Some special SU(3)⊃R(3) Wigner coefficients and their application[a)]

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Bargmann space expansions of oscillator functions are used to derive analytic expressions for SU(3)⊃R(3) Wigner coefficients for the couplings $(\lambda_1 0)\times(0\mu_2)\to(\lambda_3\mu_3)L_3=0$ and $(\lambda_1 0)\times(\lambda_2 0)\to(\lambda_3\mu_3)L_3=0$, with arbitrary $(\lambda_3\mu_3)$. These lead to expansions useful in nuclear cluster problems and are used to give a simple form for the SU(3)-irreducible tensor expansion of a scalar two-body interaction, an application which motivated this investigation.

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I. INTRODUCTION

The widespread usefulness of the group SU(3) has led to many applications of SU(3) Wigner and recoupling coefficients, and efficient computer codes for their calculation are available.[1,2] In many applications analytic expressions for certain special coefficients are very useful. In the nuclear physics applications reduced Wigner coefficients in the SU(3)⊃R(3) basis are needed. In this basis the natural subgroup labels LM are in general insufficient to label the states, and the resultant inner multiplicity leads to complicated analytic expressions. Despite this difficulty, algebraic expressions have been tabulated by Vergados[3] for many SU(3)⊃R(3) Wigner coefficients useful in nuclear shell model applications in an orthonormal basis which is closely tied to the physically relevant Elliott KLM labeling scheme.[4] In addition, many algebraic expressions have been given by Sharp, von Baeyer, and Pieper[5,6] for the reduced Wigner coefficients in the SU(3)⊃R(3) basis involving (1) representations free of inner multiplicity or (2) special states which are labeled completely by $(\lambda\mu)$, L, and M. These include (1) the couplings involving only representations $(\lambda\ 0)$, (0μ), $(\lambda\ 1)$, or (1μ) and (2) the SU(3)⊃R(3) coefficients for the "stretched" coupling $(\lambda\ 0)\times(0\mu)\to(\lambda\mu)$ and special states of $(\lambda\mu)$ such as the states $L=0$ (λ and μ both even), or $L=1$ (λ,μ = even/odd). In recent applications to problems in nuclear collective motion exploiting Sp(3,R) symmetry,[7–9] and in applications to nuclear cluster problems,[10–12] it has proved useful to expand the rotationally invariant nucleon–nucleon interaction in terms of SU(3)-irreducible tensor components. For this purpose an algebraic expression is needed for the SU(3)⊃R(3) Wigner coefficients for the coupling $(\lambda_1 0)\times(0\mu_2)$ to states $(\lambda_3\mu_3)L_3=0$ of arbitrary λ_3 and μ_3 (λ_3, μ_3 both even, but $\lambda_3\leqslant\lambda_1$, $\mu_3\leqslant\mu_2$). Similarly, the SU(3)⊃R(3) Wigner coefficients for the coupling $(\lambda_1 0)\times(\lambda_2 0)$ to states $(\lambda_3\mu_3)L_3=0$ with $\lambda_3\leqslant\lambda_1+\lambda_2$, $\mu_3\geqslant 0$ have useful applications.

It is the purpose of this note to exhibit analytic expressions for these SU(3)⊃R(3) Wigner coefficients. For ready reference the results are given in Tables I–III. The method of calculation is presented in Sec. II. It makes use of some of the methods of Sharp *et al.*[5,6] but is based on an expansion of the SU(3)-states in terms of Bargmann space polynomials.[13–15] The decomposition of an effective two-body interaction into SU(3)-irreducible tensor components can also be achieved most efficiently through the Bargmann transform of this operator. The process is illustrated in detail in Sec. III for a scalar interaction of Gaussian radial form as an illustration of the usefulness of the results of Sec. II.

II. METHOD OF CALCULATION

The notation and phase conventions will adhere to those of Ref. 1. (The latter are based on the canonical definitions of Biedenharn *et al.*[16]) For ready reference the results for SU(3)⊃R(3) reduced Wigner coefficients are collected in tabular form (Tables I-III). The method of calculation makes use of the specific construction of the state $(\lambda\mu)$ with $L=0$ by techniques similar to those used by Sharp *et al.*[5,6] However, it has proved useful to give all expansions in terms of Bargmann space variables.[13–15] With the one-dimensional real space variable x, we associate the complex Bargmann space variable K_x. With the three-dimensional real space variable $\mathbf{r}$ we associate the three-dimensional Bargmann space variable $\mathbf{K}$. Transformations from real-space square integrable functions $\phi(x)$ to the analytic Bargmann space functions $f(K_x)$ are effected by the transform

$$f(K_x)=\int dx\,A(K_x,x)\phi(x), \tag{1}$$

where

$$A(K_x,x)=\pi^{-1/4}\exp\left[-\tfrac{1}{2}K_x^2-\tfrac{1}{2}x^2+\sqrt{2}K_x x\right]. \tag{2}$$

The kernel $A(K_x,x)$ is a generating function of harmonic oscillator functions

$$A(K_x,x)=\sum_{n=0}^{\infty}\phi_n(x)^*(K_x^n/\sqrt{n!}), \tag{3}$$

that is, $K_x^n/\sqrt{n!}$ is the Bargmann transform of a normalized one-dimensional harmonic oscillator function. For single three-dimensional variables $\mathbf{r}$ and $\mathbf{K}$

$$A(\mathbf{K},\mathbf{r})=\prod_{i=x,y,z}A(K_i,x_i)=\sum_{QLM}\phi_{LM}^{(Q0)}(\mathbf{r})^*P_{LM}^{(Q0)}(\mathbf{K}). \tag{4}$$

The three-dimensional oscillator functions have been expressed in terms of SU(3) representation labels $(\lambda\mu)=(Q\,0)$, Q = total number of oscillator quanta. The ex-

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TABLE I. The coefficients $\langle (2n0)L;(02m)L \,\|(2n-2\nu,2m-2\nu)0\rangle$.

$$(-1)^{\min(m-\nu,n-\nu)}\left[\frac{(2n+2m-4\nu+2)(2L+1)}{F(2n,L)F(2m,L)(2\nu)!(2n+2m+2-2\nu)!}\right]^{1/2}$$
$$\times\sum_{l=0}^{\min(m-\nu,n-\nu)}\frac{(-1)^l(n+m-2\nu-l)!(2\nu+2l)!F(2\nu+2l,L)}{l!(n-\nu-l)!(m-\nu-l)!};$$

$$(-1)^{\min(m-\nu,n-\nu)}\left[\frac{(2n+2m-4\nu+2)(2L+1)(2\nu)!}{F(2n,L)F(2m,L)(2n+2m+2-2\nu)!}\right]^{1/2}\frac{F(2\nu,L)(n+m-2\nu)!}{(n-\nu)!(m-\nu)!}$$
$$\times{}_4F_3(\nu+1,\nu+\tfrac{1}{2},-(n-\nu),-(m-\nu);(\nu-\tfrac{1}{2}L+1),(\nu+\tfrac{1}{2}L+\tfrac{3}{2}),-(n+m-2\nu);1);$$

$$(-1)^{\min(m-\nu,n-\nu)}[(2n+2m-4\nu+2)(2L+1)F(2n,L)F(2m,L)(2\nu)!(2n+2m+2-2\nu)!]^{1/2}$$
$$\times\sum_{\alpha=\max(0,L/2-\nu)}^{[L/2]}\frac{(-1)^\alpha(2L-2\alpha)!(n-\tfrac{1}{2}L+\alpha)!(m-\tfrac{1}{2}L+\alpha)!}{\alpha!(L-\alpha)!(L-2\alpha)!(\nu-\tfrac{1}{2}L+\alpha)!(n+m-\nu-\tfrac{1}{2}L+\alpha)!(2n+2m+2-2\nu-L+2\alpha)};$$

with

$$\langle (2n0)L;(02m)L\,\|(2n-2\nu,2m-2\nu)0\rangle = f(n,m,\nu,L)$$
$$\langle (2n+1,0)L';(0,2m+1)L'\|(2n-2\nu,2m-2\nu)0\rangle = f(n+\tfrac{1}{2},m+\tfrac{1}{2},\nu+\tfrac{1}{2},L')\times(-1)^{L'}$$
$$F(a,L)\equiv(\tfrac{1}{2}a+\tfrac{1}{2}L)!/(\tfrac{1}{2}a-\tfrac{1}{2}L)!(a+L+1)!$$

pansion could have been given in terms of any convenient set of subgroup labels. For the totally symmetric representation $(Q\,0)$ the angular momentum labels LM give a complete labeling. In this SU(3)$\supset$R(3) basis the Bargmann transform of a normalized harmonic oscillator function of a single three-dimensional variable is given in terms of solid spherical harmonics, $\mathscr{Y}_{LM}(\mathbf{K})$, by

$$P^{(Q0)}_{LM}(\mathbf{K}) = [4\pi 2^L F(Q,L)]^{1/2}(\mathbf{K}\cdot\mathbf{K})^{(Q-L)/2}\mathscr{Y}_{LM}(\mathbf{K}), \quad (5a)$$

where

$$F(Q,L) = \frac{[\frac{1}{2}(Q+L)]!}{[\frac{1}{2}(Q-L)]!(Q+L+1)!} \quad (5b)$$

(see, e.g., Ref. 15, but note that the phase has been adjusted to be in agreement with Ref. 1). The solid harmonics $\mathscr{Y}_{1M}(\mathbf{K})$, e.g., are given in terms of the complex Bargmann space variables K_x,K_y,K_z by $-1/\sqrt{2}(K_x+iK_y),K_z,1/\sqrt{2}(K_x-iK_y)$ and have SU(3) transformation properties identical with those of three-dimensional SU(3) basis vectors $|(1\,0)L=1M\rangle$. Note also that the Bargmann space functions $(P_{LM}{}^{(Q0)}(\mathbf{K}))^*$ have SU(3) transformation properties identical with those of basis vectors $(-1)^M|(0Q)L-M\rangle$. The SU(3)$\supset$R(3) Wigner coefficients for the coupling $(\lambda_1 0)\times(0\mu_2)$ to states $(\lambda\mu)L=0$ are obtained from a construction of the Bargmann space functions $P^{(\lambda\mu)}_{L=0}(\bar{\mathbf{K}},\mathbf{K}^*)$ in terms of two independent three-dimensional Bargmann space variables $\bar{\mathbf{K}}$ and $\mathbf{K}$. Those for the coupling of $(\lambda_1 0)\times(\lambda_2 0)$ to states $(\lambda\mu)L=0$ are obtained from a construction $P^{(\lambda\mu)}_{L=0}(\mathbf{K}_1,\mathbf{K}_2)$ in terms of the two independent three-dimensional Bargmann space variables $\mathbf{K}_1$ and $\mathbf{K}_2$.

The first construction will be illustrated by the special case λ_1,μ_1 both even, with $(\lambda_1 0)=(2n0)$, $(0\mu_2)=(02m)$, and with $(\lambda\mu)=(2n-2\nu,\,2m-2\nu)$. Without loss of generality we shall assume $m\leqslant n$. (The construction for states with $m>n$ is trivially similar. Wigner coefficients for the case $m>n$ can also be obtained from those with $m\leqslant n$ by simple symmetry properties.[1]) The state with $(\lambda\mu)=(2n-2\nu,\,2m-2\nu)$ and with $L=0$ is constructed in terms of the expansion

$$P^{(\lambda\mu)}_{L=0}(\bar{\mathbf{K}},\mathbf{K}^*) = [P^{(2n,0)}(\bar{\mathbf{K}})\times P^{(0,2m)}(\mathbf{K}^*)]^{(\lambda\mu)=(2n-2\nu,2m-2\nu)}_{L=0} = \sum_{k=0}^{m-\nu} c_k(\bar{\mathbf{K}}\cdot\bar{\mathbf{K}})^{n-m+k}(\mathbf{K}^*\cdot\mathbf{K}^*)^k(\bar{\mathbf{K}}\cdot\mathbf{K}^*)^{2m-2k}. \quad (6)$$

The square bracket denotes SU(3) coupling. From the fact

TABLE IIA. The coefficients $\langle 2n0)L;(2m0)L\,\|(2n+2m-4\nu,2\nu)0\rangle$.

$$\frac{(-1)^{L/2+\min(\nu,n+m-2\nu)}L!}{2^L[(\frac{1}{2}L)!]^2}\left[\frac{(2n+2m-4\nu+1)(2L+1)}{F(2n,L)F(2m,L)(2n-2\nu)!(2m-2\nu)!}\right]^{1/2}$$
$$\times\frac{(n-\nu)!(m-\nu)!(n+m-\nu)!}{\nu!(2n+2m+1-2\nu)!}\sum_{k=\nu}^{\min(n,m)}\frac{(-1)^k 2^{4k-2\nu}(2n+2m-2\nu-2k)!(k!)^2F(2k,L)}{(k-\nu)!(n-k)!(m-k)!(n+m-\nu-k)!};$$

$$\frac{(-1)^{L/2+\nu+\min(\nu,n+m-2\nu)}L!}{2^L[(\frac{1}{2}L)!]^2}\left[\frac{(2n+2m-4\nu+1)(2L+1)}{F(2n,L)F(2m,L)(2n-2\nu)!(2m-2\nu)!}\right]^{1/2}$$
$$\times\frac{2^{2\nu+1}\nu!(n+m-\nu)!(2n+2m-4\nu)!(\nu+\frac{1}{2}L+1)!}{(n+m-2\nu)!(2n+2m+1-2\nu)!(\nu-\frac{1}{2}L)!(2\nu+L+2)!}$$
$$\times{}_4F_3(\nu+1,\nu+1,-(n-\nu),-(m-\nu);(\nu-\tfrac{1}{2}L+1),(\nu+\tfrac{1}{2}L+\tfrac{3}{2},-(n+m-2\nu-\tfrac{1}{2});1);$$

$$(-1)^{L/2+\nu+\min(\nu,n+m-2\nu)}[(2n+2m-4\nu+1)(2L+1)F(2n,L)F(2m,L)(2n-2\nu)!(2m-2\nu)!]^{1/2}$$
$$\times\sum_{\alpha=\max(0,L/2-\nu)}^{L/2}\frac{(-1)^\alpha 2^{2\nu+4\alpha-L}(2L-2\alpha)![(\frac{1}{2}L)!]^2}{\alpha!L!(L-\alpha)![(\frac{1}{2}L-\alpha)!]^2}$$
$$\times\frac{(2n+2m-2\nu+1)!(n-\frac{1}{2}L+\alpha)!(m-\frac{1}{2}L+\alpha)!(n+m-\nu-\frac{1}{2}L+\alpha)!\nu!}{(2n+2m-2\nu-L+2\alpha+1)!(n-\nu)!(m-\nu)!(n+m-\nu)!(\nu-\frac{1}{2}L+\alpha)!}$$

TABLE IIB. The coefficients $\langle(2n+1,0)L';(2m+1,0)L'\|(2n+2m+2-4\nu,2\nu)0\rangle$.

$$\frac{(-1)^{(L'+1)/2+\min(\nu,n+m+1-2\nu)}L'!}{2^{L'-1}[(\frac{1}{2}(L'-1))!]^2}\left[\frac{(2n+2m-4\nu+3)(2L'+1)}{F(2n+1,L')F(2m+1,L')(2n+1-2\nu)!(2m+1-2\nu)!}\right]^{1/2}$$
$$\times\frac{(n-\nu)!(m-\nu)!(n+m+1-\nu)!}{\nu!(2n+2m+3-2\nu)!}\times\sum_{k=\nu}^{\min(n,m)}\frac{(-1)^k 2^{4k-2\nu}(2n+2m+2-2\nu-2k)!(k!)^2F(2k+1,L')}{(k-\nu)!(n-k)!(m-k)!(n+m+1-\nu-k)!};$$

$$\frac{(-1)^{(L'+1)/2+\nu+\min(\nu,n+m+1-2\nu)}L'!}{2^{L'-1}[(\frac{1}{2}(L'-1))!]^2}\left[\frac{(2n+2m-4\nu+3)(2L'+1)}{F(2n+1,L')F(2m+1,L')(2n+1-2\nu)!(2m+1-2\nu)!}\right]^{1/2}$$
$$\times\frac{2^{2\nu+2}\nu!(2n+2m+1-4\nu)!(n+m+1-\nu)!(\nu+\frac{1}{2}L'+\frac{3}{2})!}{(2n+2m+3-2\nu)!(n+m-2\nu)!(\nu-\frac{1}{2}L'+\frac{1}{2})!(2\nu+L'+3)!}$$
$$\times{}_4F_3(\nu+1,\nu+1,-(n-\nu),-(m-\nu);(\nu-\tfrac{1}{2}L'+\tfrac{3}{2}),(\nu+\tfrac{1}{2}L'+2),-(n+m-2\nu+\tfrac{1}{2});1);$$

$$(-1)^{(L'+1)/2+\nu+\min(\nu,n+m+1-2\nu)}[(2n+2m+3-4\nu)(2L'+1)F(2n+1,L')F(2m+1,L')]^{1/2}$$
$$\times[(2n+1-2\nu)!(2m+1-2\nu)!]^{1/2}\sum_{\alpha=\max(0,\frac{1}{2}(L'-1)-\nu)}^{(L'-1)/2}\frac{2^{2\nu+1+4\alpha-L'}(-1)^\alpha(2L'-2\alpha)![(\frac{1}{2}(L'-1))!]^2}{\alpha!\,L'!(L'-\alpha)![(\frac{1}{2}(L'-1)-\alpha)!]^2}$$
$$\times\frac{(2n+2m+3-2\nu)!(n+\frac{1}{2}-\frac{1}{2}L'+\alpha)!(m+\frac{1}{2}-\frac{1}{2}L'+\alpha)!(n+m-\nu-\frac{1}{2}L'+\frac{3}{2}+\alpha)!\nu!}{(2n+2m+4-2\nu-L'+2\alpha)!(n-\nu)!(m-\nu)!(n+m+1-\nu)!(\nu-\frac{1}{2}L'+\frac{1}{2}+\alpha)!}$$

that the scalar product $(\bar{K}\cdot K^*)$ is also an SU(3)-scalar and from the relations

$$(\bar{\mathbf{K}}\cdot\bar{\mathbf{K}})^Q=[(2Q+1)!]^{1/2}P^{(Q0)}_{L=0}(\bar{\mathbf{K}}),$$
$$(\mathbf{K}^*\cdot\mathbf{K}^*)^Q=[(2Q+1)!]^{1/2}P^{(0Q)}_{L=0}(\mathbf{K}^*),\qquad(7)$$

we see that the k th term in the expansion (6) gives a linear combination of states with $(\lambda\mu)=(2n-2m,0)$, $(2n-2m+2,2),\ldots,(2n-2m+2k,2k)$. The coefficients c_k with $k<m-\nu$ are to be chosen to eliminate the unwanted representations with $\lambda<2n-2\nu$, $\mu<2m-2\nu$. Due to the simplicity of the Bargmann space functions this can be achieved by direct action of the SU(3) Casimir operator. In

TABLE III. Special cases.

$\langle(2n0)0;(02m)0\|(2n-2\nu,2m-2\nu)0\rangle$
$$=\left[\frac{(2n+2m+2-4\nu)(2n+2m+1-2\nu)!(2\nu)!}{(2n+2m+2-2\nu)(2n+1)!(2m+1)!}\right]^{1/2}\frac{(-1)^{\min(n-\nu,m-\nu)}n!m!}{(n+m-\nu)!\nu!}$$

$\langle(2n0)L;(02m)L\|(2n,2m)0\rangle$
$$=[(2L+1)F(2n,L)F(2m,L)(2n+2m+1)!]^{1/2}\frac{(-1)^{L/2+\min(n,m)}L!n!m!}{[(\frac{1}{2}L)!]^2(n+m)!}$$

$\langle(2n+1,0)1;(0,2m+1)1\|(2n-2\nu,2m-2\nu)0\rangle$
$$=[3F(2n+1,1)F(2m+1,1)(2n+2m-4\nu+2)(2\nu+1)!(2n+2m+3-2\nu)!]^{1/2}$$
$$\times(-1)^{1+\min(n-\nu,m-\nu)}\frac{n!m!}{\nu!(n+m+1-\nu)!}$$

$\langle(2n+1,0)L';(0,2m+1)L'\|(2n,2m)0\rangle$
$$=[(2L'+1)F(2n+1,L')F(2m+1,L')(2n+2m+3)(2n+2m+1)!]^{1/2}$$
$$\times(-1)^{(L'+1)/2+\min(n,m)}\frac{(L'+1)!n!m!}{[\frac{1}{2}(L'-1)]![\frac{1}{2}(L'+1)]!(n+m)!}$$

$\langle(2n0)(;(2m0)0\|(2n+2m-4\nu,2\nu)0\rangle$
$$=\left[\frac{(2n+2m+1-4\nu)(2n-2\nu)!(2m-2\nu)!}{(2n+1)!(2m+1)!}\right]^{1/2}\frac{(-1)^{\nu+\min(\nu,n+m-2\nu)}2^{2\nu}n!m!}{(n-\nu)!(m-\nu)!}$$

$\langle(2n0)L;(2m0)L\|(2n-2m,2m)0\rangle$
$$=\left[\frac{(2L+1)F(2m,L)(2n-2m+1)!}{F(2n,L)}\right]^{1/2}\frac{(-1)^{L/2+m+\min(m,n-m)}L!2^{2m-L}n!m!}{[(\frac{1}{2}L)!]^2(n-m)!(2n+1)!}$$

$\langle(2n+1,0)1;(2m+1,0)1\|(2n+2m+2-4\nu,2\nu)0\rangle$
$$=\left[\frac{3(2n+2m+3-4\nu)(n+1)(m+1)(2n+1-2\nu)!(2m+1-2\nu)!}{(2n+3)!(2m+3)!}\right]^{1/2}\frac{(-1)^{1+\nu+\min(\nu,n+m+1-2\nu)}2^{2\nu+1}n!m!}{(n-\nu)!(m-\nu)!}$$

$\langle(2n+1,0)L';(2m+1,0)L'\|(2n-2m+2,2m)0\rangle$
$$=\left[\frac{(2L'+1)(2n-2m+3)F(2m+1,L')(2n-2m+1)!}{F(2n+1,L')}\right]^{1/2}\frac{(-1)^{(L'+1)/2+m+\min(m,n-m+1)}2^{2\nu+2-L'}L'!(n+1)!m!}{[(\frac{1}{2}(L'-1))!]^2(2n+3)!(n-m)!}$$

$\langle(2a,2m)0;(2b,0)0\|(2a+2b,2m)0\rangle$
$$=\left[\frac{(2a)!(2a+2b+2m+1)!}{(2b+1)(2a+2b)!(2a+2m+1)!}\right]^{1/2}\frac{(a+m)!(a+b)!}{a!(a+b+m)!}(-1)^\sigma$$

$\sigma=0$ for $m<a$, $\sigma=b$ for $m>a$

the subspace spanned by the Bargmann space vectors $\overline{\mathbf{K}}$, $\mathbf{K}^*$, the U(3) generators can be expressed by

$$A_{ij} = \overline{K}_i \frac{\partial}{\partial \overline{K}_j} - K_j^* \frac{\partial}{\partial K_i^*}, \quad i,j = x,y,z, \tag{8}$$

leading to the SU(3) Casimir operator

$$\mathbf{C}_{\mathrm{SU(3)}} = \sum_{\alpha,\beta} A_{\alpha\beta} A_{\beta\alpha} - \tfrac{1}{3}(\mathrm{Tr} A)^2, \tag{9}$$

with eigenvalue

$$C_{\mathrm{SU(3)}} = \tfrac{2}{3}[\lambda^2 + \lambda\mu + \mu^2 + 3(\lambda + \mu)]. \tag{10}$$

The action of the operator $(\mathbf{C}_{\mathrm{SU(3)}} - C_{\mathrm{SU(3)}})$ on the relation (6) leads to the recursion relation

$$c_{k-1} = -\frac{k(n-m+k)}{(n-\nu+k)(m-\nu+1-k)} c_k. \tag{11}$$

The normalization is achieved by the coefficient $c_{m-\nu}$ [see the remarks following Eq. (17)],

$$c_{m-\nu} = (-1)^{m-\nu} \left[\frac{(2n+2m-4\nu+2)}{(2\nu)!(2n+2m-2\nu+2)!} \right]^{1/2} \times \frac{(n+m-2\nu)!}{(n-\nu)!(m-\nu)!}, \tag{12}$$

leading to

$$[P^{(2n,0)}(\overline{\mathbf{K}}) \times P^{(0,2m)}(\mathbf{K}^*)]^{(2n-2\nu,2m-2\nu)}_{L=0} = \left[\frac{(2n+2m-4\nu+2)}{(2\nu)!(2n+2m-2\nu+2)!} \right]^{1/2} \times \sum_{k=0}^{m-\nu} \frac{(-1)^k (n-\nu+k)!}{k!(n-m+k)!(m-\nu-k)!} \times (\overline{\mathbf{K}}\cdot\overline{\mathbf{K}})^{n-m+k} (\mathbf{K}^*\cdot\mathbf{K}^*)^k (\overline{\mathbf{K}}\cdot\mathbf{K}^*)^{2m-2k}. \tag{13}$$

The relation[10,15]

$$\frac{1}{c!} (\overline{\mathbf{K}}\cdot\mathbf{K}^*)^c = [\dim(c0)]^{1/2} [P^{(c0)}(\overline{\mathbf{K}}) \times P^{(0c)}(\mathbf{K}^*)]^{(00)}_0 = \sum_L (P^{(c0)}_L(\overline{\mathbf{K}}) \cdot P^{(0c)}_L(\mathbf{K}^*)), \tag{14}$$

together with a double application of Eq. (5), leads to

$$[P^{(2n,0)}(\overline{\mathbf{K}}) \times P^{(0,2m)}(\mathbf{K}^*)]^{(2n-2\nu,2m-2\nu)}_0 = \sum_L \langle (2n0)L;(02m)L \,\|(2n-2\nu,2m-2\nu)0\rangle (-1)^L \times \frac{(P^{(2n,0)}_L(\overline{\mathbf{K}}) \cdot P^{(0,2m)}_L(\mathbf{K}^*))}{[2L+1]^{1/2}}$$
$$= \sum_L \left[\frac{(2n+2m-4\nu+2)}{(2\nu)!(2n+2m-2\nu+2)! F(2n,L) F(2m,L)} \right]^{1/2} \times \sum_{k=0}^{m-\nu} \frac{(-1)^k (n-\nu+k)!(2m-2k)! F(2m-2k,L)}{k!(n-m+k)!(m-\nu-k)!} \times (P^{(2n,0)}_L(\overline{\mathbf{K}}) \cdot P^{(0,2m)}_L(\mathbf{K}^*)). \tag{15}$$

From this relation a first expression for the SU(3)$\supset$R(3) reduced Wigner coefficient[17] is obtained for the coupling $(2n\ 0) \times (0\ 2m) \rightarrow (\lambda\mu)L = 0$. This is listed as the first entry in Table I. In Table I the summation index has been changed from k to $l = m - \nu - k$ to gain a more symmetrical form. The summation can be expressed in terms of a Saalschutzian generalized hypergeometric function of type ${}_4F_3$ and of argument unity (see the second entry of Table I). Since no simple closed form is known for such a function, it appears that the summation cannot be carried out in closed form. For the special case $L = 0$ the hypergeometric function collapses to a Saalschutzian function of type ${}_3F_2$ for which a closed form is known; see Eq. (III.2) or Eq. (2.3.1.4) of Ref. 18. (We are indebted to Professor A. C. T. Wu for pointing out this identity to us.) This leads to a very simple expression for the coefficient $\langle (2n\ 0)0; (0\ 2m)0 \| (\lambda\mu)0\rangle$; see the first entry of Table III. The need for this coefficient formed the starting point for this investigation.

The normalization coefficient $c_{m-\nu}$, Eqs. (6), (11), and (12), was determined with the aid of the relation

$$(\overline{\mathbf{K}}\cdot\overline{\mathbf{K}})^a (\mathbf{K}^*\cdot\mathbf{K}^*)^b (\overline{\mathbf{K}}\cdot\mathbf{K}^*)^c = \sum_\gamma \langle (2a0)0;(02b)0 \| (2a-2\gamma,2b-2\gamma)0\rangle \times [(2a+1)!(2b+1)!\dim(c0)]^{1/2} c! \times [[P^{(2a,0)}(\overline{\mathbf{K}}) \times P^{(0,2b)}(\mathbf{K}^*)]^{(2a-2\gamma,2b-2\gamma)} \times [P^{(c0)}(\overline{\mathbf{K}}) \times P^{(0c)}(\mathbf{K}^*)]^{(00)}]^{(2a-2\gamma,2b-2\gamma)}_{L=0}$$
$$= \sum_\gamma \langle (2a0)0;(02b)0 \| (2a-2\gamma,2b-2\gamma)0\rangle \times \left[\frac{(2a+1)!(2b+1)!(2\gamma+c)!(2a+2b+c-2\gamma+2)!}{(2\gamma)!(2a+2b-2\gamma+2)!} \right]^{1/2} \times [P^{(2a+c,0)}(\overline{\mathbf{K}}) \times P^{(0,2b+c)}(\mathbf{K}^*)]^{(2a-2\gamma,2b-2\gamma)}_{L=0}. \tag{16}$$

In the last step of Eq. (16) the SU(3)-coupled K-space functions have been subjected to an SU(3)-recoupling transformation, and the renormalization relation

$$[P^{(\alpha 0)}(\mathbf{K}) \times P^{(\beta 0)}(\mathbf{K})]^{(\lambda\mu)}_{LM} = \delta_{(\lambda\mu)(\alpha+\beta,0)} [(\alpha+\beta)!/\alpha!\beta!]^{1/2} P^{(\alpha+\beta,0)}_{LM}(\mathbf{K}) \tag{17}$$

has been used. For the recoupling transformation, see in particular Eqs. (A18) and (A14) of Ref. 11. [Appendixes A and B of Ref. 11 contain many useful formulae for SU(3)-coupled Bargmann space functions.] When Eq. (16), with $a = n - m - k$, $b = k$, $c = 2m - 2k$, is substituted into the relation (6), only the single term with $k = m - \nu$ and $\gamma = 0$ survives in the summations. This relates the normalization coefficient $c_{m-\nu}$ and the coefficient $\langle (2n-2\nu, 0)0; (0, 2m-2\nu)0 \| (2n-2\nu, 2m-2\nu)0\rangle$ for the "stretched" coupling which can be evaluated with the further use of Eq. (15) for $L = 0$. This procedure does not determine the phase of $c_{m-\nu}$. We shall adhere to the phase conventions of Ref. 1 and choose the states $|(\lambda\mu)LM\rangle$ with $\lambda \geqslant \mu$ to have phases consistent with angular momentum projection from the intrinsic state G'_{LW} of Ref. 1, whereas those with $\mu > \lambda$ have phases consistent with angular momentum projection from the state G'_{HW}. In the notation of Ref. 1 this means that our states are of type $IJ = 0\ 1$ for $\lambda \geqslant \mu$ and of type $IJ = 1\ 0$ for $\mu > \lambda$. Note also that this insures that $P^{(Q0)}_{LM}(\mathbf{K})$ and $P^{(0Q)}_{L-M}(\mathbf{K}^*)$ are related by the simple conjugation relations spelled out in Ref. 1. With this choice of phases the states (6) are defined completely for both $m \leqslant n$, $m > n$. The sign of the coefficient $\langle (2n0)0; (02m)0 \| (\lambda\mu)0 \rangle$ is given by $(-1)^\phi$ with

$\phi = \min(\frac{1}{2}\lambda, \frac{1}{2}\mu)$ (see Table III). With the knowledge of this coefficient, Eq. (16) can be put in the form

$$(\overline{\mathbf{K}}\cdot\overline{\mathbf{K}})^a(\mathbf{K}^*\cdot\mathbf{K}^*)^b(\overline{\mathbf{K}}\cdot\mathbf{K}^*)^c = \sum_{\gamma=0}^{\min(a,b)} \frac{(-1)^{\min(a-\gamma,b-\gamma)}}{2}[(2a+2b-4\gamma+2) \times(2\gamma+c)!(2a+2b+c-2\gamma+2)!]^{1/2} \times \frac{a!b!}{\gamma!(a+b+1-\gamma)!}[P^{(2a+c,0)}(\overline{\mathbf{K}}) \times P^{(0,2b+c)}(\mathbf{K}^*)]^{(2a-2\gamma,2b-2\gamma)}_{L=0}. \quad (18)$$

This expression will prove useful in applications to nuclear cluster problems.[10–12]

The first entry of Table I is very convenient when $m-\nu$, or $n-\nu$, is a small integer since the number of terms in the l-sum is then small. For small values of ν, or for small values of L, an alternate form may prove more convenient. To obtain this form, we start with the expansion

$$(P_L^{(2n,0)}(\overline{\mathbf{K}})\cdot P_L^{(0,2m)}(\mathbf{K}^*)) = \sum_\nu \langle(2n0)L;(02m)L\,\|(2n-2\nu,2m-2\nu)0\rangle \times(-1)^L[(2L+1)]^{1/2} \times[P^{(2n,0)}(\overline{\mathbf{K}})\times P^{(0,2m)}(\mathbf{K}^*)]_0^{(2n-2\nu,2m-2\nu)}$$
$$=(2L+1)[F(2n,L)F(2m,L)]^{1/2} \times\sum_{\alpha=0}^{L/2}\frac{(-1)^\alpha(2L-2\alpha)!}{\alpha!(L-\alpha)!(L-2\alpha)!} \times(\overline{\mathbf{K}}\cdot\overline{\mathbf{K}})^{n-L/2+\alpha}(\mathbf{K}^*\cdot\mathbf{K}^*)^{m-L/2+\alpha}(\overline{\mathbf{K}}\cdot\mathbf{K}^*)^{L-2\alpha}, \quad (19)$$

where Eq. (5) has been used together with the expansion of the Legendre polynomial $P_L(\xi)$ in powers of $\xi^{L-2\alpha}$. By substituting Eq. (18) into the right-hand side of Eq. (19) the alternate form for the SU(3)⊃R(3) Wigner coefficient is obtained. This is given as the third entry in Table I in a form which can be generalized to the coupling $(\lambda_1 0)\times(0\mu_2)$ with λ_1 and μ_2 both odd.

Similar techniques can be used to calculate SU(3)⊃R(3) Wigner coefficients for the coupling $(\lambda_1 0)\times(\lambda_2 0)\rightarrow(\lambda\mu)L=0$. The SU(3)-coupled Bargmann space functions are now constructed from two independent Bargmann-space variables $\mathbf{K}_1$ and $\mathbf{K}_2$. The cases λ_1, λ_2 both even and λ_1, λ_2 both odd are now slightly different and are treated separately. With $\lambda_1 = 2n$, $\lambda_2 = 2m$, $m \leqslant n$; the coupled Bargmann space function with $(\lambda\mu) = (2n+2m-4\nu, 2\nu)$ and $L=0$ is now constructed in terms of the expansion

$$[P^{(2n,0)}(\mathbf{K}_1)\times P^{(2m,0)}(\mathbf{K}_2)]^{(2n+2m-4\nu,2\nu)}_{L=0} = \sum_{k=\nu}^{m} c_k(\mathbf{K}_1\cdot\mathbf{K}_1)^{n-k}(\mathbf{K}_2\cdot\mathbf{K}_2)^{m-k} \times[(\mathbf{K}_1\cdot\mathbf{K}_1)(\mathbf{K}_2\cdot\mathbf{K}_2)-(\mathbf{K}_1\cdot\mathbf{K}_2)^2]^k. \quad (20)$$

The last factor involves the square of the vector product $[\mathbf{K}_1\times\mathbf{K}_2]$, an SU(3) (01)-tensor, so that it carries the single SU(3) representation $(0, 2k)$. The kth term in the expansion (20) is therefore a linear combination of states with $(\lambda\mu) = (2n+2m-4k, 2k), (2n+2m-4k-4, 2k+2), \ldots, (2n-2m, 2m)$. The coefficients c_k with $k>\nu$ are to be chosen to eliminate the unwanted representations with $\lambda < 2n+2m-4\nu$ and $\mu > 2\nu$. This is again achieved by the action of the operator $(\mathbf{C}_{SU(3)} - C_{SU(3)})$, see Eqs. (9) and (10), where the U(3) generators are now given by

$$A_{ij} = K_{1i}\frac{\partial}{\partial K_{1j}} + K_{2i}\frac{\partial}{\partial K_{2j}} \quad (21)$$

in the subspace spanned by the Bargmann space vectors $\mathbf{K}_1$, $\mathbf{K}_2$. This leads to the recursion formula

$$c_{k+1} = -\frac{2(n-k)(m-k)}{(k+1-\nu)(2n+2m-2\nu-1-2k)}c_k. \quad (22)$$

The normalization is achieved by the coefficient c_ν

$$c_\nu = (-1)^{\nu+\min(\nu,n+m-2\nu)}\left[\frac{(2n+2m-4\nu+1)}{(2n-2\nu)!(2m-2\nu)!}\right]^{1/2} \times\frac{(2n+2m-4\nu)!(n+m-\nu)!}{(2n+2m-2\nu+1)!\nu!(n+m-2\nu)!}, \quad (23)$$

which follows from the term with $k=\nu$ and $\gamma=0$ in the expansion

$$(\mathbf{K}_1\cdot\mathbf{K}_1)^{n-k}(\mathbf{K}_2\cdot\mathbf{K}_2)^{m-k}([\mathbf{K}_1\times\mathbf{K}_2]\cdot[\mathbf{K}_1\times\mathbf{K}_2])^k = [(2n-2k+1)!(2m-2k+1)!]^{1/2}(2k+1)!(-i)^{2k} \times\sum_{\substack{\nu\\ \gamma=(\nu-k)}}\langle(2n-2k,0)0;(2m-2k,0)0\|(2n+2m-4k-4\gamma,2\gamma)0\rangle \times\langle(2n+2m-4k-4\gamma,2\gamma,)0;(0,2k)0\|(2n+2m-4\nu,2\nu)0\rangle \times[[P^{(2n-2k,0)}(\mathbf{K}_1)\times P^{(2m-2k,0)}(\mathbf{K}_2)]^{(2n+2m-4k-4\gamma,2\gamma)} \times[P^{(2k,0)}(\mathbf{K}_1)\times P^{(2k,0)}(\mathbf{K}_2)]^{(0,2k)}]^{(2n+2m-4\nu,2\nu)}_{L=0}$$
$$= [(2n-2k+1)!(2m-2k+1)!]^{1/2}(2k+1)!(-i)^{2k} \times\sum_\nu\langle(2n-2k,0)0;(2m-2k,0)0\|(2n+2m-4\nu,2\nu-2k)0\rangle \times\langle(2n+2m-4\nu,2\nu-2k)0;(0,2k)0\|(2n+2m-4\nu,2\nu)0\rangle\frac{1}{(2k)!} \times\left[\frac{(2\nu)!(2n+2m-2\nu+1)!}{(2k+1)(2\nu-2k)!(2n+2m-2\nu-2k+1)!}\right]^{1/2} \times[P^{(2n,0)}(\mathbf{K}_1)\times P^{(2m,0)}(\mathbf{K}_2)]^{(2n+2m-4\nu,2\nu)}_{L=0}, \quad (24)$$

where an SU(3)-recoupling transformation was used, together with relation (17), in the last step of Eq. (24); [see Eq. (A21) of Ref. 11]. Note also that the nature of the K_1, K_2-space function requires that the final SU(3) representation must correspond to a two-rowed tableau. As a result, γ is restricted to the value $\gamma = \nu - k$. Equation (24) has used Eq. (7) and the related equation

$$P^{(0,2k)}_{L=0}([\mathbf{K}_1\times\mathbf{K}_2]) = [(2k+1)!]^{1/2}(-i)^{2k}[P^{(2k,0)}(\mathbf{K}_1)\times P^{(2k,0)}(\mathbf{K}_2)]^{(0,2k)}_{L=0}. \quad (25)$$

(This relation has been given in Ref. 11; see Eqs. (B4) and

(B5); but a correction by the phase factor $(-i)^{2k}$ is needed.] In the general case, Eq. (24) can be evaluated once the coefficient $\langle(2a,0)0;(2b,0)0\|(\lambda\mu)0\rangle$ is known. The second coefficient, which is of type $\langle(2a,2b)0;(02c)0\|(2a,2b+2c)0\rangle$, is related by symmetry to $\langle(2b,2a)0;(2c,0)0\|(2b+2c,2a)0\rangle$. For $a \leqslant b$ this can be read from Eq. (13), together with Eq. (17). For $a > b$, the phase of this coefficient is $(-1)^c$; see the last entry of Table III. For the evaluation of c_ν, only the term with $k=\nu$ is needed, and in this special case Eq. (24) can be evaluated through coefficients known from our earlier discussion. This leads to

$$[P^{(2n,0)}(\mathbf{K}_1)\times P^{(2m,0)}(\mathbf{K}_2)]^{(2n+2m-4\nu,2\nu)}_{L=0}$$
$$=(-1)^{\nu+\min(\nu,n+m-2\nu)}\left[\frac{2n+2m-4\nu+1}{(2n-2\nu)!(2m-2\nu)!}\right]^{1/2}$$
$$\times\frac{(n-\nu)!(m-\nu)!(n+m-\nu)!}{\nu!(2n+2m-2\nu+1)!}$$
$$\times\sum_{k=\nu}^{\min(n,m)}\frac{(-1)^{k-\nu}2^{2k-2\nu}(2n+2m-2\nu-2k)!}{(k-\nu)!(n-k)!(m-k)!(n+m-\nu-k)!}$$
$$\times(\mathbf{K}_1\cdot\mathbf{K}_1)^{n-k}(\mathbf{K}_2\cdot\mathbf{K}_2)^{m-k}([\mathbf{K}_1\times\mathbf{K}_2]\cdot[\mathbf{K}_1\times\mathbf{K}_2])^k. \tag{26}$$

We now use Eq. (25) with the simple coefficient[6] $\langle(2k\,0)0;(2k\,0)0\|(02k)0\rangle$ (with phases chosen according to Ref. 1), to obtain

$$([\mathbf{K}_1\times\mathbf{K}_2]\cdot[\mathbf{K}_1\times\mathbf{K}_2])^k$$
$$=\sum_L (k!)^2\frac{(-1)^{L/2}2^{2k-L}L!}{[(\frac{1}{2}L)!]^2}$$
$$\times(-1)^L(P^{(2k,0)}_L(\mathbf{K}_1)\cdot P^{(2k,0)}_L(\mathbf{K}_2)). \tag{27}$$

With this relation, Eq. (26) together with a double application of Eq. (5) leads to the desired SU(3)⊃R(3) Wigner coefficient. The result is given as the first entry of Table II. The single sum can again be expressed in terms of a generalized hypergeometric function of type ${}_4F_3$; see the second entry of Table II. For $L=0$ this collapses to a ${}_3F_2$ of Saalschutzian form and leads to the simple special case shown in Table III. With this result, Eq. (24) yields

$$(\mathbf{K}_1\cdot\mathbf{K}_1)^{n-k}(\mathbf{K}_2\cdot\mathbf{K}_2)^{m-k}([\mathbf{K}_1\times\mathbf{K}_2]\cdot[\mathbf{K}_1\times\mathbf{K}_2])^k$$
$$=\sum_\nu(-1)^{\nu+\min(\nu,n+m-2\nu)}$$
$$\times[(2n+2m-4\nu+1)(2n-2\nu)!(2m-2\nu)!]^{1/2}$$
$$\times\frac{2^{2\nu-2k}(n-k)!(m-k)!\nu!(n+m-\nu-k)!(2n+2m-2\nu+1)!}{(n-\nu)!(m-\nu)!(\nu-k)!(n+m-\nu)!(2n+2m-2\nu-2k+1)!}$$
$$\times[P^{(2n,0)}(\mathbf{K}_1)\times P^{(2m,0)}(\mathbf{K}_2)]^{(2n+2m-4\nu,2\nu)}_{L=0}. \tag{28}$$

This result by itself may have useful applications in nuclear cluster problems.[10–12] It can also be used to derive an alternate form for the SU(3) Wigner coefficient more useful when L or ν (rather than $m-\nu$ or $n-\nu$) are small integers. The analog of Eq. (19) becomes

$$(P^{(2n,0)}_L(\mathbf{K}_1)\cdot P^{(2m,0)}_L(\mathbf{K}_2))$$
$$=(2L+1)[F(2n,L)F(2m,L)]^{1/2}$$
$$\times\sum_{\alpha=0}^{L/2}\frac{(-1)^{L/2+\alpha}2^{2\alpha}(2L-2\alpha)![(\frac{1}{2}L)!]^2}{\alpha!L!(L-\alpha)![(\frac{1}{2}L-\alpha)!]^2}$$
$$\times(\mathbf{K}_1\cdot\mathbf{K}_1)^{n-L/2+\alpha}(\mathbf{K}_2\cdot\mathbf{K}_2)^{m-L/2+\alpha}$$
$$\times([\mathbf{K}_1\times\mathbf{K}_2]\cdot[\mathbf{K}_1\times\mathbf{K}_2])^{L/2-\alpha}. \tag{29}$$

If this is combined with Eq. (28) the alternate form of the Wigner coefficient is obtained; see the third entry of Table II.

The SU(3)⊃R(3) Wigner coefficients for the coupling $(\lambda_1 0)\times(\lambda_2 0)\to(\lambda\mu)L=0$ with λ_1 and λ_2 both odd have a slightly different form. The key intermediate results are

$$[P^{(2n+1,0)}(\mathbf{K}_1)\times P^{(2m+1,0)}(\mathbf{K}_2)]^{(2n+2m+2-4\nu,2\nu)}_{L=0}$$
$$=(-1)^{\min(\nu,n+m+1-2\nu)}$$
$$\times\left[\frac{(2n+2m-4\nu+3)}{(2n+1-2\nu)!(2m+1-2\nu)!}\right]^{1/2}$$
$$\times\frac{(n-\nu)!(m-\nu)!(n+m+1-\nu)!}{\nu!(2n+2m-2\nu+3)!}\sum_{k=\nu}^{\min(n,m)}$$
$$\times\frac{(-1)^k2^{2k-2\nu}(2n+2m+2-2\nu-2k)!}{(k-\nu)!(n-k)!(m-k)!(n+m+1-\nu-k)!}$$
$$\times(\mathbf{K}_1\cdot\mathbf{K}_1)^{n-k}(\mathbf{K}_2\cdot\mathbf{K}_2)^{m-k}(\mathbf{K}_1\cdot\mathbf{K}_2)$$
$$\times[\mathbf{K}_1\cdot\mathbf{K}_1)(\mathbf{K}_2\cdot\mathbf{K}_2)-(\mathbf{K}_1\cdot\mathbf{K}_2)^2]^k. \tag{30}$$

Combining Eq. (27) with $(\mathbf{K}_1\cdot\mathbf{K}_2) = (P^{(1\,0)}_{L=1}(\mathbf{K}_1)\cdot P^{(1\,0)}_{L=1}(\mathbf{K}_2))$, applying Eq. (5) and standard spherical harmonic addition theorems, followed by a further application of Eq. (5), we obtain the first form of the coefficient $\langle(2n+1,0)L';(2m+1,0)L'\|(\lambda\mu)0\rangle$ tabulated in Table II. This form can again be expressed in terms of a Saalschutzian hypergeometric function of type ${}_4F_3$; see the entry in Table II. In this case the ${}_4F_3$ collapses to a ${}_3F_2$ for the special case $L'=1$, so that the coefficient with $L'=1$ can be given in closed form. This is included among the special cases of Table III. An alternate form for this coefficient is obtained from the analog of Eq. (29), which is now

$$(P^{(2n+1,0)}_{L'}(\mathbf{K}_1)\cdot P^{(2m+1,0)}_{L'}(\mathbf{K}_2)) = (2L'+1)[F(2n+1,L')F(2m+1,L')]^{1/2}\sum_{\alpha=0}^{(L'-1)/2} \times \frac{(-1)^{(L'-1)/2+\alpha}2^{2\alpha}(2L'-2\alpha)!\{[\frac{1}{2}(L'-1)]!\}^2}{\alpha!L'!(L'-\alpha)!\{[\frac{1}{2}(L'-1)-\alpha]!\}^2} \times(\mathbf{K}_1\cdot\mathbf{K}_1)^{n+1/2-L'/2+\alpha}(\mathbf{K}_2\cdot\mathbf{K}_2)^{m+1/2-L'/2+\alpha} \times(\mathbf{K}_1\cdot\mathbf{K}_2)([\mathbf{K}_1\times\mathbf{K}_2]\cdot[\mathbf{K}_1\times\mathbf{K}_2])^{(L'-1)/2-\alpha}. \tag{31}$$

Simple Wigner coefficients from Table III, together with a recoupling transformation, now give

$$(\mathbf{K}_1\cdot\mathbf{K}_1)^{n+1/2-L'/2+\alpha}(\mathbf{K}_2\cdot\mathbf{K}_2)^{m+1/2-L'/2+\alpha}(\mathbf{K}_1\cdot\mathbf{K}_2)([\mathbf{K}_1\times\mathbf{K}_2]\cdot[\mathbf{K}_1\times\mathbf{K}_2])^{(L'-1)/2-\alpha} = \sum_\nu(-1)^{\nu+\min(\nu,n+m+1-2\nu)}[(2n+2m+3-4\nu)(2n+1-2\nu)!(2m+1-2\nu)!]^{1/2} \times 2^{2\nu+2\alpha-L'+1}\frac{(2n+2m+3-2\nu)!(n+\frac{1}{2}-\frac{1}{2}L'+\alpha)!(m+\frac{1}{2}-\frac{1}{2}L'+\alpha)!\nu!}{(2n+2m+4-2\nu+2\alpha-L')!(n-\nu)!(m-\nu)!(\nu+\frac{1}{2}-\frac{1}{2}L'+\alpha)!} \times\frac{(n+m-\nu-\frac{1}{2}L'+\frac{3}{2}+\alpha)!}{(n+m-\nu+1)!}[P^{(2n+1,0)}(\mathbf{K}_1)\times P^{(2m+1,0)}(\mathbf{K}_2)]^{(2n+2m+2-4\nu,2\nu)}_{L=0}. \tag{32}$$

The combination of Eqs. (31) and (32) leads to the alternate form for the coefficient $\langle 2n+1,0)L';(2m+1,0)L'\|(\lambda\mu)0\rangle$, given as the last entry in Table II.

III. AN APPLICATION: SU(3)-IRREDUCIBLE TENSOR DECOMPOSITION OF A SCALAR INTERACTION

In recent applications to problems in nuclear collective motion exploiting Sp(3,R) symmetry[7–9] and in nuclear cluster problems[10–12] it has proved useful to expand the rotationally invariant nucleon–nucleon interaction in terms of SU(3) irreducible tensor components. If the two-body interaction with $V=\Sigma_{i<j}V_{ij}$ is given by

$$V_{12}=\sum_{ST}V_{ST}(|\mathbf{r}_1-\mathbf{r}_2|)P_{ST}, \tag{33}$$

where P_{ST} is a two-particle spin–isospin projection operator, it frequently proves convenient to expand the radial part of V in terms of Gaussians

$$V(|\mathbf{r}_1-\mathbf{r}_2|)=V_0\exp(-\beta r^2), \tag{34}$$

where $\mathbf{r}$ is the dimensionless relative coordinate

$$\mathbf{r}=[m\omega/2\hbar]^{1/2}(\mathbf{r}_1-\mathbf{r}_2). \tag{35}$$

If such a V is expanded in SU(3) irreducible tensor components

$$V=\sum_{(\lambda_0\mu_0)}V^{(\lambda_0\mu_0)}_{L_0=0}, \tag{36}$$

it is sufficient to specify V by its SU(3) reduced matrix elements, defined by[10]

$$\langle(\bar\lambda\bar\mu)\bar\kappa LM|V^{(\lambda_0\mu_0)}_{L_0=0}|(\lambda\mu)\kappa LM\rangle=(-1)^{\lambda+\mu}\sum_{\rho_0}\frac{\langle(\bar\lambda\bar\mu)\bar\kappa L;(\mu\lambda)\kappa L\|(\lambda_0\mu_0)L_0=0\rangle_{\rho_0}\langle(\bar\lambda\bar\mu)\|V^{(\lambda_0\mu_0)}\|(\lambda\mu)\rangle_{\rho_0}}{[(2L+1)]^{1/2}}, \tag{37}$$

where the SU(3) reduced matrix element is given by the last factor of Eq. (37). Note that the SU(3) reduced matrix element is defined in terms of an unconventional order of the SU(3) coupling. This definition proves convenient when the SU(3) coupling $(\bar\lambda\bar\mu)\times(\lambda\mu)\to(\lambda_0\mu_0)$ requires an outer multiplicity label ρ_0; that is, when $(\lambda_0\mu_0)$ occurs in the coupling with a multiplicity >1. For a scalar interaction of the relative coordinate $\mathbf{r}$, $V(r)$ is specified by its SU(3) reduced matrix elements in the space of oscillator functions $\phi^{(q0)}_{lm}(\mathbf{r})$ of the single three-dimensional variable $\mathbf{r}$, that is, by the numbers $\langle(\bar q0)\|V^{(\lambda_0\mu_0)}\|(q0)\rangle$. These follow at once from the Bargmann transform of $V(r)$:

$$\mathscr{V}(\bar{\mathbf{k}},\mathbf{k}^*)\equiv\int d\mathbf{r}\,A(\bar{\mathbf{k}},\mathbf{r})V(r)A(\mathbf{k}^*,\mathbf{r}) =\sum_{\bar qq}\sum_{(\lambda_0\mu_0)}\langle(\bar q0)\|V^{(\lambda_0\mu_0)}\|(q0)\rangle \times[P^{(\bar q0)}(\bar{\mathbf{k}})\times P^{(0q)}(\mathbf{k}^*)]^{(\lambda_0\mu_0)}_{L_0=0}, \tag{38}$$

where we have used the expansion (4) for $A(\bar{\mathbf{k}},\mathbf{r})$, $A(\mathbf{k}^*,\mathbf{r})$, the defining Equation (37), and the orthonormality of the Wigner coefficients. To determine the needed SU(3) reduced matrix elements, it is thus only necessary to evaluate the Bargmann transform of $V(r)$, expand it in the SU(3)-coupled Bargmann space functions, and pick off the coefficient of the $\bar{q}, q\ (\lambda_0\mu_0)$ term. For the Gaussian interaction of Eq. (34)

$$\mathscr{V}(\bar{\mathbf{k}}, \mathbf{k}^*) = \frac{V_0}{(1+\beta)^{3/2}} \times \exp\left[\frac{(\bar{\mathbf{k}}\cdot\mathbf{k}^*)}{1+\beta} - \frac{\beta}{2(1+\beta)}(\bar{\mathbf{k}}\cdot\bar{\mathbf{k}} + \mathbf{k}^*\cdot\mathbf{k}^*)\right]$$
$$= \sum_{a,b,c} \frac{1}{a!b!c!}\left(-\frac{\beta}{2}\right)^{a+b} V_0 \times \frac{1}{(1+\beta)^{a+b+c+3/2}} (\bar{\mathbf{k}}\cdot\bar{\mathbf{k}})^a (\mathbf{k}^*\cdot\mathbf{k}^*)^b (\bar{\mathbf{k}}\cdot\mathbf{k}^*)^c. \tag{39}$$

Direct application of Eq. (18) gives

$$\mathscr{V}(\bar{\mathbf{k}},\mathbf{k}^*) = V_0 \sum_{a,b,c,\nu}\left(-\frac{\beta}{2}\right)^{a+b} \times \frac{1}{(1+\beta)^{a+b+c+3/2}} \frac{(-1)^{\min(a-\nu,b-\nu)}}{2} \times \frac{[(2a+2b-4\nu+2)(2\nu+c)!(2a+2b-2\nu+c+2)!]}{c!\nu!(a+b-\nu+1)!} \times [P^{(2a+c,0)}(\bar{\mathbf{k}}) \times P^{(0,2b+c)}(\mathbf{k}^*)]^{(2a-2\nu,2b-2\nu)}_{L=0}. \tag{40}$$

With $2a+c=\bar{q}$, $2b+c=q$, $(\lambda_0\mu_0) = (2a-2\nu, 2b-2\nu)$, and Eq. (38) this gives the needed reduced matrix elements. It is convenient to name the summation index $b - \frac{1}{2}\mu_0 = m$; note also that $q - \mu_0 = \bar{q} - \lambda_0$. The result is

$$\langle(\bar{q}0)\| V^{(\lambda_0\mu_0)}\|(q0)\rangle = V_0 \frac{(-1)^{\min(\lambda_0/2,\mu_0/2)}(-1)^{(\bar{q}-q)/2}(\frac{1}{2}\beta)^{(\lambda_0+\mu_0)/2}}{2(1+\beta)^{(\bar{q}+q+3)/2}} \times [(\lambda_0+\mu_0+2)(q-\mu_0)!(\lambda_0+q+2)!]^{1/2} \times \sum_{m=0}^{[(q-\mu_0)/2]}\left(\frac{\beta}{2}\right)^{2m} \times \frac{1}{m!(q-\mu_0-2m)![\frac{1}{2}(\lambda_0+\mu_0+2)+2m]!}. \tag{41}$$

An SU(3)-recoupling transformation converts this reduced matrix element for the space of the relative coordinate $\mathbf{r}$ to the full two particle space. If the two- particle states are specified by two-particle relative motion functions $(q0)$ coupled with two-particle center of mass motion functions $(Q\,0)$ to resultant $(\lambda\mu)$, then the two-particle reduced matrix elements are

$$\langle[(\bar{q}0)\times(Q\,0)](\bar{\lambda}\bar{\mu})\| V^{(\lambda_0\mu_0)}\|[(q0)\times(Q\,0)](\lambda\mu)\rangle_{\rho_0} = U((\bar{\lambda}\bar{\mu})(0Q)(\lambda_0\mu_0)(0q);(\bar{q}0);\_\,\_;(\mu\lambda)\_\rho_0) \times(-1)^{\lambda+\mu+\bar{\lambda}+\bar{\mu}}\left[\frac{\dim(\bar{\lambda}\bar{\mu})}{\dim(\bar{q}0)}\right]^{1/2}\langle(\bar{q}0)\| V^{(\lambda_0\mu_0)}\|(q0)\rangle. \tag{42}$$

Here the U coefficient is an SU(3) Racah coefficient in unitary form.[1,11] For some applications it may be important to make a Talmi–Moshinsky–Brody transformation from the two-particle relative and center of mass motion basis to a two-particle basis expressed in terms of single-particle oscillator functions $\phi^{(q_10)}_{\cdots}(\mathbf{r}_1)$, $\phi^{(q_20)}_{\cdots}(\mathbf{r}_2)$. In the SU(3)-coupled basis the needed transformation coefficients from the $[(q0)\times(Q\,0)](\lambda\mu)$ to the $[(q_10)\times(q_20)](\lambda\mu)$ basis are simple SU(2) d-functions [see Eq. (4.1.15) of Ref. 19 for our phase convention], and

$$\langle[(\bar{q}_10)\times(\bar{q}_20)](\bar{\lambda}\bar{\mu})\| V^{(\lambda_0\mu_0)}\|[(q_10)\times(q_20)](\lambda\mu)\rangle_{\rho_0} = \sum_{\bar{q},q,Q} d^{\bar{\lambda}/2}_{(\bar{q}_1-\bar{q}_2)/2,(\bar{q}-Q)/2}(\tfrac{1}{2}\pi) d^{\lambda/2}_{(q_1-q_2)/2,(q-Q)/2}(\tfrac{1}{2}\pi) \times\langle[(\bar{q}0)\times(Q\,0)](\bar{\lambda}\bar{\mu})\| V^{(\lambda_0\mu_0)}\|[(q0)\times(Q\,0)](\lambda\mu)\rangle_{\rho_0}. \tag{43}$$

With Eqs. (41), (42), and (43) the full many-particle expression for the full interaction can then be expanded in terms of the SU(3) reduced matrix elements by

$$V = -\tfrac{1}{2}\sum_{\bar{q}_1\bar{q}_2 q_1 q_2}\sum_{(\bar{\lambda}\bar{\mu})(\lambda\mu)}\sum_{(\lambda_0\mu_0)\rho_0}\sum_{ST}[(2S+1)(2T+1)]^{1/2} \times\langle[(\bar{q}_10)\times(\bar{q}_20)](\bar{\lambda}\bar{\mu})\| V^{(\lambda_0\mu_0)}_{ST}\|[(q_10)\times(q_20)](\lambda\mu)\rangle_{\rho_0} \times[[a^+_{(\bar{q}_10)}\times a^+_{(\bar{q}_20)}]^{(\bar{\lambda}\bar{\mu})ST} \times[a_{(0q_1)}\times a_{(0q_2)}]^{(\mu\lambda)ST}]^{(\lambda_0\mu_0)\rho_0 S_0=0\,T_0=0}_{L_0=0}, \tag{44}$$

where the square brackets now denote both SU(3) and spin and isospin coupling, and where $a^+_{(q_i0)l_im_im_{s_i}m_{t_i}}$ $(a_{(0q_i)l_im_im_{s_i}m_{t_i}})$ are single particle creation (annihilation) operators for a particle in the q_ith oscillator shell.

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