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Transverse effects in the inhomogeneous beam–plasma interaction

R. Aghevli* and J. E. Rowe

Electron Physics Laboratory, Department of Electrical and Computer Engineering, The University of Michigan, Ann Arbor, Michigan 48104

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The interaction of a periodically inhomogeneous electron beam of finite transverse extent with its surrounding plasma is investigated. The hydrodynamic model is adopted for both the beam, which is of finite transverse extent, and the plasma, which can assume either an infinite or a finite transverse extent. The results indicate that for zero external magnetic field no stabilizing effects occur. This is in contrast to the results of the one-dimensional analysis previously reported. The dispersion relation for the finite magnetic field case is also presented and discussed.

I. INTRODUCTION

The stabilizing effects, introduced when a beam of longitudinally nonuniform density interacts one-dimensionally ($B_0 = \infty$) with a plasma, were reported in a previous paper.[1] The present work was inspired by the results obtained from the one-dimensional analysis and is an attempt to gain further insight into the processes which are involved in plasma stabilization. The inhomogeneity of the beam is of a periodic nature and is such that the characteristic length of the beam inhomogeneity, when measured in a frame of reference attached to the beam, is much shorter than the wavelength of the excited oscillations. The results for a zero magnetic field case indicate no suppression, possibly because of the escape of charged particles toward the outer surface of the plasma and the resultant reduced coupling between the high-frequency components of the electric field and the beam pulsation.

II. MATHEMATICAL FORMULATION

The following assumptions are made in the course of formulating the problem:

1. The hydrodynamic model applies to both the beam and the plasma.
2. The plasma is cold, stationary, and uniform with infinitely heavy ions serving as a neutralizing background.
3. The electron beam is cold, periodically inhomogeneous, and drifts with an axial velocity $\mathbf{u}_0$.
4. There is no applied static magnetic field.
5. Perturbations are assumed to be of small magnitude so as to make a linear analysis feasible.
6. The quasistatic approximation is valid.
7. Nonrelativistic mechanics applies.

On the basis of the above assumptions, the linearized equations of motion, the continuity equation and Gauss' law are written for the beam as well as the plasma particles. In the inertial frame of reference attached to the beam, these equations have the following form:

$$\frac{\partial \mathbf{v}_{1b}}{\partial t} = -\frac{e}{m}\mathbf{E}_1, \tag{1}$$

$$\frac{\partial n_{1b}}{\partial t} + \nabla \cdot (n_{0b}\mathbf{v}_{1b}) = 0, \tag{2}$$

$$\frac{\partial \mathbf{v}_{1p}}{\partial t} - \mathbf{u}_0 \cdot \nabla \mathbf{v}_{1p} = -\frac{e}{m}\mathbf{E}_1, \tag{3}$$

$$\frac{\partial n_{1p}}{\partial t} + n_{0p}\nabla \cdot \mathbf{v}_{1p} - \mathbf{u}_0 \cdot \nabla n_{1p} = 0, \tag{4}$$

and

$$\nabla \cdot \mathbf{E}_1 = -\frac{e}{\epsilon_0}(n_{1b} + n_{1p}), \tag{5}$$

where the differential vector operator ∇ is defined in the moving frame of reference attached to the beam, the subscript 1 denotes the first-order perturbation quantities, and subscripts b and p refer to the beam and plasma parameters, respectively. A steady-state solution to the above equations is sought by assuming an $\exp(j\omega t)$ variation in time for all of the first-order quantities. If this consideration is made, a combination of (1) through (5) yields the following result:

$$\left(j\omega - u_0\frac{\partial}{\partial \zeta}\right)^2 \nabla \cdot [\epsilon_b(\omega,\zeta)\mathbf{E}_1] + \omega_p^2 \nabla \cdot \mathbf{E}_1 = 0, \tag{6}$$

where $\epsilon_b(\omega, \zeta)$ is the beam dielectric constant defined as

$$\epsilon_b(\omega,\zeta) \equiv 1 - \frac{e^2 n_{0b}(\zeta)}{m\epsilon_0\omega^2}. \tag{7}$$

The differential equation, Eq. (6), cannot be solved without assuming further relations among the components of the electric field. The required relations can be established by exploiting the quasistatic approximation; on this basis the electric field can be approximated by a conservative field derivable from a scalar potential Φ and therefore (6) can be written as

$$\left(j\omega - u_0\frac{\partial}{\partial\zeta}\right)^2 \nabla\cdot[\epsilon_b(\omega,\zeta)\nabla\Phi] + \omega_p^2\nabla^2\Phi = 0. \quad (8)$$

A solution can be sought by noting that Φ is separable and can therefore be represented as $\Phi(r,\theta,\zeta) = R(r)Z(\zeta)\exp(-jm\theta)$, where the functional forms of R and Z are to be determined and m is a constant which assumes only integer values in compliance with the uniqueness of the solution. The equation obtained for R is recognized as Bessel's equation. From its two independent solutions only the function of the first kind, $J_m(Kr)$, is to be retained. The equation for Z is

$$\left(j\omega - u_0\frac{d}{d\zeta}\right)^2\left[\frac{d}{d\zeta}\left(\epsilon_b\frac{dZ}{d\zeta}\right) - K^2\epsilon_b Z\right] + \omega_p^2\left(\frac{d^2}{d\zeta^2} - K^2\right)Z = 0, \quad (9)$$

where K is a constant of separation whose value is to be determined by employing the boundary requirements.

An approximate solution to (9) can be found by employing the method of averaging as explained and demonstrated by Bogoliubov and Mitropolsky.[2] This procedure is outlined as follows. If $n_{0b}(\zeta)$ is periodic in ζ, the beam dielectric constant can be expressed in terms of its discrete Fourier components as

$$\epsilon_b(\omega,\zeta) = \sum_n f_n(\omega)\exp(-jnk_0\zeta). \quad (10)$$

Furthermore, $Z(\zeta)$ can be separated into a large-amplitude slowly varying part, $Z_0(\zeta)$, and a small-amplitude fast oscillatory part, $Z_1(\zeta)$, as

$$Z(\zeta) = Z_0(\zeta) + \sum_{n\neq 0}{}' z_n(\omega)\exp(-jnk_0\zeta). \quad (11)$$

Substitution of (10) and (11) into (9) yields the following results:

Zeroth-Order Equation

$$\left(K^2 - \frac{d^2}{d\zeta^2}\right)\left[\left(\omega + ju_o\frac{d}{d\zeta}\right)^2 f_0(\omega) - \omega_p^2\right]Z_o(\zeta) + K^2\omega^2\sum_{n\neq 0}{}' f_n(\omega)z_{-n}(\omega) = 0, \quad (12)$$

***n*th-Order Equation**

$$f_n(\omega)\left(K^2 + jnk_0\frac{d}{d\zeta} - \frac{d^2}{d\zeta^2}\right)\left(\omega + n\omega_0 + ju_0\frac{d}{d\zeta}\right)^2 Z_0(\zeta) + (K^2 + n^2k_0^2)[(\omega + n\omega_0)^2 f_0(\omega) - \omega_p^2]z_n(\omega) + (\omega + n\omega_0)^2\sum_{m\neq 0,n}{}'' (K^2 + nmk_0^2)f_{n-m}(\omega)z_m(\omega) = 0;$$
$$(n = \pm 1, \pm 2, \ldots), \quad (13)$$

where

$$\omega_0 \equiv k_0 u_0. \quad (14)$$

Application of the weak turbulence approximation[3] to the nth-order equation results in a value for $z_n(\omega)$ in terms of $Z_0(\zeta)$ and the known parameters. The dispersion relation follows immediately by eliminating $z_n(\omega)$ between (12) and (13), letting $Z_0(\zeta)$ assume an $\exp(-jk\zeta)$ dependence, and making a Doppler-shift transformation ($\omega \to \omega_D \equiv \omega - ku_0$). The result can be expressed as

$$(k^2 + K^2)\left(f_0(\omega_D) - \frac{\omega_p^2}{\omega^2} - 2f_0^{-1}(\omega_D)\frac{\omega_D^2}{\omega^2}\sum_{n=1}^{\infty}\frac{K^2}{K^2 + n^2k_0^2}|f_n(\omega_D)|^2\right) = 0. \quad (15)$$

The dispersion relation is obtained by combining the above equation with the boundary requirements. Care must be employed, however, in interpretation of the elements of (15) for the case of finite transverse geometry.

For an electron beam of finite transverse extent the space-charge field lines extend beyond the transverse extent of the beam. This corresponds to a leakage of the displacement current from the beam. This effect can properly be taken into account by reducing the beam density by a factor known as the geometric reduction factor. Therefore, the geometric reduction factor must appropriately appear in (15) in conjunction with variables whose values depend upon the beam density. Such variables are readily recognized as $f_n(\omega_D)$, $n = 0, 1, 2, \ldots$. They must be replaced by their effective values $f_{n,\text{eff}}(\omega_D)$, $n = 0, 1, 2, \ldots$, in accordance with the following definition for $f_{n,\text{eff}}(\omega)$:

$$1 - \frac{e^2 n_{0b}(\zeta)}{m\epsilon_0\omega^2}G_m\left(kb, \frac{b}{a}\right) = \sum_n f_{n,\text{eff}}(\omega)\exp(-jnk_0\zeta), \quad (16)$$

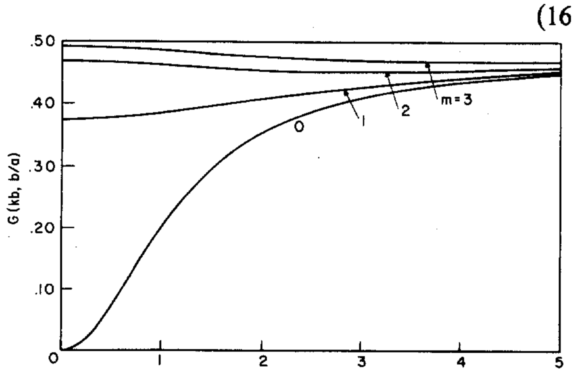

FIG. 1. Geometric reduction factor $G(kb, b/a)$ vs kb for the first four harmonics with $b/a = 0.5$, where a is the plasma radius and b the beam radius.

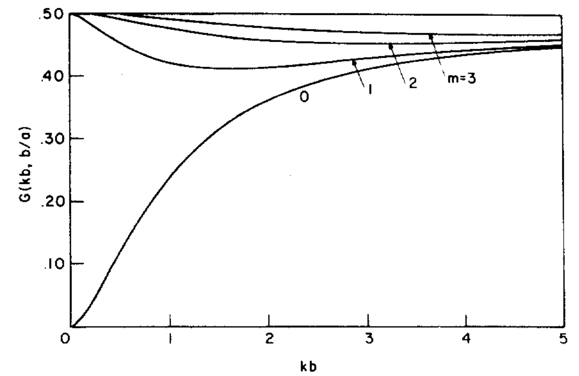

FIG. 2. Geometric reduction factor $G(kb, b/a)$ vs kb for the first four harmonics with $b/a = 0$.

where b and a are the beam and the plasma radii, respectively, and $G_m[kb, (b/a)]$ is the geometric reduction factor. A theoretical expression for G_m has been derived elsewhere[4] and it suffices to state the results:

$$G_m\left(kb, \frac{b}{a}\right) = -kb\frac{I'_m(kb)}{I_m(ka)}[K_m(ka)I_m(kb) - K_m(kb)I_m(ka)]. \quad (17)$$

The numerical values for G_m are shown in Figs. 1 and 2.

III. BOUNDARY REQUIREMENTS

The beam-plasma system that has been considered so far is composed of two distinct regions. The first is the beam-plasma region, designated as region I, which fills up a cylindrical space of a radius equal to b. The second is the surrounding region, designated as region II which contains a plasma of either infinite or finite extent contained in a perfectly conducting tube of radius a. For region I the dielectric constant and the total scalar potential function are given as

$$\epsilon_{\rm I} = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{\omega_D^2} \quad (18)$$

and

$$\phi_{\rm I}(\mathbf{r}, t) = AJ_m(Kr)\exp[j(\omega t - m\theta - kz)]; \quad (19)$$

likewise, for region II,

$$\epsilon_{\rm II} = 1 - \omega_p^2/\omega^2 \quad (20)$$

and

$$\Phi_{\rm II}(\mathbf{r}, t) = B[K_m(ka)I_m(kr) - I_m(ka)K_m(kr)] \times \exp[j(\omega t - m\theta - kz)], \quad (21)$$

where A and B are arbitrary constants and the requirement that the electric field be normal at the surface of the wall, $r = a$, has already been accounted for. ω_b is the beam-plasma frequency defined by $\omega_b^2 = e^2 n_{0b}/m\epsilon_0$. Two more requirements remain, namely that the potential and the normal component of the electric induction vector be continuous across the beam surface. A combined form of the above two requirements can be conveniently written as

$$\epsilon_{\rm I}\frac{\partial(\log \Phi_{\rm I})}{\partial r}\bigg|_{r=b} = \epsilon_{\rm II}\frac{\partial(\log \Phi_{\rm II})}{\partial r}\bigg|_{r=b}. \quad (22)$$

A direct substitution of (18) through (21) into (22) yields the result

$$Kb\frac{J'_m(Kb)}{J_m(Kb)} = \frac{1 - \omega_p^2/\omega^2}{1 - \omega_p^2/\omega^2 - \omega_b^2/\omega_D^2}F_m\left(kb, \frac{b}{a}\right), \quad (23)$$

where $F_m[kb, (b/a)]$ is defined as

$$F_m\left(kb, \frac{b}{a}\right) \equiv kb\frac{K_m(ka)I'_m(kb) - I_m(ka)K'_m(kb)}{K_m(ka)I_m(kb) - I_m(ka)K_m(kb)}. \quad (24)$$

Expression (23) is the desired boundary requirement. The dispersion relation is obtained by elimination of the constant of separation K, between (15) and (23). Furthermore, it can be shown[3] that the first multiplying factor in (15), that is, $K^2 + k^2 = 0$, corresponds to the homogeneous case and, therefore, it can be dropped to yield

$$f_{0,\rm eff}(\omega_D) - \frac{\omega_p^2}{\omega^2} - 2f_{0,\rm eff}^{-1}(\omega_D)\frac{\omega_b^2}{\omega^2} \times \sum_{n=1}^{\infty} \frac{K^2}{K^2 + n^2k_0^2}|f_{n,\rm eff}(\omega_D)|^2 = 0. \quad (25)$$

The combination of (23) and (25) form the desired dispersion relation. The mathematical equivalence of (25) in the limit of vanishing inhomogeneity ($f_{n,\rm eff} \to 0, n \neq 0$) with the dispersion relation derived for the homogeneous case, derived elsewhere,[5] is easily verified.[3]

IV. RESULTS FOR THE ZERO MAGNETIC FIELD CASE

From the form of (23) and (25) it is evident that a straightforward mathematical technique for elimination of K does not exist. An iteration technique can be developed, however, by noting that the constant of separation K has a value equal to $\pm jk$ for the homogeneous beam case. In the first approximation the growth rate, when normalized to that for a homogeneous beam, can be expressed as

$$\frac{\delta}{\delta_0} \cong \mu_0^{1/2} - 2\frac{\omega_b^2}{\omega_p^2}\mu_0^{-1/2}G_m\left(\frac{b\omega}{u_0}, \frac{b}{a}\right) \times \sum_{n=1}^{\infty} \frac{\omega^2}{n^2\omega_0^2 - \omega^2}|\mu_n|^2, \quad (26)$$

where δ and δ_0 are the imaginary parts of the propagation constant k for the homogeneous and inhomogeneous cases, respectively, and the μ_n's are defined as

$$n_{0b}(\zeta) = n_{0b} \sum_{\nu} \mu_\nu \exp(-j\nu k_0 \zeta). \tag{27}$$

A measure of suppression of the convective instabilities due to the beam inhomogeneity can be found by using (26). Care must be employed, however, in the manner in which the rate of suppression is defined. For a beam of rectangular density profile the dc power carried by the beam is reduced from that of the homogeneous beam by a factor equal to the duty cycle. This, in turn, results in a decrease of the growth rate by a factor of $\Delta^{1/2}$, where Δ is the duty cycle. Therefore, the first term in (26) does not truly represent the suppression of convective instabilities due to an inhomogeneity and is merely an indication of the fact that the input dc power has been reduced by a factor of Δ. The rate of suppression, Γ, of the convective instabilities due to axial inhomogeneity of beam density can then be represented as

$$\Gamma = 1 - \frac{\delta/\delta_0}{\Delta^{1/2}} = \frac{2}{\pi^2}\frac{\omega_b^2}{\omega_p^2}\frac{G_m}{\Delta}\sum_{n=1}^{\infty}\frac{\omega^2}{n^2\omega_0^2-\omega^2}\left(\frac{\sin n\pi\Delta}{n}\right)^2. \tag{28}$$

Figures 3 and 4 show the variation of the rate of suppression Γ as a function of the duty cycle Δ, with the normalized frequency $\Omega \equiv \omega/\omega_p$ taken as a parameter.

For the sinusoidal case the above-mentioned difficulty does not arise since the spatial average of the beam density does not change with the modulation and remains equal to n_{0b}. The rate of suppression Γ can therefore be expressed as

$$\Gamma = 1 - \frac{\delta}{\delta_0} = \frac{\alpha^2}{2}\frac{\omega_b^2}{\omega_p^2}\frac{\omega^2}{n^2\omega_0^2-\omega^2}G_m\left(\frac{b\omega}{u_0},\frac{b}{a}\right), \tag{29}$$

where α is the depth of modulation. Figures 5 and 6 show the variation of the rate of suppression Γ as a function of the depth of modulation α with the normalized frequency Ω taken as a parameter.

From Figs. 3–6 it is evident that the rate of suppression remains negligibly small throughout the region of instability; thus, it is concluded that an inhomogeneity in beam density, which was previously shown[1] to suppress the convectively unstable modes in a one-dimensional model, has no suppressive effects on the convective instabilities excited in the system if the externally imposed dc magnetic field is removed. The physical reason is believed to lie in the fact that, at low values of the dc magnetic flux density, escape of charges toward the outer surface of the plasma occurs. This results in reduced coupling between the high-frequency components of the electric field and the beam pulsations. In addition to the above-mentioned effect the weak field associated with the high-frequency oscillations in the inhomogeneous beam does not appreciably extend into the plasma and thus lowers the effectiveness of the coupling.

V. FINITE-MAGNETIC FIELD CASE

Since the procedure involved in the derivation of the dispersion relation is quite similar, and in view of the fact that the mathematics is cumbersome and requires a great deal of space for presentation, it is believed sufficient to state the final result. The dispersion relation for the two-dimensional system when an axially directed dc magnetic field of flux density B_0 is applied is given as

$$k^2\left(f_0(\omega_D) - \frac{\omega_p^2}{\omega^2}\right) + K^2\left(f_0(\omega_{1D}) - \frac{\omega_p^2}{\omega^2-\omega_c^2}\right) - 2\frac{\omega_b^2-\omega_c^2}{\omega^2-\omega_c^2} \times \sum_{n=1}^{\infty}\frac{K^2[k^2 f_n^*(\omega_D) + K^2 f_n^*(\omega_{1D})]f_n(\omega_{1D})}{n^2k_0^2[f_0(\omega_D) - \omega_p^2/n^2\omega_0^2] + K^2[f_0(\omega_{1D}) - \omega_p^2/(n^2\omega_0^2-\omega_c^2)]} = 0, \tag{30}$$

where $\omega_c \equiv eB_0/m$ is the electron-cyclotron frequency $\omega_{1D}^2 \equiv \omega_b^2 - \omega_c^2$ and the other parameters were defined previously. The constant of separation K is to be eliminated between the above equation and the boundary requirement

$$\epsilon_{\perp,\mathrm{I}} Ka\frac{J_m'(Ka)}{J_m(Ka)} + m\epsilon_{\parallel,\mathrm{I}} = \epsilon_{\perp,\mathrm{II}} T_2 a\frac{N_m(T_2a)J_m'(T_2b) - J_m(T_2a)N_m'(T_2b)}{N_m(T_2a)J_m(T_2b) - J_m(T_2a)N_m(T_2b)} + m\epsilon_{\parallel,\mathrm{II}}. \tag{31}$$

The following definitions were used in (31):

$$T_2^2 \equiv -k^2\epsilon_{\parallel,\mathrm{II}}/\epsilon_{\perp,\mathrm{II}}, \quad \epsilon_{\perp,\mathrm{I}} = 1 - \frac{\omega_p^2}{\omega^2-\omega_c^2} - \frac{\omega_b^2}{\omega_D^2-\omega_c^2},$$

$$\epsilon_{\parallel,\mathrm{I}} = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{\omega_D^2}, \quad \epsilon_{\perp,\mathrm{II}} = 1 - \frac{\omega_p^2}{\omega^2-\omega_c^2},$$

and

$$\epsilon_{\parallel,\mathrm{II}} = 1 - \frac{\omega_p^2}{\omega^2}.$$

Although an iteration technique to solve the above equations has been suggested,[3] the formidable task of tracing the roots of the dispersion relation has not been undertaken. It has been shown (one-dimension-

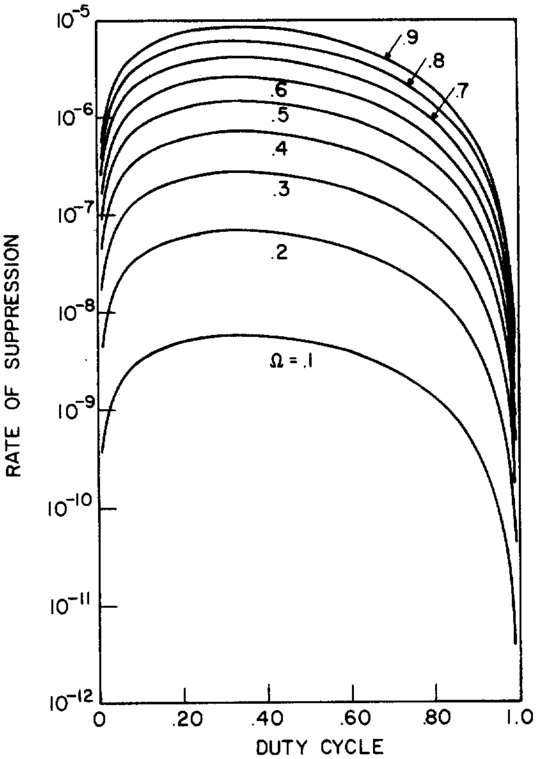

FIG. 3. The rate of suppression vs duty cycle with normalized frequency taken as a parameter for $b/a = 0$ and $m = 0$ (rectangular inhomogeneity) ($\omega_b/\omega_p = 0.1$, $b\omega_p/u_0 = 1$, $\omega_0/\omega_p = 10$).

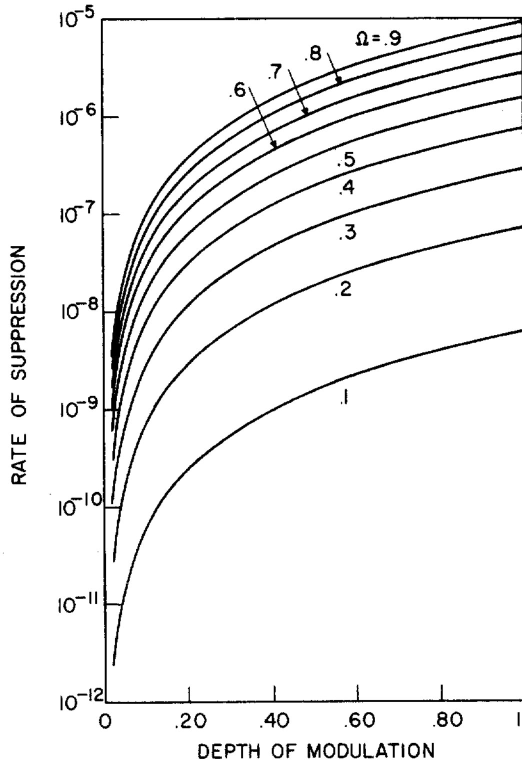

FIG. 5. The rate of suppression vs depth of modulation with normalized frequency taken as a parameter for $b/a = 0$ and $m = 0$ (sinusoidal inhomogeneity) ($\omega_b/\omega_p = 0.1$, $b\omega_p/u_0 = 1$, $\omega_0/\omega_p = 10$).

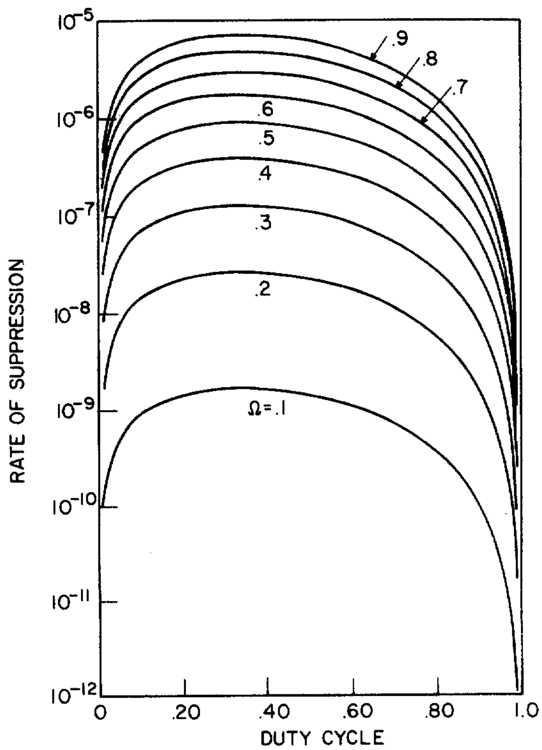

FIG. 4. The rate of suppression vs duty cycle with normalized frequency taken as a parameter for $b/a = 0.5$ and $m = 0$ (rectangular inhomogeneity) ($\omega_b/\omega_p = 0.1$, $b\omega_p/u_0 = 1$, $\omega_0/\omega_p = 10$).

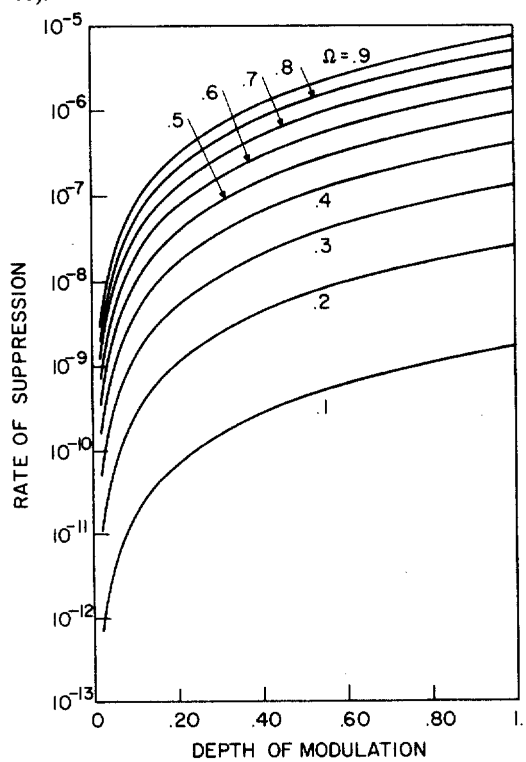

FIG. 6. The rate of suppression vs depth of modulation with normalized frequency taken as a parameter for $b/a = 0.5$ and $m = 0$ (sinusoidal inhomogeneity) ($\omega_b/\omega_p = 0.1$, $b\omega_p/u_0 = 1$, $\omega_0/\omega_p = 10$).

al analysis with $B_0 = \infty$) that in a beam-plasma system a periodic inhomogeneity in beam density has the effect of stabilizing convective instabilities if the system is immersed in an infinitely large, static magnetic field. On the other hand, there will be essentially no reduction in the value of the growth rate of convectively unstable waves due to beam inhomogeneity if the applied magnetic field is removed from the system. In numerical terms the rate of suppression is of the order of 10^{-1} for the one-dimensional case and zero for the two-dimensional, zero magnetic field case. The error bound in both cases is $(\omega_p/\omega_0)^2$. Having realized that the electron inertia sets an upper limit on the value of ω_0, the error bound cannot be made arbitrarily small while maintaining a reasonably dense plasma. Values as high as 3000 MHz for ω_0 have been reported in some experiments.[6] For such values of ω_0 and moderately dense plasmas (densities in the range of 10^8 to 10^9 cm^{-3}) the numerical value of the error bound $(\omega_p/\omega_0)^2$ equals 10^{-2}.

On the other hand, it is plausible to expect some measure of suppression for the finite magnetic field case. The value of a static magnetic field is expected to have a minimum threshold below which suppressive effects are either negligible or nonexistent. Above the threshold, however, suppression of convective instabilities is expected with suppression rates smaller than 10^{-1}. If it is noted that the dispersion relation (31) is valid within an error of ω/ω_0, it can easily be concluded that the expected suppression is of the same order of magnitude as the error bound in the more important parts of the spectrum, $\omega \sim \omega_c$ and $\omega \sim \omega_p$. Consequently, the task of solving the obtained dispersion relation numerically under the said approximations would not lead to any conclusive results and therefore is not undertaken here. Instead, a computer simulation of the entire problem is suggested to study this particular case.

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*Present address: University of Tehran, Tehran, Iran.

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