## III. DISPERSION RELATION

For waves of the form  $b(z, t) = b_0 e^{i(kz - \omega t)}$ ,  $c(z, t) = c_0 e^{i(kz - \omega t)}$ , Eq. (1) becomes

$$\begin{aligned} -\Omega^2 b &= -\omega_e^2 (b - c) - 2\alpha_e (-i\Omega b), \\ -\omega^2 c &= -\omega_i^2 (c - b) - 2\alpha_i (-i\omega c). \end{aligned} \quad (2)$$

In Eq. (2),  $\Omega = \omega - vk$  arises from the action of the convective derivative on the electron terms for which  $d/dt = \partial/\partial t + v \partial/\partial z$ , where  $v$  is the electron beam velocity. In order to consider the boundary value problem, we consider the case where  $\omega$  is real. The quantities  $b$  and  $c$  can be eliminated from Eq. (2), yielding the dispersion relation

$$k = \frac{1}{v} \left[ \omega + i\alpha_e \pm \omega_e \left( 1 - A_e^2 + \frac{\omega_i^2}{\omega^2 - \omega_i^2 + 2i\omega\alpha_i} \right)^{1/2} \right]. \quad (3)$$

The modes where  $\omega \operatorname{Re}(k) > 0$  and  $\operatorname{Im}(k) < 0$  describe hose oscillations that propagate downstream and grow, resulting in a convective instability. Resonant growth occurs for  $\omega = \pm \omega_i$ .

For  $0.1 \leq A_e \approx A_i \leq 0.4$ , the resonant waves that propagate downstream obey

$$|\operatorname{Re}(k)| \sim -\operatorname{Im}(k) \sim \omega_e / v. \quad (4)$$

Thus these waves have wavelengths and  $e$ -folding lengths on the order of the beam-averaged betatron wavelength for which  $k = \omega_e / (2^{1/2} v)$ . This is consistent with the results of numerical spread-mass modeling<sup>7</sup> as well as experimental observations of transverse beam-channel displacements after propagation distances on the order of the betatron wavelength.<sup>3-5</sup>

With sufficiently large damping coefficients ( $A_e \approx A_i \geq 0.5$ ), the modes propagating downstream will be damped [ $\operatorname{Im}(k) > 0$ ]. This suggests that it may be possible to stabilize the IFR ion hose instability by using a beam and channel system with sufficiently large levels of phase-mix damping and/or kinetic (Landau) damping.

#### IV. GROUP VELOCITY

A disturbance arising from the ion hose instability will be dominated by the resonant modes for which  $\omega = \pm \omega_i$ . The group velocity of such a growing wave packet may be approximated by  $v_g \approx [\text{Re}(dk/d\omega)|_{\pm \omega_i}]^{-1}$ . Differentiating the dispersion relation (3) with respect to  $\omega$  yields

$$\frac{dk}{d\omega} \Big|_{\omega = \pm \omega_i} = \frac{1}{v} \left[ 1 \pm \frac{\omega_e}{\omega_i} \left( 1 - A_e^2 + \frac{\omega_i^2}{2i\omega\alpha_i} \right)^{-1/2} \times \left( \frac{\omega_i^2}{2\omega\alpha_i} \right)^2 \left( \frac{\omega + i\alpha_i}{\omega_i} \right) \right] \Big|_{\omega = \pm \omega_i}, \quad (5)$$

so that, for downstream-propagating waves with  $A_i, A_e \ll 1$ ,

$$\text{Re} \left( \frac{dk}{d\omega} \right) \Big|_{\pm \omega_i} \approx \frac{\omega_e}{4v\omega_i} A_i^{-3/2}. \quad (6)$$

Thus the group velocity of an ion hose disturbance with  $A_i, A_e \ll 1$  will obey

$$v_g \approx 4v(\omega_i/\omega_e) A_i^{3/2} = 4v(\gamma m_e/fm_i)^{1/2} A_i^{3/2}, \quad (7)$$

where  $f = n_i/n_e$ . From Eq. (7) we observe that the group velocity goes to zero as  $A_i \rightarrow 0$ , recovering the absolute instability of the rigid beam model with no damping. For realistic damping coefficients ( $A_i \sim 1$ ), the approximate group velocity is comparable to the value  $[4v(\gamma m_e/fm_i)^{1/2}]$  obtained with a numerical solution to the spread-mass model.<sup>7</sup>

A disturbance arising from the ion hose instability, traveling at  $v_g \sim 4v\omega_i/\omega_e$ , will propagate an  $e$ -folding length ( $\sim 2\pi v/\omega_e$ ) in a time on the order of  $\pi/2\omega_i$ . Thus the  $e$ -folding time is on the order of  $1/\omega_i$ , consistent with the results of a numerical spread-mass model.<sup>7</sup>

The reasonable agreement with numerical results for the wavelength,  $e$ -folding length, and group velocity suggests that the rigid beam model with damping terms may provide a useful analytic description of the ion hose instability.

#### V. IMPULSE RESPONSE

An important quantity in determining the response of the beam-channel system to a perturbation is the impulse response function. We consider an impulsive force on the electrons, since they will tend to respond more than the relatively massive ions. In terms of the variables  $Z = \omega_e z/v$  and  $\xi = \omega_i(t - z/v)$ , Eq. (1) with an impulse becomes

$$\frac{\partial^2 b}{\partial Z^2} = -(b - c) - 2A_e \frac{\partial b}{\partial Z} + \delta(Z)\delta(\xi), \quad (8a)$$

$$\frac{\partial^2 c}{\partial \xi^2} = -(c - b) - 2A_i \frac{\partial c}{\partial \xi}. \quad (8b)$$

We consider the case where  $A_e, A_i < 1$ , which describes underdamped electron and ion oscillations. The initial condition,  $b = c = 0$  for  $Z, \xi < 0$ , describes the retarded response of a beam-channel system that is not perturbed upstream of

the impulse and downstream of the resultant disturbance. With this initial condition, the Laplace transforms  $\hat{b}(p, s)$  and  $\hat{c}(p, s)$  [where  $(Z, \xi) \rightarrow (p, s)$ ] obey

$$\begin{aligned} p^2 \hat{b} &= -(\hat{b} - \hat{c}) - 2A_e p \hat{b} + 1, \\ s^2 \hat{c} &= -(\hat{c} - \hat{b}) - 2A_i s \hat{c}, \end{aligned} \quad (9)$$

so that

$$\begin{aligned} \hat{b} &= [p^2 + 2A_e p + 1 - (s^2 + 2A_i s + 1)^{-1}]^{-1}, \\ \hat{c} &= \hat{b}(s^2 + 2A_i s + 1)^{-1}. \end{aligned} \quad (10)$$

We can invert  $\hat{b}(p, s)$  with respect to  $p$  by letting  $p' = p + A_e$

$$\begin{aligned} \frac{1}{2\pi i} \int dp \hat{b} e^{pZ} &= \frac{e^{-A_e Z}}{2\pi i} \int \frac{dp' e^{p'Z}}{p'^2 + f(s)} \\ &= [e^{-A_e Z}/f(s)^{1/2}] \sin[f(s)^{1/2} Z], \end{aligned} \quad (11)$$

where  $f(s) \equiv 1 - A_e^2 - (s^2 + 2A_i s + 1)^{-1}$ . Expanding the sin, we obtain

$$\begin{aligned} b(Z, \xi) &= \frac{1}{2\pi i} \int ds e^{s\xi} e^{-A_e Z} \frac{\sin[f(s)^{1/2} Z]}{f(s)^{1/2}} \\ &= e^{-A_e Z} \sum_{n=0}^{\infty} \frac{(-1)^n Z^{2n+1}}{(2n+1)!} \\ &\quad \times \left( \frac{1}{2\pi i} \int ds e^{s\xi} f(s)^n \right). \end{aligned} \quad (12)$$

In this expression, the final term is the inverse Laplace transform of  $f(s)^n$ . For  $n = 0$ , this equals  $\delta(\xi)$ . For  $n \geq 1$ , the inversion can be found by using the binomial formula to expand  $f(s)^n$ :

$$\begin{aligned} f(s)^n &= (1 - A_e^2)^n (1 - [(1 - A_e^2)(s^2 + 2A_i s + 1)]^{-1})^n \\ &= (1 - A_e^2)^n \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)! k!} \\ &\quad \times (1 - A_e^2)^{-k} (s^2 + 2A_i s + 1)^{-k}. \end{aligned} \quad (13)$$

We now invert the expansion of Eq. (13) term by term, letting  $s' = s + A_i$ , and using the table value for the inversion of  $(s'^2 + a^2)^{-k}$ :

$$\begin{aligned} \frac{1}{2\pi i} \int ds e^{s\xi} (s^2 + 2A_i s + 1)^{-k} \\ &= e^{-A_i \xi} \left( \frac{1}{2\pi i} \int ds' e^{s'\xi} (s'^2 + 1 - A_i^2)^{-k} \right) \\ &= \frac{e^{-A_i \xi} \sqrt{\pi}}{(k-1)!} \left( \frac{\xi}{2\sqrt{1-A_i^2}} \right)^{k-1/2} J_{k-1/2}(\xi \sqrt{1-A_i^2}). \end{aligned} \quad (14)$$

Substituting into Eq. (12), and noting that

$$\sum_{n=0}^{\infty} \sum_{k=0}^n \cdots = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \cdots,$$

we have

$$\begin{aligned} b(Z, \xi) &= \frac{\delta(\xi) e^{-A_e Z}}{\sqrt{1-A_e^2}} \sin(Z\sqrt{1-A_e^2}) + e^{-A_e Z} e^{-A_i \xi} \\ &\times \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{(-1)^{n-k} n! (Z\sqrt{1-A_e^2})^{2n+1} \sqrt{\pi}}{(2n+1)!(n-k)!k!(k-1)!(1-A_e^2)^{k+1/2}} \left(\frac{\xi}{2\sqrt{1-A_e^2}}\right)^{k-1/2} J_{k-1/2}(\xi\sqrt{1-A_e^2}) \\ &= \frac{\delta(\xi) e^{-A_e Z}}{\sqrt{1-A_e^2}} \sin(Z\sqrt{1-A_e^2}) + e^{-A_e Z} e^{-A_i \xi} \sum_{k=1}^{\infty} \frac{\pi}{k!(k-1)!} \left(\frac{\xi}{2\sqrt{1-A_e^2}}\right)^{k-1/2} \\ &\times J_{k-1/2}(\xi\sqrt{1-A_e^2}) \left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{k+1/2} J_{k+1/2}(Z\sqrt{1-A_e^2}). \end{aligned} \quad (15)$$

A similar calculation yields

$$c(Z, \xi) = e^{-A_e Z} e^{-A_i \xi} \sum_{k=1}^{\infty} \frac{\pi}{((k-1)!)^2} \left(\frac{\xi}{2\sqrt{1-A_e^2}}\right)^{k-1/2} J_{k-1/2}(\xi\sqrt{1-A_e^2}) \left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{k-1/2} J_{k-1/2}(Z\sqrt{1-A_e^2}). \quad (16)$$

The first term of Eq. (15),

$$\frac{\delta(\xi) e^{-A_e Z}}{\sqrt{1-A_e^2}} \sin(Z\sqrt{1-A_e^2}),$$

is the solution to Eq. (8a) with the constraint  $c(Z, \xi) \equiv 0$ . Thus it describes the response of the electrons to an impulse when the ions are immobile. The remaining terms of Eqs. (15) and (16) arise from the coupling of electron and ion oscillations. Consider the first of these terms in Eq. (15):

$$\left[ e^{-A_i \xi} e^{-A_e Z} / 2\sqrt{1-A_e^2} (1-A_e^2)^{3/2} \right] \sin(\xi\sqrt{1-A_e^2}) \left[ \sin(Z\sqrt{1-A_e^2}) - Z\sqrt{1-A_e^2} \cos(Z\sqrt{1-A_e^2}) \right]. \quad (17)$$

We recall that  $Z = \omega_e z/v$  and  $\xi = \omega_i(t-z/v)$ , so that this term (as well as the remaining terms) displays the behavior expected for coupling of ion and electron oscillations, exhibiting oscillations at the natural frequencies of damped ion and electron motion.

## VI. SUMMARY AND CONCLUSIONS

The rigid beam model with "realistic" damping coefficients gives a simple description of the IFR ion hose instability with the following properties.

(a) The wavelength and  $e$ -folding length of the resonant modes are on the order of the betatron wavelength, in agreement with numerical spread-mass results, and consistent with experimental observations of transverse motion at distances on the order of the betatron wavelength.

(b) The group velocity of an ion hose disturbance is comparable to that found with a numerical spread-mass model.

(c) The impulse response function exhibits coupled oscillations at the natural frequencies of damped electron and ion motion.

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el with no damping,<sup>10</sup> and would like to acknowledge a helpful discussion with that author, K. J. O'Brien.

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