A semi-Euclidean approach to boson–fermion model theories*

David Brydges†

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104

Paul Federbush‡§

Institute for Advanced Study, Princeton, New Jersey 08540
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A formulation is presented for the study of semiboundedness of coupled boson–fermion model field theories. Euclidean–boson fields and ordinary fermion fields are employed. Expansion steps used to derive estimates are presented.

I. INTRODUCTION

There is a great deal of interest at present in discovering techniques for treating boson—fermion model field theories parallel to the use of Euclidean boson fields in studying purely boson models. We address ourselves here solely to the question of semiboundedness of the energy (as the first problem usually encountered for any model) although there is no reason to exclude further applications of the machinery discussed. There are three superrenormalizable models available, Y_2, Y_3, and the generalized Yukawa model in one space dimension (hereafter called GY_2). The treatment of Y_2 and GY_2 is in some sense just practice for the tackling of Y_3. Four-dimensional theories so far appear impregnable.

Glimm obtained semiboundedness of the energy for Y_2 in Ref. 1. Schrader extended this result to show the linear dependence of the bound on the volume.[2] One of the authors showed the semiboundedness of the GY_2 energy.[3] There are studies under way attempting to study boson—fermion field theory models by eliminating the fermi fields initially, using the closed form expression involving a Fredholm determinant, similar to the corresponding expression in the variational approach to field theory.[4,5] Here we continue the development initiated in Ref. 3. A unified treatment of Y_2 and GY_2 is obtained,[6] whose basic line is here presented. Whether these methods, or the methods in Ref. 5, will be successful in studying Y_3 must be decided in the future. Other paths of evolution, or unifications, cannot be excluded, such as the work of Gross.[7] We are enthusiastic about the usefulness of the present program since it captures for boson—fermion models analogs of all the techniques used by Glimm and Jaffe in obtaining semiboundedness for ϕ_3^4, including localization.[8]

II. FEYNMAN-KAC FORMULA

Any Hamiltonian we consider is of the form

$$H = H_{0B} + H_{0F} + G(\phi) + \int dx \int dy\, Q(x,y,\phi)\,\bar{\psi}(x)\,\psi(y) \quad (1)$$

$$= H_{0B} + K_F. \quad (2)$$

There are volume and momentum cutoffs in the interaction and renormalization terms in the $G(\phi)$. Subscripts F will often denote expressions in terms of Foch space operators. Using the Trotter product formula, we have

$$\langle 0|\exp(-HT)|0\rangle_F = \lim_{n\to\infty}\langle 0|\,[\exp(-H_{0B}T/n)\exp(-K_F T/n)]^n\,|0\rangle_F. \quad (3)$$

$|0\rangle_F$ denotes the Foch vacuum. We introduce a total Hilbert space $\mathcal{H}$, the tensor product of Euclidean boson Hilbert space $\mathcal{H}_{EB}$ and $\mathcal{H}_F$ the Foch fermi Hilbert space

$$\mathcal{H} = \mathcal{H}_{EB} \times \mathcal{H}_F \quad (4)$$

and Euclidean boson fields $\phi(x,t)$. We also introduce dummy variables into the Fermi fields

$$\psi(x,t) = \psi(x). \quad (5)$$

These dummy t variables will only be used to define a time ordering operation. (One may alternatively say we are developing a Euclidean Fermi field theory—translation invariant but not rotation invariant—with the zero operator generating time translations. The Fermi kinetic energy terms are included in the interaction; in form, they and the interaction terms do not appreciably differ. This may be contrasted with the boson situation where the energy contains π's and the interaction does not.) K_F is replaced by $K(t)$ by substituting the time dependent fields for the $(t=0)$ Foch field:

$$K_F = H_{0F}(\bar{\psi},\psi) + G(\phi) + \int dx \int dy\, Q(x,y,\phi)\,\bar{\psi}(x)\,\psi(y)$$

$$K(t) = H_{0F}(\bar{\psi}(t),\psi(t)) + G(\phi(t)) + \int dx \int dy\, Q(x,y,\phi(t))\,\bar{\psi}(x,t)\,\psi(y,t). \quad (6)$$

Equation (3) becomes

$$\langle 0|\exp(-HT)|0\rangle_F = T\langle 0|\exp[-\textstyle\int_0^T K(t)\,dt]|0\rangle. \quad (7)$$

Here $|0\rangle$ is the vector in $\mathcal{H}$ that is the product of the boson Euclidean space vacuum with the Fermi Foch space vacuum. T indicates a time ordering in the t variables in the $\phi(x,t)$ and $\psi(x,t)$. All of our efforts are directed to finding techniques for estimating the right side of Eq. (7).

III. THE DUHAMEL EXPANSION

The process we have for removing parts of the exponent is the Duhamel expansion. We decompose $K(t)$ into two parts:

$$K(t) = K_0(t), \quad K(t) = K_i(t) + R_i(t), \quad i = 1, 2, \ldots, \quad (8)$$

where $K_i(t)$ and $R_i(t)$ are functions of $\psi(t)$, $\bar{\psi}(t)$, and $\phi(t)$ $\phi(t)$—all the fields at the fixed time t. Often in applications the $K_i(t)$ and $R_i(t)$ are picked to have no explicit time dependence. The Duhamel expansion assumes the form

$$T\langle 0|\exp[-\textstyle\int_0^T K(t)\,dt]R|0\rangle$$

$$= \sum_0^\infty (-1)^n \int_0^T dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1$$

$$\times T\langle 0|\prod_{i=1}^{n} R_i(t_i)\exp\left[-\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} K_j(s)\,ds\right] R|0\rangle, \tag{9}$$

where $t_0=0$ and $t_{n+1}=T$. [The $K(t)$ in (8) and (9) is not necessarily the same as in (6), but may be a similar expression such as one of the $K_i(t)$ arising in an inductive procedure.] An example of this expansion for $P(\phi)_2$ is found in Ref. 9 and for GY_2 in Ref. 3.

If space–time is divided into regions and a separate Duhamel expansion is developed for the interaction in each region, then the different Duhamel expansions can be combined into a sum of single Duhamel expansions such as (9). This is a primary device for localization.

IV. THE PULL THROUGH EXPANSION

The "pull through" operation was introduced in Ref. 10. Like the Duhamel expansion it is purely algebraic and applies alike to Fermi and boson fields. An operator in some $R_i(t_i)$ in (9) is decomposed into creation and annihilation operators which are "pulled through" until they either annihilate on the vacuum, contract on the exponent, contract on some other $R_j(t_j)$, or until the operator being pulled through has moved far enough to collect some desirable time factor, $\exp(-ut)$, and then stopped. This last operation is not used for bosons. It is possible to iteratively use pull through operations and Duhamel expansions, to generate an inductive procedure.

After any number of applications of the two operations above one has an expression

$$\langle 0|\exp(-HT)|0\rangle_F=\sum_\alpha T_\alpha$$

where a typical term T_α has the form

$$T_\alpha=\int_0^T dt_r\int_0^{t_r} dt_{r-1}\cdots\int_0^{t_2} dt_1 T\langle 0|R_\alpha\exp(-K_\alpha)|0\rangle \tag{10}$$

with

$$K_\alpha=\sum_{j=0}^{r}\int_{t_j}^{t_{j+1}} K_j(s)\,ds \tag{11}$$

and

$$R_\alpha=\int dx_1\cdots dx_s\int dy_1\cdots dy_s\,\bar\psi(x_1,t_{\alpha(1)})\cdots\bar\psi(x_s,t_{\alpha(s)})$$
$$\times\cdots\psi(y_s,t_{\alpha(2s)})\,Q_\alpha(x_1,\ldots,x_s,y_1,\ldots,y_s,\phi,t). \tag{12}$$

Summation over Fermion field indices is always implicit. As many of the time integrations as possible are included in Q_α. Time variables arising from contractions from the exponent in which both fermions are contracted fall in this category. Combination of terms in T_α is also advantageous; in the pull through procedure it is possible to construct time and space locally averaged boson fields as in Ref. 8 which would appear in Eq. (12).

V. ESTIMATES AND DEFERMIATION

When the algebraic operations of the last two sections are completed, estimates are required for $K_j(s)$ and R_α. Assume estimates of the form

$$K_j(s)\geq C_j(\phi,s) \tag{13}$$

and

$$|Q_\alpha(x,y,\ldots)|_1\leq d_\alpha(\phi,t_1,\ldots,t_r), \tag{14}$$

where $|f(x_1,\ldots,x_n)|_1$ is the inf of $\sum_1^\infty|a_i|$ over a_i satisfying

$$f=\sum_i a_i\prod_{j=1}^{n} g_{ij}(x_j)$$

with $|g_{ij}(x)|_2=1$. The "defermiation" step is then the estimate:

$$\langle 0|\exp(-HT)|0\rangle_F\leq\sum_\alpha|T_\alpha|, \tag{15}$$

$$|T_\alpha|\leq\int_0^T dt_r\cdots\int_0^{t_2} dt_1\langle 0|d_\alpha(\phi)$$
$$\times\exp-\sum_{j=0}^{r}\int_{t_j}^{t_{j+1}} C_j(\phi,s)\,ds\,|0\rangle. \tag{16}$$

In (16) only boson fields remain, and all the techniques for estimating such a purely boson expression are available. Unlike the algebraic operations discussed above the defermiation can be performed just once in the procedure, it is a decisive step.

In Ref. 3 is an illustrative use of a Duhamel expansion, pull throughs, the estimates of Eq. (13), defermiation, and estimation of Eq. (16). There is one important technical improvement here over Ref. 3, the use of $\|_1$ estimates for Q_α. The estimate procedure in Ref. 3 is adequate to obtain semiboundedness for Y_2 or GY_2 in a finite volume, but yields an incorrect volume dependence. The present procedure behaves correctly under localization and therefore is the correct one to use for obtaining the volume dependence and attempting Y_3.

The statement that the estimates behaves correctly under localization is easiest to explain in the case when all the K_i contain only the fermion kinetic energies (a heuristic example). Then in estimates (15) and (16) the terms involving only contractions between operators lying in the same space–time squares contribute to the sum $\sum_\alpha|T_\alpha|$ an expression of the form

$$\sum_{\text{nonoverlap}}|T_\alpha|\leq\langle 0|\prod_\Delta D_\Delta(\phi)|0\rangle, \tag{17}$$

where $D_\Delta(\phi)$ are corresponding estimates for the squares Δ. When the K_i contain other than just energies the localization property imposes conditions on the form of estimate (13)—the right side must be a sum of adequate estimates for the individual squares cut at $t=s$. Localization methods as used in Ref. 2 are valuable to achieve this.

VI. DISCUSSION

We say a few words about the treatment of Y_2 and GY_2. In these models in each unit space time block the Duhamel expansion may be performed just once—no induction is necessary. The interaction terms are included in the K_i with an upper momentum cutoff on the fermions increasing with i. (Alternate developments are possible.) The pull throughs are used to exhibit the renormalization cancellation. Additional pull throughs are required also; those for GY_2 are slightly different from those in Ref. 3 since $\|_1$ estimates are used. In particular each vertex (basic interaction term not in the exponent) must be connected to at least one other vertex by a fermion line, however the number of contractions is to be limited. In any expansion in which no fermion operators other than the kinetic energy appear in the exponents, the whole procedure could have been performed using Osterwalder–Schrader fields.[4]

It is interesting to consider what special properties of fermions are used in the above program. One could have derived the same formulas for a boson ψ field, except Eq. (14). The fermion nature has been used so far in three ways (two of these ways only implicit in this paper):

(1) To derive Eq. (14) one has used that $|\psi(f)| \leq |f|_2$.

(2) To derive in Eq. (13) a useful estimate for the Y_2 or GY_2 scattering terms the free $N_{0F}^{-1/2}$ factor in N_τ estimates with fermions is useful.

(3) To derive in Eq. (13) a useful estimate for the Y_2 or GY_2 creation and annihilation terms, employing as in Ref. 1 a partial dressing for the fermions (see Ref. 3), the sign of a term arising from the anticommutativity is crucial. This sign is available in other models and other dressings.

Possibly to treat Eq. (13) for Y_3 more properties will be discovered, though this may not be necessary. In any case the exchange of boson commutativity for these three properties, a three for one deal, may not be a bad trade.

We feel that the approach of this paper provides sufficiently powerful machinery to consider an attack on the Y_3 problem and that it may be as close as one can come to realizing for fermions a Euclidean formulation for performing estimates.

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†Michigan Junior Fellow.
‡John Simon Guggenheim Memorial Fellow.
§Permanent address: Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104.

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