ENGINEERING RESEARCH INSTITUTE
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A STUDY OF RECIRCULATION AS A FREQUENCY-MEMORY DEVICE

Technical Report No. 46
Electronic Defense Group
Department of Electrical Engineering

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Project 2262

TASK ORDER NO. EDG-1
CONTRACT NO. DA-36-039 sc-63203
SIGNAL CORPS, DEPARTMENT OF THE ARMY
DEPARTMENT OF ARMY PROJECT NO. 3-99-04-042
SIGNAL CORPS PROJECT NO. 194B

February, 1955

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ABSTRACT

The output spectrum of a device which stores frequency information by a recirculation technique is analyzed in detail. The error in output frequency is shown to depend upon the frequency of the input signal and to reach a maximum value of $1/2\tau$ cycles, where is the delay time in seconds. The frequency error decreases to a minimum value of zero when the product of input angular frequency and delay time is a multiple of 2π.

If the spectral efficiency is defined as the percentage of the total power that resides at the peak of the output spectrum, this efficiency increases with the number of recirculations. With a given number of recirculations the spectral efficiency varies with the input frequency and reaches a maximum of $(n\tau/T)^2$ when the product of input angular frequency and delay time is an integral multiple of 2π. Here, n is equal to the number of recirculations plus one and T is the reciprocal of repetition rate.

A STUDY OF RECIRCULATION AS A FREQUENCY-MEMORY DEVICE

1. INTRODUCTION

This paper deals with a method of storing frequency information utilizing a recirculation technique to make available an output signal for a longer period of time than that allowed for observation of the input frequency. The error that is introduced in the output frequency of the memory device and the spectral efficiency of the output are both analyzed in some detail. The error in the frequency of the output arises from the fact that the peak in the output spectrum is not necessarily at the frequency of the input signal. For the purposes of this report, the spectral efficiency is defined as the percentage of the power residing in the peak of the output spectrum.

One way of achieving this method of frequency storage is shown in the block diagram of Figure 1.

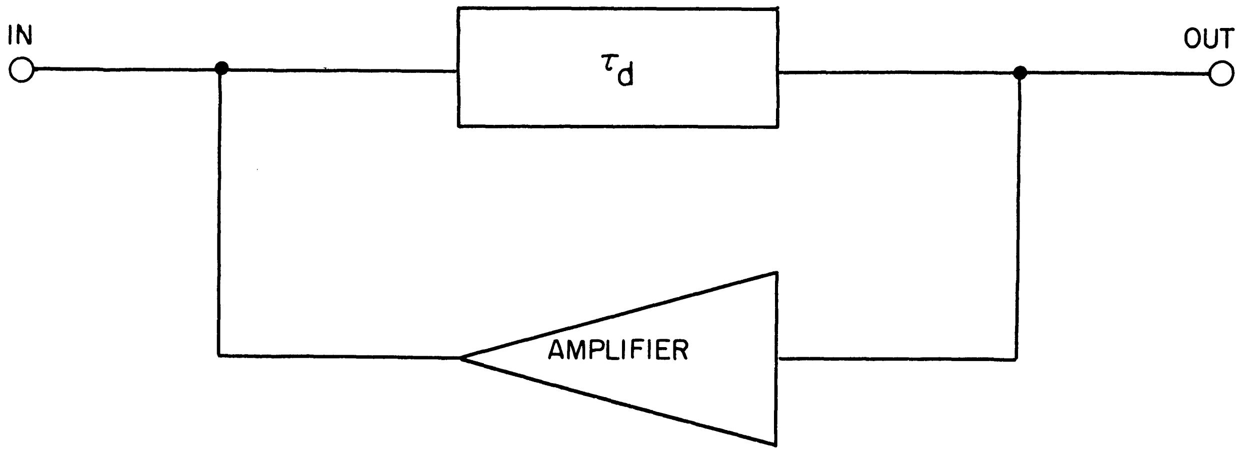

FIG. I. SIMPLE RECIRCULATION SYSTEM.

Here, the input signal is fed into a delay line which has a delay of τ seconds; the output of the delay is divided so that half of the signal goes to the output terminal and the other half is fed into the input of the recirculation amplifier. The duration of the input signal is assumed to be equal to or less than the delay time. The output of the recirculation amplifier feeds the delayed signal back to the input of the delay line. The gain of the recirculation amplifier is such that the delayed signal has the same amplitude as the original signal. Hence, the input signal is recirculated through the delay for a number of times, this number being determined by the length of time that the recirculation amplifier is turned on. With this system, the output of the recirculation device will be continually available for a length of time equal to the duration of the "on-period" of the recirculation amplifier. The amplitude of the output will remain constant between recirculations. In order to determine the effect of recirculation on the spectrum of the input pulse, the spectrum of the output pulse with recirculation is analyzed below.

2. MATHEMATICAL INVESTIGATION OF THE OUTPUT OF A RECIRCULATION SYSTEM

For the purposes of this investigation the delay time is assumed equal to the duration of the input signal; the recirculation amplifier is assumed to have negligible delay and to have a gain equal to the attenuation in the delay. The waveform at the output of the recirculation system is assumed to have a constant frequency with time, but the phase is assumed to change with each recirculation by an amount that depends upon the input frequency and the delay time.

The equation of the output voltage can be written as

$$\begin{aligned} f(t) &= E_m \sin \omega_o (t-\tau) \quad , \quad \tau \le t \le 2\tau \\ &= E_m \sin \omega_o (t-2\tau) \quad , \quad 2\tau \le t \le 3\tau \\ &\vdots \\ &= E_m \sin \omega_o (t-n\tau) \quad , \quad n\tau \le t \le (n+1)\tau . \end{aligned} \tag{1}$$

Here τ is the delay time and ω_o is the input angular frequency. The envelope of the output spectrum can be obtained from the Fourier Transform of this time function.

$$g(\omega) = \frac{E_m}{2\pi} \sum_{K=1}^{n} \int_{K\tau}^{(K+1)\tau} \sin \omega_o (t-K\tau) e^{-j\omega t} dt \tag{2}$$

If r is the number of recirculations, then n = r+1. The result of integrating Equation 2 (see Appendix A), substituting the limits, and simplifying, is

$$g(\omega) = - \frac{E_m}{2\pi(\omega_o^2-\omega^2)} \sum_{K=1}^{n} (e^{-jK\omega\tau}) \left[e^{-j\omega\tau} (\omega_o \cos \omega_o\tau + j\omega \sin \omega_o\tau) -\omega_o \right] \tag{3}$$

Since the expression in brackets does not contain K, it can be factored out of the series. Summing this series (see Appendix B), we have

$$\sum_{K=1}^{n} e^{-jK\omega\tau} = e^{-j(n+1)\frac{\omega\tau}{2}} \frac{\sin(\frac{n\omega\tau}{2})}{\sin(\frac{\omega\tau}{2})} \tag{4}$$

When this result is substituted into Equation 3, Equation 5 is obtained for the envelope of the output spectrum.

$$g(\omega) = -\frac{1}{2\pi}\frac{E_m}{(\omega_o{}^2 - \omega^2)} \; e^{-j(n+1)\frac{\omega\tau}{2}} \; \frac{\sin\left(\frac{n\omega\tau}{2}\right)}{\sin\left(\frac{\omega\tau}{2}\right)} \left[e^{-j\omega\tau}(\omega_o \cos\omega_o\tau + j\omega\sin\omega_o\tau) - \omega_o\right] \tag{5}$$

It can be seen that Equation 5 involves a factor times the envelope of the spectrum of the input pulse to the system,

$$g_o(\omega) = -\frac{E_m}{2\pi(\omega_o{}^2 - \omega^2)}\left[e^{-j\omega\tau}(\omega_o\cos\omega_o\tau + j\omega\sin\omega_o\tau) - \omega_o\right] \tag{6}$$

Hence, the envelope of the spectrum of the output of the recirculation system can be written as

$$g(\omega) = \frac{\sin\left(\frac{n\omega\tau}{2}\right)}{\sin\left(\frac{\omega\tau}{2}\right)} \; e^{-j(n+1)\frac{\omega\tau}{2}} \; g_o(\omega) \tag{7}$$

where $g_o(\omega)$ is the envelope of the input pulse spectrum.

2.1 Result of Recirculation

Thus, the effect of recirculation of a pulse signal is to modify the spectrum of the input pulse by the factor

$$R(\omega\tau) = \frac{\sin\left(\frac{n\omega\tau}{2}\right)}{\sin\left(\frac{\omega\tau}{2}\right)} \; e^{-j(n+1)\frac{\omega\tau}{2}} \tag{8}$$

From an examination of $R(\omega\tau)$ it can be seen that its character is not influenced by the frequency of the input pulse, and also that the peaks of its amplitude occur when $\omega\tau$ is an integral number of 2π's. Hence, for a 1 microsecond delay, peaks in $R(\omega\tau)$ occur every megacycle; for a 10 microsecond delay, peaks occur every 100 kc, etc. The height of these peaks is determined by n, which is equal to one plus the number of recirculations, and the width of the peaks is inversely proportional to n. $R(\omega\tau)$ is plotted in Figure 2 for a specific case where the delay time is 1 microsecond.

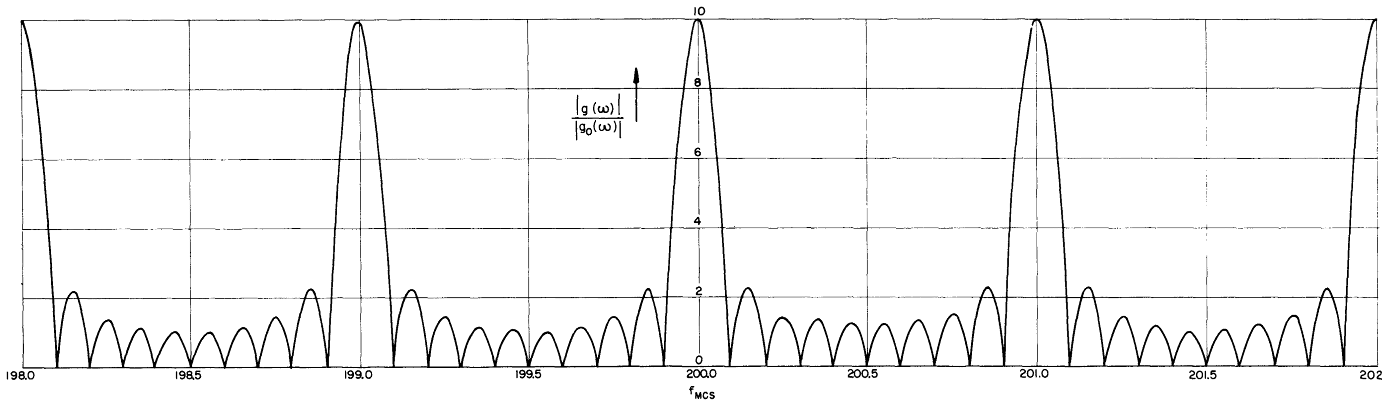

FIG. 2 SCANNING FUNCTION : K = 9 RE-CIRCULATIONS, τ = 1μ SEC.

2.2 Prediction of the Envelope of the Output Spectrum Using a Scanning Technique

Calculation of the envelope of the output spectrum from Equation 5 is a rather tedious procedure if the input frequency is allowed to vary. However, this calculation can be simplified to some degree by making use of the fact that $R(\omega\tau)$ is independent of the frequency of the input, and that it is one factor of the output spectrum. The other factor in the output spectrum is the spectrum of the input pulse. If these two functions are plotted on separate sheets and one is used as a overlay, the shape of the output spectrum can be determined by inspection. Another graphic method of finding the shape of the output spectrum is to plot the two functions one above the other and determine their product point by point (see Figs. 3, 4, and 5). Figure 3 shows the spectrum for f_o = 200 mc and τ = 1 microsecond. Now, if ω_o changes, all that need be done to determine the change in the spectrum is to move the input envelope until it is centered at the new input frequency. Again, a point-by-point multiplication gives the output spectrum. This movement along the curve gives rise to the notion of "scanning technique". Figure 4 shows the output spectrum for f_o = 200.25 mc, while Fig. 5 is for an input frequency of 200.5 mc. For an input frequency which is a multiple of 1 megacycle (i.e., $\omega\tau = 2\pi$), the output will have a peak centered at the input frequency. For an input frequency which is an odd multiple of 1/2 megacycle, there will be zero output at the frequency of the input. Obviously, this recirculation system does not give the desired result for all input frequencies.

2.3 Experimental Verification of Calculated Results

A block diagram of the experimental set up is shown in Figure 6. The input signal is fed through a 200 mc amplifier, which is gated by the pulse generator. The gated signal is fed into a 1 microsecond delay line (649 feet of RG-58/U cable) and a portion of the delayed signal is returned to the input of the delay line. The recirculation amplifier is gated-on when the input amplifier is gated-off (shown in Figure 6). Another 200 mc amplifier is used in conjunction with a scope and a spectrum analyzer to observe the output signal. All of the amplifiers have a bandwidth of 10 mc. The gain of the recirculation amplifier was adjusted to be equal to the attenuation of the cable

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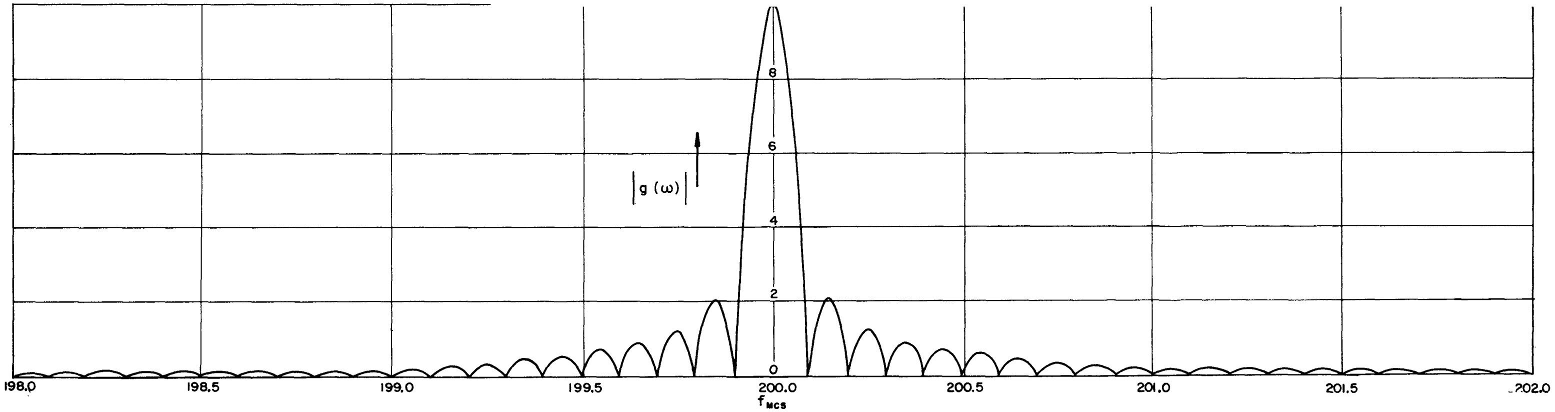
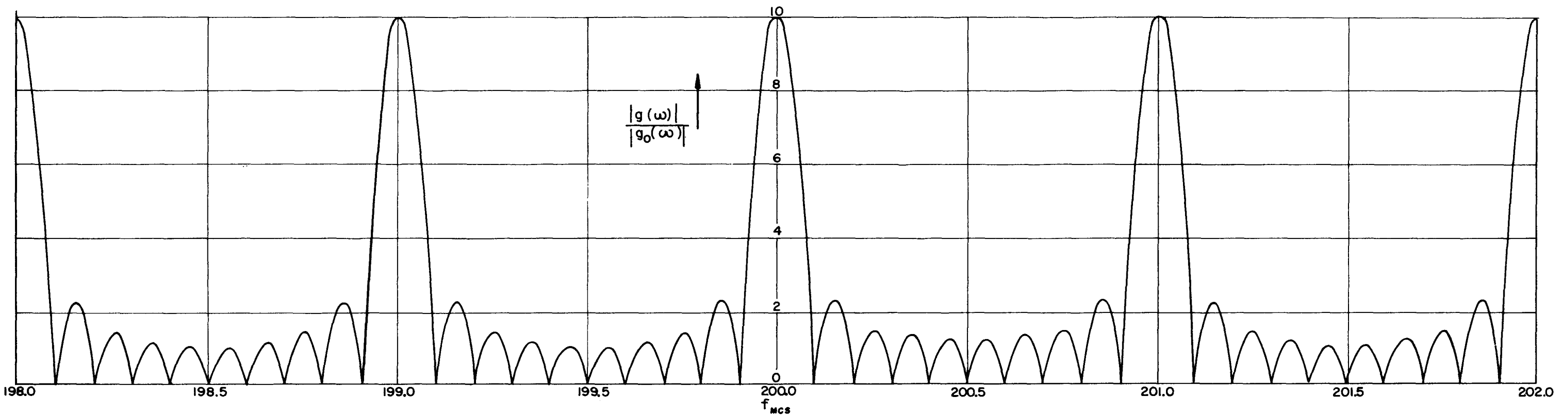
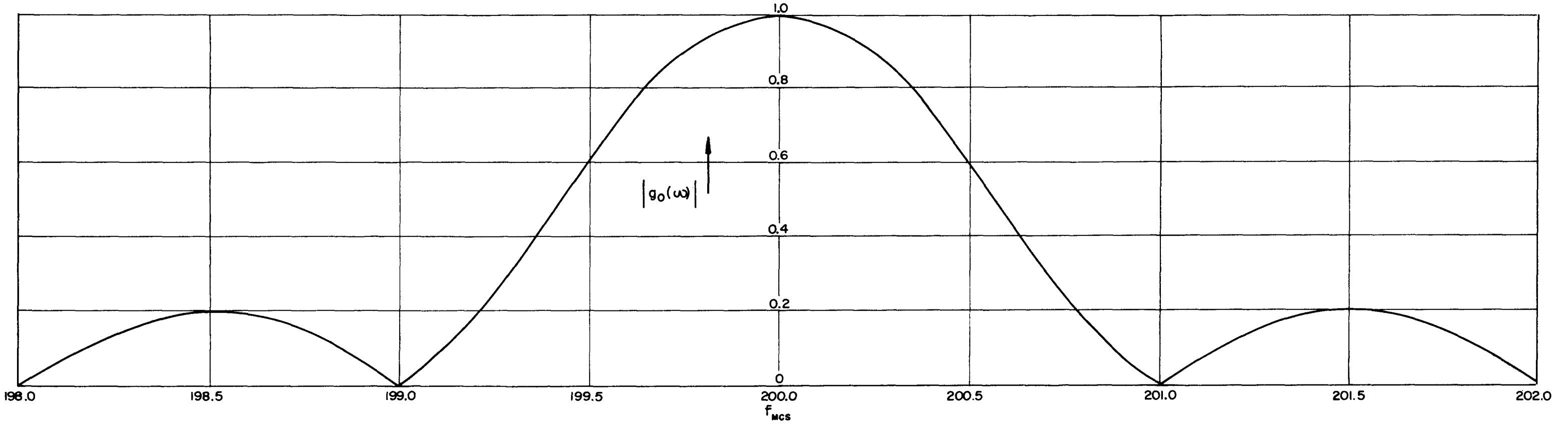


3 (a). FREQUENCY SPECTRUM OF OUTPUT.

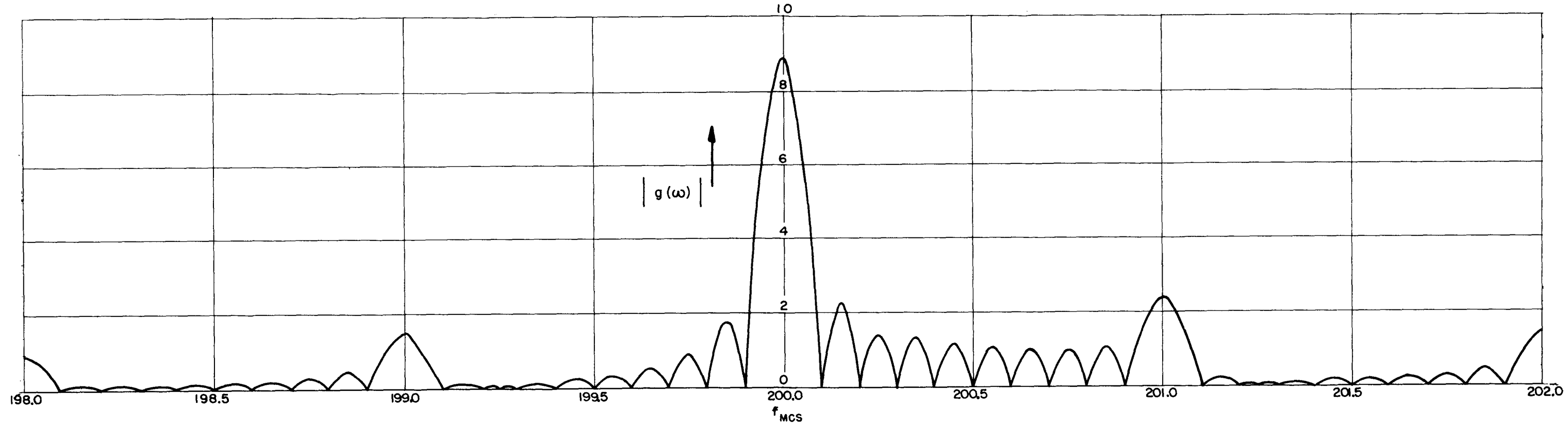


3 (b). SCANNING FUNCTION : K=9 RE-CIRCULATIONS, $\tau = 1\mu$ SEC.

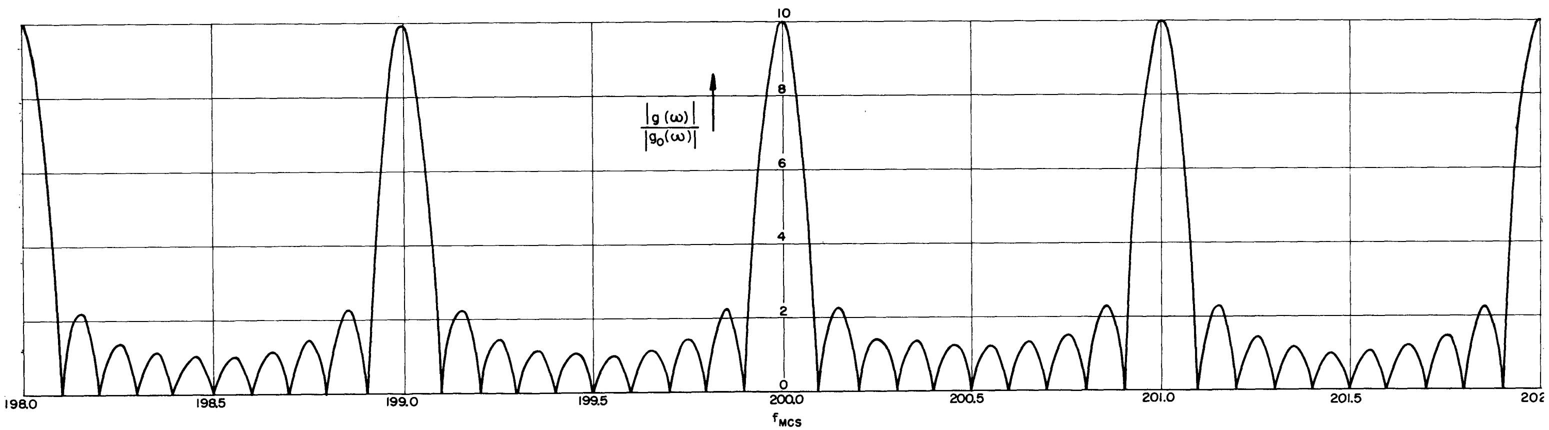


3 (c). FREQUENCY SPECTRUM OF INPUT PULSE : f_0 = 200 MCS, $\tau = 1\mu$ SEC.

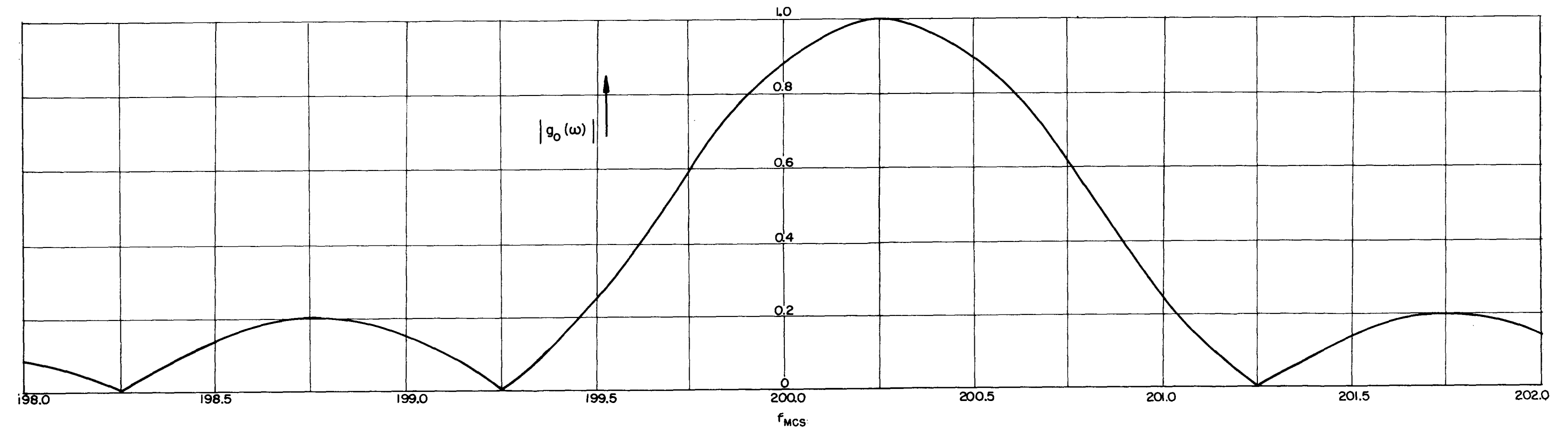
FIG. 3. DERIVATION OF OUTPUT FREQUENCY SPECTRUM : f_0 = 200MCS.



4 (a). FREQUENCY SPECTRUM OF OUTPUT.



4 (b). SCANNING FUNCTION : K = 9 RE-CIRCULATIONS, τ = 1μ SEC.



4 (c). FREQUENCY SPECTRUM OF INPUT PULSE : f_0 = 200.25 MCS, τ = 1μ SEC.

FIG. 4. DERIVATION OF OUTPUT FREQUENCY SPECTRUM : f_0 = 200.25 MCS.

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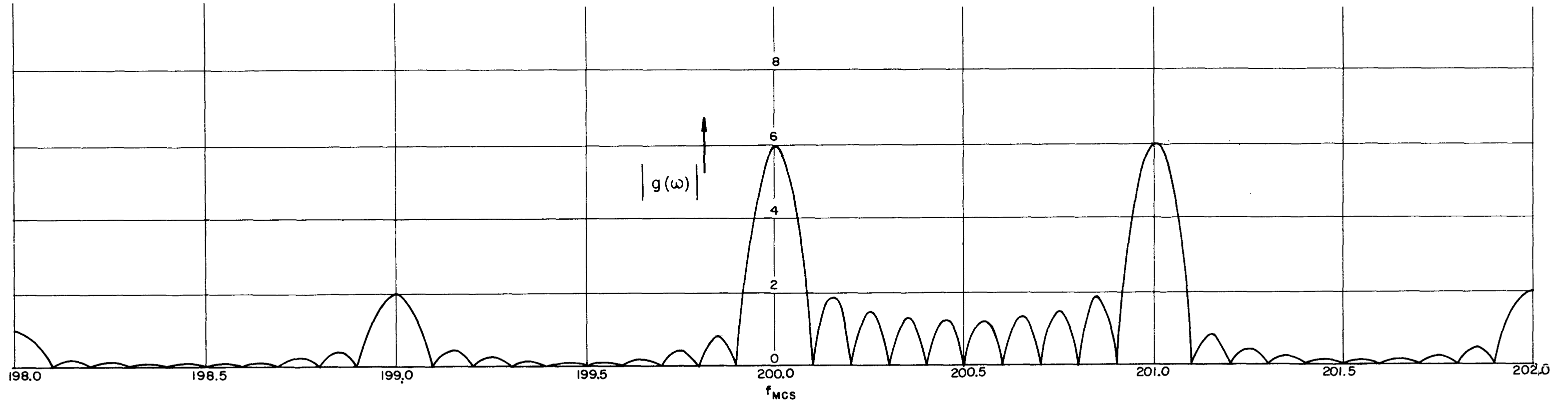
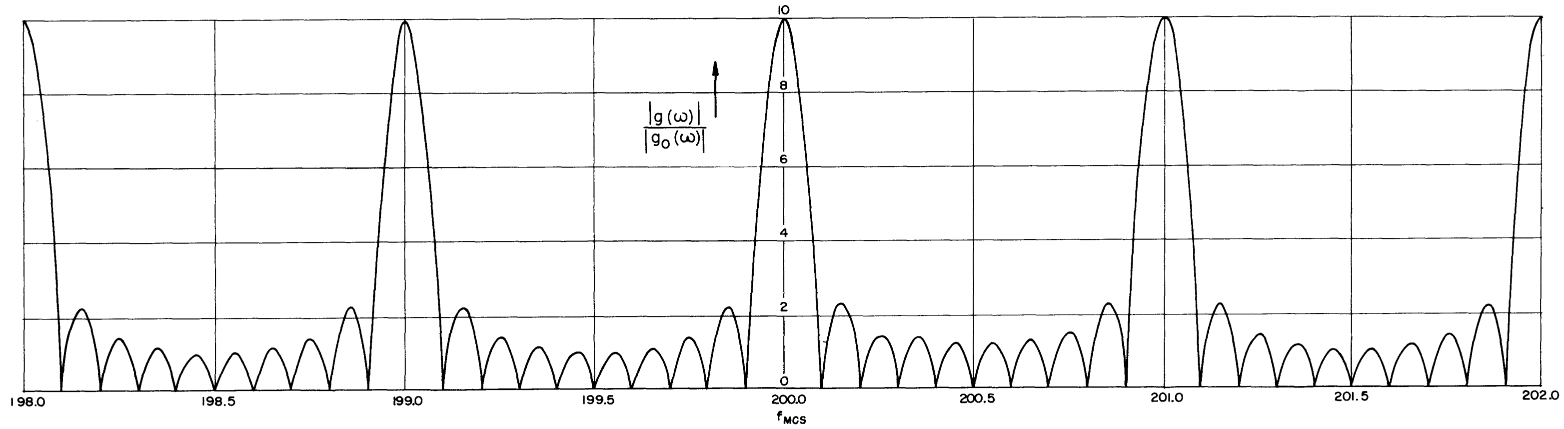
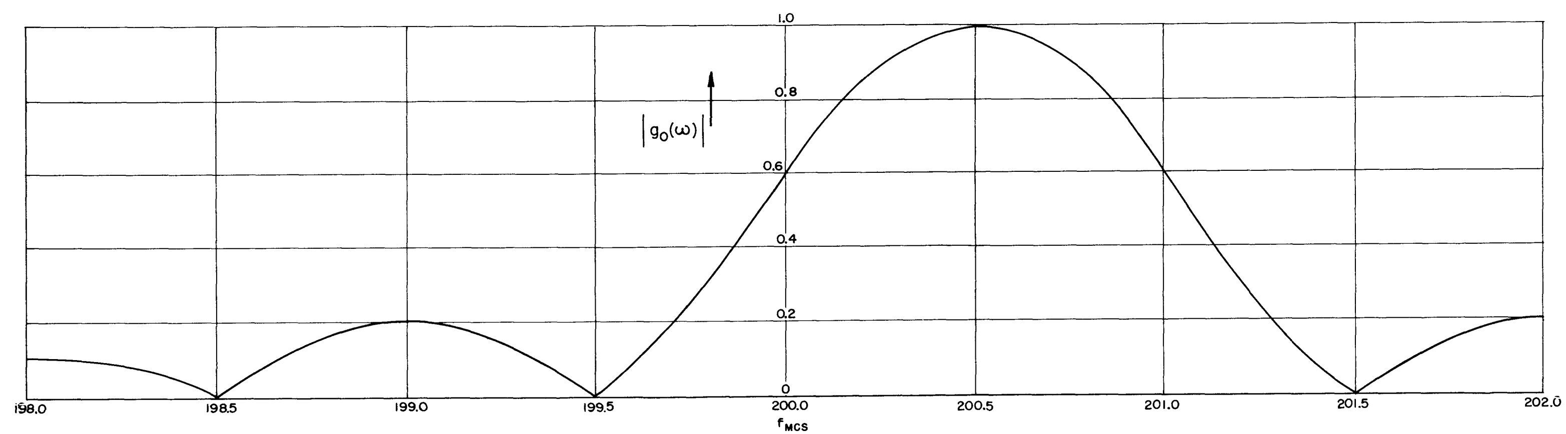


5 (a). FREQUENCY SPECTRUM OF OUTPUT.



5 (b). SCANNING FUNCTION: K = 9 RE-CIRCULATIONS, $\tau = 1\ \mu$ SEC.



5 (c). FREQUENCY SPECTRUM INPUT PULSE: f_0 = 200.5 MCS, $\tau = 1\ \mu$ SEC.

FIG. 5. DERIVATION OF OUTPUT FREQUENCY SPECTRUM : f_0 = 200.5 MCS.

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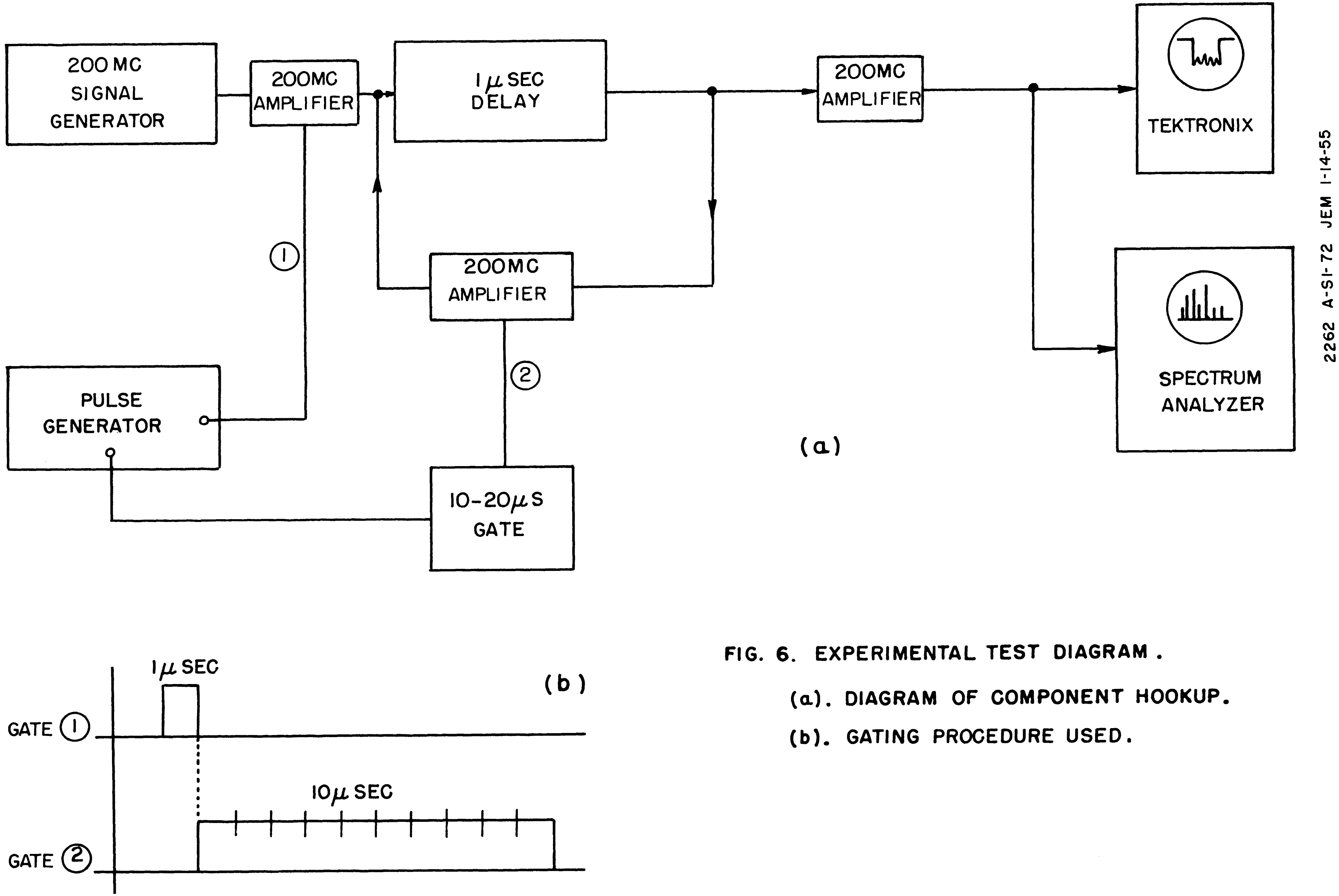
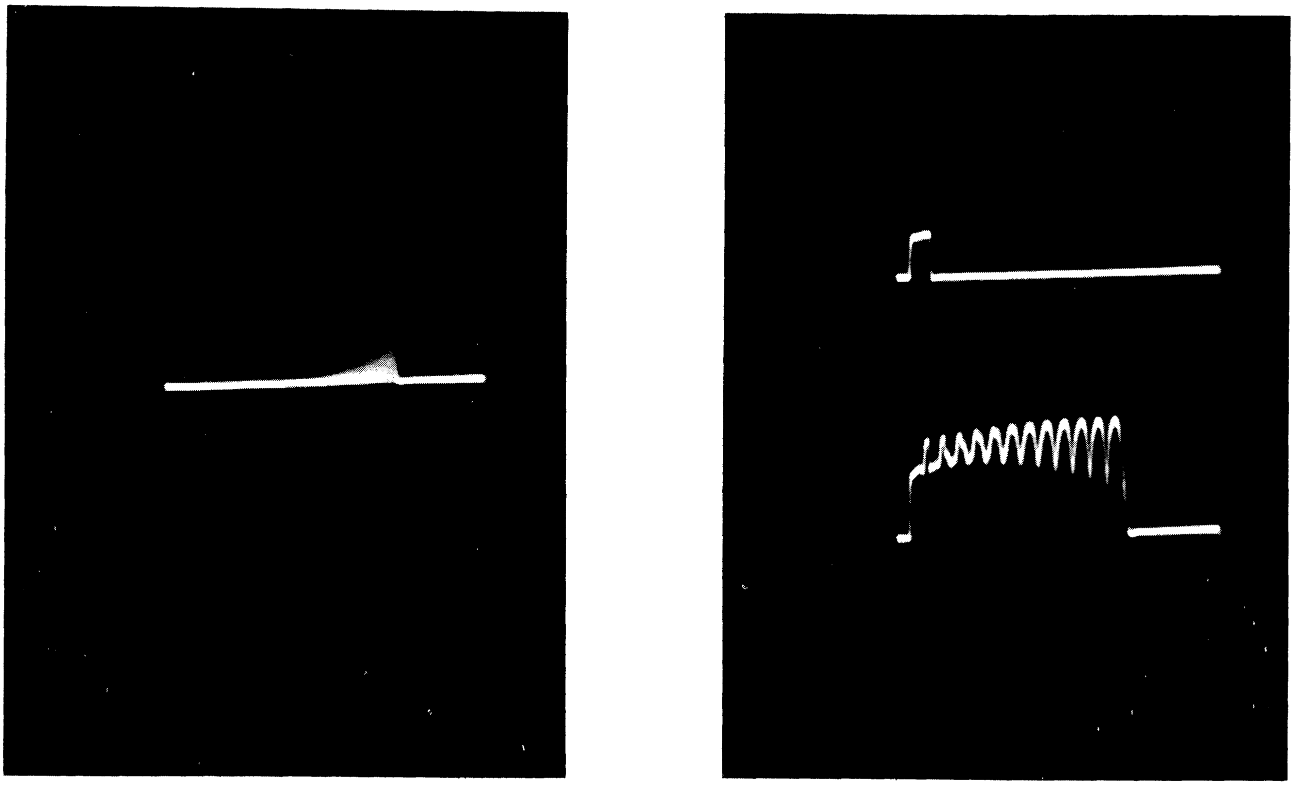


FIG. 6. EXPERIMENTAL TEST DIAGRAM.

(a). DIAGRAM OF COMPONENT HOOKUP.

(b). GATING PROCEDURE USED.

by observing the output vs. time on the oscilloscope. If the gain is greater than the loss, a noise buildup can be observed, as shown in Figure 7 (a). A 200 mc input signal was fed into the system and the output observed on the spectrum analyzer. The output spectrum corresponding to an input frequency of 200 mc is shown in Figure 8 (b). The input frequency differential between Figures 8 (a), (b), (c), and (d) is 0.5 mc. It can be seen that the experimental results bear out the mathematical calculations. For an input frequency of 200 mc, there is just one peak in the output spectrum centered at the input frequency. As the input frequency is changed by 0.5 mc, the output spectrum shows that the signal frequency drops to zero while there are sidebands of output 1 mc apart. It will be noticed that in Figures 8 (a) and (c) the peaks on either side of the input frequency are not of the same amplitude. In the mathematical analysis, these peaks were of the same height. This discrepancy can be explained by a consideration of the scanning technique. With the input frequency exactly at a 0.5 mc point, the amplitude of these peaks will be the same. However, if the frequency is slightly to one side of this point, it is obvious that these peaks will be of different amplitude. This is undoubtedly the case in Figures 8 (a) and (c). If the input frequency is again shifted by 0.5 mc (to 201 mc), then the output consists of an individual peak at the frequency of the input. This is predicted from the calculations. These pictures substantiate the deductions made on the basis of the mathematical analysis. For certain frequencies, the recirculation system will retain the information given it. These frequencies will be 200 mc $\pm$ n mc where n is an integer for a one microsecond delay. At other frequencies, the output will consist of sidebands on either side of the input frequency, while the input frequency is lost.



(a) (b)

FIG. 7. OUTPUT VS. TIME SCOPE DISPLAY.

(a) NOISE BUILDUP WHEN GAIN OF AMPLIFIER IS GREATER THAN ATTENUATION OF CABLE.

(b) OUTPUT OF RECIRCULATION SYSTEM. TOP PORTION IS OUTPUT PULSE WITHOUT RECIRCULATION. BOTTOM HALF SHOWS OUTPUT WITH RECIRCULATION.

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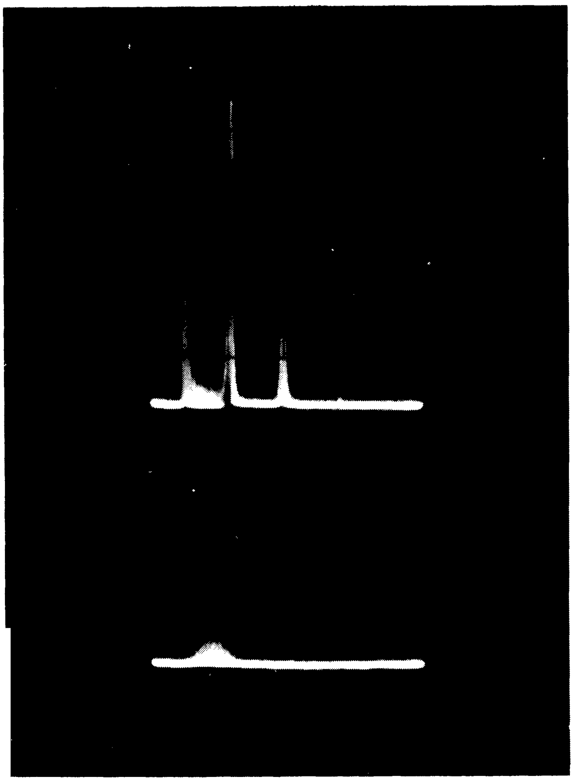
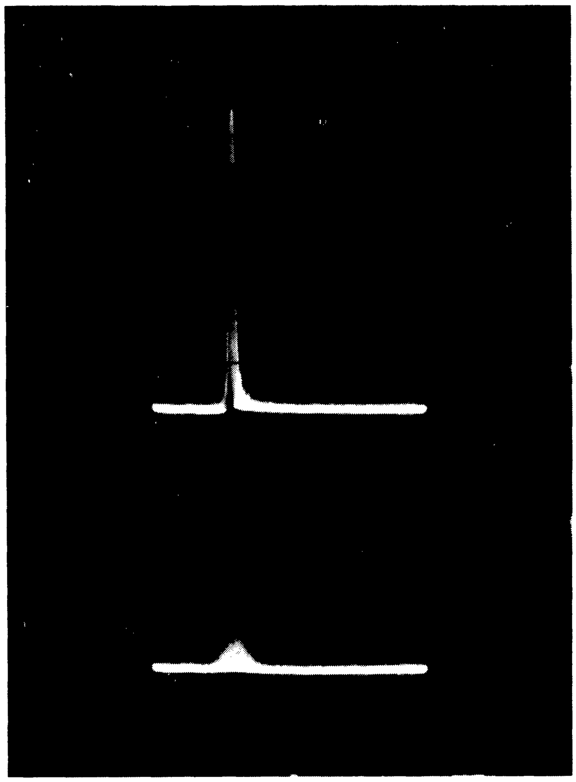
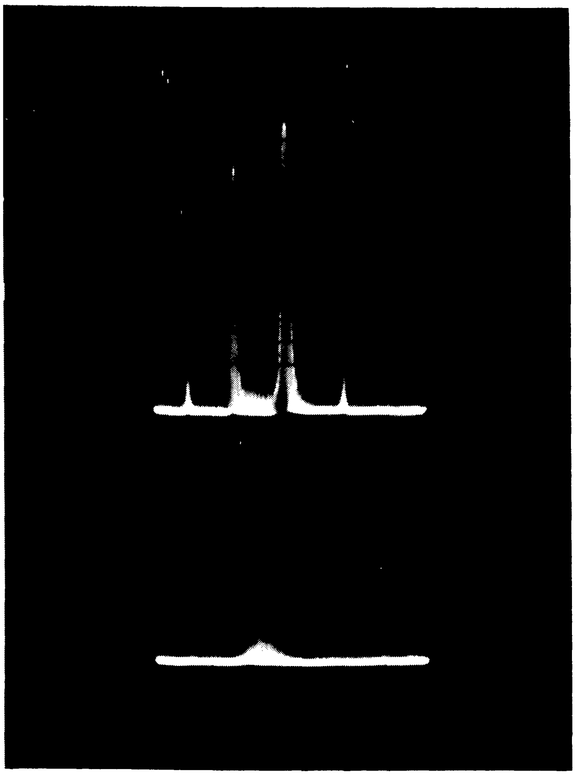


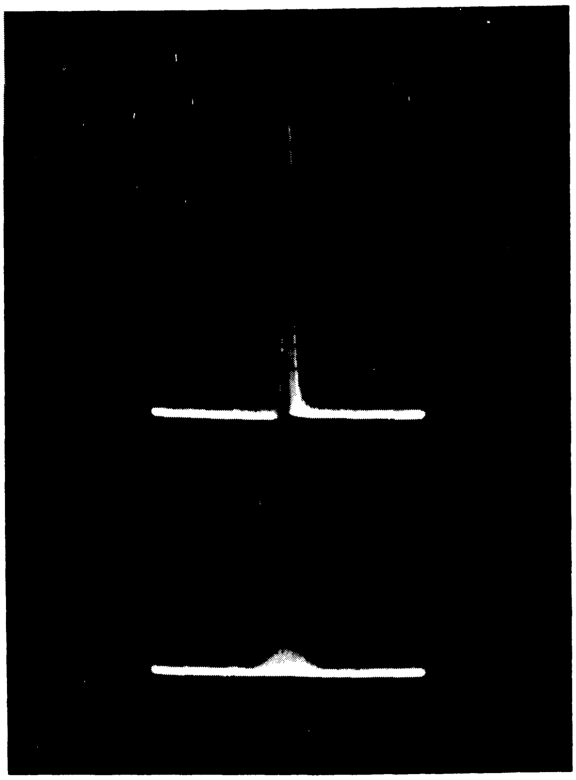
(a) 199.5 mc



(b) 200.0 mc



(c) 200.5 mc



(d) 201.0mc

FIG. 8. OUTPUT SPECTRUM OF RECIRCULATION SYSTEM.

BOTTOM PORTION OF EACH PICTURE IS OUTPUT SPECTRUM WITHOUT RECIRCULATION.
TOP PORTION IS SYSTEM SPECTRUM WITH RECIRCULATION.

3. CONCLUSIONS

The recirculation device described in this report will not remember the exact frequency introduced into it, except at certain frequencies where the product of delay time and input angular frequency is a multiple of 2π. If the error of the output is defined as the difference between the frequency of the input and the frequency at which the output amplitude is maximum, then the maximum error is $\frac{1}{2\tau}$ cps. This error arises from the fact that with some input frequencies the output consists of sidebands on either side of the input frequency (not necessarily centered at the input frequency), while the output at the input frequency is small and in some cases is zero. Hence, this memory device is not applicable to systems requiring very close tolerances on frequency. However, the tolerances in many systems would allow this device to be used to advantage.

The spectral efficiency, which is defined as the percentage of the total power output that is at the peak of the output spectrum, is increased by recirculation. However, the spectral efficiency will vary with the input frequency and will repeat when the product of input angular frequency and delay time is a multiple of 2π. It reaches a maximum of $(\frac{n\tau}{T})^2$, where T is the reciprocal of the repetition rate of the system.

APPENDIX A

Integration of Equation 2.

$$g(\omega) = \frac{E_m}{2\pi} \sum_{K=1}^{n} \int_{K\tau}^{(K+1)\tau} \sin \omega_o (t-K\tau) e^{-j\omega t} \, dt \qquad (2)$$

$$\text{Since} \quad \sin \omega_o(t-K\tau) = \frac{e^{j\omega_o(t-K\tau)} - e^{-j\omega_o(t-K\tau)}}{2j}$$

$$= \frac{E_m}{2\pi} \sum_{K=1}^{n} \left[\int_{K\tau}^{(K+1)\tau} \frac{e^{j\left[\omega_o(t-K\tau)-\omega t\right]}}{2j} dt - \int_{K\tau}^{(K+1)\tau} \frac{e^{-j\left[\omega_o(t-K\tau)+\omega t\right]}}{2j} dt \right] \qquad (A\text{-}1)$$

$$= \frac{E_m}{2\pi} \sum_{K=1}^{n} \left[\frac{e^{j\left[\omega_o(t-K\tau)-\omega t\right]}}{2j(j\left[\omega_o-\omega\right])} - \frac{e^{-j\left[\omega_o(t-K\tau)+\omega t\right]}}{2j(-j\left[\omega_o+\omega\right])} \right]_{K\tau}^{(K+1)\tau} \qquad (A\text{-}2)$$

$$= \frac{E_m}{2\pi} \sum_{K=1}^{n} \frac{e^{j\left\{\omega_o\left[(K+1)\tau-K\tau\right]-\omega(K+1)\tau\right\}} - e^{j\left[\omega_o(K\tau-K\tau)-\omega K\tau\right]}}{2j\left[j(\omega_o-\omega)\right]}$$

$$- \frac{e^{-j\left\{\omega_o\left[(K+1)\tau-K\tau\right] + \omega(K+1)\tau\right\}} - e^{-j\left[\omega_o(K\tau-K\tau)+\omega K\tau\right]}}{2j\left[-j(\omega_o+\omega)\right]} \qquad (A\text{-}3)$$

Simplifying the above equation results

$$g(\omega) = \frac{E_m}{2\pi} \sum_{K=1}^{n} \left[e^{-j\omega(K+1)\tau} \left\{ \frac{e^{j\omega_o\tau}}{2j\left[j(\omega_o-\omega)\right]} - \frac{e^{-j\omega_o\tau}}{2j\left[-j(\omega_o+\omega)\right]} \right\} \right.$$

$$\left. -e^{-j\omega K\tau} \left\{ \frac{1}{2j\left[j(\omega_o-\omega)\right]} - \frac{1}{2j\left[-j(\omega_o+\omega)\right]} \right\} \right] \qquad \text{(A-4)}$$

$$= \frac{E_m}{2\pi} \sum_{K=1}^{n} \left[e^{-j\omega(K+1)\tau} \left\{ \frac{-j(\omega_o+\omega)e^{j\omega_o\tau} \quad -j(\omega_o-\omega)e^{-j\omega_o\tau}}{2j(\omega_o^2-\omega^2)} \right\} \right.$$

$$\left. -e^{-j\omega K\tau} \left\{ \frac{-j(\omega_o+\omega) \quad -j(\omega_o-\omega)}{2j(\omega_o^2-\omega^2)} \right\} \right] \qquad \text{(A-5)}$$

Canceling out like terms leaves

$$g(\omega) = \frac{E_m}{2\pi} \sum_{K=1}^{n} \left[e^{-j\omega(K+1)\tau} \left\{ -\omega_o \left[\frac{e^{j\omega_o\tau} + e^{-j\omega_o\tau}}{2(\omega_o^2-\omega^2)} \right] -\omega \left[\frac{e^{j\omega_o\tau} - e^{-j\omega_o\tau}}{2(\omega_o^2-\omega^2)} \right] \right\} \right.$$

$$\left. + \frac{\omega_o}{\omega_o^2-\omega^2} e^{-jK\omega\tau} \right] \qquad \text{(A-6)}$$

$$= -\frac{E_m}{2\pi(\omega_o^2-\omega^2)} \sum_{K=1}^{n} \left(e^{-jK\omega\tau} \right) \left[e^{-j\omega\tau}(\omega_o\cos\omega_o\tau + j\omega\sin\omega_o\tau) - \omega_o \right] \qquad (3)$$

APPENDIX B

Derivation of equation (4)

$$S = \sum_{K=1}^{n} e^{-jK\omega\tau} = e^{-j\omega\tau} + e^{-j2\omega\tau} + e^{-j3\omega\tau} + \ldots e^{-jn\omega\tau}$$

$$S\, e^{-j\omega\tau} = + e^{-j2\omega\tau} + e^{-j3\omega\tau} + \ldots e^{-jn\omega\tau} + e^{-j(n+1)\omega\tau}$$

$$S - Se^{-j\omega\tau} = S(1-e^{-j\omega\tau}) = e^{-j\omega\tau} - e^{-j(n+1)\omega\tau} \qquad \text{(B-1)}$$

$$S = \frac{e^{-j\omega\tau} - e^{-j(n+1)\omega\tau}}{1-e^{-j\omega\tau}}$$

It can be seen that this is the same as

$$S = \frac{e^{-j\omega\tau} \times e^{-j\frac{n\omega\tau}{2}}}{e^{-j\frac{\omega\tau}{2}}} \times \frac{e^{jn\frac{\omega\tau}{2}} - e^{-jn\frac{\omega\tau}{2}}}{e^{j\frac{\omega\tau}{2}} - e^{-j\frac{\omega\tau}{2}}} \qquad \text{(B-2)}$$

Since $\sin \frac{n\omega\tau}{2} = \dfrac{e^{jn\frac{\omega\tau}{2}} - e^{-jn\frac{\omega\tau}{2}}}{2j}$

and $\sin \frac{\omega\tau}{2} = \dfrac{e^{j\omega\tau} - e^{-j\omega\tau}}{2j}$

the above reduces to

$$S = e^{-j(n+1)\frac{\omega\tau}{2}} \frac{\sin\left(\frac{n\omega\tau}{2}\right)}{\sin\left(\frac{\omega\tau}{2}\right)} . \qquad (4)$$

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