Introduction to the Problem of the Isochronous Hairspring

S. Goudsmit and Ming-chen Wang

Department of Physics, University of Michigan, Ann Arbor, Michigan

Deviations from isochronism for the system of hairspring and balance wheel are treated by a method due to Haag. The results are given a geometrical interpretation. It is shown that the deviations are related to the displacement of the end point of the hairspring in a set-up in which this end point is free to move keeping its tangent constant in direction. Such an arrangement can be realized with Bouasse's pendulum. For the so-called "helical" spring a more accurate solution than Phillips' has been found for the shape of the terminals. Extensive numerical calculations have been made for flat spiral springs, and a new simple terminal has been designed.

I. The Problem

1. Introduction

THE system of hairspring and balance wheel in use to regulate watches and chronometers does not perform an exact simple harmonic motion and the period of the vibrations depends upon the amplitude of the balance wheel. In watches this amplitude may vary from 270° to each side, immediately after winding, down to 150° at the time the watch has to be rewound. For the proper functioning of the watch it is necessary to reduce the lack of isochronism of the vibrations as much as possible.

One end of the hairspring is fastened to the axis of the balance wheel, the other end by means of the "stud" to the frame of the watch. If the balance wheel is turned out of its equilibrium position, a reaction force is set up in the two end points. The mathematical analysis of the problem shows that this reaction force is the cause of the deviations from isochronism. The simplest and most rigorous mathematical treatment of the problem has been given by J. Haag[1] of the Institut de Chronométrie at Besançon, France. Unfortunately this beautiful method appears to be described only in a very short note in the *Proceedings of the 3rd International Congress for Applied Mechanics*, which does not contain any reference to a more detailed publication. The present paper is based almost entirely on the method used in Haag's note and the mathematical part gives a detailed derivation of some of the formulas given there.

2. Method of Phillips

If the shape of the spring is designed so as to make the reaction force at the end points as small as possible the deviations from isochronism will be correspondingly small. This is, at least in principle, the approach to the problem as given in the famous classical article by Phillips.[2] For the spring of finite length fastened to balance axis and stud, the reaction force can never be made exactly zero, but by using a special shape for the end portions of the spring, the "terminals," this force can be reduced considerably.

3. Method of Résal and Caspari

It is also possible to design the shape of the hairspring so that the reaction force changes the period by an amount which is approximately independent of the amplitude. This is, in principle, the solution given by Résal and Caspari.[3] In this case the reaction force does not need to be small. The Phillips' type of solution, however, seems to be preferred in practical applications.

4. End point displacement of free-end spring

Bouasse has pointed out that perfect isochronism for a flat spring would result if the outer end point were not fixed at the stud, but were free to move in the plane of the spring, provided the direction of the tangent at that end point remained the same. This parallel displace-

[1] J. Haag, *Proceedings of the 3rd International Congress for Applied Mechanics, II* (1931), pp. 96–98.

[2] M. Phillips, Ann. des Mines, Mémoires 20, 1–107 (1861).

[3] See the excellent survey of the problem in *Pendule, Spiral, Diapason* by H. Bouasse, Vol. II, Chap. V (Delgrave, Paris, 1920). Unfortunately Bouasse does not give any references and we have been unable to locate the original articles by Résal and Caspari.

ment can, of course, not be realized in a watch, but it may be visualized by means of the large scale model, shown in Fig. 1. Bouasse proposes that the end point of the model spring be fastened to the end of a long rod which can move as a spherical pendulum. The end point will, indeed, move parallel with itself and, if the pendulum is well balanced by means of the counter weight at the top, it will not create a reaction force on the end of the spring. We shall call this arrangement a "free-end spring."

If the axis to which the inner end of the spring is fastened is turned over a certain angle, the outer end point will experience a displacement. This displacement is approximately proportional to the reaction force which would exist at the end point if it were fixed at the stud instead of free to move. The Phillips solution is therefore equivalent to designing a spring for which the free-end displacement is as small as possible.

This approach reduces the dynamical problem to a geometrical one. Furthermore a crude large scale model like that of Fig. 1 can actually be used to give some information about the end point displacement and the isochronism for various shapes (see Fig. 2); whereas measurements of the reaction forces and periods are more difficult to perform. We shall, in the following, obtain the dynamical properties of the actual spring with both ends fixed by studying the end point displacement of the same spring considered as a free-end spring.
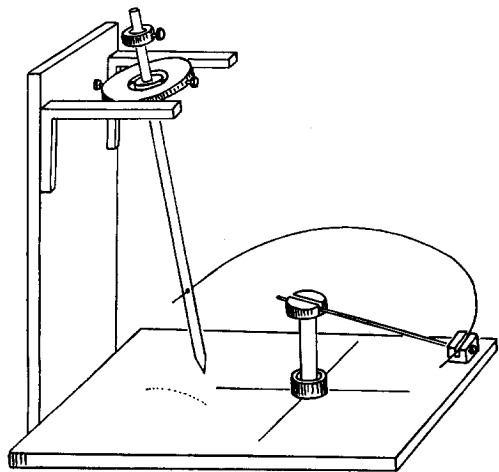

FIG. 1. Bouasse's pendulum.

II. RESULTS AND APPLICATIONS

1. The "helical" spring

The calculations of Phillips deal with the so-called "helical" spring. As, however, the height of the helix is neglected, they are strictly valid only for a fictitious flat spring of several windings, all having exactly the same radius. Nevertheless the consideration of the idealized spring is useful as an illustration of the problem, because the
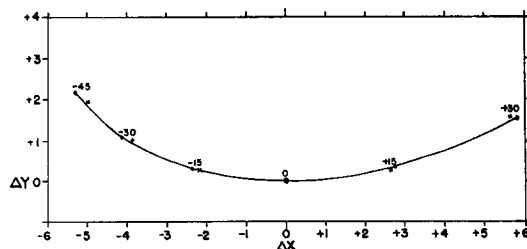

FIG. 2. End point displacement in cm for a semicircle of radius 15.5 cm; (●) calculated, (×) obtained by the model shown in Fig. 1. Numbers on curve indicate the angle of rotation of the balance wheel. Positive values mean unwinding and negative mean winding up.

calculations can be executed explicitly. Figure 3 shows the actual path of the end point for a "helical" spring of 10 windings and total length L. The line from the balance wheel axis to the end point in the rest position is chosen as the positive x axis. Figure 4 is for $10\frac{1}{4}$ windings.

The end point displacement is greatly reduced when the "terminals" of the spring are given a different shape fulfilling the first approximation calculated by Phillips. The magnitude of the displacement is roughly 20 times smaller, which reduces the deviation from isochronism by a factor of the order 400. If the terminals fulfill both the first and second approximations, the displacement will be about 300 times smaller.

2. Résal-Caspari solution for the helical spring

The mathematical treatment shows that the period of the vibrations of the fastened spring will be changed by an almost constant amount if the shape is such that for the free-end spring the ratio $(|\Delta|/\alpha)^2$ is a constant plus an odd function of α, where $|\Delta|$ is the magnitude of the end point displacement, and α the angle turned by the balance wheel. This is realized for a "helical" spring with $n+\frac{1}{4}$ and $n-\frac{1}{4}$ windings,

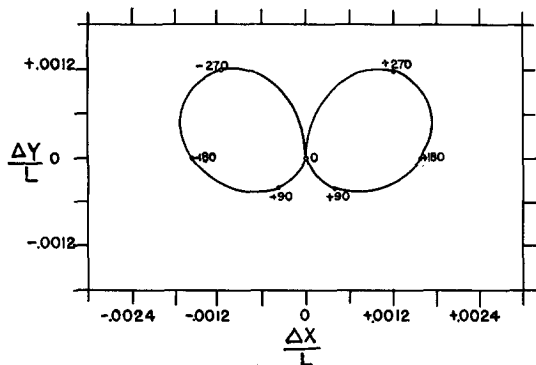FIG. 3. End point displacement for "helical" spring of 10 windings (displacement decreases with the square of the number of windings), total length L.

according to Résal and Caspari, as we shall prove below.

3. Flat spiral spring

For the hairsprings used in watches the calculation must be done numerically. The terminals usually applied to a flat spiral are the same as those calculated by Phillips for a "helical" spring with the same radii as the inner and out-most windings. Figures 5 and 7 indicate the smaller end point displacement. A disadvantage is that the outer terminal has to be bent out of the plane of the spiral, because it comes nearer to the axis than the outer windings. This deflection is not considered in the calculation.

An ingenious solution has been proposed by Van den Broek,[4] who considers a spring which first winds outward and then spirals inward again: the two spirals unfortunately must lie in different planes. Figure 6 shows that this double spiral has indeed a small end point displacement.

Figures 5 and 7 indicate also the reduction in end point displacement for a new terminal. It is certainly as satisfactory as the Phillips' terminal and has the great advantage that it is completely in the same plane as the spiral. Its construction is shown in Fig. 8.

4. Center of gravity

The mathematical analysis shows that the deviation from isochronism depends not only upon the end point displacement Δ but also upon $d\Delta/d\alpha$. The deviation is in fact proportional

[4] J. A. Van den Broek, Trans. Am. Soc. Mech. Eng. 53, APM page 247 (1931).

to the product $\alpha |\Delta| d|\Delta|/d\alpha$ averaged over a period of the balance wheel. Considering Δ as a vector it is possible to show that $d\Delta/d\alpha$ is connected with the position of the center of gravity of the free-end spring. If the vector $\mathbf{G}$ indicates the position of the center of gravity as a function of the rotation angle we shall show later that the above-mentioned product can also be written as $\alpha |[\mathbf{G} \times \Delta]|$.

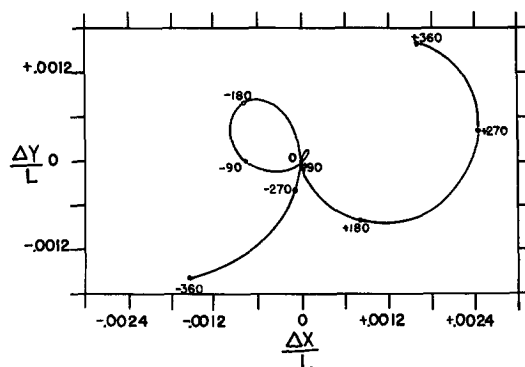It is obviously advantageous to have the center

FIG. 4. End point displacement for "helical" spring of $10\frac{1}{4}$ windings, total length L.

of gravity as near as possible to the balance axis in order to make this product small. However, it is not necessarily best to place the balance axis at the center of gravity of the undistorted spring. This will certainly make the product small for small angles α, but for the usual large amplitudes $\alpha |[\mathbf{G} \times \Delta]|$ may become larger, as is the case for Archimedes spirals.

The vectors Δ and especially $\mathbf{G}$ are rather complicated functions of the rotation angle α. The occurrence of a vector product adds to the complication. It is, therefore, advisable to disregard the interpretation in terms of the center of gravity of the free-end spring and to consider the much simpler quantities $|\Delta|$ and $d|\Delta|/d\alpha$. Figure 7 shows the change in $|\Delta|$ for an Archimedes spiral of 13.5 windings after addition of Phillips' terminals, and also for the new outer terminal.

III. MATHEMATICAL

1. Notation

We choose a coordinate system with its origin at the axis of the balance wheel. It is advan-

808

tageous, though not necessary, to take either the inner or the outer end point of the spring on the positive x axis. The shape of the spring is characterized by giving at each point the angle ψ which the tangent makes with the x axis as a function of s, the arc length, measured from the inner end point (Fig. 9). The formulas are simplified by combining the x and y coordinates into a complex number

$$z = x + iy.$$

In this notation we have for the equation of the spring

$$x(s) = x(0) + \int_0^s ds \cos \psi(s),$$

$$y(s) = y(0) + \int_0^s ds \sin \psi(s)$$

or

$$z(s) = z(0) + \int_0^s ds e^{i\psi(s)}, \tag{1}$$

where $z(0)$ determines the position of the inner end point. Differentiation after the arc length s shall be indicated by a prime, thus

$$z'(s) = e^{i\psi(s)}. \tag{2}$$

The curvature $1/\rho$ at each point is given by

$$1/\rho = \psi'(s). \tag{3}$$

When the balance wheel is rotated over an angle α, the spring will be distorted and the new shape can be characterized by giving the change in $\psi(s)$. This change of angle for each element is denoted by $\phi(s)$. It depends upon the amount α the balance wheel is turned, and also upon the conditions at the end point. It will be a different function for the "free-end" spring and the spring
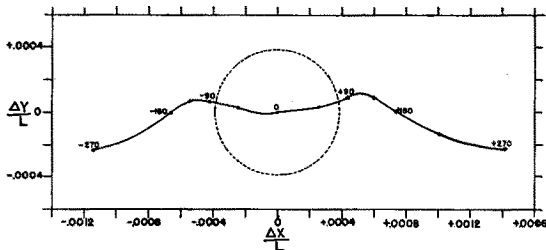
FIG. 5. End point displacement for Archimedes spiral of $13\frac{1}{2}$ windings, inner radius 1, outer radius 2.35, length 45.2π. The dotted circle indicates the range of the end point path for this spiral plus Phillips' terminals, or an outer terminal as shown in Fig. 8.

fastened at the stud. The change in curvature of each element of the spring due to the distortion is given by

$$\text{change in curvature} = \phi'(s). \tag{4}$$

2. Elastic energy

If the deformation is within Hooke's law the energy stored in each element of a distorted spring is proportional to the square of the
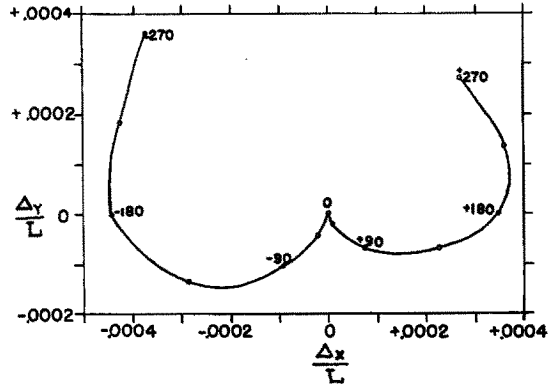
FIG. 6. End point displacement for Van den Broek double spiral of 6 windings in and out, inner radius 1, outer radius 2.2, total length $L = 37.2\pi$.

change in curvature. The total potential energy V is therefore

$$V = \frac{k}{2} \int_0^L \phi'^2(s) ds. \tag{5}$$

The constant k depends upon the material of the spring and its cross section, L is the total length of the spring.

When the balance wheel is turned, the distorted shape of the spring will be such that the energy V is a minimum, subject to the proper conditions at the two end points. The determination of $\phi(s)$ is thus a standard problem of the calculus of variations.

3. The free-end spring

We denote by $z_f(s)$ and $\phi_f(s)$ the functions $z(s)$ and $\phi(s)$ of the "free-end" spring after distortion. When the balance wheel is turned over an angle α, the inner end point moves in a circle over the same arc, thus

$$z_f(0) = z(0)e^{i\alpha}. \tag{6}$$

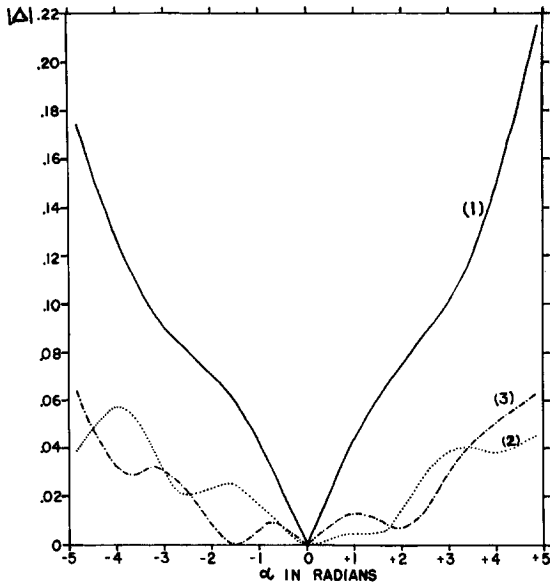The position of the outer end point is not prescribed. The free-end spring was defined such

FIG. 7. Magnitude of end point displacement $|\Delta|$ as function of rotation angle α. (1) For Archimedes spiral, inner radius 1, outer radius 2.35, 13.5 windings, (2) with Phillips' terminals, (3) with new outer terminal of Fig. 8.

that the tangent at the outer end point remains the same as before the distortion. At the inner end point the tangent turns over the angle α, giving

$$\phi_f(L) = 0, \tag{7}$$

$$\phi_f(0) = \alpha. \tag{8}$$

With these boundary conditions the minimum of the potential energy of Eq. (5) is simply determined by

$$\phi_f''(s) = 0, \tag{9}$$

which has the solution

$$\phi_f(s) = -\alpha s/L + \alpha. \tag{10}$$

It also follows that

$$\phi_f'(s) = -\alpha/L. \tag{11}$$

The latter result means that for the free-end spring the change in curvature is the same everywhere along the spring and proportional to the angle of rotation α.

4. End point displacement

The end point displacement Δ is given by

$$\Delta = z_f(L) - z(L) = z(0)e^{i\alpha}$$

$$+ \int_0^L ds\, e^{i(\psi+\phi)} - z(L). \tag{12}$$

A partial integration, using Eqs. (2), (7), and (8) gives

$$\Delta = -i \int_0^L z(s)e^{i\phi(s)}\phi'(s)ds. \tag{13}$$

This relation is valid in general, not only for the free-end spring, because Eqs. (7) and (8) are also true for the fastened spring. Substituting $\phi_f(s)$ for the free-end spring, we find

$$\Delta = i\frac{\alpha}{L}e^{i\alpha} \int_0^L z(s)e^{-i\alpha s/L}ds. \tag{14}$$

This is the principal formula of the present paper. The problem consists in designing the undistorted spring, $z(s)$, so that this integral becomes as small as possible.

As an example we can take a "helical" spring of radius R, consisting of n complete windings and a partial winding of arc θ. Its equation is
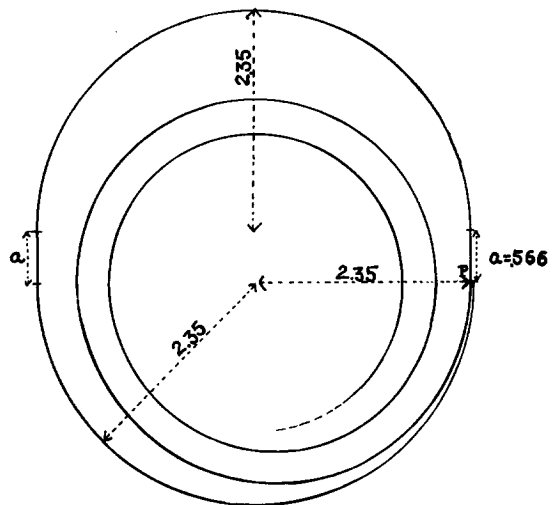

FIG. 8. Plane terminal, consisting of two semicircles joined by two straight portions a with dimension indicated. P is the point where it joins the main spiral.

$$z(s) = Re^{is/R}; \quad L = (2\pi n + \theta)R. \tag{15}\ \ (16)$$

This yields for Δ

$$\Delta = \frac{R^2\alpha}{L - R\alpha}(e^{i\theta} - e^{i\alpha}) \sim \frac{L\alpha}{4\pi^2 n^2}(e^{i\theta} - e^{i\alpha}). \tag{17}$$

Figures 3 and 4 show some examples of this.

5. Terminals

We divide the spring of length L into three parts l_1, $L - l_1 - l_2$, l_2 and take the "terminals"

810

l_1 and l_2 short compared to the total length L. We can consider the end point displacements due to each part separately. The contribution arising from the inner terminal l_1 is

$$\Delta_1 = i\frac{\alpha}{L}e^{i\alpha}\int_0^{l_1} z(s)e^{-i\alpha s/L}ds$$
$$= i\frac{\alpha}{L}e^{i\alpha}e^{-i\alpha l_1/L}\int_0^{l_1} z(\sigma)e^{+i\alpha\sigma/L}d\sigma. \quad (18)$$

The latter integral results if we measure the arc length σ along this terminal in the opposite direction, $\sigma = l_1 - s$. Similarly the outer terminal l_2 contributes

$$\Delta_2 = i\frac{\alpha}{L}e^{i\alpha}\int_{L-l_2}^{L} z(s)e^{-i\alpha s/L}ds$$
$$= i\frac{\alpha}{L}e^{i\alpha l_2/L}\int_0^{l_2} z(\sigma)e^{-i\alpha\sigma/L}d\sigma. \quad (19)$$

The second integral is obtained if the arc length σ is measured from the beginning of the terminal, $\sigma = s - L + l_2$. The terminals are supposed to be short compared to the total length, the exponentials in the integrands may be expanded in powers of α/L. The problem consists in choosing the shape of the terminals in such a way that Δ_1 and Δ_2 approximately cancel the contribution to the end point displacement caused by the main central portion of the spring.

6. Terminals for "helical" spring

Returning to the classical example we take for the central portion a "helical" spring. Its equation is conveniently taken as

$$z(s) = Re^{i(s-l_1)/R}. \quad (20)$$

The arc length over the helical part goes from l_1 to $L - l_2$ and we choose again n complete windings plus an arc θ

$$L - l_1 - l_2 = (2\pi n + \theta)R. \quad (21)$$

Its contribution to the end point displacement is

$$\Delta_c = i\frac{\alpha}{L}e^{i\alpha}\int_{l_1}^{L-l_2} z(s)e^{-i\alpha s/L}ds$$
$$= \frac{R^2\alpha}{L-R\alpha}(e^{i\theta}\cdot e^{i\alpha l_2/L} - e^{i\alpha}\cdot e^{-i\alpha l_1/L}). \quad (22)$$

In order that $\Delta_1 + \Delta_2 + \Delta_c \sim 0$ we compare Eq. (22) with Eqs. (18) and (19) and set

$$e^{i\theta}R^2\alpha/(L-R\alpha) \sim -i\frac{\alpha}{L}\int_0^{l_2} z(\sigma)e^{-i\alpha\sigma/L}d\sigma, \quad (23)$$

$$R^2\alpha/(L-R\alpha) \sim i\frac{\alpha}{L}\int_0^{l_1} z(\sigma)e^{+i\alpha\sigma/L}d\sigma. \quad (24)$$

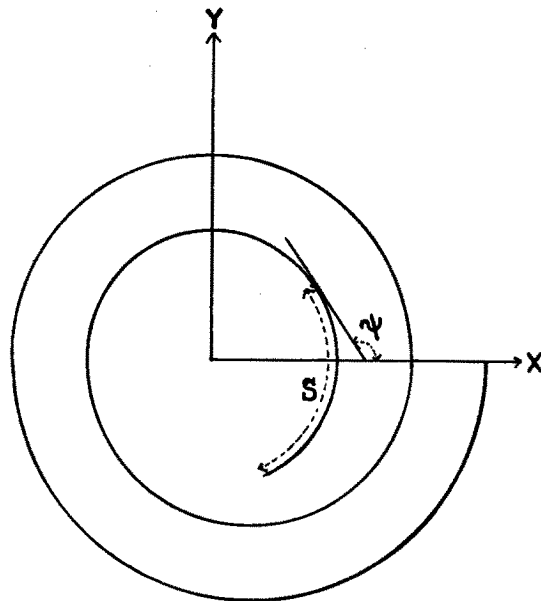The factor $e^{i\theta}$ is immaterial, it merely means a rotation of the whole outer terminal over the

FIG. 9. Coordinates characterizing the spiral.

angle θ, the incompleted winding. Disregarding this by taking $\theta = 0$, we see that Eqs. (23) and (24) are just complex conjugates. The two terminals are thus the same, except for a reflection with respect to the x axis. Expanding Eq. (24) in powers of α/L, the first term gives

$$R^2 = i\int_0^l z(\sigma)d\sigma \quad (25)$$

or

$$R^2 = -\int_0^l yd\sigma; \quad (25a)$$

$$0 = \int_0^l xd\sigma. \quad (25b)$$

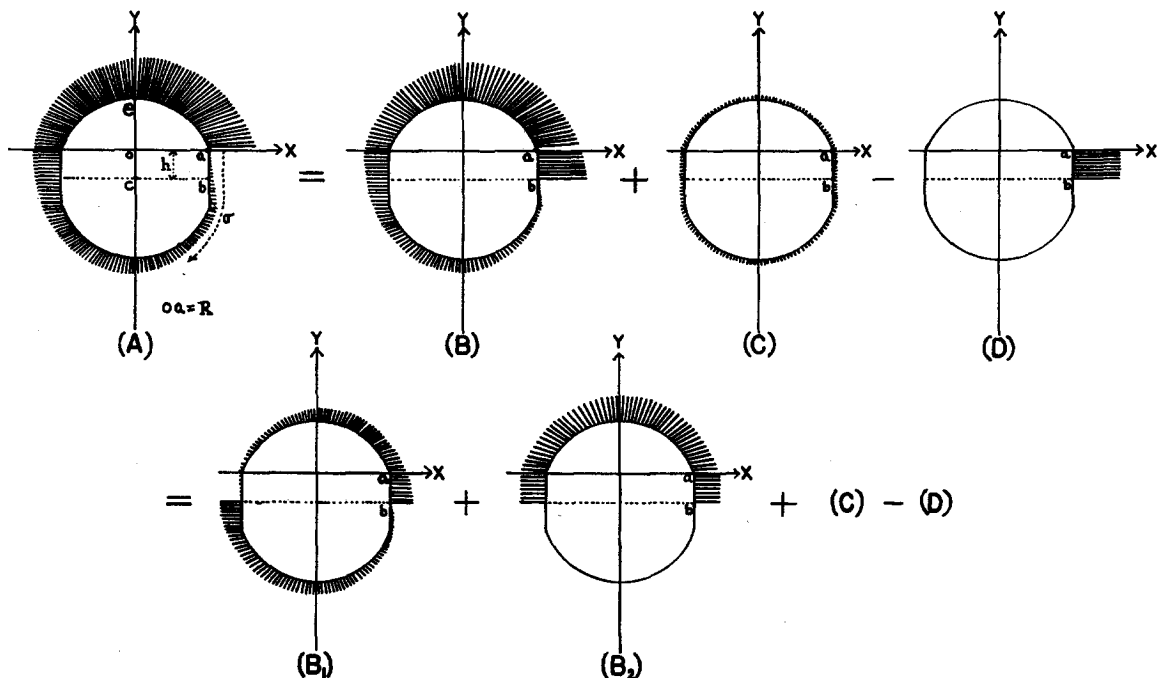These are the famous conditions obtained by Phillips. They mean that the terminal must have

FIG. 10. The width of the shaded part is proportional to the fictitious density, and the solid line is the outline of the terminal. The x (or y) moment of the required distribution (A) is equal to the sum of x (or y) moments of (B₁), (B₂) and (C) minus that of (D). (A) Weighted terminal with starting point a to be analyzed; (B) similarly weighted terminal with starting point b, which is in turn analyzed into (B₁) and (B₂); (C) uniform strip of density h along the whole terminal; (D) uniform strip of density l along the portion ab only.

its center of gravity on the negative y axis, a distance R^2/l from the origin. As Phillips pointed out, one can satisfy these conditions in an infinite number of ways.

7. The second approximation

Comparing the coefficients of $(\alpha/L)^2$ in the expansion of Eq. (24) we find for the next requirement

$$R^3 = -\int_0^l \sigma z(\sigma) d\sigma \qquad (26)$$

or

$$R^3 = -\int_0^l \sigma x d\sigma; \qquad (26a)$$

$$0 = \int_0^l \sigma y d\sigma. \qquad (26b)$$

We must thus find a terminal fulfilling both Eqs. (25) and (26) and moreover

$$z(\sigma) = R \quad \text{for } \sigma = 0, \qquad (27)$$

because the terminal has to fit onto the helical

part of the spring. Haag[5] states that he has been unable to find a solution. However, here follows a solution found by one of us (Miss W.).

A terminal, as shown in Fig. 10(A), with the following specifications: (1) Symmetrical with respect to the y axis, (2) symmetrical with respect to a line through its center of gravity c perpendicular to the y axis, (3) with center of gravity at a distance of $h(=R^2/l)$ below the x axis, (4) with ab and its corresponding portions parallel to the y axis, will fulfill conditions (25a) (25b) and (26a) simultaneously. Since the form of ae and three similar arcs is not specified, one can adjust it to satisfy condition (26b).

Now conditions (25a) and (25b) are taken care of by specifications No. 3 and 1, respectively. The geometrical interpretation of condition (26a) is as follows. If we give each element of the terminal a fictitious density equal to its arc length σ from the point a, where it joins the main part of the spring, the x moment $(\sum_i x_i dm_i)$ of this weighted terminal must be $-R^3$. Such a

[5] Cf. reference 1. See also F. Keelhoff, Mémoires de L'Académie Royale de Belgique 6, (1922).

weighted terminal can be analyzed into several simple density distributions, as shown in Fig. 10, so that the x moment of the former can be obtained from those of the equivalent distributions.

By virtue of specifications No. 1 and 2 the x moment of the density distributions (B_1) (B_2) and (C) are all zero. (D) will give an x moment of amount $l \cdot h \cdot R$, therefore the net x moment of the original distribution (A) is the negative of (D), i.e., $-lhR$. Combining this with the condition $lh = R^2$, we have the x moment of the weighted terminal equal to $-R^3$.

As a simple example, one can assume the portions ae, etc., to be corners of a rectangle, which fit onto the straight parts ab, etc., to make the terminal a perfect rectangle of width $2R$ and height $2g$. We can adjust the length g in order to fulfill (26b). This gives for g the equation

$$g^4 + 4g^3R + 4g^2R^2 - 17R^4/16 = 0 \qquad (28)$$

with the solution
$$g = 0.425R. \qquad (29)$$

8. The Archimedes spiral

In polar coordinates, the equation of the Archimedes spiral is

$$r = a\theta. \qquad (30)$$

It follows that in our notation

$$z = a\theta e^{i\theta}. \qquad (31)$$

The hairsprings are not complete Archimedes spirals which begin at the origin, but several inner windings are omitted. Therefore, the angle does not start from 0, but begins at θ_0, which is usually of the order 15π. There may be from 10 to 15 windings in the spiral, thus θ goes up to about 50π. The arc length is given by

$$s = \tfrac{1}{2}a\left[\theta(1+\theta^2)^{\frac{1}{2}} + \log\{\theta + (1+\theta^2)^{\frac{1}{2}}\}\right]_{\theta_0}^{\theta}$$
$$\sim \tfrac{1}{2}a(\theta^2 - \theta_0^2). \qquad (32)$$

The latter approximation is certainly sufficient in view of the range of values of θ. For the end point displacement we obtain, using θ as the variable,

$$\Delta = i\frac{\alpha}{L}e^{i\alpha}\int_{\theta_0}^{\theta_L} a^2\theta^2 e^{i\theta}e^{-(i\alpha a/2L)(\theta^2-\theta_0^2)}d\theta. \qquad (33)$$

This integral has to be evaluated numerically with the help of tables or asymptotic formulas for Fresnel integrals. Results are shown in Figs. 5 and 7.

9. Spring with fixed end point

We have studied above the end point displacement of the "free-end" spring and must now prove that this displacement is a measure of the reaction forces set up in the end points of the actual hairspring which is fastened to the stud. We denote by $z_a(s)$ and $\phi_a(s)$ the coordinates z and the function ϕ for the actual spring *after* rotation of the balance wheel over an angle α. We now have the additional requirement that the end point remain fixed

$$z_a(L) = z(L) = \text{constant} \qquad (34)$$

or in components

$$x_a(L) = x_a(0)$$
$$+ \int_0^L ds \cos(\psi + \phi_a) = \text{constant}, \qquad (35a)$$

$$y_a(L) = y_a(0)$$
$$+ \int_0^L ds \sin(\psi + \phi_a) = \text{constant}. \qquad (35b)$$

We must now find the minimum of the potential energy V, Eq. (5), subject to these constraints. Using Lagrange multipliers 2λ and 2μ which are proportional to the x component and y component of the reaction force caused by the constraint, the Euler-Lagrange equation now becomes

$$\phi_a''(s) + \lambda \sin(\psi + \phi_a) - \mu \cos(\psi + \phi_a) = 0. \qquad (36)$$

This equation for $\phi_a(s)$ cannot, in general, be solved, but we can derive some useful conclusions from it. Anticipating that the distribution will differ little from that of the free-end spring we write

$$\phi_a(s) = \phi_f(s) + \epsilon(s) = -\alpha s/L + \alpha + \epsilon(s). \qquad (37)$$

This changes the differential equation into

$$\epsilon''(s) + \lambda y_a' - \mu x_a' = 0 \qquad (38)$$

with the boundary values

$$\epsilon(0) = \epsilon(L) = 0. \qquad (39)$$

Integrating once we get

$$\epsilon'(s)+\lambda y_a-\mu x_a=\text{constant.} \qquad (40)$$

The constant is determined by integrating once more, now from 0 to L, using the boundary values for $\epsilon(s)$. We finally obtain

$$\epsilon'(s)=-\lambda(y_a-Y_a)+\mu(x_a-X_a) \qquad (41)$$

with X_a and Y_a the coordinates of the center of gravity of the *distorted* spring, defined by

$$LX_a=\int_0^L x_a ds; \qquad (42a)$$

$$LY_a=\int_0^L y_a ds. \qquad (42b)$$

We must now connect this with the displacement Δ by observing that

$$\Delta=z_f(L)-z(L)=z_f(L)-z_a(L)$$

$$=\int_0^L ds(e^{i\psi_f}-e^{i\psi_a})=\int_0^L ds e^{i\psi_a}(e^{-i\epsilon}-1)$$

$$=i\int_0^L z_a(s)e^{-i\epsilon}\epsilon' ds \sim i\int_0^L z_a(s)\epsilon' ds. \qquad (43)$$

The last line is obtained by partial integration and in the final approximate expression the exponential is replaced by unity. Next substituting Eq. (41) for ϵ' and writing the components, we obtain

$$\Delta_x=\int [\lambda(y_a^2-y_aY_a)-\mu(x_ay_a-y_aX_a)]ds$$

$$=\lambda LA-\mu LF, \qquad (44a)$$

and

$$\Delta_y=-\lambda LF+\mu LB, \qquad (44b)$$

where A, B, and F are the moments and products of inertia, with respect to the center of gravity of the distorted spring, defined as

$$LA=\int_0^L y_a^2 ds-LY_a^2; \qquad (45a)$$

$$LB=\int_0^L x_a^2 ds-LX_a^2; \qquad (45b)$$

$$LF=\int_0^L x_ay_a ds-LX_aY_a. \qquad (45c)$$

We find at once for the square of the reaction force

$$\lambda^2+\mu^2=\frac{(B\Delta_x+F\Delta_y)^2+(F\Delta_x+A\Delta_y)^2}{L^2(AB-F^2)^2}. \qquad (46)$$

Usually $A\sim B$ and $F=0$ so that a first approximation gives

$$\lambda^2+\mu^2\sim |\Delta|^2/A^2L^2. \qquad (47)$$

This result proves that the reaction force is approximately proportional to the displacement Δ.

10. Deviation from isochronism

We finally consider the energy of the distorted spring with a fixed end point

$$V=\frac{k}{2}\int_0^L \phi_a'^2(s)ds=\frac{k}{2}\int_0^L (\phi_f'+\epsilon')^2 ds$$

$$=\frac{k\alpha^2}{2L}+\frac{k}{2}\int_0^L \epsilon'^2 ds. \qquad (48)$$

The last result is obtained by using the known solution for ϕ_f, Eq. (10) and the boundary values for $\epsilon(s)$, Eq. (39). Substituting the formula for ϵ' found above we can write

$$V=\frac{k\alpha^2}{2L}+\frac{k}{2}(B\Delta_x^2+A\Delta_y^2+2F\Delta_x\Delta_y)/L(AB-F^2)$$

$$\sim\frac{k\alpha^2}{2L}+\frac{k|\Delta|^2}{2LA}=\frac{k}{2L}(\alpha^2+\xi(\alpha)). \qquad (49)$$

The correction term, abbreviated $\xi(\alpha)$, causes the deviation from isochronism. The balance wheel is very much heavier than the hairspring and is alone responsible for the kinetic energy, which is thus proportional to $(d\alpha/dt)^2$. The equation of motion of the balance wheel has, therefore, the form

$$d^2\alpha/dt^2=-\omega^2(\alpha+\tfrac{1}{2}d\xi/d\alpha), \qquad (50)$$

where $\omega=2\pi/T$, T being the period if the correction term were absent as in the free-end spring. Writing

$$\alpha\sim\alpha_0 \sin \omega(1+\delta)t+\cdots, \qquad (51)$$

we find by well-known methods the first approxi-

mation of δ, the fractional increase in frequency

$$\delta \sim \frac{1}{\alpha_0{}^2 T} \int_0^T \frac{d\xi}{d\alpha} \alpha dt$$
$$\sim \frac{2}{\alpha_0{}^2 TA} \int_0^T |\Delta| \frac{d|\Delta|}{d\alpha} \alpha dt. \quad (52)$$

We shall not discuss this expression in detail but call attention to the fact that if $\xi(\alpha)$ is an odd function of α, the integral vanishes. However, $\xi(\alpha)$, being proportional to the square of $|\Delta(\alpha)|$, can never be a pure odd function of α. If the even part of ξ is α^2 only, δ will be some constant value independent of the amplitude α_0. Therefore if $|\Delta|^2/\alpha^2$ is a constant plus an odd function of α, the period will change by a constant amount. For example, in the case of a "helical" spring

$$\Delta \sim \alpha(e^{i\theta} - e^{i\alpha}), \quad (53)$$

which gives

$$|\Delta|^2/\alpha^2 \sim [1 - \cos(\alpha - \theta)]. \quad (54)$$

When $\theta = \pm \pi/2$, $\cos(\alpha - \theta) = \pm \sin \alpha$, and $|\Delta|^2/\alpha^2$ fulfills the requirement for constant change of period, which proves the theorem of Résal and Caspari.

Because $\xi(\alpha)$ is proportional to $|\Delta|^2$, the integral representing δ will be small if $|\Delta|$ is small, provided, however, that $d|\Delta|/d\alpha$ does not become too large. This confirms our point of view that the best way to achieve isochronism is by making the end point displacement Δ be as small as possible. This result can be connected with the motion of the center of gravity of the free-end spring. Using Eqs. (10) and (12) we obtain

$$\frac{d\Delta}{d\alpha} = iz(0)e^{i\alpha} + i \int_0^L e^{i\psi_f} ds - \frac{i}{L} \int_0^L se^{i\psi_f} ds$$
$$= \frac{i}{L} \int_0^L z_f(s) ds = iG. \quad (55)$$

G is the complex coordinate of the center of gravity of the free-end spring and changes its position with different rotations α. We may now write for Eq. (52)

$$\delta = \frac{2}{\alpha_0{}^2 TA} \int_0^T |[G \times \Delta]| \alpha dt. \quad (56)$$

It is obviously advantageous to have G as small as possible for all values of α. To have the center of gravity of the undistorted spring ($\alpha = 0$) fall exactly on the axis is, however, not necessarily the best solution (except for small amplitudes), because G is a complicated function of α. We verified this numerically for a few examples.