Motion of a Sphere in a Rotating Fluid at Small Reynolds Numbers

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The slow steady rise of a solid sphere along the axis of a uniformly rotating viscous liquid is studied when the Reynolds number, based on the translational velocity along the axis of rotation, is less than unity. The terminal velocity of a sphere rising slowly along the axis of a uniformly rotating viscous liquid has been determined experimentally. Spheres of varying sizes with a density less than that of the fluid were released at the bottom of a rotating cylinder filled with castor oil. The subsequent terminal velocities along the axis of rotation were measured. It has been found that the rise velocity decreases from the Stokes velocity as the angular speed of rotation is increased. The governing equations show that this is due to the Coriolis acceleration.

INTRODUCTION

THE motion of solid bodies in rotating, inviscid and incompressible fluids was studied by Taylor,[1] Proudman[2] and Grace,[3] who published several papers during the period 1915–1926. Since then many other researchers have made contributions to this field. In contrast, comparatively little work in this field has taken the effects of viscosity into account. It seems desirable, therefore, to investigate the motion of solid bodies in a rotating viscous fluid. A logical beginning to such an investigation is a study of the motion of a symmetrical body, such as a sphere, at low Reynolds numbers. Proudman and Taylor predicted by theory and later Taylor confirmed by experiment that the slow motion of a sphere along the axis of a rotating inviscid fluid is two dimensional. That is, the velocity components are independent of the coordinate z measured along the axis of rotation. Just what effects does viscosity have on this motion? From a different point of view, what effect will rotation have on the terminal velocity of a sphere in the Stokes range? These questions have motivated this study.

It has been found that viscosity acts to destroy the two-dimensionality predicted by Taylor and Proudman for weak steady motion of an inviscid fluid under rotation, and that the effect of rotation is to reduce the speed of fall or rise of the sphere in the Stokes range. Specific data are presented in the results section of this paper. It has also been found that, in general, spheres which are less dense than the surrounding liquid move to the axis of rotation and that spheres more dense than the liquid spiral out from the axis of rotation.

THE GOVERNING EQUATIONS

The problem considered here is the motion of a sphere along the axis of an infinitely long cylinder which is filled with a viscous liquid and rotating about its axis. It is convenient to use a Cartesian reference frame which rotates at the same angular speed ω as the cylinder and which has its origin at the center of the sphere. The coordinates with respect to this frame are denoted by x, y, z, with the z axis coinciding with the axis of the cylinder. The coordinates with respect to the inertial frame of reference with origin at the center of the sphere are denoted by X, Y, and Z, as shown in Fig. 1. The components of the velocity of the fluid relative to the rotating frame are denoted by u, v, and w. At a time that the coordinates x, y, and z momentarily coincide with

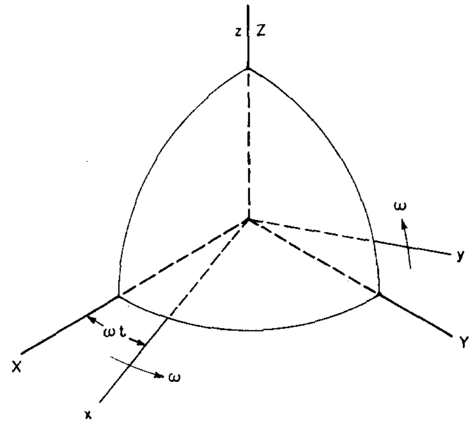

FIG. 1. Reference frames.

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[1] G. I. Taylor, Proc. Roy. Soc. (London) **A93**, 99 (1917); Proc. Cambridge Phil. Soc. **20**, 326 (1921); Proc. Roy. Soc. (London) **A100**, 114 (1922), **A102**, 180 (1922), **A104**, 213 (1923); *Proceedings of the First International Congress on Applied Mechanics*, edited by C. B. Biezeno and J. M. Burgers (Waltman, Delft, The Netherlands, 1924), pp. 89–96.

[2] J. Proudman, Proc. Roy. Soc. (London) **A92**, 408 (1916).

[3] S. F. Grace, Proc. Roy. Soc. (London) **A102**, 89 (1922), **A104**, 278 (1923), **A105**, 532 (1924), **A113**, 46 (1926).

X, Y, and Z, the velocity components in the inertial frame of reference are

$$U = u - \omega y, \tag{1}$$

$$V = v + \omega x, \tag{2}$$

and

$$W = w. \tag{3}$$

For a flow which is independent of time, the choice of the origin of time at which the rotating reference frame coincides with the inertial reference frame is immaterial.

The differential system for the fluid medium to be satisfied between the boundaries are the Navier–Stokes equations along with the continuity equation. Substitution of (1), (2), and (3) into these equations gives[4]

$$\frac{Du}{Dt} - 2\omega v - \omega^2 x = -\frac{1}{\rho_1}\frac{\partial p}{\partial x} + F_x + \nu\nabla^2 u, \tag{4}$$

$$\frac{Dv}{Dt} + 2\omega u - \omega^2 y = -\frac{1}{\rho_1}\frac{\partial p}{\partial y} + F_y + \nu\nabla^2 v, \tag{5}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_1}\frac{\partial p}{\partial z} + F_z + \nu\nabla^2 w, \tag{6}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{7}$$

where D/Dt and ∇^2 represent the substantial derivative and the Laplacian operator, respectively, with respect to the rotating reference, F represents the body forces per unit mass, ν and ρ_1 are the kinematic viscosity and mass density of the liquid, respectively.

The size and density of the sphere and the viscosity and density of the liquid may all be chosen such that the translational velocity (W_ω) of the sphere along the z axis will be small. It is assumed here that the Reynolds number based on W_ω and the sphere diameter, d, is less than unity. Hence u, v, and w are small compared with ν/d for all values of ω. Further, we consider here only the motion after terminal velocity of the sphere has been reached. The experiments show that the sphere reaches a terminal velocity rapidly. The unsteady-flow problem may now be looked upon as a steady-flow problem by considering the center of the sphere as fixed, with a uniform velocity equal and opposite to W_ω superposed on the rotating liquid and cylinder. With these assumptions, one can neglect the substantial derivatives of the velocity components in Eqs. (4)–(6). The only body force present in this treatment is that due to gravity. Therefore, if

$$p_d = p - p_s,$$

in which p_s is the hydrostatic part of the pressure and if

$$P \equiv \frac{p_d}{\rho_1} - \frac{\omega^2}{2}(x^2 + y^2), \tag{8}$$

the differential equations (4) through (6) reduce to

$$-2\omega v = -\partial P/\partial x + \nu\nabla^2 u, \tag{9}$$

$$2\omega u = -\partial P/\partial y + \nu\nabla^2 v, \tag{10}$$

$$0 = -\partial P/dz + \nu\nabla^2 w. \tag{11}$$

Equations (7), and (9)–(11) along with the appropriate boundary conditions constitute the boundary-value problem which governs the flow of fluid between the boundaries of the sphere and the cylinder. Although these differential equations have not yet been integrated, much insight to the problem can be obtained by examining them. Once u, v, w, and P are determined, the pressure and shear drag on the sphere surface may be obtained by integration.

It is worthwhile here to point out the similarities of this problem for the special case of a cylinder with infinite radius to the classical problem solved by Stokes for the slow motion of a sphere in a nonrotating infinite medium. If the Coriolis acceleration components are neglected in Eqs. (9) and (10), the system reduces to a set of equations quite similar to those of Stokes with the only difference being that p in the Stokes problem is now replaced by P and the velocity components here are relative velocities whereas they represent absolute values in the Stokes problem. Integration of the resulting pressure and shear stresses over the surface of the sphere gives results identical to Stokes' for the pressure drag and shear drag. Thus it is the effect of the Coriolis acceleration that leads to a sphere terminal velocity which is less than the Stokes velocity. The physical explanation of this phenomenon can be seen by considering a single fluid particle which is attempting to flow radially outward in order to pass down and around the sphere. The particle is not as free to flow out radially as it is in Stokes flow due to the force associated with the Coriolis acceleration resulting from the relative swirl velocity.

The angular rotation of the sphere is governed by Euler's equation of motion for a rigid body. For the steady-state case under consideration here, the sphere rotates at a constant angular velocity. Thus the angular acceleration and hence the torque on the sphere are zero. It has been verified experimentally by the writer (for angular speeds ranging from 60 to 400 rpm) that for translational velocities along the

[4] H. Lamb, *Hydrodynamics* (Dover Publications, Inc., New York, 1945), pp. 318 and 722.

axis within the Stokes range the sphere rotates at the same angular speed as the cylinder. This was accomplished by means of a Strobotac light. However, this is true only for very low values of W_ω, and at any rate one should not have to rely upon this experimental fact in the mathematical formulation of the problem. The speed of rotation of the sphere is determined by the condition that the torque on it should be zero. At higher Reynolds numbers, the spread of the center streamline over the sphere and the motion of the fluid adjacent to this streamline is likely to make the sphere rotate less fast than the ambient fluid.

The governing system may be further reduced by combining (7), (9)–(11) to obtain a single equation in terms of the dependent variable w, i.e.,

$$\nu^2\nabla^2\nabla^2\nabla^2 w + 4\omega^2\,\partial^2 w/\partial z^2 = 0. \tag{12}$$

This equation is difficult to integrate unless one resorts to the very tedious method of relaxation.

DIMENSIONAL ANALYSIS

By using the sphere diameter d as a reference length and ωd as a reference velocity, the nondimensional form of Eq. (12) is

$$\nabla^{*2}\nabla^{*2}\nabla^{*2}w^* + 4\left(\frac{\omega\, d^2\rho_l}{\mu}\right)^2 \frac{\partial^2 w^*}{\partial z^{*2}} = 0, \tag{13}$$

where the asterisks indicate dimensionless quantities and μ is the liquid viscosity.

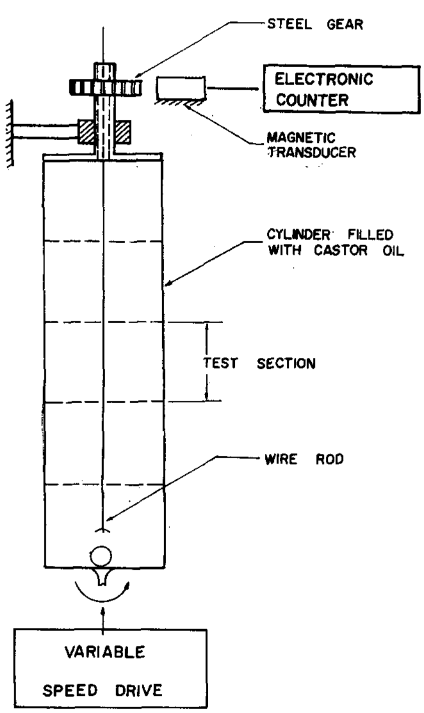

FIG. 2. Equipment diagram.

Newton's second law applied to the sphere shows that the drag is constant. By using $\rho_l(\omega d)^2 d^2$ to nondimensionalize the drag, one obtains

$$(\gamma_l - \gamma_s)/\rho_l\omega^2\, d = \text{const}, \tag{14}$$

where γ_l and γ_s represent the specific weight of the liquid and sphere, respectively.

The boundaries of the fluid medium are described by the sphere diameter d and the cylinder diameter D. Also, the velocity at infinity in the z direction must be equal to the terminal velocity of the sphere W_ω. These conditions introduce two more dimensionless parameters, i.e., d/D and $W_\omega/\omega d$.

The dependent variable and the variable to be measured experimentally is the terminal velocity of the sphere along the axis of rotation. The functional equation is

$$W_\omega = f(d, D, \omega, \gamma_l - \gamma_s, \mu, \rho_l). \tag{15}$$

There are several sets of nondimensional ratios (π-terms) which the variables in Eq. (15) may be arranged but it is desired here to have W_ω and ω appear in the first power in only one π-term each so that the data can be advantageously presented. Using d, $\gamma_l - \gamma_s$, and μ as repeating variables gives

$$\pi_1 = \frac{W_\omega}{d^2}\left(\frac{\mu}{\gamma_l - \gamma_s}\right), \qquad \pi_2 = \frac{\omega}{d}\left(\frac{\mu}{\gamma_l - \gamma_s}\right), \qquad \pi_3 = \frac{d}{D}, \qquad \pi_4 = \frac{(\gamma_l - \gamma_s)}{\rho_l}\frac{d^3}{\nu^2}, \tag{16}$$

for the four π-terms according to the Buckingham π-theorem. Note that these π-terms are merely combinations of the nondimensional parameters obtained from the governing equations and the boundary conditions. Equation (15) now reduces to

$$W_\omega = d^2\,\frac{(\gamma_l - \gamma_s)}{\mu}\, f\left[\frac{d}{D}, \frac{\omega}{d}\left(\frac{\mu}{\gamma_l - \gamma_s}\right), \left(\frac{\gamma_l - \gamma_s}{\rho_l}\right)\frac{d^3}{\nu^2}\right]. \tag{17}$$

Equation (17) furnishes the guide for experimentation. For a particular sphere, cylinder and liquid, π_3 and π_4 are constant if the temperature is held constant, π_2 can be controlled by controlling the angular speed of the cylinder and π_1 can be measured so that the results can then be presented by two-dimensional plots.

EXPERIMENTAL EQUIPMENT AND PROCEDURE

The apparatus consisted mainly of a clear plastic cylinder 30 in. long with a 5-in. i.d. filled with castor oil and mounted on an adjustable-speed turntable as shown schematically in Fig. 2. The speed

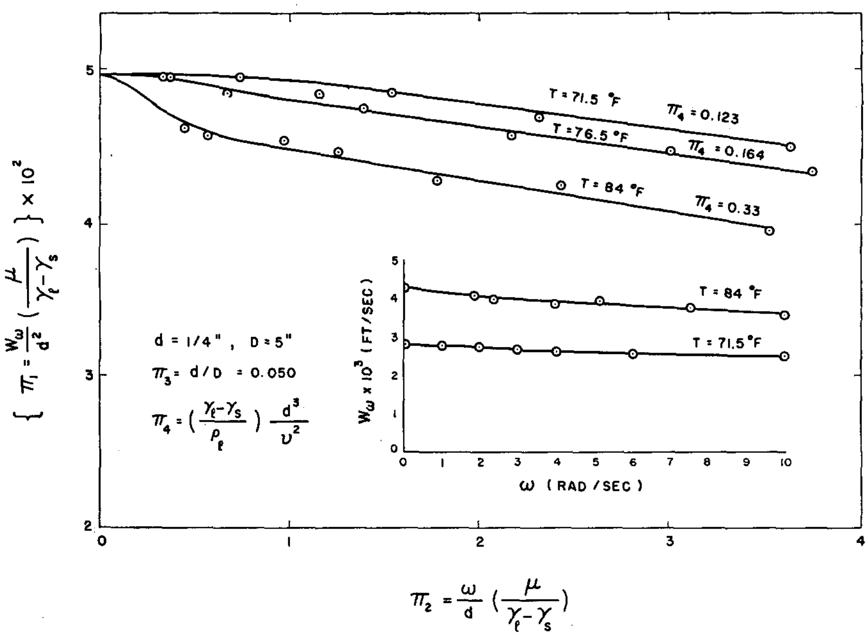

FIG. 3. Curves of constant π_4 for $\pi_3 = 0.050$.

of rotation was determined by use of an electronic counter and magnetic transducer. The transducer was mounted close to a steel gear which turned with the cylinder. The gear teeth actuated the transducer which in turn supplied the counter with an input signal.

Castor oil was found desirable for these experiments because its viscosity is high and its change of viscosity with temperature is relatively large in comparison with other common oils and liquids. The high change in viscosity with temperature made it easy to vary the viscosity simply by varying the ambient temperature. The variation of the physical properties with temperature for the U.S.P.-grade castor oil used is shown in the Appendix.[5]

For motion along the axis, it was necessary to use spheres which were slightly less dense than the castor oil so that they would remain on the axis and the motion would be small enough for the Reynolds number to be less than unity. Polyethylene spheres (specific gravity $\cong$ 0.92) were found satisfactory for this purpose. The sphere was first pushed to the bottom center of the cylinder with a wire rod. The cylinder of liquid was set into rotation at the desired speed and a few seconds were allowed for the castor oil to attain solid body rotation. The rod was then pulled out setting the sphere free to move along the axis. By the time the sphere reached the test section, the distrubance created by the wire rod had decayed and the sphere moved at essentially the terminal speed. The test section was one half foot long and was located in the central portion of the cylinder in order to minimize end effects. Since the motion was very slow, the time for the sphere to travel one-half foot along the axis was easily measured with a stopwatch. Data could be taken directly above and below the main test section to verify that the sphere was not accelerating.

Starting the cylinder from rest to the desired speed of rotation created shear stresses in the liquid until it reached a solid body rotation. Shear stresses were again created while bringing the liquid back to rest after the data were taken for a particular value of ω. These shear stresses created small amounts of heat which affected the viscosity to some degree. Consequently, it was desirable to measure the viscosity between each data point taken. This was done when the liquid was quiescent by checking the rise velocity W_0 of the same sphere being used for the experiment and by using the well-known Stokes relation for an infinite fluid

$$\mu = d^2(\gamma_l - \gamma_s)/18W_0, \tag{18}$$

along with the Francis equation[6] for wall-effect correction

$$\frac{W_0}{W_\infty} = \left\{\frac{1 - d/D}{1 - 0.475\, d/D}\right\}^4 \qquad (d/D < 0.9). \tag{19}$$

[5] This information was kindly supplied to the writer by the manufacturer, The Baker Castor Oil Company.

[6] V. Fidleris and R. L. Whitmore, Brit. J. Appl. Phys. **12**, 490 (1961).

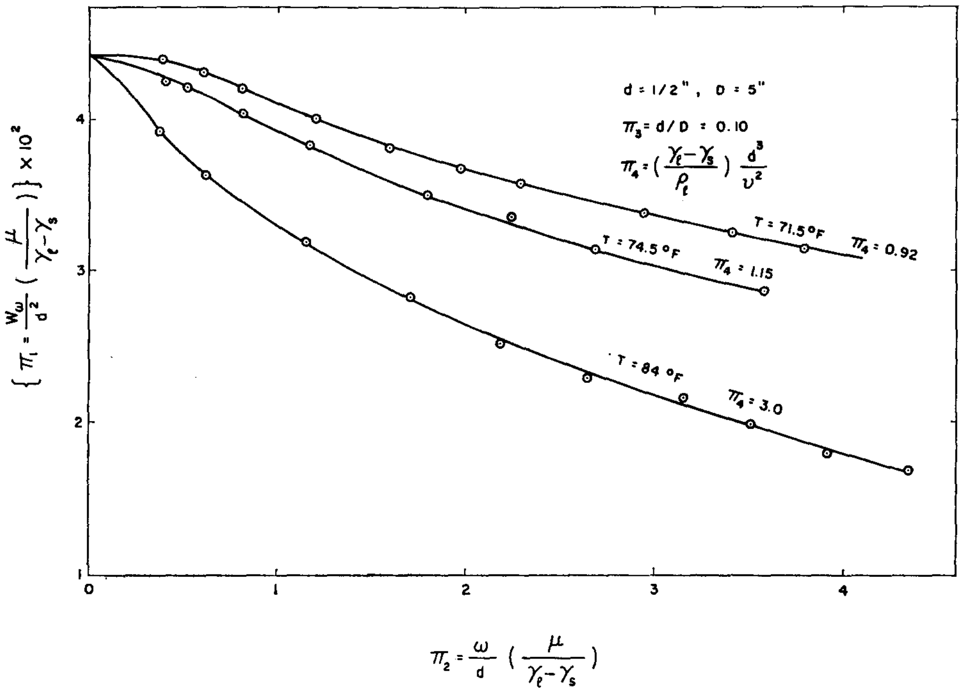

FIG. 4. Curves of constant π_4 for $\pi_3 = 0.100$.

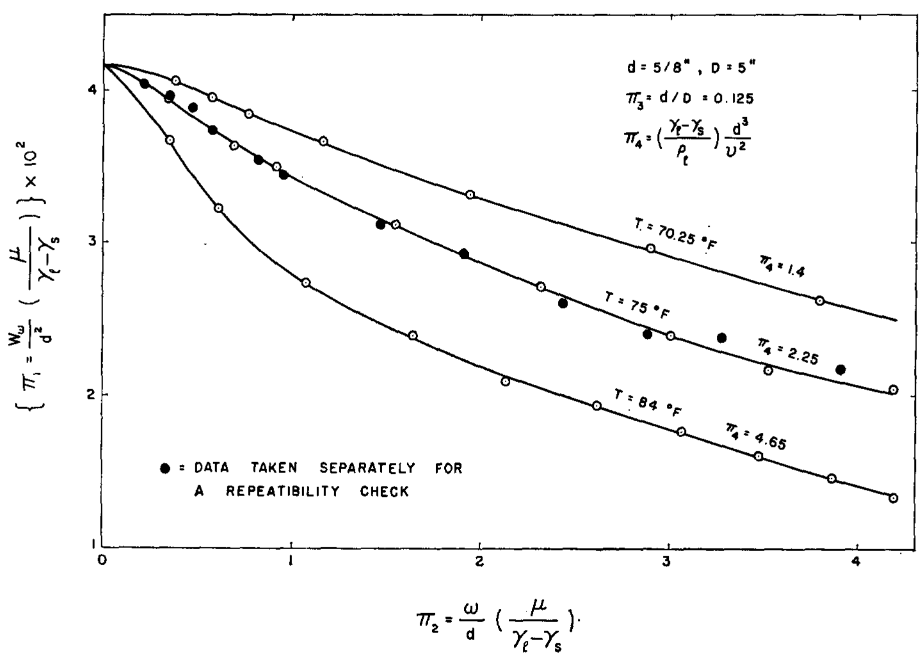

FIG. 5. Curves of constant π_4 for $\pi_3 = 0.125$.

The nondimensional ratios involving viscosity are then

$$\pi_1 = \frac{W_\omega}{18W_0}\left\{\frac{1 - d/D}{1 - 0.475\, d/D}\right\}^4, \tag{20}$$

$$\pi_2 = \frac{\omega\, d}{18W_0}\left\{\frac{1 - d/D}{1 - 0.475\, d/D}\right\}^4, \tag{21}$$

and

$$\pi_4 = \frac{(18W_0)^2}{dg(1 - \gamma_s/\gamma_l)}\left\{\frac{1 - 0.475\, d/D}{1 - d/D}\right\}^8. \tag{22}$$

RESULTS

Figures 3–7 represent the data taken for five spheres ranging in size from $\frac{1}{4}$ to $1\frac{1}{8}$ in. in diameter.

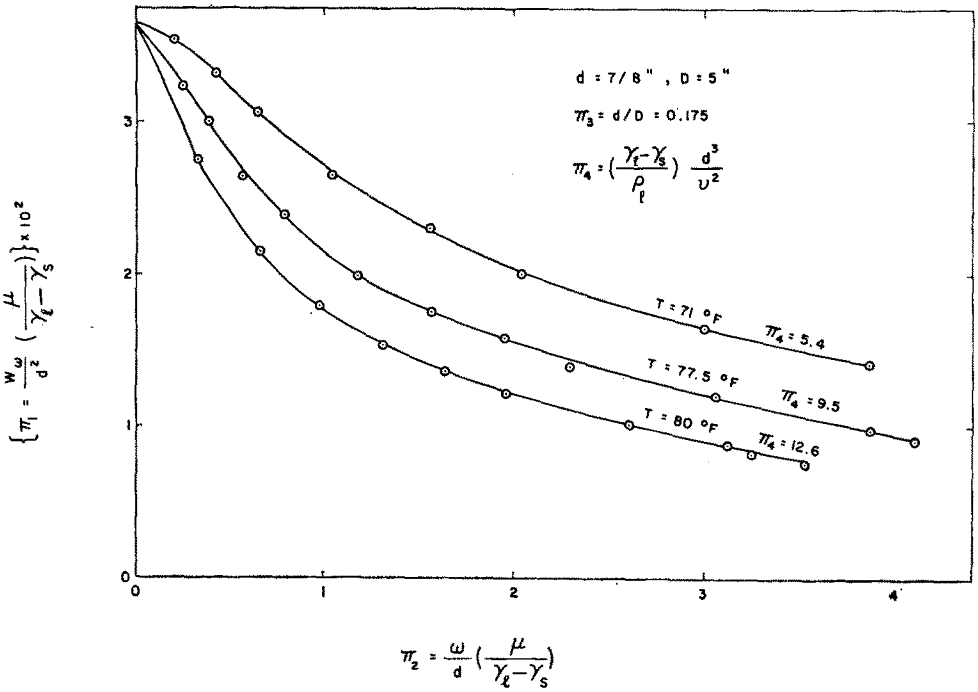

FIG. 6. Curves of constant π_4 for $\pi_3 = 0.175$.

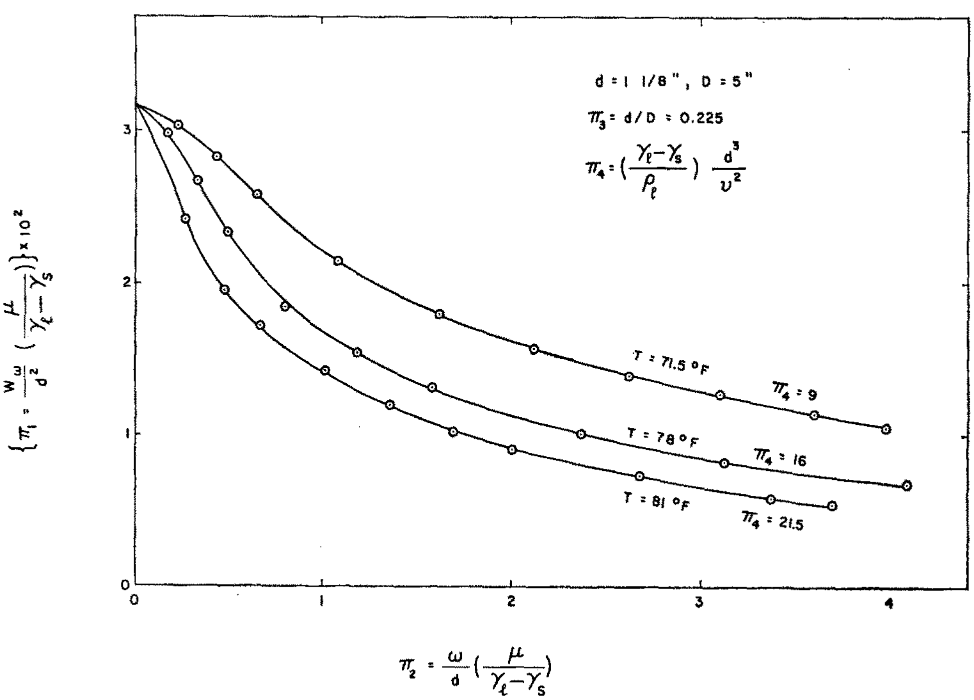

FIG. 7. Curves of constant π_4 for $\pi_3 = 0.225$.

The range of π_2 for each sphere is from 0 to 4. The data for each sphere are shown in a plot of π_1 vs π_2 for three constant values of π_4. π_4 was varied by varying the temperature of the castor oil.

Figure 8 represents the data taken for eight values of π_3 when π_4/d^3 was held approximately constant. The relations given in the Appendix for γ_1 and ρ_1, with a quadratic approximation for the data given

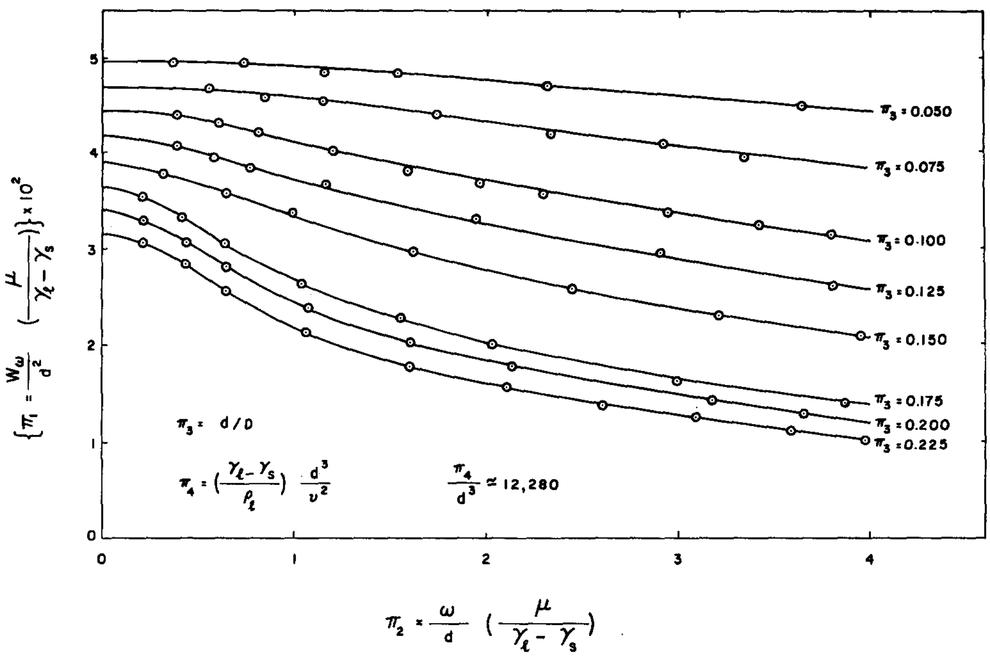

Fig. 8. Curves of constant π_3 for $\pi_4/d^3 \cong 12{,}280$.

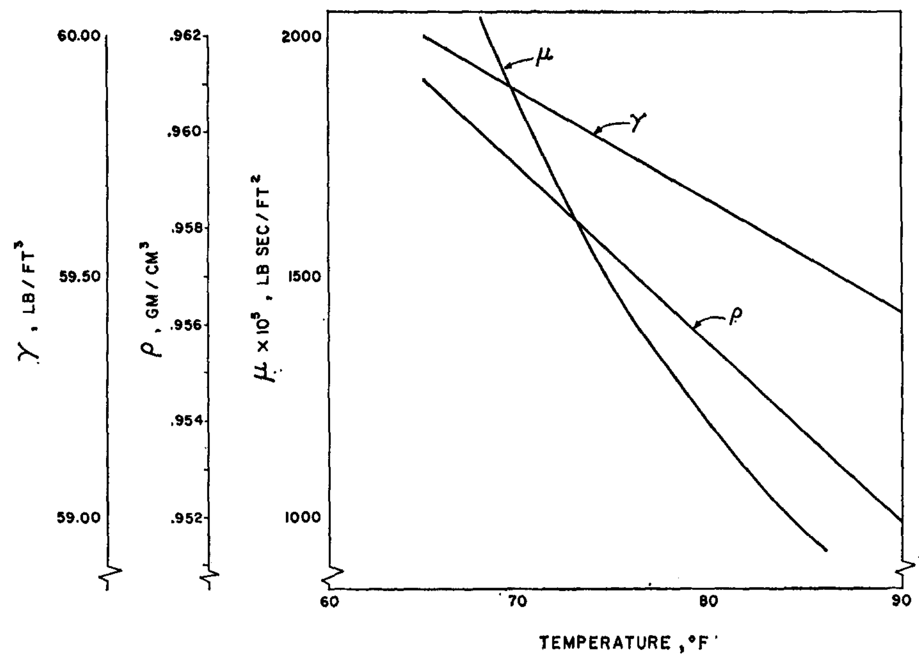

Fig. 9. Plot of the physical properties of castor oil vs temperature.

for μ in the Appendix (Fig. 9), give

$$\frac{\pi_4}{d^3} = \frac{(3.739 - 2.289 \times 10^2\, T)(1.911 - 7.113 \times 10^{-4}\, T)}{(160.494 \times 10^{-5}\, T^2 - 308.609 \times 10^{-5}\, T + 15{,}602.183 \times 10^{-5})^2} \qquad (68°\text{F} \leq T \leq 86°\text{F}), \qquad (23)$$

where T represents the temperature in degrees Fahrenheit. Equation 23 shows that π_4/d^3 is a function of temperature only. The oil temperature, which depended upon the ambient temperature of the laboratory at the time of a test, varied from a minimum of 70.25° F for $\pi_3 = 0.125$ to a maximum of 71.5° F for the values of π_3 of 0.225, 0.200, and 0.100, with the average temperature for the eight values being 71.3° F. Hence π_4/d^3

was held approximately constant, as stated earlier. The value of π_4 for five of the runs shown in Fig. 8 may be found on the corresponding curve in Figs. 3–7 which represent the same data.

The writer has taken data similar to that represented in Figs. 3–7 for eight different sphere diameters which amounts to twenty four runs. In thirteen of these runs the viscosity decrease from beginning of the run to the end (due to heat created by starting and stopping the rotation of the fluid) was less than 5%, less than 10% in eight of the runs and less than 15% in three of the runs. An average viscosity was used for each run to calculate π_4. π_4 was also calculated by use of the data in the Appendix and the mean measured temperature for the run.

The maximum Reynolds number, based on W_0 and the sphere diameter, was less than 0.7 for all the data taken.

The run for $d = \frac{5}{8}$ in. and $T = 75°$ F was chosen at random for a repeatability check. The data acquired from both runs lie on the same curve as shown in Fig. 5.

DISCUSSION OF RESULTS

The plots of the data shown in Figs. 4–12 verify Eq. (17) which was derived by dimensional analysis.

For the special case of a cylinder with infinite radius, Eq. (17) holds if $\pi_3 = d/D$ is deleted. To discuss this special case, we look at Fig. 3 which represents the data with the smallest amount of wall effect assuming that the wall effect is a function of geometry only ($W_0/W_\infty = 0.895$ here for Stokes flow). The velocity is only slightly decreased by rotation for $\omega < 1$. This is as expected since the Coriolis acceleration is very small and the problem is little different from the Stokes problem. The data here show that the velocity decreases from the Stokes velocity, as ω is increased, more rapidly as the rotating liquid becomes less viscous. This suggests that as the viscosity of the liquid is decreased the amount of liquid affected or moved along with the sphere is increased with the limiting case being a cylindrical column as first found by Taylor. The writer performed an experiment with a slowly moving sphere ($W_\omega \approx 0.04$ ft/sec, $\omega \approx 150$ rpm) in rotating water and, with the use of potassium permaganate as a dye, observed the column of fluid that the sphere pushed along. The Reynolds number for this experiment was approximately 300. A similar experiment with castor oil is difficult. The writer, however, by injecting a mixture of castor oil and linseed oil, which had been dyed black, along the axis of rotation, did observe that some fluid near the sphere was pushed along with the sphere but the results were inconclusive as to the amount and exact shape of this fluid.

CONCLUSIONS

The following conclusions can be drawn for the slow steady motion of a sphere along the axis of a rotating viscous liquid: (a) If the sphere is less dense than the liquid and is released on the axis of rotation, the free motion will remain along the axis of rotation. (b) The sphere rotates at very much the same angular velocity as the cylinder. (c) The velocity of the sphere in a liquid with constant temperature decreases from the Stokes velocity as the angular rotation is increased. Specific results are given in Figs. 3–8. (d) The rate of decrease of velocity, as the angular rotation is increased, increases as the viscosity of the liquid medium is decreased. Thus, the effect of viscosity is to decrease the amount of fluid affected or moved along with the sphere. (e) From the equations of motion, it is concluded that it is the effect of the Coriolis acceleration that leads to a terminal velocity less than the Stokes velocity for the sphere provided that the assumed slow motion is attained. (f) The velocity of the sphere along the axis is given by

$$W_\omega = d^2 \frac{(\gamma_l - \gamma_s)}{\mu} f\left[\frac{d}{D}, \frac{\omega}{d}\left(\frac{\mu}{\gamma_l - \gamma_s}\right), \left(\frac{\gamma_l - \gamma_s}{\rho_l}\right)\frac{d^3}{\nu^2}\right]. \tag{17}$$

The function is given graphically in Figs. 3–8.

APPENDIX

The coefficient of expansion of castor oil in terms of density is 0.00066. Also, for the range of temperatures used herein,

$$\rho = 0.9849 - 3.667 \times 10^{-4}\, T \text{ gm/cm}^3,$$

$$\rho = 1.9107 - 7.113 \times 10^{-4}\, T \text{ slug/ft}^3,$$

and

$$\gamma = 61.4873 - 2.2891 \times 10^{-2}\, T \text{ lb/ft}^3,$$

where T is measured in degrees Fahrenheit.

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